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Analytical Description of the Effects of System Nonlinearities on Output Frequency Responses: A Case Study

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The output spectrum of a linear dynamic system is equal to the input spectrum multiplied by the system frequency response function. This well known relationship analytically exposes the effects of the system parameters on the output frequency response. In this paper, the extension of this relationship to the nonlinear case is investigated via a case study where an analytical relationship between the output frequency response and the nonlinear damping characteristic parameters is derived for a SDOF spring damper system. The analysis is based on the frequency domain analysis of nonlinear systems, the basic idea can be extended to general situations. Simulation studies are included to verify the theoretical analysis and demonstrate the effectiveness of the new relationship. The results provide an important basis for the analytical study and design of nonlinear engineering systems and structures in the frequency domain.

1. Introduction

The output frequency response of engineering systems has been widely used in many areas to investigate and study system behaviours. If the underlying system is linear the relationship between the system output frequency response and the input is well known; the output spectrum $Y(j\omega)$ is equal to the input spectrum $U(j\omega)$ multiplied by the system frequency response function (FRF) $H(j\omega)$. The FRF is the frequency domain description for linear systems, and the simple linear frequency domain relationship $Y(j\omega) = H(j\omega)U(j\omega)$ analytically describes the effect of system properties on the output frequency response. This analytical relationship has been applied in control engineering for controlled plant analysis and controller design, in electronics and communications for the synthesis of analogue and digital filters, and in mechanical and civil engineering for the analysis of vibrations in the analysis and design of associated structures.

Nonlinear systems have been widely studied by many authors and significant progress towards understanding these systems has been made. Many of these studies have been based in the time domain with results relating to the Volterra series [1] [2] [3], NARMAX (Nonlinear AutoRegresive Moving Average with eXogenous input) models [4] [5], neural networks and fuzzy systems [6], and classical nonlinear models such as the Duffing equation [7] [8] and the Van der Pol oscillator [9]. The study of nonlinear systems in the frequency domain is based on the concept of generalised frequency response functions (GFRFs) [10] that extend the linear FRF concept to the nonlinear case. Many studies in the frequency domain have been focused on system modelling which involves the determination of the GFRFs from input output data or the establishment of system frequency domain models from input output spectra [11]-[16]. Output frequency responses of nonlinear systems were recently studied by Lang and Billings [17] [18] [19] and Billings and Lang [20] [21] [22]. These studies extended the above basic linear relationship between the input and output spectra and introduced explicit relationships between input and output frequencies of nonlinear systems. Based on these relationships, Billings and Lang [23] proposed the concept of energy transfer filters and developed a general procedure for the design of the energy transfer filters which can be implemented using the NARX (Nonlinear AutoRegressive with eXogenous input) model with input nonlinearity.

Unlike linear systems, the relationship between the input and output spectra of nonlinear systems is much more complicated. The relationship involves sophisticated multi-dimensional integration known as association of variables and a summation with a possibly infinite number of terms [2]. This complicates the effect of the system parameters on the output frequency response. Consequently, the linear system frequency domain analysis and design approaches cannot be easily extended to the nonlinear case.

As an attempt to solve this problem, in this paper a case study is conducted on a single-degree-of-freedom (SDOF) spring damper system with a nonlinear damping characteristic. In this study, an analytical relationship between the system output frequency response and the characteristic parameters of the system damping nonlinearity is derived for the first time using the frequency domain theories of nonlinear systems. The results explicitly reveal how the system output frequency response depends on the damping characteristic parameters which define the system nonlinearity. Simulation studies are performed to evaluate the accurate system output frequency responses at different input frequencies and magnitudes, and to compare these with the analytically determined results. The results verify the effectiveness and significance of the theoretical derivations. The study presented in the paper is focused on a relatively simple SDOF system to emphasize the idea of the basic approach, but the results can be extended to very general cases. The work provides an important basis for analytical studies and the design of nonlinear engineering systems and structures in the frequency domain.

2. System description

In order to demonstrate the idea of analytical analysis of the effects of system nonlinearity on the output frequency response, a simple SDOF system will be considered, as shown in Figure 1. The mass, m, supported on the nonlinear damper and parallel spring, is subject to a harmonic disturbance of amplitude, F_d, and frequency, Ω. The nonlinear damping effect is described by a third order polynomial such that

$$f(.) = a_1(.) + a_2(.)^2 + a_3(.)^3 \tag{1}$$

where a_1, a_2, a_3 are the parameters of the damping characteristic, and a_2 and a_3 represent the system nonlinearity. The analysis of the effects of parameters a_2 and a_3

on the system output frequency response is the focus of the present study. The spring with characteristic parameter k in parallel with the nonlinear damper f(.) provides an isolation between the disturbance force, $F_d \sin \Omega t$, and the force transmitted to the support, $F_s(t)$.

The equilibrium equation for the system in Figure 1, and the corresponding force at the support, can be expressed as

$$m\ddot{x}(t) + a_1\dot{x}(t) + a_2\dot{x}^2(t) + a_3\dot{x}^3(t) + kx(t) = F_d \sin(\Omega t) \tag{2}$$

$$F_s(t) = a_1\dot{x}(t) + a_2\dot{x}^2(t) + a_3\dot{x}^3(t) + kx(t) \tag{3}$$

For convenience of analysis, denote

$$y_1(t) = x(t) \tag{4}$$

$$y_2(t) = F_s(t) \tag{5}$$

and

$$u_1(t) = F_d \sin(\Omega t) \tag{6}$$

The system can then be described by a single input two output system

$$m\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_2\dot{y}_1^2(t) + a_3\dot{y}_1^3(t) + ky_1(t) = u_1(t) \tag{7}$$

$$y_2(t) = a_1\dot{y}_1(t) + a_2\dot{y}_1^2(t) + a_3\dot{y}_1^3(t) + ky_1(t) \tag{8}$$

What is interesting in this study is how the spectrum of the second system output $y_2(t)$ depends on the parameters a_2, a_3 of the nonlinear damping characteristic f(). Although this appears to be a relatively simple problem, surprisingly, there are no results in the literature that can address this fundamental problem. The reason for this omission is the complexity that is introduced by the nonlinearities even for this apparently simple system. The objective therefore is to establish an analytical relationship between the output spectrum and the system parameters.

3. Volterra modelling of the system in the time and frequency domain

The output $y(t)$ of a single input single output analytical system can be expressed as a Volterra functional polynomial of the input $u(t)$ [24] to give

$$y(t) = \sum_{n=1}^{N} y^{(n)}(t) \tag{9}$$

where N is the maximum order of the system nonlinearity, the nth order output of the system $y^{(n)}(t)$ is given by

$$y^{(n)}(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) d\tau_i \qquad n > 0 \tag{10}$$

and $h_n(\tau_1,\cdots,\tau_n)$ is a real valued function of $\tau_1,\cdots,\tau_n$ called the nth order impulse response function or Volterra kernel of the system.

The multi-dimensional Fourier transform of the nth order impulse response function yields the nth order transfer function or generalised frequency response function (GFRF)

$$H_n(j\omega_1,\cdots,j\omega_n)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,\cdots,\tau_n)e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \qquad (11)$$

Using the concept of GFRF, the relationship between the input spectrum $U(j\omega)$ and output spectrum $Y(j\omega)$, i.e., the frequency domain input output description of the system, can be obtained as [17]

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1/\sqrt{n}}{(2\pi)^{n-1}}\int_{\omega_1+\cdots+\omega_n=\omega}H_n(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{\omega n} \qquad (12)$$

where $\int_{\omega_1+\cdots+\omega_n=\omega}(.)d\sigma_{\omega n}$ denotes the integration of (.) over the n-dimensional hyperplane $\omega_1+\cdots+\omega_n=\omega$.

When the system is subject to a multi-tone input such that

$$u(t)=\sum_{i=1}^{K}|A_i|\cos(\omega_i t+\angle A_i) \qquad (13)$$

Lang and Billings [17] showed that equation (12) can be expressed as

$$Y(j\omega)=\sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(\omega_{k_1})\cdots A(\omega_{k_n}) \qquad (14)$$

where

$k_l\in\{-K,\cdots,-1,1,\cdots,K\},\ l=1,\cdots,n,$

$$A(\omega)=\begin{cases}|A_k|e^{j\angle A_k} & \text{if } \omega\in\{\omega_k, k=\pm1,\cdots,\pm K\}\\ 0 & \text{otherwise}\end{cases},$$

