Article:
Watling, DP (2002) A second order stochastic network equilibrium model. I: theoretical foundation. Transportation Science, 36 (2). 149 - 166. ISSN 0041-1655

https://doi.org/10.1287/trsc.36.2.149.560

A SECOND ORDER STOCHASTIC NETWORK EQUILIBRIUM MODEL
I: THEORETICAL FOUNDATION

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Abstract – Existing models of stochastic network equilibrium route choice in transport networks are able to represent exogenously-specified variations in drivers' actual or perceived travel costs, but assume throughout that flows are deterministic. In this paper, a new notion of equilibrium is presented based on stochastic flow variables, in which the impact of variable flows on the variability in travel costs is endogenously handled. Firstly, a very general notion of equilibrium is deduced as a fixed point condition on the joint probability distribution of network flows. Then, an approximation to this condition is derived, which operates by equilibrating moments of order n and below of the joint flow probability distribution, and is termed a Generalised Stochastic User Equilibrium of order n, being denoted GSUE(n). The GSUE(1) model is seen to be a conventional Stochastic User Equilibrium. The paper goes on to focus on the second order model, GSUE(2). Conditions are presented to guarantee the existence of GSUE(2) solutions. Conditions are deduced to guarantee (a) uniqueness of solutions in networks with a single inter-zonal movement, and (b) proximity of solutions in networks with multiple inter-zonal movements. Finally, a simple example is presented.

INTRODUCTION

The conventional stochastic network equilibrium problem is concerned with predicting link flows and travel times/costs over a network, corresponding to a fixed point solution to a problem in which:

(a) actual link travel times/costs are dependent on the link flows, and

(b) drivers' route choices are made according to a random utility model, their perceptual differences in evaluating cost (=–utility) represented by a known probability distribution.

Daganzo & Sheffi (1977) appear to be the first to have formally proposed the term Stochastic User Equilibrium (SUE) to describe such a fixed point solution, the word 'user' intended to emphasise the non-cooperative behaviour of individuals in their route selection. The properties of this model have been extensively investigated by Sheffi (1985).

An important point to note about the SUE approach is that the flow variables—whether link or route flows—are assumed to be deterministic quantities. Provided this assumption is considered reasonable, then this paper makes no dispute about the appropriateness of the SUE model. In reality, however, it is well-known that the flows on roads may vary considerably from day-to-day, and so it is not difficult to make a case that flows are more appropriately represented as stochastic variables. In such a setting, it is reasonable to ask whether the SUE model still characterises some useful feature of long-term behaviour.

An apparently reasonable, intuitive argument in support of the SUE model runs thus. The choice probabilities corresponding to the random utility component of an SUE model (component (b) above) describe the aggregate proportions of drivers on each inter-zonal movement that will use the alternative routes available at given mean cost levels. Let us suppose that this description of deterministic, aggregate-level behaviour may be extended to characterise a particular form of stochastic, disaggregate-level behaviour, where individuals independently and at random choose between the alternative routes with common probabilities equal to the choice proportions of the random utility model. Intuition suggests that as, in the disaggregate model, individuals are choosing at random with common probabilities, then the aggregate stochastic behaviour of route flows on each inter-zonal movement will be characterised by a multinomial probability distribution. Therefore, following this reasoning, the link with the aggregate deterministic behaviour of the SUE model is that this latter is simply representing the mean aggregate flows from such a multinomial distribution.

In the special case in which actual travel times/costs are independent of flow and deterministic (the so-called Stochastic Network Loading Problem), the reasoning above is well-founded (Daganzo, 1977). In this case, the coincidence with multinomial mean flows is exact, and such mean route flows—once derived—may be used to compute higher order moments such as variances and covariances, through standard properties of the multinomial distribution.

However, with a weakening of these assumptions to permit actual costs that are flow-dependent, the justification for assuming SUE to represent mean flows is no longer clear. Indeed, there are strong theoretical arguments to suggest that it will be a systematically biased estimator of mean conditions, and that this bias will produce different effects on different parts of the network. In brief, there are two reasons why this may occur.

