Monograph:
Li, L.M. and Billings, S.A. (2006) Volterra Series and Reduction in the Frequency Domain for Weakly Nonlinear System. Research Report. ACSE Research Report 939 . Department of Automatic Control and Systems Engineering

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L.M Li and S.A.Billings

Department of Automatic Control
and Systems Engineering,
University of Sheffield, Sheffield
Post Box 600 S1 3JD
UK

Research Report No. 939
August 2006

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L.M.Li and S.A.Billings*
Department of Automatic Control and Systems Engineering
University of Sheffield
Sheffield S1 3JD
UK
* S.Billings@sheffield.ac.uk

Abstract: New results for model order reduction, for weakly nonlinear systems in the frequency domain, are derived based on a parametric modelling approach.

1. Introduction

Model order reduction has long been a topic in linear system theory. It is desirable in many control and analysis problems. The basic idea of linear model reduction is to reduce the dynamic order of linear systems subject to certain performance criteria, so that the resulting reduced models have a similar behaviour to the original model under certain operating conditions. The performance criteria used are generally purely in the time domain.

Nonlinear systems have much more diverse features than linear systems. For example, harmonic distortion, hysteresis, limit cycles, bifurcations and chaos, just to name a few. There is no universal theory for all types of nonlinearities. One classification of nonlinear systems is weakly (mildly) and severely nonlinear systems. The most significant feature separating these classes is that the former systems can be represented by Volterra series models, which are the topic of the current study.

Nonlinear system modelling with Volterra series was first proposed in the 1930s and was enhanced by Wiener's contribution to nonlinear system analysis. From the late 1950s, there has been a continuous effort in the application of Volterra series to nonlinear systems theory. Summaries of major contributions in the application of Volterra series modelling for the representation, analysis and design of nonlinear systems can be found in Schetzen(1980), Rugh(1981) and Sandberg(1984).

A big advantage of the Volterra based representations is that they can be readily transformed into the frequency domain using Generalised Frequency Response Functions (GFRF's). The inherent features of the underlying nonlinear systems can then be studied using the GFRF's(Bedrosian and Rice, 1971; Bussgang, et. al., 1974; Lang and Billings, 2000), and this provides an analogous theory to linear frequency response analysis, which is so important for linear systems. Many nonlinear phenomena have been analysed and interpreted in terms of the GFRF's, including gain compression, intermodulation effects, harmonics and desensitisation.

The GFRF's are of practical use only when the representation and analysis of the underlying system can be done using a finite number and order of frequency functions. This is called Volterra/frequency truncation. Billings and Lang(1997) studied the order and terms of the Volterra series expansion in such a truncation in the frequency domain.

In this paper, for the first time, the problem of model order reduction for weakly nonlinear systems in the Volterra/frequency domain is addressed. Unlike the linear model order reduction problem in which the term 'order' refers to the order of the dynamics, here in the nonlinear Volterra/frequency model order reduction, the 'order' of a frequency domain Volterra expansion refers to the order of the GFRF's.

2. Volterra series in the time and the frequency domain

Volterra series modelling (Volterra, 1930) has been widely studied for the representation, analysis and design of nonlinear systems. The Volterra series is a nonlinear functional series that can be expanded as a polynomial functional series and is a direct generalisation of the linear convolution integral, therefore providing an intuitive representation in a simple and easy to apply way. For a SISO nonlinear system, with $u(t)$ and $y(t)$ the input and output respectively, the Volterra series can be expressed as

$$y(t)=\sum_{n=1}^{\infty} y_n(t) \qquad (1.a)$$

where $y_n(t)$ is the 'n-th order output' of the system

$$y_n(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)d\tau_i \qquad n>0 \qquad (1.b)$$

$h_n(\tau_1,\cdots,\tau_n)$ is called the 'nth-order Kernel' or 'nth-order impulse response function'. If $n=1$, this reduces to the familiar linear convolution integral.