$\omega_{-k}=-\omega_k$

and

$|A_{-k}|e^{j\angle A_{-k}}=|A_k|e^{-j\angle A_k}$

The extension of the above theoretical results to the single input two output nonlinear system case is straightforward. The results in the time domain, which are the extensions of equations (9) and (10), are given in [25][26]

$$y_{j_1}(t) = \sum_{n=1}^{N} y_{j_1}^{(n)}(t) \qquad j_1 = 1,2 \tag{15}$$

where

$$y_{j_1}^{(n)}(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n^{j_1:\underbrace{1\cdots1}_{n}}(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u_1(t-\tau_i)d\tau_i \qquad j_1 = 1,2 \tag{16}$$

and $h_n^{j_1:\underbrace{1\cdots1}_{n}}(\tau_1,\cdots,\tau_n)$ is the nth order Volterra kernel of the system corresponding to the j_1th output. The results in the frequency domain, which are the extensions of equations (12) and (14), are

$$Y_{j_1}(j\omega) = \sum_{n=1}^{N} \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} H_n^{j_1:\underbrace{1\cdots1}_{n}}(j\omega_1,\cdots,j\omega_n)\prod_{i=1}^{n} U(j\omega_i)d\sigma_{\omega n} \qquad j_1 = 1,2 \tag{17}$$

and

$$Y_{j_1}(j\omega) = \sum_{n=1}^{N} \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^{j_1:\underbrace{1\cdots1}_{n}}(j\omega_{k_1},\cdots,j\omega_{k_n})A(\omega_{k_1})\cdots A(\omega_{k_n}) \qquad j_1 = 1,2 \tag{18}$$

where

$$H_n^{j_1:\underbrace{1\cdots1}_{n}}(j\omega_1,\cdots,j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n^{j_1:\underbrace{1\cdots1}_{n}}(\tau_1,\cdots,\tau_n)e^{-j(\omega_1\tau_1+\cdots+\omega_n\tau_n)}d\tau_1\cdots d\tau_n \tag{19}$$

It is obvious that the Volterra time domain model of the system (2) (3) is given by equation (15), and the output frequency response of the system when subject to the input in (6) is given by equation (18) with

$$k_l \in \{-1,+1\} \qquad l = 1,\ldots,n \tag{20}$$

$$A(\omega) = \begin{cases} |A_k|e^{j\angle A_k} & \text{if } \omega \in \{\omega_k, k = \pm 1\}, \text{ where } |A_{\pm 1}| = F_d,\ \omega_{\pm 1} = \pm\Omega, \text{ and } \angle A_{\pm 1} = \mp\pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

Equation (18) is the starting point for the derivation of an analytical expression for the effects of nonlinearity on the output frequency response of the system (2) (3).

4. The effects of system nonlinearity on the output frequency response

The focus of this section is to investigate the effects of the nonlinear damping characteristic of the system (2) (3) on the output frequency response when the system is subject to a multi-tone or a harmonic input under the condition that the system can be described by the frequency domain Volterra model (18). This study involves two steps. First the GFRF matrices of the system

$$\left[H_n^{1:\underbrace{1\cdots 1}_{n}}(j\omega_1,\cdots,j\omega_n),\ H_n^{2:\underbrace{1\cdots 1}_{n}}(j\omega_1,\cdots,j\omega_n) \right] \quad n=1,2,3,\ldots$$

are derived using the probing method [27]. Then an analytical relationship between the system output frequency response $Y_2(j\omega)$ and the parameters of the nonlinear damping characteristic is determined.

4.1 The probing method

Given a parametric model of a nonlinear system, the GFRFs of the system can be derived analytically using the probing method. In the case of single input single output nonlinear systems, the basic idea of the probing method can be introduced as below.

It was shown by Rugh [2] that for nonlinear systems which are described by the Volterra model (9) (10) and excited by a combination of exponentials

$$u(t)=\sum_{i=1}^{R} e^{j\omega_i t} \qquad 1 \le R \le N \tag{22}$$

the output response can be written as

$$\begin{aligned} y(t) &= \sum_{n=1}^{N} \sum_{i_1=1}^{R} \cdots \sum_{i_n=1}^{R} H_n\left(j\omega_{i_1},\cdots,j\omega_{i_n}\right) e^{j(\omega_{i_1}+\cdots+\omega_{i_n})t} \\ &= \sum_{n=1}^{N} \sum_{m(n)} G_{m_1(n)\cdots m_R(n)}\left(j\omega_1,\cdots,j\omega_R\right) e^{j[m_1(n)\omega_1+\cdots+m_R(n)\omega_R]t} \end{aligned} \tag{23}$$

where $\sum_{m(n)}$ indicates a R-fold sum over all integer indices $m_1(n),\cdots,m_R(n)$ such that

$0 \le m_i(n) \le n$, $m_1(n)+\cdots+m_R(n)=n$, and

$$G_{m_1(n)\cdots m_R(n)}\left(j\omega_1,\cdots,j\omega_R\right)=\frac{n!}{m_1(n)!\cdots m_R(n)!} H_n(\underbrace{j\omega_1,\cdots,j\omega_1}_{m_1(n)},\cdots,\underbrace{j\omega_R,\cdots,j\omega_R}_{m_R(n)}) \tag{24}$$

Notice that in (24) when n=R, $m_i(n)=1, i=1,\ldots,R$, therefore

$$G_{m_1(R)\cdots m_R(R)}(j\omega_1,\cdots,j\omega_R)=R!H_R(j\omega_1,\cdots,j\omega_R) \tag{25}$$

Considering (25), (23) can be written as

$$\begin{aligned} y(t)=\sum_{n=1,n\neq R}^{N}\sum_{m(n)}G_{m_1(n)\cdots m_R(n)}(j\omega_1,\cdots,j\omega_R)e^{j[m_1(n)\omega_1+\cdots+m_R(n)\omega_R]t} \\ +R!H_R(j\omega_1,\cdots,j\omega_R)e^{j(\omega_1+\cdots+\omega_R)t} \end{aligned} \tag{26}$$

For nonlinear systems which have a parametric model with parameter vector θ

$$y(t)=f_0(t,\theta,y(t),u(t)) \tag{27}$$

and which can also be described by the Volterra model (9) (10), substituting (22) and (26) into (27) for $y(t)$ and $u(t)$, and extracting the coefficient of $e^{j(\omega_1+\cdots+\omega_R)t}$ from the resulting expression will produce an equation from which the GFRF $H_R(j\omega_1,\cdots,j\omega_R)$ can be obtained.

For single input two output systems, which are described by the Volterra model (15)(16), and excited by input (22), it can be shown, based on the same idea as used for the single input single output system case above, that the output response is given by

$$\begin{aligned} y_{j_1}(t)=\sum_{n=1,n\neq R}^{N}\sum_{m(n)}G_{m_1(n)\cdots m_R(n)}^{j_1:\overbrace{1\cdots1}^{n}}(j\omega_1,\cdots,j\omega_R)e^{j[m_1(n)\omega_1+\cdots+m_R(n)\omega_R]t} \qquad j_1=1,2 \\ +R!H_R^{j_1:\overbrace{1\cdots1}^{R}}(j\omega_1,\cdots,j\omega_R)e^{j(\omega_1+\cdots+\omega_R)t} \end{aligned} \tag{28}$$

where

$$G_{m_1(n)\cdots m_R(n)}^{j_1:\overbrace{1\cdots1}^{n}}(j\omega_1,\cdots,j\omega_R)=\frac{n!}{m_1(n)!\cdots m_R(n)!}H_n^{j_1:\overbrace{1\cdots1}^{n}}(\underbrace{j\omega_1,\cdots,j\omega_1}_{m_1(n)},\cdots,\underbrace{j\omega_R,\cdots,j\omega_R}_{m_R(n)}) \tag{29}$$

If the system can be described by the parametric model

$$\begin{cases} y_1(t)=f_1(t,\theta,y_1(t),y_2(t),u_1(t)) \\ y_2(t)=f_2(t,\theta,y_1(t),y_2(t),u_1(t)) \end{cases} \tag{30}$$

then substituting $u_1(t)=\sum_{i=1}^{R}e^{j\omega_i t}$, and $y_1(t)$ and $y_2(t)$ given by (28) into (30), and extracting the coefficient of $e^{j(\omega_1+\cdots+\omega_R)t}$ from the resulting expressions will produce two coupled equations for which the GFRF matrix

$$\left[H_R^{\overbrace{1:1\cdots1}^{R}}(j\omega_1,\cdots,j\omega_R),\ H_R^{\overbrace{2:1\cdots1}^{R}}(j\omega_1,\cdots,j\omega_R) \right]$$

can be obtained.

4.2 Derivation of the system GFRF matrices

Following the probing method for single input two output systems introduced above, the GFRF matrices of the system (2) (3) up to third order are determined in the following.

