Proceedings Paper:

Fewster, Chris orcid.org/0000-0001-8915-5321, Ford, L. H. and Roman, Thomas A. (2014) Probability distributions for quantum stress tensors in two and four dimensions. In: Bičák, Jiří and Ledvinka, Tomáš, (eds.) Relativity and Gravitation:100 years after Einstein in Prague. Springer Proceedings in Physics . Springer , pp. 489-496.

Probability distributions for quantum stress tensors in two and four dimensions

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September 26, 2014

Abstract

The probability distributions for the smeared energy densities of quantum fields, in the two and four-dimensional Minkowski vacuum are discussed. These distributions share the property that there is a lower bound at a finite negative value, but no upper bound. Thus arbitrarily large positive energy density fluctuations are possible. In two dimensions we are able to give an exact unique analytic form for the distribution. However, in four dimensions, we are not able to give closed form expressions for the probability distribution, but rather use calculations of a finite number of moments to estimate the lower bound, and the asymptotic form of the tail of the distribution. The first 65 moments are used for these purposes. All of our four-dimensional results are subject to the caveat that these distributions are not uniquely determined by the moments. One can apply the asymptotic form of the electromagnetic energy density distribution to estimate the nucleation rates of black holes and of Boltzmann brains.

1 Introduction

There has been extensive work in recent decades on the definition and use of the expectation value of a quantum stress tensor operator. However, the semiclassical theory does not describe the effects of quantum fluctuations of the stress tensor around its expectation value.

One way to examine these fluctuations is through the probability distribution for individual measurements of a smeared stress tensor operator. This distribution was given recently for Gaussian averaged stress tensors operators in two-dimensional flat spacetime [3] using analytical methods, and more recently for averaged stress tensors in four-dimensional spacetime from calculations of a finite set of moments. (Throughout our discussion, all quadratic operators are understood to be normal-ordered with respect to the Minkowski vacuum state.)

1.1 Quantum Inequalities

Quantum inequalities are lower bounds on the *expectation values* of the smeared energy density operator in arbitrary quantum states [9], [10], [6], [8], [5], [2]. If we sample in time along the worldline of an inertial observer, the quantum inequality takes the form

$$\int_{-\infty}^{\infty} f(t)\, \langle T_{\mu\nu} u^\mu u^\nu \rangle \, dt \geq -\frac{C}{\tau^d}\,, \tag{1}$$

where $T_{\mu\nu} u^\mu u^\nu$ is the normal-ordered energy density operator, which is classically non-negative, t is the observer's proper time, and $f(t)$ is a sampling function with characteristic width τ. Here C is a numerical constant, typically small compared to unity, d is the number of spacetime dimensions, and we work in units where $c = \hbar = 1$.

Although quantum field theory allows negative expectation values of the energy density, quantum inequalities place strong constraints on the effects of this negative energy for violating the second law of thermodynamics [9], maintaining traversable wormholes [7] or warpdrive spacetimes [11]. The implication of Eq. (1) is that there is an inverse power relation between the magnitude and duration of negative energy density.

For a massless scalar field in two-dimensional spacetime, Flanagan [5] has found a formula for the constant C for a given $f(t)$ which makes Eq. (1) an optimal inequality. This formula is

$$C = \frac{1}{6\pi} \int_{-\infty}^{\infty} du \left(\frac{d}{du} \sqrt{g(u)} \right)^2 \,, \tag{2}$$

where $f(t) = \tau^{-1} g(u)$ and $u = t/\tau$. In four-dimensional spacetime, Fewster and Eveson [2] have derived an analogous formula for C, but in this case the bound is not necessarily optimal.

