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MOMENT APPROXIMATION TO A MARKOV MODEL OF BINARY ROUTE CHOICE

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ABSTRACT

The paper considers a discrete-time, Markov, stochastic process model of drivers' day-to-day evolving route choice, the evolving 'state' of such a system being governed by the traffic interactions between vehicles, and the adaptive behaviour of drivers in response to previous travel experiences. An approximating deterministic process is proposed, by approximating both the probability distribution of previous experiences—the "memory filter" —and the conditional distribution of future choices. This approximating process includes both flow means and variances as state variables. Existence and uniqueness of fixed points of this process are examined, and an example used to contrast these with conventional stochastic equilibrium models. The elaboration of this approach to networks of an arbitrary size is discussed.

1. INTRODUCTION

Stochastic network equilibrium approaches to modelling driver route choice in congested traffic networks are concerned with predicting network link flows and travel costs, corresponding to a fixed point solution to a problem in which (Sheffi, 1985):

— actual link travel costs are dependent on the link flows; and
— drivers' route choices are made according to a random utility model, their perceptual differences in cost represented by a known probability distribution.

The term Stochastic User Equilibrium (SUE) is typically used to describe such a fixed point.

A radically different approach to modelling this interaction between flow-dependent travel costs and travel choices was proposed by Cascetta (1989), and studied further by Davis & Nihan (1993), Cantarella & Cascetta (1995) and Watling (1996). This Stochastic Process (SP) approach differs in two major ways from SUE. Firstly, the SP model is specified as a dynamical process of the day-to-day adjustments in route choice made by drivers, in response to travel experiences encountered on previous days. The concept of long-term "equilibrium" is therefore related to an explicit adjustment process. Secondly, in the SP approach flows are modelled as stochastic quantities, in contrast to SUE in which they are regarded throughout as deterministic quantities. Equilibrium in the SP sense, if achieved, relates to an equilibrium probability distribution of network flows, relating to the probabilities of the alternative discrete flow states.

Certainly, since it is well-known that the flows on roads may vary considerably from day-to-day, it is not difficult to make a case that flows are more appropriately represented as stochastic variables. However, it is recognised that the SP approach makes a radical departure from conventional network equilibrium wisdom. Moreover, if we are interested only in the equilibrium behaviour of the SP, there seems no clear way of directly estimating it, without simulating the actual dynamical behaviour of the process (as suggested by Cascetta, 1989).

The work in the present paper was therefore motivated by two broad objectives: firstly, to gain an improved understanding of the relationship between the SUE and SP approaches, and secondly to examine the extent to which the equilibrium behaviour of the SP model may be estimated directly. Due to the limitations on space, and in order to make the analysis more accessible, the paper will restrict attention to the simplest case of a binary choice. (In section 6 the extension to general networks is briefly discussed—further details are available on request from the author).

2. STOCHASTIC PROCESS MODEL: SPECIFICATION AND NOTATION

Consider a network serving a single origin and destination with a fixed integer demand of q over some given time period of the day (e.g. peak-hour) of duration τ hours. Here, q is a known, deterministic, integer quantity that does not vary between days. Suppose that the origin and destination are connected by two parallel links/routes, link 1 and link 2. We wish to examine how the flows on these links will vary over time, i.e. days. We therefore let the random variables $F^{(n)}$ (n=1,2,…) denote the flow on link 1 on day n. Clearly the flow on link 2 will then be q- $F^{(n)}$, and so knowledge of $F^{(n)}$ is sufficient to characterise the state of the network on any given day. We suppose that these flows arise from the choices of individual drivers, and so may only take integer values. The demand-feasible flows on link 1 (denoted $f \in D$) are therefore 0,1,2,…,q.