The discrete time domain counterpart of the continuous time domain SISO Volterra expression (1) is

$$y(k)=\sum_{n=1}^{\infty} y_n(k) \qquad (2.a)$$

where

$$y_n(k)=\sum_{-\infty}^{\infty}\cdots\sum_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(k-\tau_i) \qquad n>0, k\in \mathrm{Z} \qquad (2.b)$$

In practical problems only a finite Volterra series can be used, on the assumption that the contribution of the higher order kernels falls off rapidly. This is called the truncated Volterra series. Systems that can be adequately represented by a truncated Volterra series with just a few terms are called weakly or mildly nonlinear systems.

For discrete-time systems the truncated, discrete-time Volterra series is given as

$$y_n(k)=\sum_{n=1}^{K}\sum_{\tau_n=0}^{k}\cdots\sum_{\tau_1=0}^{k} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(k-\tau_i) \qquad n>0, k\in \mathrm{Z} \qquad (3)$$

A discrete time Volterra series is also called a NX (Nonlinear model with eXogenous inputs) model.

The multi-dimensional Fourier transform of $h_n(\cdot)$ yields the 'nth-order frequency response function' or the Generalised Frequency Response Function (GFRF):

$$H_n(\omega_1,\cdots,\omega_n) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\exp(-j(\omega_1\tau_1+\cdots+\omega_n\tau_n))d\tau_1\cdots d\tau_n \quad (4)$$

The generalised frequency response functions represent an inherent and invariant property of the underlying system, and have proved to be an important analysis and design tool for characterising nonlinear phenomena. In practice, the GFRF's can be estimated using non-parametric or parametric methods. The parametric method involves mapping a nonlinear differential equation(Billings and Peyton Jones, 1990) or mapping a nonlinear difference equation(Peyton Jones and Billings, 1989) into the frequency domain using the probing method. The steady-state response of a mildly nonlinear system subject to sinusoidal inputs can be computed using the GFRF's ($H_n^{'s}(\cdot)$) as

$$\begin{aligned} y(t) &= \sum_{n=1}^{\infty} y_n(t) \\ &= \sum_{n=1}^{\infty}\frac{1}{(2\pi)^n}\int\cdots\int H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)e^{j\sum_{i=1}^{n}\omega_i t}d\omega_1\cdots d\omega_n \end{aligned} \quad (5)$$

Like time-domain Volterra series truncation, the generalised frequency response functions are of practical use when the characteristics of the underlying nonlinear system can be efficiently explained by the first few orders of $H_n^{'s}(\cdot)$. But in many nonlinear systems it is very common that higher order GFRF's are necessary in order to obtain a satisfactory truncation error, especially around the resonance frequencies.. This sometimes makes the application of the GFRF's quite complicated, tedious and computationally demanding. In this study a new method is presented to address the problem of order reduction of the Volterra/frequency domain representation. The method will be described using the example of a Duffing oscillator to demonstrate the approach in the simplest manner.

3. Volterra series reduction in the frequency domain

The example which will be analysed in detail in this section is the Duffing oscillator, which is described by

$$\ddot{y} + 0.2\dot{y} + y + 0.5y^3 = A\cos(\omega t) \quad (6)$$

where the working amplitude of the driving input was A=0.12. The Duffing equation (6) has a resonant frequency at around $\omega = 1\, rad/\sec$. The approach adopted for this specific example can be repeated for other model forms. Focusing on one specific example illustrates the idea in a much more transparent way than if the method was introduced for a general class of models.

The first step in analysing a nonlinear system behaviour in the frequency domain can be taken by plotting the Response Spectrum Map(RSM), developed by Billings and Boaghe(2001). This is done by showing the FFT of the response with a varying system parameter(for example, driving input amplitude or frequency). Here the RSM of system (6) against varying frequency ω is shown in Figure 1.