2 Shifted Gamma Distributions - 2D Case

In two-dimensional Minkowski spacetime, we determined the probability distribution for individual measurements, in the vacuum state, of the Gaussian sampled energy density to be

$$\rho = \frac{1}{\sqrt{\pi}\,\tau} \int_{-\infty}^{\infty} T_{tt}\, \mathrm{e}^{-t^2/\tau^2}\, dt\,. \tag{3}$$

This was achieved by finding a closed form expression for the generating function of the moments $\langle \rho^n \rangle$ of ρ, from which the probability distribution was obtained. The definition of the n'th moment of the distribution of a variable x is given by

$$a_n = \int x^n \, P(x)\, dx\,. \tag{4}$$

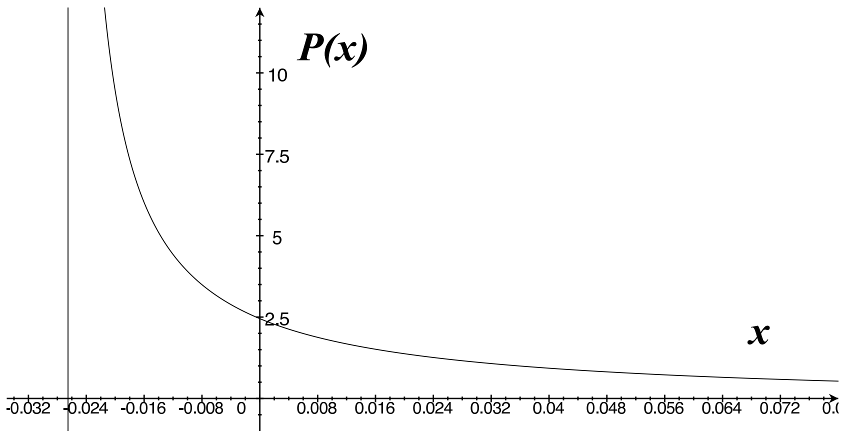

Figure 1: The graph of $P(x)$ vs x of the probability distribution function for the energy density, ρ, of a massless scalar field sampled in time with a Gaussian of width τ. Here $x = \rho\tau^2$. The distribution has an integrable singularity at the optimal quantum inequality bound $x = -x_0 = -1/12\pi$.

The resulting distribution is conveniently expressed in terms of the dimensionless variable $x = \rho\,\tau^2$ and is a shifted Gamma distribution:

$$P(x) = \vartheta(x + x_0)\frac{\beta^\alpha (x + x_0)^{\alpha-1}}{\Gamma(\alpha)} \exp(-\beta(x + x_0))\,, \tag{5}$$

with parameters

$$x_0 = \frac{1}{12\pi}, \qquad \alpha = \frac{1}{12}, \qquad \beta = \pi\,. \tag{6}$$

Here $x = -x_0$ is the lower bound of the distribution.

The lower bound, $-x_0$, for the probability distribution for energy density fluctuations in the vacuum is exactly Flanagan's optimum lower bound, Eq.(2), on the Gaussian sampled expectation value. As was argued in Ref. [3], this is a general feature, giving a deep connection between quantum inequality bounds and stress tensor probability distributions. The quantum inequality bound is the lowest eigenvalue of the sampled operator, and is hence the lowest possible expectation value and the smallest result which can be found in a measurement. That the probability distribution for vacuum fluctuations actually extends down to this value is more subtle and depends upon special properties of the vacuum state, and is implied by the Reeh-Schlieder theorem.

There is no upper bound on $P(x)$, as arbitrarily large values of the energy density can arise in vacuum fluctuations. Nonetheless, for the massless scalar field, negative values are much more likely; 84% of the time, a measurement of the Gaussian averaged energy density will produce a negative value. However, the positive values found the remaining 16% of the time will typically be much larger, and the average first moment of $P(x)$ will be zero.

Furthermore, the probability distribution for the two-dimensional stress tensor is uniquely determined by its moments, as a consequence of the Hamburger moment theorem [12]. This condition is a sufficient, although not necessary, condition for uniqueness, and is fulfilled by the moments of the shifted Gamma distribution.

3 The 4D Case

In four dimensions, the operators ρ_S, and ρ_{EM} all have dimensions of $length^{-4}$. Their probability distributions $P(x)$ are taken to be functions of the dimensionless variable

$$x = (4\pi\,\tau^2)^2\,A\,, \tag{7}$$

where A is the Lorentzian time average of ρ_S, and ρ_{EM}, where ρ_S and ρ_{EM} are the smeared energy density operators for the massless scalar field, and electromagnetic fields, respectively.