To determine the first order GFRF matrix

$$\left[H_1^{1:1}(j\omega_1),\ H_1^{2:1}(j\omega_1) \right]$$

the probing input

$$u_1(t) = e^{j\omega_1 t} \tag{31}$$

is used and, by taking R=1, equation (28) can be written as

$$\begin{cases} y_1(t) = H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + \cdots \\ y_2(t) = H_1^{(2:1)}(j\omega_1)e^{j\omega_1 t} + \cdots \end{cases} \tag{32}$$

Substituting (31) and (32) into (7) (8) for $u_1(t)$, $y_1(t)$, and $y_2(t)$, and extracting the coefficient of $e^{j(\omega_1)t}$ from the resulting expressions yields two equations for $[H_1^{1:1}(j\omega_1),\ H_1^{2:1}(j\omega_1)]$ which can be expressed in a matrix form such that

$$\begin{bmatrix} m(j\omega_1)^2 & 1 \\ -k - a_1(j\omega_1) & 1 \end{bmatrix} \begin{bmatrix} H_1^{1:1}(j\omega_1) \\ H_1^{2:1}(j\omega_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{33}$$

Consequently the first order GFRF matrix is determined as

$$\begin{bmatrix} H_1^{1:1}(j\omega_1) \\ H_1^{2:1}(j\omega_1) \end{bmatrix} = \begin{bmatrix} 1/\left(m(j\omega_1)^2 + a_1 j\omega_1 + k\right) \\ (a_1 j\omega_1 + k)/\left(m(j\omega_1)^2 + a_1 j\omega_1 + k\right) \end{bmatrix} \tag{34}$$

To determine the second order GFRF matrix

$$\left[H_2^{1:11}(j\omega_1, j\omega_2),\ H_2^{2:11}(j\omega_1, j\omega_2) \right]$$

the probing input

$$u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \tag{35}$$

is used and, by taking R=2, equation (28) can be written as

$$\begin{cases} y_1(t) = H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(1:1)}(j\omega_2)e^{j\omega_2 t} + 2H_2^{1:11}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + \cdots \\ y_2(t) = H_1^{(2:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(2:1)}(j\omega_2)e^{j\omega_2 t} + 2H_2^{2:11}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + \cdots \end{cases} \tag{36}$$

Substituting (35) and (36) into (7) (8) for $u_1(t)$, $y_1(t)$, and $y_2(t)$, and extracting the coefficient of $e^{j(\omega_1+\omega_2)t}$ from the resulting expressions yields two coupled equations for $[H_2^{1:11}(j\omega_1, j\omega_2),\ H_2^{2:11}(j\omega_1, j\omega_2)]$

$$\begin{cases} mH_2^{1:11}(j\omega_1, j\omega_2)(j\omega_1 + j\omega_2)^2 + H_2^{2:11}(j\omega_1, j\omega_2) = 0 \\ H_2^{2:11}(j\omega_1, j\omega_2) - kH_2^{1:11}(j\omega_1, j\omega_2) - a_1 H_2^{1:11}(j\omega_1, j\omega_2)(j\omega_1 + j\omega_2) \\ \qquad - a_2 H_1^{1:1}(j\omega_1)H_1^{1:1}(j\omega_2)(j\omega_1)(j\omega_2) = 0 \end{cases} \tag{37}$$

So the second order GFRF matrix is obtained as

$$\begin{cases} H_2^{1:11}(j\omega_1, j\omega_2) = -\dfrac{a_2 H_1^{1:1}(j\omega_1)H_1^{1:1}(j\omega_2)(j\omega_1)(j\omega_2)}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k} \\ H_2^{2:11}(j\omega_1, j\omega_2) = \dfrac{m a_2 H_1^{1:1}(j\omega_1)H_1^{1:1}(j\omega_2)(j\omega_1)(j\omega_2)(j\omega_1 + j\omega_2)^2}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k} \end{cases} \tag{38}$$

To determine the third order GFRF matrix

$[H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3),\ H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3)]$

the probing input

$$u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t} \tag{39}$$

is used and, by taking R=3, equation (28) can be written as

$$\begin{cases} y_1(t) = \begin{bmatrix} H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(1:1)}(j\omega_2)e^{j\omega_2 t} + H_1^{(1:1)}(j\omega_3)e^{j\omega_3 t} + 2H_2^{1:11}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + \\ 2H_2^{1:11}(j\omega_2, j\omega_3)e^{j(\omega_2+\omega_3)t} + 2H_2^{1:11}(j\omega_1, j\omega_3)e^{j(\omega_1+\omega_3)t} + 6H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3)e^{j(\omega_1+\omega_2+\omega_3)t} \\ + \cdots\cdots \end{bmatrix} \\ y_2(t) = \begin{bmatrix} H_1^{(2:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(2:1)}(j\omega_2)e^{j\omega_2 t} + H_1^{(2:1)}(j\omega_3)e^{j\omega_3 t} + 2H_2^{2:11}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + \\ 2H_2^{2:11}(j\omega_2, j\omega_3)e^{j(\omega_2+\omega_3)t} + 2H_2^{2:11}(j\omega_1, j\omega_3)e^{j(\omega_1+\omega_3)t} + 6H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3)e^{j(\omega_1+\omega_2+\omega_3)t} \\ + \cdots\cdots \end{bmatrix} \end{cases} \tag{40}$$

Substituting (39) and (40) into (7) (8) for $u_1(t)$, $y_1(t)$, and $y_2(t)$, and extracting the coefficient of $e^{j(\omega_1+\omega_2+\omega_3)t}$ from the resulting expressions yield two coupled equations for $[H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3),\ H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3)]$

$$\begin{cases} H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3) - m[j(\omega_1+\omega_2+\omega_3)]^2 H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3) = 0 \\ 6H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3) - 6kH_3^{1:111}(j\omega_1, j\omega_2, j\omega_3) \\ \quad - 6a_1 H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3)[j(\omega_1+\omega_2+\omega_3)] \\ \quad - 2a_2 H_1^{1:1}(j\omega_1) H_2^{1:11}(j\omega_2, j\omega_3)(j\omega_1)(j\omega_2 + j\omega_3) \\ \quad - 2a_2 H_1^{1:1}(j\omega_2) H_2^{1:11}(j\omega_1, j\omega_3)(j\omega_{21})(j\omega_1 + j\omega_3) \\ \quad - 2a_2 H_1^{1:1}(j\omega_3) H_2^{1:11}(j\omega_1, j\omega_2)(j\omega_3)(j\omega_1 + j\omega_2) \\ \quad - a_3 H_1^{1:1}(j\omega_1) H_1^{1:1}(j\omega_2) H_1^{1:1}(j\omega_3)(j\omega_1)(j\omega_2)(j\omega_3) = 0 \end{cases} \tag{41}$$

Thus the third order GFRF matrix is obtained as

$$\begin{cases} H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3) = \dfrac{1}{\{m(j\omega_1+j\omega_2+j\omega_3)^2 + a(j\omega_1+j\omega_2+j\omega_3) + k\}} \times \\ \qquad \begin{Bmatrix} -\dfrac{a_2}{3}\begin{bmatrix} H_1^{1:1}(j\omega_1) H_2^{1:11}(j\omega_2, j\omega_3)(j\omega_1)(j\omega_2 + j\omega_3) \\ + H_1^{1:1}(j\omega_2) H_2^{1:11}(j\omega_1, j\omega_3)(j\omega_{21})(j\omega_1 + j\omega_3) \\ + H_1^{1:1}(j\omega_3) H_2^{1:11}(j\omega_1, j\omega_2)(j\omega_3)(j\omega_1 + j\omega_2) \end{bmatrix} \\ -\dfrac{a_3}{6} H_1^{1:1}(j\omega_1) H_1^{1:1}(j\omega_2) H_1^{1:1}(j\omega_3)(j\omega_1)(j\omega_2)(j\omega_3) \end{Bmatrix} \\ H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3) = -m[j(\omega_1+\omega_2+\omega_3)]^2 H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3) \end{cases}$$

(42)

Denote

$$H_2^{1:11}(j\omega_1, j\omega_2) = -\frac{a_2 H_1^{1:1}(j\omega_1) H_1^{1:1}(j\omega_2)(j\omega_1)(j\omega_2)}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k} = -a_2 F_0(j\omega_1, j\omega_2) \tag{43}$$

and

$$\frac{1}{3}\begin{bmatrix} H_1^{1:1}(j\omega_1) H_2^{1:11}(j\omega_2, j\omega_3)(j\omega_1)(j\omega_2 + j\omega_3) \\ + H_1^{1:1}(j\omega_2) H_2^{1:11}(j\omega_1, j\omega_3)(j\omega_{21})(j\omega_1 + j\omega_3) \\ + H_1^{1:1}(j\omega_3) H_2^{1:11}(j\omega_1, j\omega_2)(j\omega_3)(j\omega_1 + j\omega_2) \end{bmatrix} =$$

$$-\frac{a_2}{3}\begin{bmatrix} H_1^{1:1}(j\omega_1) F_0(j\omega_2, j\omega_3)(j\omega_1)(j\omega_2 + j\omega_3) \\ + H_1^{1:1}(j\omega_2) F_0(j\omega_1, j\omega_3)(j\omega_2)(j\omega_1 + j\omega_3) \\ + H_1^{1:1}(j\omega_3) F_0(j\omega_1, j\omega_2)(j\omega_3)(j\omega_1 + j\omega_2) \end{bmatrix} = -a_2 F_1(j\omega_1, j\omega_2, j\omega_3)$$

(44)

where

$$F_0(j\omega_1, j\omega_2) = \frac{H_1^{1:1}(j\omega_1)H_1^{1:1}(j\omega_2)(j\omega_1)(j\omega_2)}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k}$$

and

$$F_1(j\omega_1, j\omega_2, j\omega_3) = \frac{1}{3}\begin{bmatrix} H_1^{1:1}(j\omega_1)F_0(j\omega_2, j\omega_3)(j\omega_1)(j\omega_2 + j\omega_3) \\ + H_1^{1:1}(j\omega_2)F_0(j\omega_1, j\omega_3)(j\omega_2)(j\omega_1 + j\omega_3) \\ + H_1^{1:1}(j\omega_3)F_0(j\omega_1, j\omega_2)(j\omega_3)(j\omega_1 + j\omega_2) \end{bmatrix}$$

and define

$$F_2(j\omega_1, j\omega_2, j\omega_3) = \frac{1}{6}H_1^{1:1}(j\omega_1)H_1^{1:1}(j\omega_2)H_1^{1:1}(j\omega_3)(j\omega_1)(j\omega_2)(j\omega_3) \tag{45}$$