Related to a flow f in a period of duration τ >0 hours, we define $f\tau^{-1}$ vehicles/hour to be the flow rate (similarly, $q\tau^{-1}$ is the origin-destination demand rate). We shall suppose that the actual cost (e.g. travel time) of travelling along link j at a given flow rate $f\tau^{-1}$ on link 1 is given by $c_j\left(f\tau^{-1}\right)$ (j=1,2), the $c_j(\cdot)$ being time-independent, known, deterministic functions. Denote the mean perceived cost of travel on link j at the end of day n by $U_j^{(n)}$ $(j = 1,2;\ n = 1,2,\ldots)$. These are random variables, whose stochasticity is—we shall assume—related entirely to the stochasticity in the flows, from the learning model:

$$U_j^{(n-1)} = \frac{1}{m}\sum_{i=1}^{m} c_j\left(F^{(n-i)}\,\tau^{-1}\right) \quad (j = 1,2;\ n = 1,2,\ldots) \tag{2.1}$$

where m is some given, finite, positive integer. That is to say, $U_j^{(n-1)}$ is an average cost over all drivers, based on the actual costs from previous days. Finally, suppose that conditional on the past, the q drivers select a route at the beginning of each day n independently of one another. The probability of a randomly selected driver choosing route 1 on day n, given mean perceived costs at the end of day n-1 of $(u_1^{(n-1)}, u_2^{(n-1)})$, is then assumed to be given by $p(u_1^{(n-1)}, u_2^{(n-1)})$, where $p(\cdot,\cdot)$ is a time-independent, known, deterministic function.

For a given initial condition $F^{(0)} = f^{(0)} \in D$, the evolution of this process can be written:

$$F^{(n)} \left| \left\{F^{(n-i)} : i = 1,2,\ldots,m\right\} \right. \sim \text{Binomial}\left(q,\, p(U_1^{(n-1)}, U_2^{(n-1)})\right) \tag{2.2}$$

where the $U_j^{(n-1)}$ $(j = 1,2)$ are related to the $\left\{F^{(n-i)} : i = 1,2,\ldots,m\right\}$ by (2.1). Cascetta (1989) has shown that under mild conditions, the above process converges to a unique stationary probability distribution regardless of the initial conditions. These are basically conditions for the process to be irreducible and Markov, for which it is sufficient that m be finite, and $p(\cdot,\cdot)$ give values strictly in the open interval (0,1).

3. APPROXIMATING DETERMINISTIC PROCESS AND EQUILIBRIUM CONDITIONS

3.1 Approximation of memory filter

Now for any f∈D, by standard laws of conditional probabilities we have:

$$\begin{aligned}\Pr(F^{(n)} = f) = \sum_{f_1\in D}\sum_{f_2\in D}\cdots\sum_{f_m\in D} &\Pr(F^{(n)} = f \,|\, F^{(n-1)} = f_1, F^{(n-2)} = f_2, \ldots, F^{(n-m)} = f_m) \times \\ &\Pr(F^{(n-1)} = f_1, F^{(n-2)} = f_2, \ldots, F^{(n-m)} = f_m).\end{aligned} \tag{3.1}$$

Since the dependence on the past is, from (2.1), only in the form of dependence on the $U_j^{(n-1)}$ $(j = 1,2)$, then if we define the set of implied demand-feasible mean perceived costs as

$$\Omega \equiv \left\{(u_1, u_2) : u_j = \tfrac{1}{m}\sum_{i=1}^{m} c_j(f_i\,\tau^{-1}) \ \ (j = 1,2)\ \text{ for } f_i \in D \ \ (i = 1,2,\ldots,m)\right\}$$

we can write (3.1) as

$$\Pr(F^{(n)} = f) = \sum_{(u_1,u_2)\in\Omega} \Pr(F^{(n)} = f \,|\, (U_1^{(n-1)}, U_2^{(n-1)}) = (u_1,u_2))\ \Pr((U_1^{(n-1)}, U_2^{(n-1)}) = (u_1,u_2)). \tag{3.2}$$

Now, it is proposed that an approximation to the joint distribution of $(U_1^{(n-1)}, U_2^{(n-1)})$ is, as $m \to \infty$,

$$\Pr((U_1^{(n-1)}, U_2^{(n-1)}) = (u_1, u_2)) \approx \begin{cases} 1 & \text{if } (u_1, u_2) = (E[U_1^{(n-1)}], E[U_2^{(n-1)}]) \\ 0 & \text{otherwise} \end{cases} . \tag{3.3}$$

Although intuition suggests that, from (2.1), as $m \to \infty$ then $\mathrm{var}(U_j^{(n-1)}) \to 0$ $(j = 1,2)$, which would support the approximation (3.3), a formal proof has yet to be obtained, although some suggestive points should be noted. In stationarity, $F^{(n-1)}, F^{(n-2)}, \ldots, F^{(n-m)}$ will have a common marginal probability distribution, and hence so will $c_j\left(F^{(n-1)}\tau^{-1}\right), c_j\left(F^{(n-2)}\tau^{-1}\right), \ldots, c_j\left(F^{(n-m)}\tau^{-1}\right)$ (j=1,2). If these were independent, then applying a Central Limit Theorem would establish the required result, but they are clearly correlated.