Firstly, if flows are stochastic then actual costs will also be stochastic (through the link performance functions), regardless of any additional perceptual differences between drivers. Since it is reasonable to assume drivers form their own estimates of actual cost from experience, then these estimates will also be stochastic, and so therefore will the choice probabilities. Thus, although there is a case for aggregate choice behaviour conditional upon experienced costs to be multinomially distributed, the unconditional probability distribution of choices/flows (which must surely underlie any notion of equilibrium in this setting) will not in general be multinomial, since the choice probabilities are not fixed.

Secondly, even if drivers' experience of the network is sufficiently long that the mean of their experiences approximates the long run expected costs, then—for networks with non-linear link cost-flow performance functions—the SUE flows and costs will not represent the appropriate long-run expectations under the model assumed (as noted previously by Cascetta, 1989). Since, even if SUE link flows were the appropriate mean flows, then the SUE costs (which are obtained by substituting these flows in the cost-flow relationships) would represent the costs at mean flows, which for non-linear cost-flow functions are different from the mean costs. Conversely, if the appropriate mean costs were used in the SUE choice model, the flows arising would not be the appropriate mean flows. The cost bias will not, in general, balance out when taking route cost differences (as one does in a random utility model), since the magnitude of the bias in estimating the mean costs by SUE costs is dependent on the form of the cost-flow relationships, which varies across the network.

Therefore, the objective of this paper is to formulate a generalised version of SUE, in which stochastic variation in both flow and actual cost variables is represented, with the stochasticity endogenous to the equilibration process. Although previous authors have incorporated the effect of stochastic actual costs (Mirchandani and Soroush, 1987), or of stochastic capacities that affect the actual costs (Arnott et al, 1991; Van Berkum & Van der Mede, 1993; and Emmerink et al, 1995), these random elements were all assumed to be exogenously specified, and no relationship was made between the variability in actual costs and the variability in flows (the latter being assumed deterministic). Bell (1991), in an unpublished note, studied the use of independent Poisson distributions to represent the variability in link/route flows in a network with two parallel routes, and used a second order approximation to approximate the effect of this variability on mean costs. He did not, however, operationalise this approach for general networks; in particular, it does not seem possible with this approach to address the issue of covariances in flows between links that logically arises because of links that are part of the same route (positive correlation) or part of alternative routes (negative correlation). Cascetta (1989) and Cantarella & Cascetta (1995) did indeed take proper, consistent account of flow and cost variability for general networks, but at the expense of deviating from fixed point notions of static network equilibrium. They formulated their model as a discrete time stochastic process, with the corresponding “equilibrium” behaviour now a rather more complex condition on the dynamics of sequences of stochastic choices. Such an approach must therefore be regarded as an alternative to, rather than an extension of, SUE; in general, little is known of the relationship between stochastic process and SUE approaches.

The paper is structured as follows. Following basic notation and definitions (section 1), in section 2 equilibrium conditions (i.e. fixed point conditions for the joint probability distribution of network flows) are deduced for a generalised version of SUE in which flows are stochastic. A generalised concept of stochastic equilibrium, denoted GSUE(n), is subsequently introduced in section 3, based on an approximation to these conditions in which the order n and lower moments of the joint flow probability distribution are equilibrated. The remainder of the paper is devoted to studying a particular member of this family, GSUE(2), detailed attention being paid to the existence (section 4) and uniqueness (section 5) of GSUE(2) solutions. Finally, a simple illustrative example is given, and paths for further research identified.