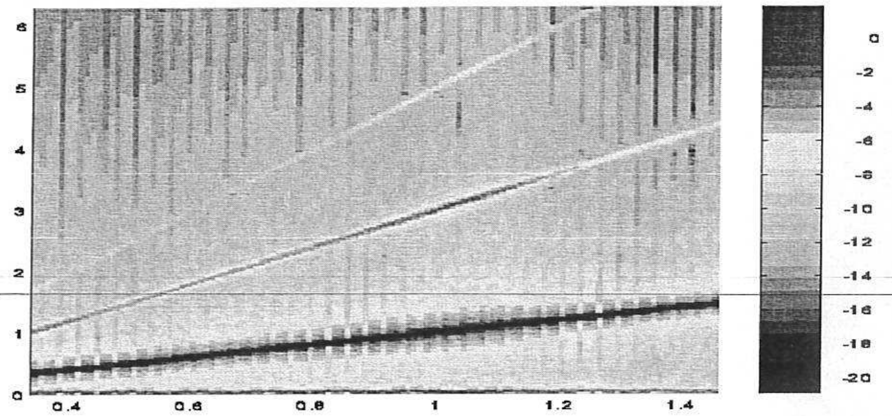

Figure 1. Response Spectrum Map of Duffing Oscillator (6)

Figure 1 shows that for the input frequency range 0.34-1.46 rad/sec, only fundamental and odd order harmonic components are present in the response, suggesting that a Volterra representations over this whole frequency range exists.

Because the Duffing equation (6) contains a cubic nonlinear term y^3, the number of Volterra series terms in (1) or the number of the GFRF's in (5) may become infinite. Also the even orders of GFRF's are zero and make no contribution to the system response. Therefore the steady-state response using equation (5) with nonlinearities up to the 5th order truncation is

$$y(t) \approx \sum_{n=1}^{5} y_n(t) = y_1(t) + y_3(t) + y_5(t) \tag{7}$$

where the response for the various orders are

$$y_1(t) = 2(\frac{A}{2})\mathrm{Re}\{H_1(j\omega)e^{j\omega t}\} \tag{8.a}$$

$$y_3(t) = 2(\frac{A}{2})^3\mathrm{Re}\{H_3(j\omega,j\omega,j\omega)e^{3j\omega t}\} + 6(\frac{A}{2})^3\mathrm{Re}\{H_3(j\omega,j\omega,-j\omega)e^{j\omega t}\} \tag{8.b}$$

$$y_5(t) = 2(\frac{A}{2})^5\mathrm{Re}\{H_5(j\omega,j\omega,j\omega,j\omega,j\omega)e^{5j\omega t}\} + 10(\frac{A}{2})^5\mathrm{Re}\{H_5(j\omega,j\omega,j\omega,j\omega,-j\omega)e^{3j\omega t}\}$$

$$+ 20(\frac{A}{2})^5\mathrm{Re}\{H_5(j\omega,j\omega,j\omega,-j\omega,-j\omega)e^{j\omega t}\} \tag{8.c}$$

However, the orders of the GFRF's needed for a required truncation error vary for different frequency ranges. To illustrate this, (6) was simulated at $T = 60/\pi$ and the response $y(t)$ was compared for different levels of approximation using (7) and (8) in terms of the truncation error e_i, defined by $e_i = (y - \sum_{j=1}^{i(odd)} y_j)/\max(y)$ as a percentage. The truncation errors for different input frequency values are presented in Table 1,

ω	e_1	e_3	e_5
$\omega = 0.2$	0.8929	0.0286	0.0021
$\omega = 0.6$	1.8168	0.0920	0.0148
$\omega = 0.9$	22.2387	13.8128	10.7461
$\omega = 1.0$	27.4282	12.5687	6.1998
$\omega = 1.3$	1.3590	0.0249	0.0373

Table 1. Truncation Error $e_i[\%]$ for Duffing Equation (5) at Different Frequencies

The Volterra representation of the Duffing equation (5) is convergent, as shown in Table 1. It is shown that in the lower frequency range (for example, $\omega \in [0\ 0.6]$ rad/sec), and the higher frequency range (for example, $\omega > 1.3$ rad/sec), the truncation errors fall off rapidly, with the 3rd order GFRF's being sufficient to describe the corresponding responses. While for $\omega = 0.9$, which is around the resonant frequency, the truncation error is relatively high, and a much higher order of GFRF's will be required to obtain a satisfactory truncation error. These observations agree well with the RSM in Figure 1, where the response has a significant 5th order harmonic presence around the resonant frequency, while at other frequency ranges these effects are quite weak with negligible 5th order harmonic contributions.