However,

$$\mathrm{var}(U_j^{(n-1)}) = \frac{1}{m^2}\sum_{i=1}^{m} \mathrm{var}\left(c_j(F^{(n-i)}\tau^{-1})\right) + \frac{2}{m^2}\sum_{i=1}^{m}\sum_{\substack{k=1 \\ (k<i)}}^{m} \mathrm{cov}\left(c_j(F^{(n-i)}\tau^{-1}), c_j(F^{(n-k)}\tau^{-1})\right).$$

The first term is the variance that would arise if the costs were independent over time, and the second term is the sum of cost covariances i-k time periods apart (i-k=1,2,…,m-1). Now for a typical choice probability function for $p(\cdot,\cdot)$ such as a random utility model, in which the probability of choosing an alternative is a decreasing function of the cost of that alternative, and cost functions in which $c_j(\cdot)$ is an increasing function of the flow rate on alternative j, we could guarantee that costs one time period apart will be negatively correlated. This is because an increased cost of alternative j on day i, will tend to reduce the number of users choosing j on day i+1, which in turn will tend to reduce the cost of alternative j on day i+1. Although this does not extend to other, more distant costs, which may be positively correlated, there would appear to be some possibilities here for future work examining conditions under which cost covariances may decay in magnitude over time, and/or of using the variance obtained in the independent case as some sort of bound.

The proposed approximation (3.3) then gives rise to the approximating process of:

$$\Pr(F^{(n)} = f) \approx \Pr(F^{(n)} = f \,|\, (U_1^{(n-1)}, U_2^{(n-1)}) = (E[U_1^{(n-1)}], E[U_2^{(n-1)}])) . \tag{3.4}$$

3.2 Approximation to the flow probability distribution

The approximation proposed in section 3.1 simplifies the process by saying that we need not condition on the whole cost probability distribution of the previous day (which describes the memory of all previous days) in order to determine approximately the flow probability distribution for the current day. The approximating process (3.4)/(2.1) does, however, still require computation of the evolution of the individual state probabilities, i.e. the whole flow probability distribution. Here we introduce a further approximation, which allows us to consider only the evolution of the first two moments of the flow probability distribution.

Now, by (2.1), and using a second order Taylor series approximation to $c_j(\cdot)$ in the neighbourhood of $E[F^{(n-i)}\tau^{-1}]$, we obtain:

$$\begin{aligned} E[U_j^{(n-1)}] = \frac{1}{m}\sum_{i=1}^{m} E[c_j(F^{(n-i)}\tau^{-1})] &\approx \frac{1}{m}\sum_{i=1}^{m}\left(c_j(E[F^{(n-i)}\tau^{-1}]) + \tfrac{1}{2}\mathrm{var}(F^{(n-i)}\tau^{-1}).c_j''(E[F^{(n-i)}\tau^{-1}])\right) \\ &= \frac{1}{m}\sum_{i=1}^{m}\left(c_j(\mu^{(n-i)}) + \tfrac{1}{2}\phi^{(n-i)}c_j''(\mu^{(n-i)})\right) \end{aligned} \tag{3.5}$$

where the mean flow rate, and variance in the flow rate, are given by

$$\mu^{(n)} = E[F^{(n)}\tau^{-1}] = \tau^{-1}E[F^{(n)}] \qquad \text{and} \qquad \phi^{(n)} = \mathrm{var}(F^{(n)}\tau^{-1}) = \tau^{-2}\,\mathrm{var}(F^{(n)}).$$