Substituting these results into (42) yields

$$\begin{cases} H_3^{1:111}(j\omega_1, j\omega_2, j\omega_3) = \dfrac{1}{\beta(j\omega_1 + j\omega_2 + j\omega_3)} \\ \qquad \left\{a_2^2 F_1(j\omega_1, j\omega_2, j\omega_3) - a_3 F_2(j\omega_1, j\omega_2, j\omega_3)\right\} \\ H_3^{2:111}(j\omega_1, j\omega_2, j\omega_3) = \dfrac{-m[j(\omega_1 + \omega_2 + \omega_3)]^2}{\beta(j\omega_1 + j\omega_2 + j\omega_3)} \\ \qquad \left\{a_2^2 F_1(j\omega_1, j\omega_2, j\omega_3) - a_3 F_2(j\omega_1, j\omega_2, j\omega_3)\right\} \end{cases} \tag{46}$$

where $\beta(j\omega_1 + j\omega_2 + j\omega_3) = \left\{m(j\omega_1 + j\omega_2 + j\omega_3)^2 + a_1(j\omega_1 + j\omega_2 + j\omega_3) + k\right\}$.

Substituting the definition of $F_0(j\omega_1, j\omega_2)$ into equation (38) yields

$$\begin{cases} H_2^{1:11}(j\omega_1, j\omega_2) = -a_2 F_0(j\omega_1, j\omega_2) \\ H_2^{2:11}(j\omega_1, j\omega_2) = m a_2 (j\omega_1 + j\omega_2)^2 F_0(j\omega_1, j\omega_2) \end{cases} \tag{47}$$

giving a more concise expression of $H_2^{1:11}(j\omega_1, j\omega_2)$ and $H_2^{2:11}(j\omega_1, j\omega_2)$.

Equations (34) (47) and (46) give the system GFRF matrices up to third order. Notice that $F_0(.,.)$, $F_1(.,.,.)$, $F_2(.,.,.)$, and $\beta(.)$ only depend on m, a_1, k, the parameters which describe the system linear characteristics. Therefore, given the system linear characteristics, equations (47) and (46) explicitly reveal how the second and third order GFRF matrices depend on the parameters a_2 and a_3 of the system nonlinear damping characteristic.

4.3 The effects of system nonlinearity on the output frequency response

Having derived expressions for the system GFRF matrices in terms of the nonlinear damping characteristic parameters a_2 and a_3, how the output spectrum $Y_2(j\omega)$ of the system depends on the parameters when subjected to a general multi-tone input can be revealed by substituting equations (47) and (46) into (18) for $H_2^{2:11}(j\omega_1, j\omega_2)$ and $H_2^{2:111}(j\omega_1, j\omega_2, j\omega_3)$ to yield

$$Y_2(j\omega) = \sum_{n=1}^{N} \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n^{2:\overbrace{1\cdots1}^{n}}\left(j\omega_{k_1}, \cdots, j\omega_{k_n}\right) A(\omega_{k_1}) \cdots A(\omega_{k_n})$$

$$= \frac{1}{2} H_1^{2:1}(j\omega) A(\omega) + \frac{1}{2^2} \sum_{\omega_{k_1}+\omega_{k_2}=\omega} H_2^{2:11}\left(j\omega_{k_1}, j\omega_{k_2}\right) A(\omega_{k_1}) A(\omega_{k_2}) +$$

$$\frac{1}{2^3} \sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} H_3^{2:111}\left(j\omega_{k_1}, j\omega_{k_2}, j\omega_{k_3}\right) A(\omega_{k_1}) A(\omega_{k_2}) A(\omega_{k_3}) + \cdots\cdots$$

$$= \frac{1}{2} H_1^{2:1}(j\omega) A(\omega) + \frac{m a_2 (j\omega)^2}{2^2} \sum_{\omega_{k_1}+\omega_{k_2}=\omega} A(\omega_{k_1}) A(\omega_{k_2}) F_0(j\omega_{k_1}, j\omega_{k_2})$$

$$- \frac{m(j\omega)^2 a_2^2}{2^3 \beta(j\omega)} \sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_1(j\omega_{k_1}, j\omega_{k_2}, j\omega_{k_3}) A(\omega_{k_1}) A(\omega_{k_2}) A(\omega_{k_3})$$

$$+ \frac{m(j\omega)^2 a_3}{2^3 \beta(j\omega)} \sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_2(j\omega_{k_1}, j\omega_{k_2}, j\omega_{k_3}) A(\omega_{k_1}) A(\omega_{k_2}) A(\omega_{k_3})$$

$$+ \cdots\cdots$$

$$= p_1(j\omega) + p_2(j\omega) a_2 - p_3(j\omega) a_2^2 + p_4(j\omega) a_3 + \cdots\cdots \qquad (48)$$

where

$$p_1(j\omega) = \frac{1}{2} H_1^{2:1}(j\omega) A(\omega)$$

$$p_2(j\omega) = \frac{m(j\omega)^2}{2^2} \sum_{\omega_{k_1}+\omega_{k_2}=\omega} A(\omega_{k_1}) A(\omega_{k_2}) F_0(j\omega_{k_1}, j\omega_{k_2})$$

$$p_3(j\omega) = \frac{m(j\omega)^2}{2^3 \beta(j\omega)} \sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_1(j\omega_{k_1}, j\omega_{k_2}, j\omega_{k_3}) A(\omega_{k_1}) A(\omega_{k_2}) A(\omega_{k_3})$$

$$p_4(j\omega) = \frac{m(j\omega)^2}{2^3 \beta(j\omega)} \sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_2(j\omega_{k_1}, j\omega_{k_2}, j\omega_{k_3}) A(\omega_{k_1}) A(\omega_{k_2}) A(\omega_{k_3})$$

depend on the applied multi-tone input and the parameters which describe the system linear characteristics but are independent of a_2 and a_3.

Equation (48) is a very important result which describes the relationship between the system frequency response and the characteristic parameters of the system nonlinearity. As far as we are aware, little effort if any has previously been made to arrive at such an explicit description for this relationship. The result extends the fundamental analytical relationship between the linear characteristic parameters and the output frequency response to the nonlinear case for the system (2) (3) when the system is subject to a multi-tone input, and can be further extended to very general situations.

Under the condition that the system can be well approximated by the terms in the Volterra model up to third order of nonlinearity, equation (48) can be written as

$$Y_2(j\omega) = p_1(j\omega) + p_2(j\omega)a_2 - p_3(j\omega)a_2^2 + p_4(j\omega)a_3 \quad (49)$$

If $a_2 = a_3 = 0$, then the system is a simple linear SDOF spring and damper system, and

$$Y_2(j\omega) = p_1(j\omega) = \frac{1}{2}H_1^{2:1}(j\omega)A(\omega) = \frac{(a_1 j\omega_1 + k)}{2\left(m(j\omega_1)^2 + a_1 j\omega_1 + k\right)}A(\omega) \quad (50)$$

Given an applied multi-tone input, equation (50) shows an explicit analytical relationship between the system output frequency response and the system linear characteristic parameters m, a_1, and k. This relationship is well-known and is used for the analysis and design of linear SDOF spring and damper systems.

When $a_2 \neq 0$ and/or $a_3 \neq 0$, the system behaves nonlinearily. In the case of nonlinear systems, it has generally been believed that the relationship between the system output frequency response and the system characteristic parameters is very complicated, and researchers and engineers basically relied on numerical analysis, rather than analytical studies as in the linear case, to investigate the effect of system parameters on the output frequency response. Equation (49), however, reveals the relationship, and shows that the analysis of the nonlinear SDOF system can be achieved via two steps if the system description is valid over the considered operating conditions. In the first step, the analysis of the effect of the system linear characteristic parameters on the output frequency response is conducted based on equation (50). This can be achieved in the case where the system is excited by an input with a low amplitude such that the system basically works over the linear regime.. Secondly, the analysis of the effects of the system nonlinear characteristic parameters on the output frequency response is performed based on equation (49) with fixed linear characteristic parameters. This covers the operating scenarios where the system works under a regime where the description (49) is valid or approximately valid. The first step is straightforward and is the same as the widely applied linear system analysis approach. The second step deals with the nonlinear system analysis issue in a very novel way. Based on this idea the frequency domain design of nonlinear systems can be performed in a totally new and systematic manner, this will be the subject of a series of our later publications. In the following discussions, however, the emphasis will be focused on studying how well equation (49) can be used to represent the effect of the nonlinear characteristic parameters a_2 and a_3 on the system output spectrum to reveal the implications of this important relationship for the analysis and design of nonlinear systems.