The Duffing system (6) was then excited at different frequencies and the excitation-response pairs were collected accordingly. NX models can be built based on these excitation-response data sets at each frequency. For example,

for ω=0.6 the corresponding NX model is

$$y(k) = 4.1945\,u(k-4) - 2.6988\,u(k-3) + 0.95122u(k-4)u(k-4)u(k-4) - 0.18801\,u(k-3)u(k-4)u(k-4) \tag{9}$$

and for ω=0.9

$$y(k) = 17.551\,u(k-4) - 14.672\,u(k-2) + 57.086u(k-4)u(k-4)u(k-4) - 56.217\,u(k-3)u(k-4)u(k-4) \tag{10}$$

By repeating the above NX modelling procedure throughout the whole frequency range a series of models can be obtained. These NX models can then be mapped into the frequency domain and the GFRF's up to third order can be derived.

By collecting the H_1 data at each frequency point from each corresponding NX model in the above step, an $H_1(\omega)$ over the whole frequency range can be constructed, as shown in Figure 2. The H_1 from the original Duffing equation (6) is also plotted in Figure 2. From Figure 2 it can be seen that the new H_1 constructed from the NX modelling shares similar frequency features as the original Duffing equation at low and high frequency ranges. This reflects the fact as explained before that in the low and high frequency ranges the Duffing equation already provides excellent Volterra/frequency truncation.

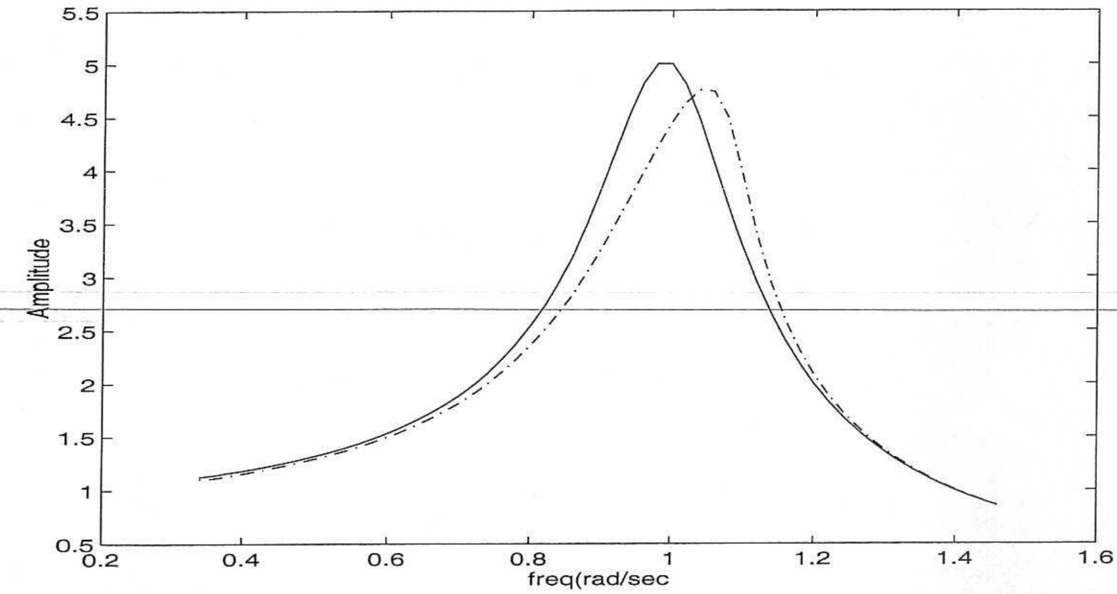

Figure 2. H_1 from original Duffing equation (6) –Solid and H_1 from NX modelling—Dot Dashed