Therefore, with such an approximation, the right hand side of (3.4) can be written in terms of the evolution of the flow rate means and variances alone. In other words, in order to know the evolution of this approximating process, there is no need to compute the whole flow probability distribution, only its first two moments. Now, from (2.2) and the approximation (3.4), combined with standard properties of the Binomial, we have on the demand-side:

$$\begin{aligned} \mu^{(n)} &= \tau^{-1}E[F^{(n)}] \approx \tau^{-1}q\; p(E[U_1^{(n-1)}], E[U_2^{(n-1)}]) \\ \phi^{(n)} &= \tau^{-2}\mathrm{var}(F^{(n)}) \approx \tau^{-2}q\; p(E[U_1^{(n-1)}], E[U_2^{(n-1)}])\;\left(1 - p(E[U_1^{(n-1)}], E[U_2^{(n-1)}])\right) \end{aligned} \tag{3.6}$$

and so, letting $\bar{q} = q\tau^{-1}$ denote the origin-destination demand rate and combining (3.5)/(3.6), we end up with the approximating deterministic process given by the linked expressions:

$$\begin{cases} \mu^{(n)} = \bar{q}\; p\left(\frac{1}{m}\sum_{i=1}^{m} h_1(\mu^{(n-i)},\phi^{(n-i)}), \frac{1}{m}\sum_{i=1}^{m} h_2(\mu^{(n-i)},\phi^{(n-i)})\right) \\ \phi^{(n)} = \bar{q}\tau^{-1}\; p\left(\frac{1}{m}\sum_{i=1}^{m} h_1(\mu^{(n-i)},\phi^{(n-i)}), \frac{1}{m}\sum_{i=1}^{m} h_2(\mu^{(n-i)},\phi^{(n-i)})\right)\left(1 - p\left(\frac{1}{m}\sum_{i=1}^{m} h_1(\mu^{(n-i)},\phi^{(n-i)}), \frac{1}{m}\sum_{i=1}^{m} h_2(\mu^{(n-i)},\phi^{(n-i)})\right)\right) \end{cases} \tag{3.7}$$

where

$$h_1(\mu,\phi) = c_1(\mu) + \frac{\phi}{2}.c_1''(\mu) \qquad \text{and} \qquad h_2(\mu,\phi) = c_2(\bar{q}-\mu) + \frac{\phi}{2}.c_2''(\bar{q}-\mu) \tag{3.8}$$

where we have used (in the expression for h_2) the fact that the mean flow rates on the two routes must sum to $\bar{q}$, and that by the binomial assumption the (absolute) flow variances on the two routes must be the same, hence the flow rate variances must also be the same (ϕ).

3.3 Equilibrium conditions

Stationary points of the process (3.7) are obtained by setting

$$\mu^{(n-i)} = \mu^* \quad \text{and} \quad \phi^{(n-i)} = \phi^* \qquad (i = 1,2,\ldots,m)$$

and in this case the summations in (3.7) drop out to yield:

$$\begin{cases} \mu^* = \bar{q}\, p\left(h_1(\mu^*,\phi^*), h_2(\mu^*,\phi^*)\right) \\ \phi^* = \bar{q}\tau^{-1}\, p\left(h_1(\mu^*,\phi^*), h_2(\mu^*,\phi^*)\right)\left(1 - p\left(h_1(\mu^*,\phi^*), h_2(\mu^*,\phi^*)\right)\right) \end{cases} \tag{3.9}$$

A flow allocation (μ^*,ϕ^*) satisfying the fixed point conditions (3.9) will be termed a Second Order Stochastic User Equilibrium, or Second Order SUE for short. It will also be denoted SUE(2), the 2 relating to the 2nd order approximation made in forming (3.5). It is trivial to show that had a first order approximation been made here instead, we would have obtained a conventional SUE for μ^*, i.e. a SUE(1) is a conventional SUE.

4. Example

Consider a problem with an origin-destination demand rate of $\bar{q}$ = 200 vehicles/hour over a period of duration $\tau > 0$ hours, and relationships between cost and flow rate of the form:

$$c_1\left(f\tau^{-1}\right) = 10\left(\frac{f\tau^{-1}}{100}\right)^6 \qquad c_2\left(\bar{q} - f\tau^{-1}\right) = 2. \tag{4.1}$$

(For information, this problem has a unique Wardrop equilibrium, at a flow rate on route 1 of approximately 76.5.) Let us suppose that the choice probability has the logit form:

$$p\left(u_1, u_2\right) = (1 + \exp(\theta(u_1 - u_2)))^{-1} \qquad (\theta > 0) \tag{4.2}$$

with dispersion parameter $\theta = 0.3$. This problem has a unique SUE at $f \approx 82.6$.