From the definitions of $p_i(j\omega)$ $i = 1,2,3,4$, it is known that given an applied multi-tone input and the linear characteristic parameters m, a_1, k, $p_i(j\omega)$ $i = 1,2,3,4$ are known functions of frequency ω. Therefore equation (49) indicates that the system output spectrum is a polynomial function of the nonlinear damping characteristic parameters a_2 and a_3 at each frequency. In the case where the system is subject to the harmonic input (6), and the output frequency of interest in the analysis is the same as the input frequency Ω, $p_i(j\omega)$ $i=1,2,3,4$ can be further written as

$$p_1(j\Omega)=\frac{1}{2}H_1^{2:1}(j\Omega)A(\Omega) \tag{51}$$

$$\begin{aligned} p_2(j\Omega)&=\frac{m(j\Omega)^2}{2^2}\sum_{\omega_{k_1}+\omega_{k_2}=\Omega}A(\omega_{k_1})A(\omega_{k_2})F_0(j\omega_{k_1},j\omega_{k_2})\\ &=\frac{m(j\Omega)^2}{2^2}\sum_{k_1=-1}^{1}A(\omega_{k_1})A(\Omega-\omega_{k_1})F_0(j\omega_{k_1},j(\Omega-\omega_{k_1}))\\ &=\frac{m(j\Omega)^2}{2^2}\begin{bmatrix}A(\omega_{-1})A(\Omega-\omega_{-1})F_0(j\omega_{-1},j(\Omega-\omega_{-1}))+\\ A(\omega_1)A(\Omega-\omega_1)F_0(j\omega_1,j(\Omega-\omega_1))\end{bmatrix}\\ &=\frac{m(j\Omega)^2}{2^2}\begin{bmatrix}A(-\Omega)A(2\Omega)F_0(-\Omega j,j2\Omega)+\\ A(\Omega)A(0)F_0(j\Omega,j0)\end{bmatrix}=0 \end{aligned} \tag{52}$$

$$\begin{aligned} p_3(j\Omega)&=\frac{m(j\Omega)^2}{2^3\beta(j\Omega)}\sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\Omega}F_1(j\omega_{k_1},j\omega_{k_2},j\omega_{k_3})A(\omega_{k_1})A(\omega_{k_2})A(\omega_{k_3})\\ &=\frac{m(j\Omega)^2}{2^3\beta(j\Omega)}\sum_{k_1=-1}^{1}\sum_{k_2=-1}^{1}F_1(j\omega_{k_1},j\omega_{k_2},j(\Omega-\omega_{k_1}-\omega_{k_2}))A(\omega_{k_1})A(\omega_{k_2})A(\Omega-\omega_{k_1}-\omega_{k_2})\\ &=\frac{m(j\Omega)^2}{2^3\beta(j\Omega)}\begin{bmatrix}F_1(-\Omega j,-j\Omega,j3\Omega)A(-\Omega)A(-\Omega)A(3\Omega)+\\ F_1(-j\Omega,j\Omega,j\Omega)A(-\Omega)A(\Omega)A(\Omega)+\\ F_1(j\Omega,-\Omega j,j\Omega)A(\Omega)A(-\Omega)A(\Omega)+\\ F_1(j\Omega,j\Omega,-j\Omega)A(\Omega)A(-\Omega)\end{bmatrix}\\ &=\frac{3m(j\Omega)^2}{2^3\beta(j\Omega)}F_1(-j\Omega,j\Omega,j\Omega)\left|A(\Omega)\right|^2A(\Omega) \end{aligned} \tag{53}$$

$$p_4(j\Omega)=\frac{3m(j\Omega)^2}{2^3\beta(j\Omega)}F_2(-j\Omega,j\Omega,j\Omega)\left|A(\Omega)\right|^2A(\Omega) \tag{54}$$

Substituting

$$\begin{aligned} F_1(-j\Omega,j\Omega,j\Omega)&=\frac{1}{3}H_1^{1:1}(-j\Omega)F_0(j\Omega,j\Omega)(2j\Omega)(-j\Omega)\\ &=-\frac{2\left|H_1^{1:1}(j\Omega)\right|^2H_1^{1:1}(j\Omega)(\Omega^4)}{3\beta(2j\Omega)} \end{aligned}$$

and

$$F_2(-j\Omega, j\Omega, j\Omega) = \frac{1}{6}\left|H_1^{1:1}(j\Omega)\right|^2 H_1^{1:1}(j\Omega)(j\Omega^3)$$

into (53) and (54) yields

$$p_3(j\Omega) = \frac{\Omega^6 m\left|H_1^{1:1}(j\Omega)\right|^2 H_1^{1:1}(j\Omega)}{2^2 \beta(j\Omega)\beta(2j\Omega)}\left|A(\Omega)\right|^2 A(\Omega) \tag{55}$$

and

$$p_4(j\Omega) = -\frac{j\Omega^5 m\left|H_1^{1:1}(j\Omega)\right|^2 H_1^{1:1}(j\Omega)}{2^4 \beta(j\Omega)}\left|A(\Omega)\right|^2 A(\Omega) \tag{56}$$

So, in this case, equation (49) can be written as

$$Y_2(j\Omega) = p_1(j\Omega) - p_3(j\Omega)a_2^2 + p_4(j\Omega)a_3 \tag{57}$$

with $p_i(j\Omega), i = 1,3,4$ defined by (51), (55) and (56).

Simulation studies will be conducted in the next section for the system (2)(3) to evaluate the output frequency responses for a harmonic input (6) under different values of a_2 and a_3. The results are then compared with the output spectrum $Y_2(j\Omega)$ determined using (57), an analytical relationship between the system nonlinear damping characteristic parameters and the output frequency response when the system is subject to the harmonic input. The objective is to verify the effectiveness of the theoretically derived analytical relationship and to show the potential of using the relationship in system analysis and design

5. Simulation studies and discussions

Consider the system (2) (3) subject to the harmonic input (6), and take the system linear characteristic parameters as

$a_1 = 2960$

$m = 240\,kg$

$k = 16000\,N/m$

Over a considerable range of a_2 and a_3, conduct simulation studies for the system to generate the output frequency response $Y_2(j\Omega)$ and compare this with the result analytically determined from (57) in the following five cases:

(i) $\Omega = 20\,rad/s,\ F_d = 50,\ a_3 = 0$;

(ii) $\Omega = 20\,rad/s,\ F_d = 150,\ a_3 = 0$;

(iii) $\Omega = 50\ rad/s$, $F_d = 100$, $a_3 = 0$;

(iv) $\Omega = 10\ rad/s$, $F_d = 100$, $a_3 = 0$;

(v) $\Omega = 20\ rad/s$, $F_d = 100$, $a_3 \neq 0$;

Figures 2, 3, 4, 5 show the results in cases (i) (ii) (iii) and (iv) and Figure 6 (a) (b) and (c) show the results in case (v). Notice that instead of $|Y_2(j\Omega)|$, $2|Y_2(j\Omega)|$ is used to show the output spectrum. This is because $2|Y_2(j\Omega)|$ represents the physical magnitude of the system output $y_2(t)$ at frequency Ω.

The results in cases of (i)-(iv) reflect how the magnitude of the system output frequency response changes with the single nonlinear damping characteristic parameter a_2. The solid line shows the magnitudes of the output spectrum $2Y_2(j\Omega)$ determined using the analytical description (57) over a range of values of a_2. The circles show the computed results of the spectrum over a set of discrete points of a_2, which are obtained by performing a FFT operation on the time domain output $y_2(t)$ to give the numerical simulation results for the system. The effects of the amplitude F_d of the harmonic input on the relationship between $Y_2(j\Omega)$ and a_2 can be observed from Figures 2 and 3 where the input frequency is fixed to be $\Omega = 20$, and the input amplitudes are $F_d = 50$ and $F_d = 150$ respectively. The effect of the frequency Ω of the harmonic input on the relationship between $Y_2(j\Omega)$ and a_2 can be observed from Figures 4 and 5 where the input amplitude is fixed to be $F_d = 100$, and the input frequencies are $\Omega = 50$ and $\Omega = 10$ respectively.

Figure 6 (a) shows how the magnitude of $Y_2(j\Omega)$ changes with the nonlinear damping characteristic parameters a_2 and a_3 according to the analytical expression (57). Figure 6 (b) shows the same result but from a different perspective to reveal the effect of parameter a_3 on the output frequency response. Figure 6 (c) shows a comparison between the magnitude of the analytically determined output spectrum and the magnitude of the spectrum obtained using the FFT from the numerically simulated system output over a set of discrete points of (a_2, a_3).

The results in Figures 2-6 indicate that the analytically determined system output frequency responses match the simulation results quite well over a considerable range of values of a_2 and a_3 in the five different cases. This verifies the theoretical analysis in the previous sections and demonstrates the effectiveness of the derived analytical description for the effect of system nonlinearity on the output frequency response.

An observation of Figures 2 and 3 indicates that the analytically determined output spectra match the simulation results well over a wider range of values of a_2 when the applied harmonic input has a smaller amplitude. A similar observation for Figures 4 and 5 indicates that the analytical results match the simulated spectra well over a wider range of values of a_2 when the applied harmonic input has a higher input frequency. The difference between the theoretical and the simulation results is due to the effects of the system nonlinearity that can not be covered by the analytical expression (57), which simply assumes that the system can be approximated by a Volterra model with terms up to third order of nonlinearity. Inputs with a greater amplitude can obviously cause more severe nonlinear effects in the system than just third order. The reason the input with a lower frequency induces a more severe nonlinear effect is that the lower frequency is much nearer to the system resonant frequency, which is 8.16 rad/s for this partriculer system. At resonant frequencies, the system frequency responses due to both linear and nonlinear effects are amplified.