The distributions were calculated numerically from 65 moments [4] The situation here is less straightforward. In this case, the moments grow too rapidly to satisfy the Hamburger moment criterion. Unfortunately, this means that we cannot be guaranteed of finding a unique probability distribution $P(x)$ from these moments. These probability distributions share some of the main characteristics of their two-dimensional counterparts. They have a lower bound but no upper bound. Our our techniques allow us to give approximate lower bounds and the asymptotic forms of the tails of the distributions.

Our estimates for the lower bounds are

$$-\,x_0(\rho_{EM}) \approx -0.0472 \qquad -\,x_0(\rho_S) \approx -0.0236\,. \tag{8}$$

These are also estimates of the optimal quantum inequality bounds for each field. In contrast, the non-optimal bound for ρ_S, given by the method of Fewster and Eveson [2], is $-x_0(FE) = -27/128 \approx -0.21$, which is an order of magnitude larger.

It is of interest to note that the magnitudes of the dimensionless lower bounds, given in Eq. (8) are small compared to unity. The fact that the probability distribution has a long positive tail, and must have a unit zeroth moment and a vanishing first moment, implies that the total probability of a negative value to be substantial. The small magnitudes of $x_0(\rho_S)$ and $x_0(\rho_{EM})$ imply strong constraints on the magnitude of negative energy which can arise either as an expectation value in an arbitrary state, or as a fluctuation in the vacuum. They also imply that an individual measurement of the sampled energy density in the vacuum state is very likely to yield a negative value.

One can show that the asymptotic behavior of the tail of the probability distribution is determined by the moments, even if the exact probability distribution is not uniquely determined. Our fitted tail decreases asymptotically as

$$P_{\rm fit} \sim e^{-a x^{1/3}}\,, \tag{9}$$

where a is a constant. We are also able to show that no distribution with the same moments can have a tail which decreases at a faster rate than ours.

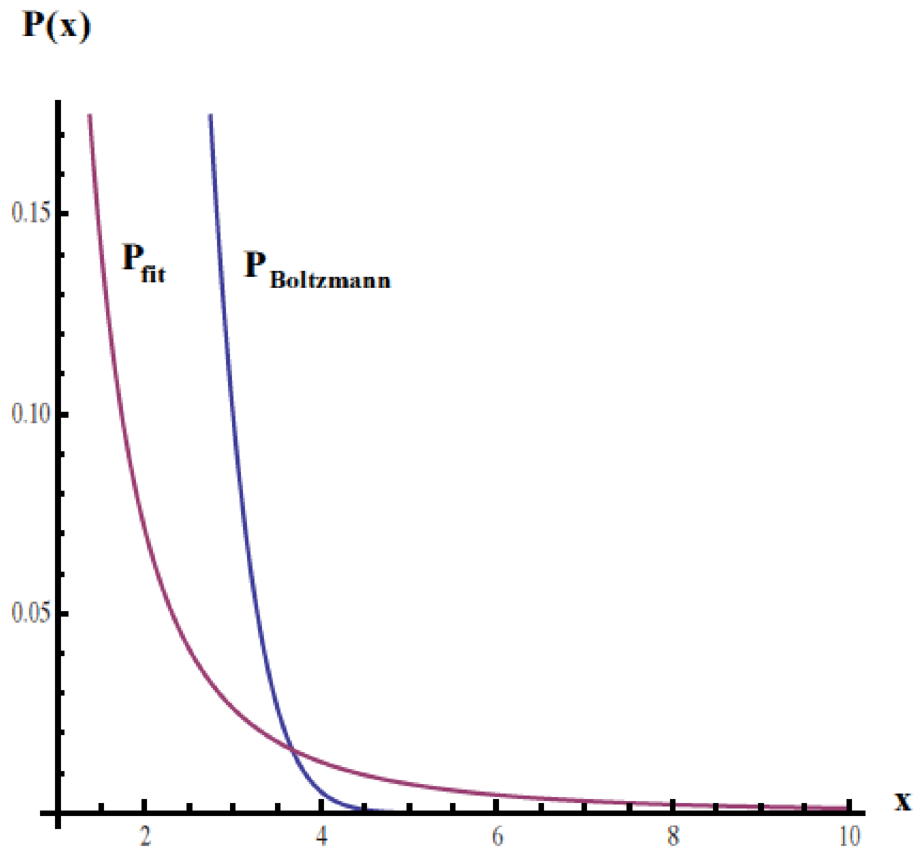

Figure 2: The figure shows a comparison of the asymptotic form of the tails of both our fitted distribution for vacuum fluctuations and for the thermal fluctuations described by the Boltzmann distribution. At high energies, vacuum fluctuations outweigh thermal fluctuations.