Now, this problem does not have a unique SUE(2) solution, but rather one that varies with the value of the time period duration τ ; for a given τ, there is a unique SUE(2) solution. Such solutions have been determined numerically, by a fine grid search technique, for various given values of τ, and the resulting mean flow rates are illustrated in Figure 1. The horizontal dashed line in the figure is the SUE solution, which is invariant to τ. As we might have expected from (3.9), as $\tau \to \infty$ then the SUE(2) mean flow rate approaches the SUE solution. This may be regarded as an asymptotic ("law of large numbers") result, in a similar spirit to that of Davis and Nihan (1993), since for fixed $\bar{q}$ by letting $\tau \to \infty$, we are effectively letting the number of drivers tend to infinity.

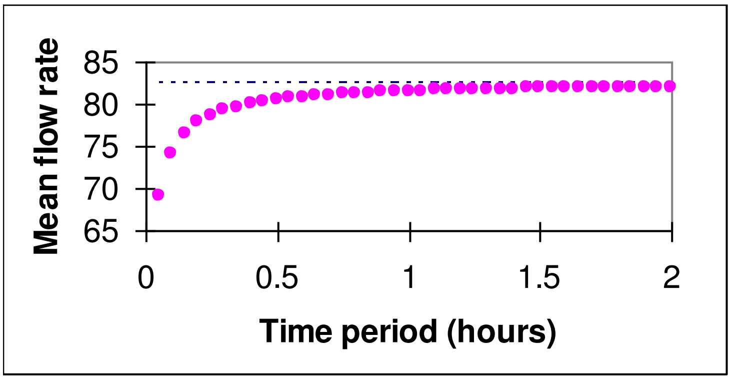

Figure 1: SUE(2) mean flow rate as a function of time period duration τ

5. Existence and Uniqueness Conditions

It is pertinent to ask under what conditions will there exist solutions to the SUE(2) model proposed in (3.9), and under what conditions will there be a unique solution. The existence question is quite straightforward to answer, using tools similar to those used by Smith (1979) for deterministic user equilibrium and Cantarella & Cascetta (1995) for conventional SUE. Basically, if all the functions involved—i.e. the choice probability function $p(u_1, u_2)$ and the modified cost-flow functions given by (3.8)—are continuous, then Brouwer's fixed point theorem (Baiocchi & Capelo, 1984) will ensure the existence of at least one solution to the fixed point problem (3.9), since the mapping implied by the right hand side of (3.9) is to a closed, bounded, convex set. In order for the modified cost-flow functions to be continuous, it is clearly sufficient that the original cost-flow functions are twice continuously differentiable throughout their range.

The uniqueness question needs a little more thought. In the introduction of the SUE(2) problem (expressions (3.7)-(3.9)), it was notationally convenient to express the problem purely in terms of the flow mean and variance on route 1. However, let us now assume the functions h_1 and h_2 given by (3.8) are instead replaced by functions of (μ_1, ϕ_1) and (μ_2, ϕ_2), respectively the flow rate mean and variance on routes 1 and 2, i.e.

$$\tilde{h}_j(\mu_j, \phi_j) = c_j(\mu_j) + \frac{\phi_j}{2} c_j''(\mu_j) \qquad (j = 1,2)\,. \tag{5.1}$$

Letting

$$\Omega(\boldsymbol{\mu}, \boldsymbol{\phi}) = p(\tilde{h}_1(\mu_1, \phi_1), \tilde{h}_2(\mu_2, \phi_2)) \text{ where } \boldsymbol{\mu} = (\mu_1, \mu_2) \text{ and } \boldsymbol{\phi} = (\phi_1, \phi_2) \tag{5.2}$$

then the SUE(2) conditions on $(\boldsymbol{\mu}, \boldsymbol{\phi})$ are:

$$\mu_1^* = \bar{q}\ \Omega(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*) \qquad \phi_1^* = \bar{q}\tau^{-1}\,\Omega(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*)(1 - \Omega(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*)) \tag{5.3}$$

with

$$\mu_2^* = \bar{q} - \mu_1^* \quad \text{and} \quad \phi_2^* = \phi_1^* \ . \tag{5.4}$$