In order to increase the accuracy of the analytical description for the output frequency response when more severe nonlinear effects on the system response have to be considered, higher order system nonlinearities need to be included in the analytical description, and other nonlinear behaviours such as subharmonics may also need to be included. These problems are currently being investigated and the results will be reported in a later publication. However, the considerable significance of an analytical description for the output frequency response of a nonlinear system has been demonstrated in the present case study.

This study has shown that although the analytical expression (57) only includes terms up to third order of nonlinearity, given a_1, m, k, F_d, and Ω, the expression can be used to represent the relationship between $Y_2(j\Omega)$ and a_2, a_3 over a considerable range of the parameters. Because of this, over this range of a_2 and a_3, the analytical description (57) can be used directly in system analysis to study how the parameters affect the system output frequency response. Moreover, equation (57) can also be used for the design of the damping characteristic parameters. The basic idea is straightforward. Given a desired output frequency response which can be realised within the range of the parameters where the analytical relationship (57) is valid, the values of the parameters which make the system output reach the desired response can be determined from (57) via an optimisation procedure.

6. Conclusions

In this paper, an analytical description of the effects of system nonlinearities on the output frequency response has been investigated. A case study has been conducted where an analytical relationship between the output frequency response and the nonlinear damping characteristic parameters has been derived for a SDOF spring damper system. The derivations are based on the frequency domain analysis of nonlinear systems. Results from the simulation studies verify the theoretical analysis and demonstrate the effectiveness of the derived analytical relationship.

The basic ideas of this work can be extended to general situations to arrive at a comprehensive analytical description for the relationship between nonlinear system output frequency responses and model parameters. As demonstrated in the present study, this analytical relationship can be very powerful and can be used to considerably facilitate the analysis and design of a wide range of nonlinear engineering systems and structures.

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References

[1] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems*, New York: John Wiley Interscience Publications, 1980.

[2] W. J. Rugh, *Nonlinear System Theory – the Volterra /Wiener Approach*, Baltimore, MD: Johns Hopkins University Press, 1981.

[3] S. Boyd and L. O. Chua, Fading memory and the problem of approximating nonlinear operators with Volterra series. *IEEE Transactions on Circuits and Systems* 30 (1985) 1150-1161.

[4] S. A. Billings and S. Chen, Extended model set, global data and threshold model identification of severely nonlinear systems. *International Journal of Control* 50 (1989) 1897-1923.

[5] S. Chen and S. A. Billings, Representation of nonlinear systems: the NARMAX model. *International Journal of Control* 49 (1989) 1013-1032.

[6] D. Nauck, F. Klawonn, and R. Kruse, *Foundations of Neuro-Fuzzy Systems*, John Wiley & Sons, 1997.

[7] G. Duffing, *Erzwungene schwingungen bei weranderlicher eigenfrequenz*, Vieweg: Braunschweig, 1918.

[8] T. Matsumoto, M. Komuro, H. Kokubu, and R. Tokunaga, *Bifurcations: Sights, Sounds, and Mathematics*, Springer-Verlag, 1993.

[9] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra,* Academic Press, 1974.

[10] D. A. George, Continuous nonlinear systems. MIT Res. Lab. Electron., Tech. Rep.335, 1959.

[11] L. J. Tick, The estimation of transfer functions of quadratic systems. *Technom* 3 (1961) 503-507.

[12] K. I. Kim and E. J. Powers, A digital method of modelling quadratically nonlinear systems with a general random input. *IEEE Trans. Acoustics, Speech, Signal Processing*, ASSP-36 (1988) 1758-1769.

[13] J. C. Peyton-Jones and S. A. Billings, A recursive algorithm for computing the frequency response of a class of nonlinear difference equation models. *International Journal of Control* 50 (1989) 1925-1940.

[14] S. A. Billings and J. C. Peyton-Jones, Mapping nonlinear integro-differential equation into the frequency domain. *International Journal of Control* 54 (1990) 863-879.

[15] S. W. Nam and E. J. Powers, Application of higher order spectral analysis to cubicaly nonlinear system identification, *IEEE Transactions on Signal Processing* 42 (1994) 1746-1745.

[16] D. E. Adams, Frequency domain ARX model and multi-harmonic FRF estimators for nonlinear dynamic systems. *Journal of Sound and Vibration* 250 (2002) 935-950.

[17] Z. Q. Lang and S. A. Billings, Output frequency characteristics of nonlinear systems. *International Journal of Control* 64 (1996) 1049-1067.

[18] Z. Q. Lang and S. A. Billings, Output frequencies of nonlinear systems. *International Journal of Control* 67 (1997) 713-730.

[19] Z. Q. Lang and S. A. Billings, Evaluation of output frequency response of nonlinear systems under multiple inputs. *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing* 47 (2000) 28-38.

[20] S. A. Billings and Z. Q. Lang, A bound for the magnitude characteristics of nonlinear output frequency response functions, Part 1: Analysis and computation. *International Journal of Control* 65 (1996) 309-328.

[21] S. A. Billings and Z. Q. Lang, A bound for the magnitude characteristics of nonlinear output frequency response functions, Part 2: Practical computation of the bound for systems described by the NARX model. *International Journal of Control* 65 (1996) 365-384.

[22] S. A. Billings and Z. Q. Lang, Truncation of nonlinear system expansions in the frequency domain. *International Journal of Control* 68 (1997) 1019-1042.

[23] S. A. Billings and Z. Q. Lang, Nonlinear systems in the frequency domain: energy transfer filters. *International Journal of Control* 75 (2002) 1066-1081.

[24] V. Volterra, *Theory of Functions*, Blackie, 1930.

[25] A. K. Swain and S. A. Billings, Generalised frequency response function matrix for MIMO nonlinear systems. *International Journal of Control* 74 (2001) 829-844.

[26] K. Worden, G. Manson, and G. R. Tomlinson, A harmonic probing algorithm for the multi-input Volterra series. *Journal of Sound and Vibration* 201 (1997) 67-84.

[27] E. Bedrosian and S. O. Rice, The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian input. *Proceedings of the IEEE* 59 (1971) 1688-1707.

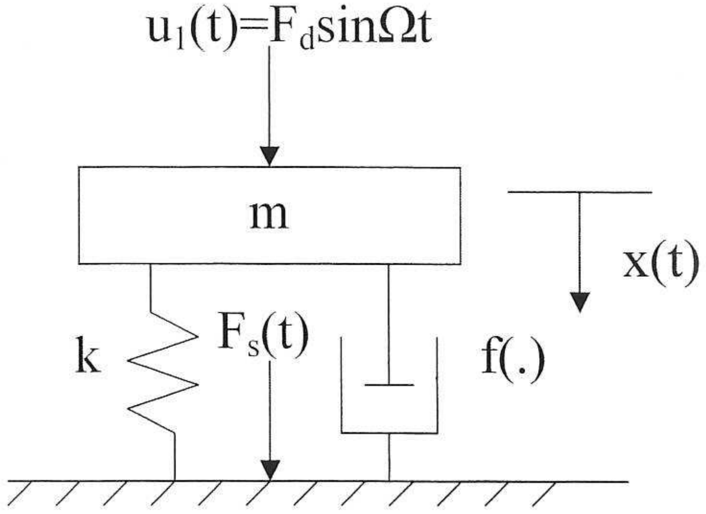

Figure 1 The single degree of freedom spring damper system considered in the study