By contrast, the tail of a Boltzmann distribution for thermal fluctuations falls off as

$$P_{\text{Boltzmann}} \sim e^{-\beta x}\,, \tag{10}$$

where β is a constant. Therefore vacuum fluctuations outweigh thermal fluctuations at high energies.

3.1 Application: Black Hole Nucleation

The fact that the energy density probability distribution has a long positive tail implies a finite probability for the nucleation of black holes out of the Minkowski vacuum via large, though infrequent positive fluctuations (see Ref [4]). This probability cannot be too large, of course, or it will conflict with observation. Our estimate of the probability depends only on the asymptotic form of the tail. (One can use similar arguments to estimate the probability of "Boltzmann brains" [1] nucleating out of the vacuum.)

4 Summary

We have found that the probability distribution for vacuum fluctuations of the Gaussian-smeared energy density for a massless scalar field in two-dimensional spacetime is *uniquely* defined by a shifted gamma distribution. The distribution has a negative lower bound but no upper bound. It has an integrable singularity (i.e., a "spike") at the lower bound. In addition, we find that there is a deep connection between the lower bound of the distribution and the quantum inequalities. In fact the lower bound of the distribution coincides *exactly* with the *optimal* quantum inequality bound for a Gaussian sampling function, derived earlier by Flanagan.

The lower bound is very small in magnitude, but the probability distribution is large in the region between zero and the lower bound. As a result, rather surprisingly, the probability of obtaining a negative result in an individual measurement is 84%! Although the negative fluctuations are very frequent, they are small in magnitude. As a result, one would not expect to see large effects of negative energy (e.g., violations of the second law, wormholes, warpdrives, etc.) nucleating out of the vacuum. However, the distribution has a long positive tail, which guarantees that the frequent but small negative energy density fluctuations are balanced by the much rarer but larger positive energy fluctuations. Therefore, the expectation value of the energy density in the Minkowski vacuum state is zero. It is quite remarkable that the quantum inequalities which are bounds on the *expectation value* of the energy density in an *arbitrary* quantum state, should be so intimately related to the probability distribution of *individual* measurements of the energy density made in the *vacuum* state.

In four dimensions, we find similarities with the two-dimensional case, in that there is a lower bound but no upper bound. We are able to give numerical estimates of the lower bounds, i.e., the optimal bounds, and the asymptotic form of the tails. The lower bounds are negative with small magnitudes. However, our methods do not allow us to determine whether there is a "spike" at the lower bound, as in two dimensions. Nonetheless, the low magnitudes of the lower bounds indicate that a significant fraction of the probability must lie in the negative region. Therefore, as in the two-dimensional case, the probability of obtaining a negative value in an individual measurement is quite high. The long positive tail drops off more slowly than that of a Boltzmann distribution, which implies that vacuum fluctuations dominate over thermal fluctuations at high energies.

Unfortunately, it seems likely that it is not possible to uniquely determine the four-dimensional distributions from the moments alone, as the latter do not obey the Hamburger moment condition. Nonetheless, we are able to glean some information from the moments. For example, we can determine that no distribution with the same moments as ours can have a tail which decreases faster than ours. The asymptotic forms of the long positive tail allow us to estimate the probability of nucleation of (small) black holes and "Boltzmann brains" out of the vacuum.

Clearly further work can be done on this subject. One topic would be to see what additional information can be obtained from our calculated four-dimensional probability distributions, even if they cannot be uniquely determined from their moments. For example, does the "spike" behavior persist in four dimensions as well as in two, and what is its physical significance? Another would be to determine what the optimal quantum inequality bounds actually are. It would also be useful to try various sampling functions. Can the probability distributions and optimal bounds can be obtained by other methods which do not have the limitation of the ambiguities in the moment methods? There is more to do to explore the physical content of stress-tensor fluctuations.

5 References

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