Writing the condition on ϕ_1^* in (5.3) in terms of μ_1^* (similarly, for ϕ_2^* and μ_2^* from (5.4)), yields:

$$\phi_j^* = \frac{\mu_j^*}{\tau} - \frac{(\mu_j^*)^2}{\bar{q}\tau} \qquad (j = 1,2) \ . \tag{5.5}$$

Now, (5.5) holds only at SUE(2) solutions, i.e. at any solution to the fixed point problem (3.10). However, if this condition were imposed at all points, i.e.:

$$\phi_j = \frac{\mu_j}{\tau} - \frac{\mu_j^{\,2}}{\bar{q}\tau} \qquad (j = 1,2) \tag{5.6}$$

then under (5.6), we would then be able to regard the modified cost-flow functions (5.1) as functions of the μ_j only:

$$\frac{\partial \tilde{h}_j}{\partial \mu_j} = c_j'(\mu_j) + \frac{1}{2}\frac{d\phi_j}{d\mu_j} c_j''(\mu_j) + \frac{\phi_j}{2} c_j'''(\mu_j) \qquad (j = 1,2) \tag{5.7}$$

which, in view of (5.6), may be expressed after some rearrangement as:

$$\frac{\partial \tilde{h}_j}{\partial \mu_j} = \left(c_j'(\mu_j) - \frac{\mu_j}{\bar{q}\tau} c_j''(\mu_j) \right) + \tfrac{1}{2}\tau^{-1} c_j''(\mu_j) + \frac{\phi_j}{2} c_j'''(\mu_j) \qquad (j = 1,2) \ . \tag{5.8}$$

Now let us consider separately two possibilities for the cost-flow functions, firstly that they are linear and secondly that they are non-linear. If they are linear and increasing, then all derivatives but the first are zero, and (5.8) reduces to one term which is positive. In the second case, let us assume they are non-linear on link j (say) and satisfy the following conditions:

$$c_j''(x) > 0 \quad \text{for all} \ \ x > 0 \tag{5.9}$$

$$c_j'''(x) \geq 0 \quad \text{for all} \ \ x \geq 0 \tag{5.10}$$

$$\frac{x\, c_j''(x)}{c_j'(x)} \leq \bar{q}\tau \quad \text{for all} \ \ x \geq 0 \tag{5.11}$$

(For example, in the case of power-law functions $c_j(x) = \alpha_a + \gamma_a x^{n_a}$, then conditions (5.9)-(5.11) hold if $\alpha_a \geq 0$, $\gamma_a > 0$, and $n_a = 1$ or $2 \leq n_a \leq 1 + \bar{q}\tau$.) Under conditions (5.9)-(5.11), it can be seen that the first (bracketed) term and third term in (5.8) are non-negative, and the second term positive, and so the derivative overall is positive.

Hence, under (5.6), in both the linear and non-linear cases, the modified cost-flow functions are strictly increasing in the mean flow rates, and so the vector function

$$\left(\tilde{h}_1(\mu_1,\phi_1), \tilde{h}_2(\mu_2,\phi_2)\right) = \left(\tilde{h}_1\left(\mu_1, \frac{\mu_1}{\tau} - \frac{\mu_1^{\,2}}{\bar{q}\tau} \right), \tilde{h}_2\left(\mu_2, \frac{\mu_2}{\tau} - \frac{\mu_2^{\,2}}{\bar{q}\tau} \right) \right) = (\hat{h}_1(\mu_1), \hat{h}_2(\mu_2)) \ \text{(say)} \tag{5.12}$$

is strictly monotonically increasing in (μ_1,μ_2). This holds for any $(\boldsymbol{\mu},\boldsymbol{\phi})$ related by (5.6), and so certainly holds at a SUE(2) solution $(\boldsymbol{\mu}^*,\boldsymbol{\phi}^*)$, which is guaranteed to satisfy (5.5). A slight complication is that in view of (5.9), this result only applies to the case where all μ_j^* are non zero ((5.9) notably excludes the case x=0). However, this complication may be overcome by requiring the path choice probability function p to produce a result strictly in the open interval (0,1), thus excluding the possibility of an assignment with some $\mu_j^* = 0$ being a SUE(2) solution. Such a condition is satisfied by conventional logit and probit choice models, for example.