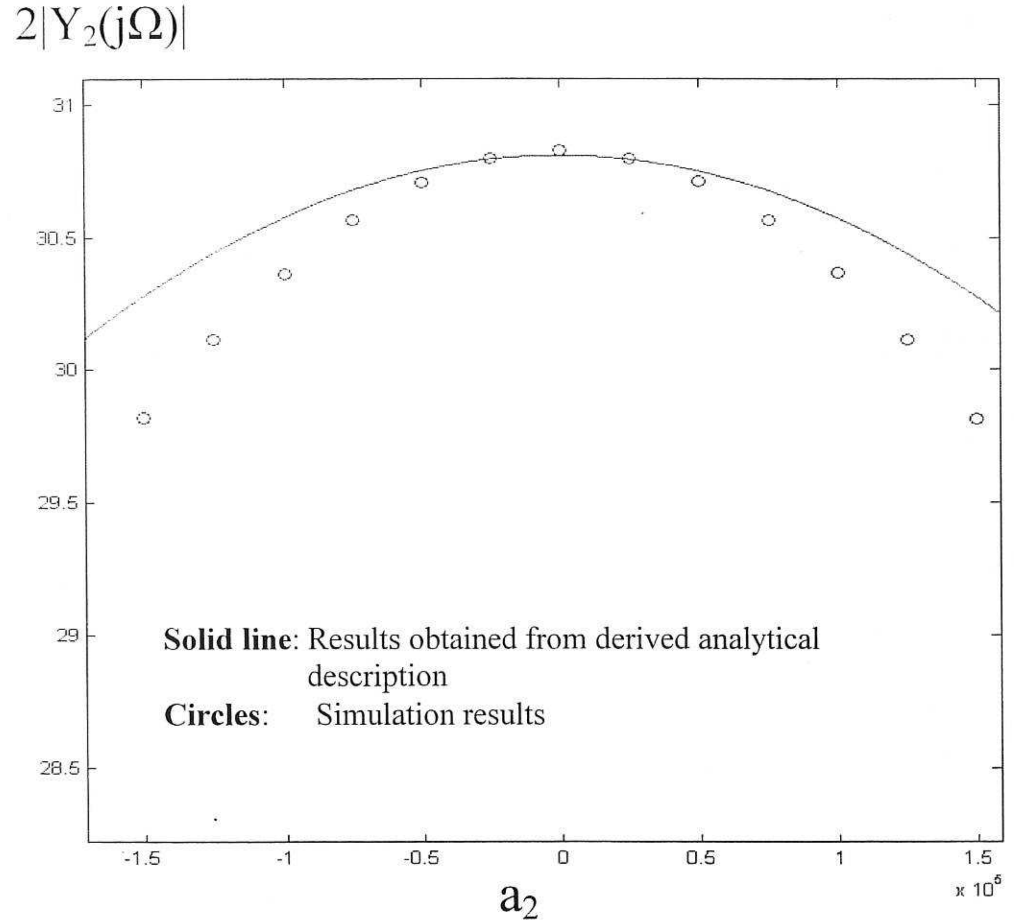

Figure 2 The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 20\, rad/s$, $F_d = 50$, and $a_3 = 0$

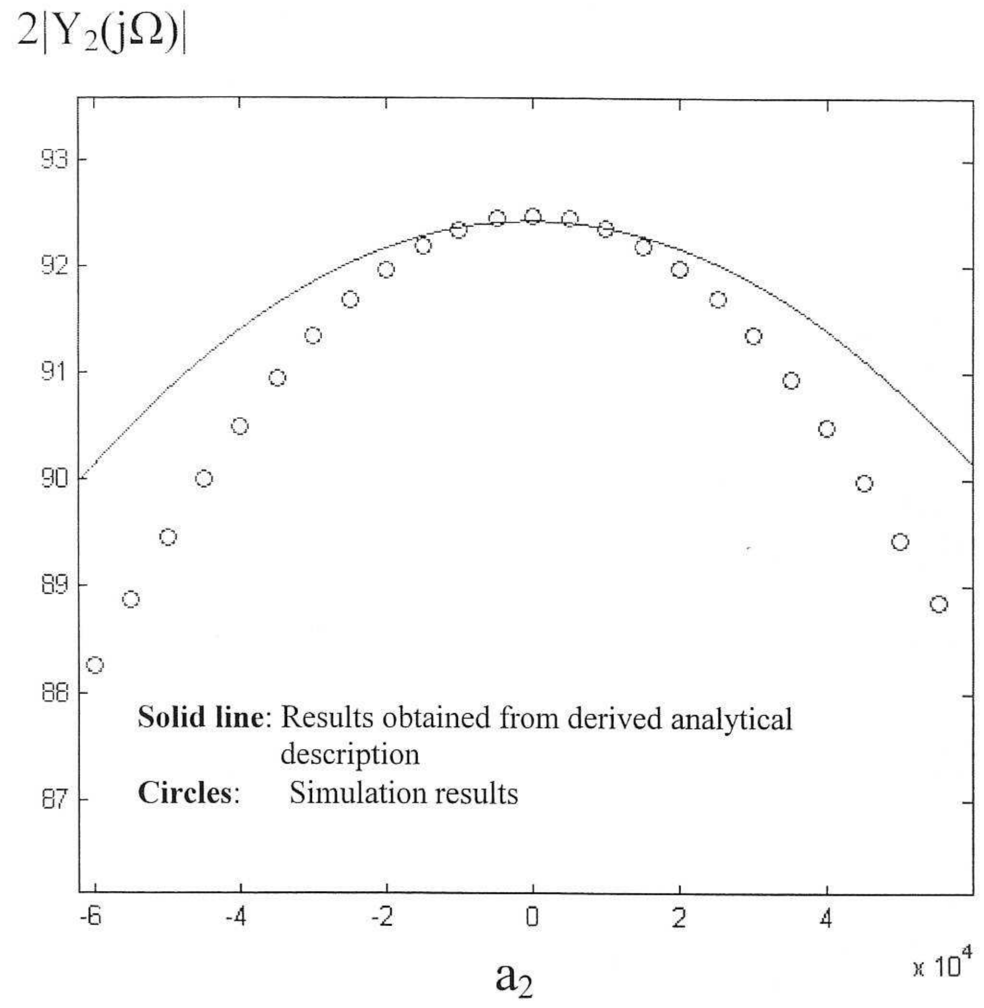

Figure 3. The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 20\,rad/s$, $F_d = 150$, and $a_3 = 0$

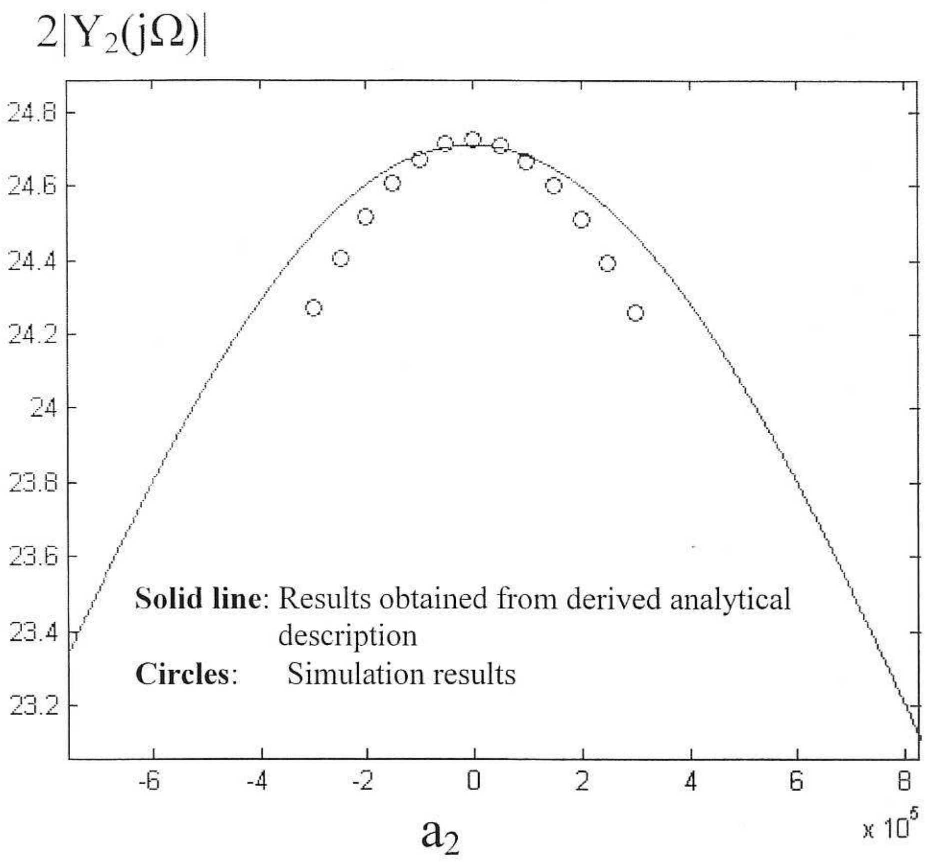

Figure 4 The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 50\, rad/s$, $F_d = 100$, and $a_3 = 0$