The $H_3(\cdot)$ data at each frequency points from each corresponding NX model can also be collected. Once the $H_1(\cdot)$ and $H_3(\cdot)$ data sets are obtained, the approach of re-constructing a nonlinear continuous time model proposed by Li and Billings(2001) can be applied. An important advantage of Li and Billings's algorithm is that the non-linear model can be constructed sequentially by building in the linear model terms, followed by the quadratic terms and so on. In current example, only linear and cubic nonlinear model terms are required.

First, a continuous time linear differential model can be re-constructed from the H_1 data in Figure 2 in the NX modelling by using the method of Li and Billings(2001). The model is given as

$$\begin{aligned} &y + 0.20405\frac{dy}{dt} + 2.6731\frac{d^2y}{dt^2} + 0.35139\frac{d^3y}{dt^3} + 2.4127\frac{d^4y}{dt^4} + 0.15566\frac{d^5y}{dt^5} \\ &+ 0.73464\frac{d^6y}{dt^6} - 0.99845u + 0.00033778\frac{du}{dt} - 1.7046\frac{d^2u}{dt^2} + 0.0010263\frac{d^3y}{dt^3} \\ &- 0.74912\frac{d^4y}{dt^4} + 0.0022\frac{d^5y}{dt^5} = 0 \end{aligned} \tag{11}$$

Next, continuous time third order nonlinear terms can be re-construct from $H_3(\cdot)$ in the NX modelling using the method of Li and Billings(2001). The final re-constructed third order nonlinear differential model is given as

$$\begin{aligned} &y + 0.20405\frac{dy}{dt} + 2.6731\frac{d^2y}{dt^2} + 0.35139\frac{d^3y}{dt^3} + 2.4127\frac{d^4y}{dt^4} + 0.15566\frac{d^5y}{dt^5} \\ &+ 0.73464\frac{d^6y}{dt^6} - 0.99845u + 0.00033778\frac{du}{dt} - 1.7046\frac{d^2u}{dt^2} + 0.0010263\frac{d^3y}{dt^3} \\ &- 0.74912\frac{d^4y}{dt^4} + 0.0022\frac{d^5y}{dt^5} + 0.0018994y^3 - 0.08239y^2\frac{dy}{dt} + 0.58942y^2\frac{d^2y}{dt^2} \\ &+ 1.7497y\left(\frac{dy}{dt}\right)^2 - 0.42462y\frac{dy}{dt}\frac{d^2y}{dt^2} + 4.7878y\left(\frac{d^2y}{dt^2}\right)^2 - 0.084502\left(\frac{dy}{dt}\right)^3 \\ &+ 14.352\left(\frac{dy}{dt}\right)^2\frac{d^2y}{dt^2} + 0.16924\left(\frac{d^2y}{dt^2}\right)^2\frac{d^3y}{dt^3} - 0.027329\left(\frac{d^3y}{dt^3}\right)^3 = 0 \end{aligned} \tag{12}$$

The $H_1(\omega)$ from the re-constructed nonlinear differential equation model (12) was compared with the $H_1(\omega)$ from the NX modelling in Figure 3, and shows a perfect match. This is also true for third order GFRF's comparison.