With this monotonicity condition, we can apply similar arguments to those used by Cantarella and Cascetta (1995) to establish SUE uniqueness. We shall require the choice probability function $\boldsymbol{\zeta}(\mathbf{u}) = (\, p(u_1,u_2),1-p(u_1,u_2))$ to be monotonically non-increasing in $\mathbf{u} = (u_1,u_2)$, as satisfied by random utility models that are 'regular' (in the sense that the probability distribution of perceptual errors is independent of (u_1,u_2), as in logit choice or probit choice with a constant covariance matrix). Letting $\hat{\mathbf{h}}(\boldsymbol{\mu}) = (\hat{h}_1(\mu_1),\hat{h}_2(\mu_2))$ be given by the right hand side of (5.12), then let us suppose that two SUE(2) solutions do, on the contrary, exist:

$$\mathbf{h}' = \hat{\mathbf{h}}(\boldsymbol{\mu}') \text{ and } \mathbf{h}'' = \hat{\mathbf{h}}(\boldsymbol{\mu}'') \quad \text{where } \boldsymbol{\mu}' = \bar{q}\boldsymbol{\zeta}(\mathbf{h}') \text{ and } \boldsymbol{\mu}'' = \bar{q}\boldsymbol{\zeta}(\mathbf{h}'').$$

Then

$$(\boldsymbol{\mu}'-\boldsymbol{\mu}'')^T(\mathbf{h}'-\mathbf{h}'') = \bar{q}\,(\boldsymbol{\zeta}(\mathbf{h}')-\boldsymbol{\zeta}(\mathbf{h}''))^T(\mathbf{h}'-\mathbf{h}'') \le 0 \qquad (5.13)$$

since $\boldsymbol{\zeta}$ is monotonically non-increasing, by hypothesis. But we also have:

$$(\boldsymbol{\mu}'-\boldsymbol{\mu}'')^T(\mathbf{h}'-\mathbf{h}'') = (\boldsymbol{\mu}'-\boldsymbol{\mu}'')^T(\hat{\mathbf{h}}(\boldsymbol{\mu}')-\hat{\mathbf{h}}(\boldsymbol{\mu}'')) > 0 \qquad (5.14)$$

since we have established above that $\hat{\mathbf{h}}(\boldsymbol{\mu})$ is monotonically increasing (and note that here, we only use the monotonicity condition at a SUE(2) mean flow solution) . Since (5.13) and (5.14) together give a contradiction, we can conclude that only one SUE(2) solution may exist.

6. CONCLUSION AND FURTHER RESEARCH

The SUE(2) model derived, in approximating the equilibrium behaviour of a SP approach, goes some way to achieving the objectives stated in the Introduction. Firstly, it gives the potential to derive efficient algorithms for directly estimating equilibrium behaviour, without regard to the underlying dynamical process. Secondly, by providing an intermediate modelling paradigm, it provides an insight into the relationship between the SP approach and conventional stochastic network equilibrium (an SUE(1)).

The extension of this work to general networks is a natural step to consider. The author has strong evidence that this is achievable, though with rather greater complexity: for example, (2.2) becomes a multinomial route flow distribution, (3.6) then involves route flow covariance terms, and the step to (3.7) (transformation from the route flow to link flow domain) is no longer trivial. Existence and uniqueness of SUE(2) may be established in an analogous way, but with the $\hat{h}$ expressions in (5.12) now being considered functions of the mean link flows disaggregated by origin-destination movement. One efficient, heuristic solution algorithm is derived by repeated application of the "method of successive averages" (Sheffi, 1985) to the modified costs (5.1), conditional on the link flow variances.

In parallel with this extension to more general networks, more elaborate approximations to the SP model may be considered—for example, relating equilibrium conditions to the assumed learning model (a dependence noted by Cascetta, 1989, in the context of the SP approach).

A third area of further research is to extend the modelling capabilities, such as the incorporation of stochastic variation in travel demand, by including a no-travel option with a user-specified choice probability, and the extension to multiple driver classes with different learning and predictive capabilities (as may occur if some drivers are provided with travel information).

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