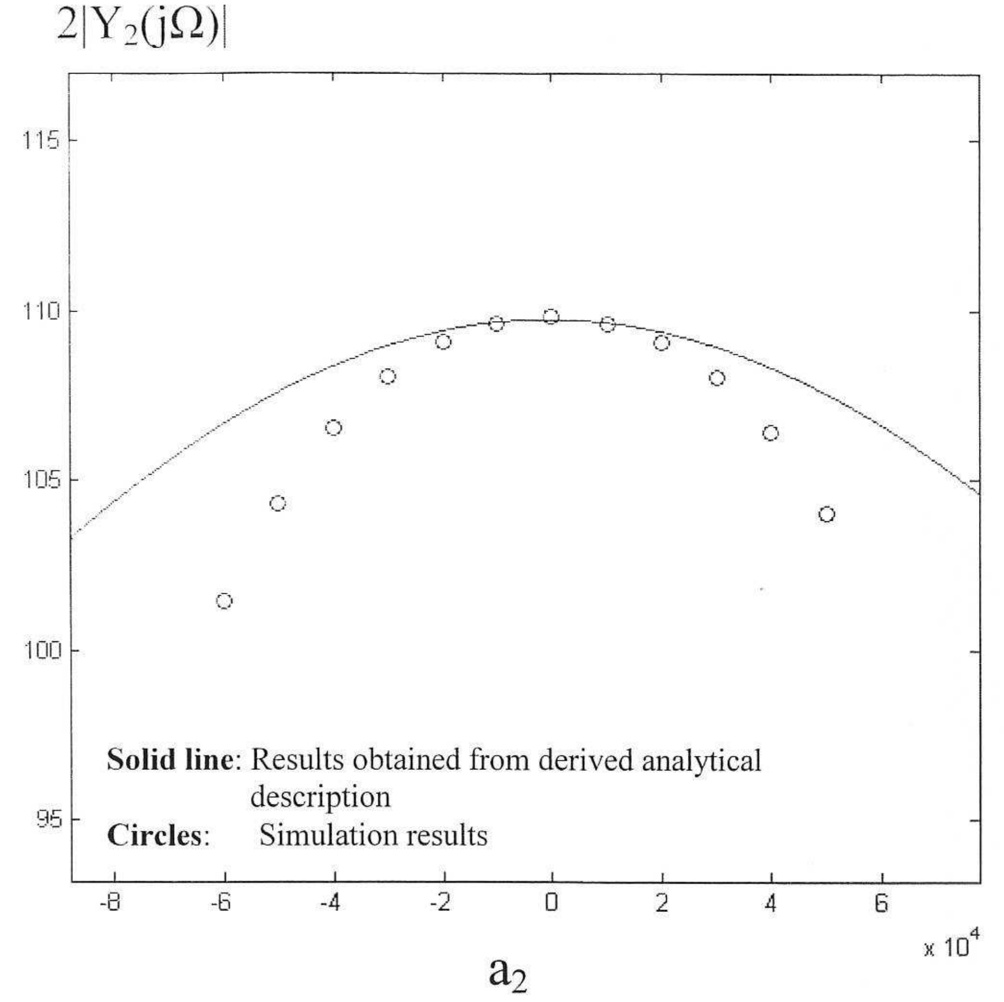

Figure 5 The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 10\, rad/s$, $F_d = 100$, and $a_3 = 0$.

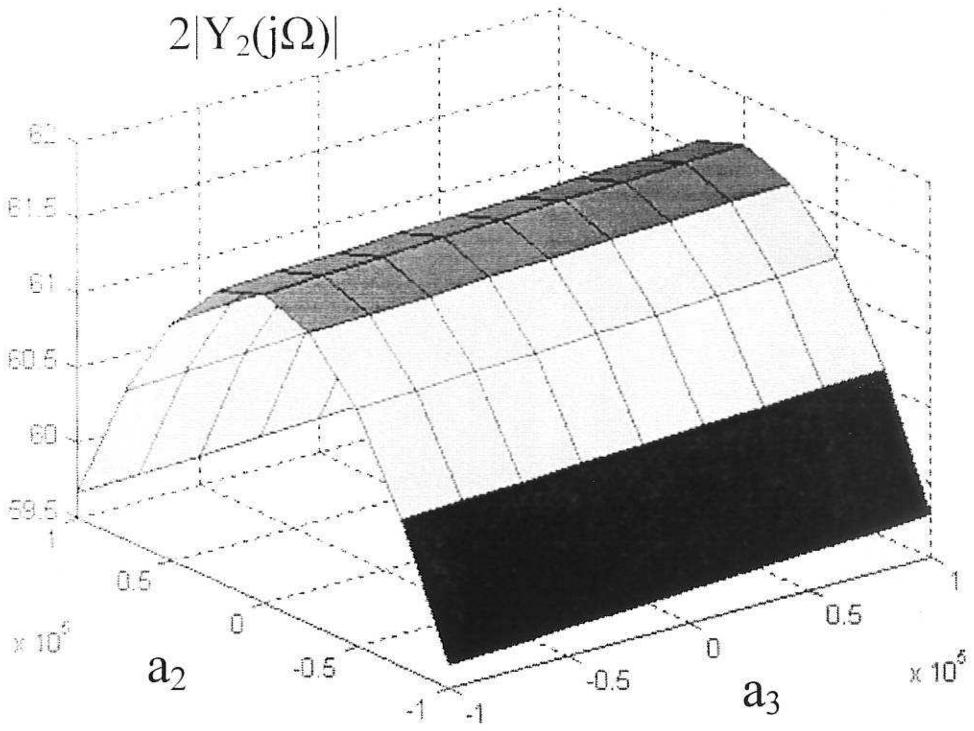

Figure 6 (a) The analytically determined relationship between the nonlinear damping characteristic parameters a_2, a_3 and the system output frequency response when $\Omega = 20\,rad/s$ and $F_d = 100$

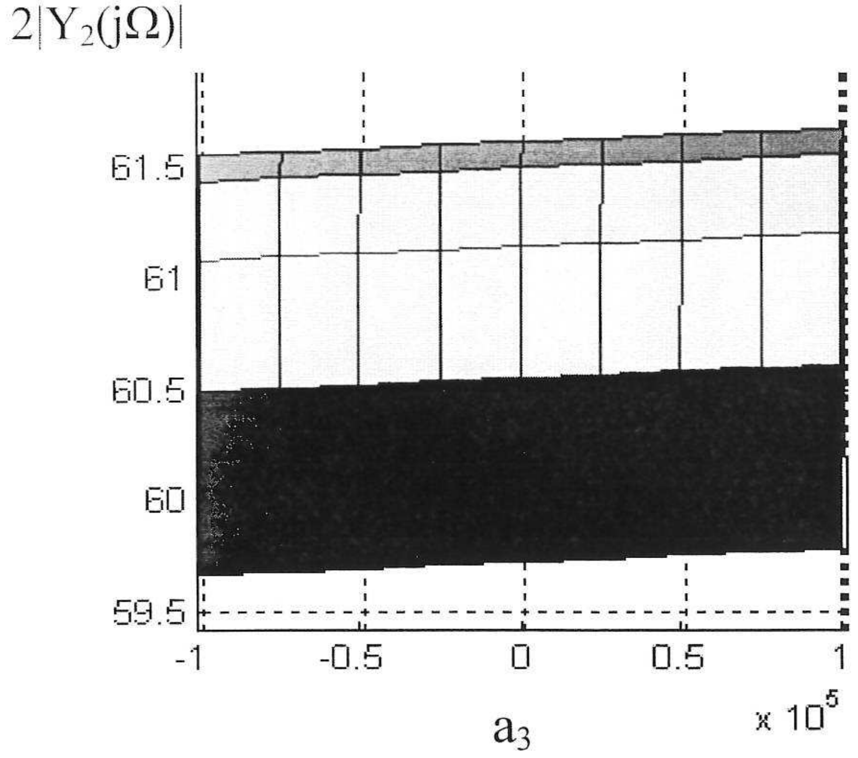

Figure 6 (b) The analytically determined relationship between the nonlinear damping characteristic parameters a_2, a_3 and the system output frequency response when $\Omega = 20\, rad/s$ and $F_d = 100$ observed from a different perspective

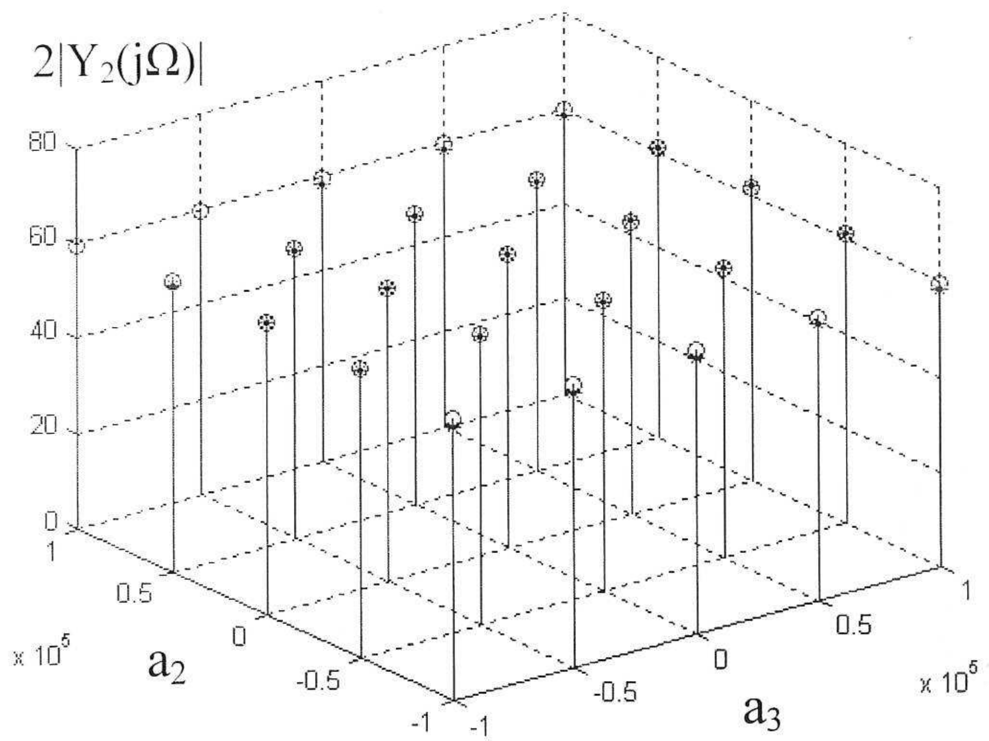

Figure 6 (c) A comparison of the analytically determined (circles) and the simulated (stars) system output frequency responses over a set of discrete points of (a_2, a_3) when $\Omega = 20\, rad/s$ and $F_d = 100$.