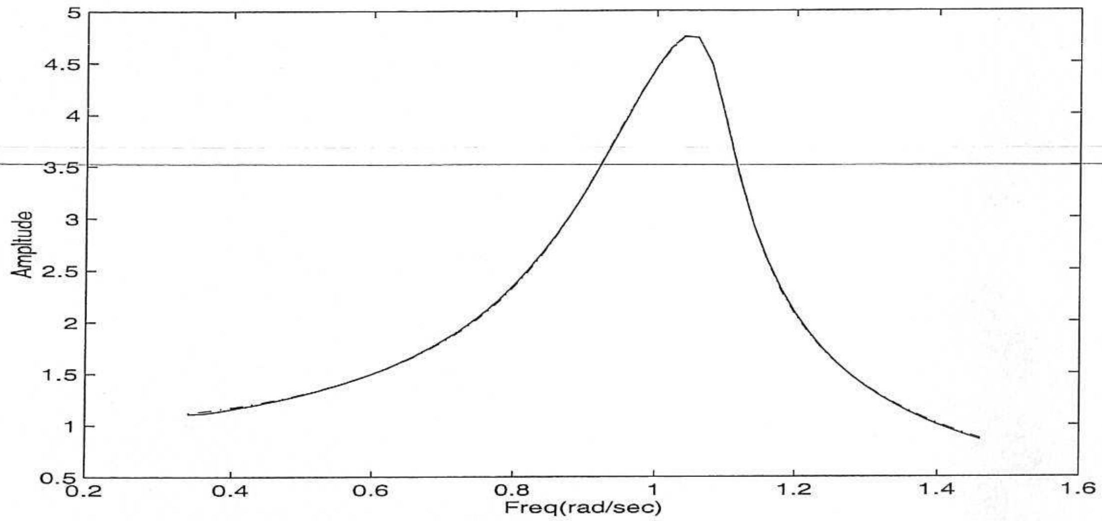

Figure 3. H_1 from NX modelling –Solid, and H_1 from reconstructed linear model (12)

This re-constructed nonlinear differential equation (12) will have the same time domain response as the original Duffing equation (6). Although generally the re-constructed nonlinear differential equation model will have a more complicated time domain expression, it enjoys a much simpler expression in terms of the associated Volterra series expression, which means that the complexity of the frequency domain analysis can be expected to reduce significantly. In this example, the truncation errors for ω =0.9 using the first and third order GFRF's from (12) are e_1 =0.5040% and e_3=0.1957% respectively, a big reduction compared with the results in Table 1, which means that a third order Volterra representation, which can be obtained from (12), is adequate in terms of the truncation error. This is true for the whole frequency range under (12). This means therefore that the entire frequency domain analysis of the original Duffing equation (6) can be performed based on the single re-constructed nonlinear differential equation (12). The proposed parametric approach illustrated using the specific Duffing oscillator can be effectively repeated on other nonlinear cases where the complexity of frequency domain analysis needs to be reduced.

4. Conclusions

For a linear system, the frequency response function, which is the Fourier transform of the first order convolution, will be naturally sufficient to represent the input-output relation for the whole frequency range, irrespective of the amplitude of the input signal. But for a nonlinear system, the behaviour of the system will depend heavily on the amplitude of the input. A nonlinear system can exhibit severely nonlinear phenomena, such as hysteresis, limit cycles, subharmonics and chaos, while the input excitation varies. It is a well known fact that the traditional Volterra series cannot represent the severe class of nonlinear systems, therefore the frequency domain transfer functions are not available for these cases. But even for weakly nonlinear systems where a convergent Volterra series exists, analysis may become impractical because the order of the Volterra series might become very high in order to obtain the required truncation accuracy.

It has been shown in this study that a new parametric modelling approach can provide the frequency features in a reduced Volterra model. Generally the resulting continuous time models which have a reduced Volterra expansion in the frequency domain will have an extended time domain expression so that the physical relation with the underlying system is lost. However this should help to significantly simplify the analysis of the underlying nonlinear system in the frequency domain.

In practical nonlinear continuous time system identification problems, when only input-output observations are available, there will be a balance between a physically meaningful identification, where the resulting model coefficients can be related back to the underlying system but which may not be fully frequency domain efficient, and a meaningful frequency domain identification, where the resulting model can be used as a basis for frequency domain analysis with limited orders of GFRF's but which may not have a simple physical interpretation. This will be investigated in a future study.

Acknowledgement: SAB gratefully acknowledges that this research was supported by the UK Engineering and Physical Sciences Research Council.

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