1. NOTATION

A network is considered as a directed graph consisting of A links indexed a=1,2,...,A, and W inter- zonal (origin-destination) movements indexed k=1,2,…,W. The N possible routes that pass through a link at most once, across all such inter-zonal movements, are indexed by the set {1,2,...,N}, in such a way that the subset of N_k routes relating to inter-zonal movement k are indexed by the set

$$R_k = \left\{ r + \sum_{j=1}^{k-1} N_j \; : \; r = 1,2,\ldots, N_k \right\}.$$

The demand rates for each of the W inter-zonal movements are held in the column vector $\mathbf{q}$ of dimension W, with elements q_k (k=1,2,...,W). Without loss of generality, we assume throughout that all $q_k > 0$, since any inter-zonal movement with zero demand could be removed from the problem without effect. The units of q_k will be referred to as vehicles per hour, although more generally they could be equivalent passenger car units per hour. For notational purposes, it will on occasion be convenient to express these demand rates in an alternative form. This is obtained by defining $\boldsymbol{\Gamma}$ to be an $N \times W$ path-movement incidence matrix with elements

$$\Gamma_{rk} = \begin{cases} 1 & \text{if route } r \in R_k \text{, the movement } k \text{ route set} \\ 0 & \text{otherwise} \end{cases} \qquad \text{(r=1,2,...,N; k=1,2,...,W).}$$

Then $\mathrm{diag}(\mathbf{\Gamma q})$ is an $N \times N$ diagonal matrix such that each row relates to a route, and the diagonal entry for that row is the inter-zonal demand appropriate to that route.

The column vector **f** of dimension N denotes an assignment of flow to each of the possible routes, with the convex set of demand-feasible non-negative route flow rates denoted by $\Omega_1 = \{\mathbf{f} \in \Re_+^N : \mathbf{\Gamma}^T \mathbf{f} = \mathbf{q}\}$, where $\Re_+^N$ denotes N-dimensional non-negative real space, and $\mathbf{A}^T$ denotes the transpose of the matrix **A**. The convex set of demand feasible link flow rates is $\Omega_2 = \{\mathbf{v} \in \Re_+^A : \mathbf{v} = \mathbf{\Delta f} \text{ where } \mathbf{f} \in \Omega_1\}$, where $\mathbf{\Delta}$ is an $A \times N$ link-path incidence matrix with elements:

$$\Delta_{ar} = \begin{cases} 1 & \text{if link a is part of route r} \\ 0 & \text{otherwise} \end{cases} \qquad (a=1,2,...,A;\ r=1,2,...,N).$$

The cost of travelling along link a at a given link flow vector **v** is denoted by $t_a(\mathbf{v})$. These functions may also themselves be arranged in a column vector, $\mathbf{t}(\mathbf{v})$. These link performance functions imply corresponding route cost-flow performance functions, $\mathbf{c}(\mathbf{f}) = \mathbf{\Delta}^T \mathbf{t}(\mathbf{\Delta f})$.

Suppose further that for each movement k, $\{p_r(\mathbf{u}) : r \in R_k\}$ is a route choice model describing the fraction of drivers on inter-zonal movement k that would choose each of the alternative routes in R_k when the perceived route costs (averaged across the driver population) are **u**. Let $\mathbf{p}(\mathbf{u})$ denote the vector of these functions across all movements, arranged in a column vector of dimension N. We may then state the following well-known definition (Sheffi, 1985):

Definition The route flow vector $\mathbf{f} \in \Omega_1$ is a stochastic user equilibrium (SUE) if and only if

$$\mathbf{f} = \mathrm{diag}(\mathbf{\Gamma q}).\mathbf{p}(\mathbf{c}(\mathbf{f})) \tag{1.1}$$

Alternatively, the link flow vector $\mathbf{v} \in \Omega_2$ is termed a SUE if and only if

$$\mathbf{v} = \mathbf{\Delta}.\mathrm{diag}(\mathbf{\Gamma q}).\mathbf{p}(\mathbf{\Delta}^T \mathbf{t}(\mathbf{v})). \tag{1.2}$$

Typically, for each movement k, it is assumed that $\{p_r(\mathbf{u}) : r \in R_k\}$ is a random utility model. This is achieved by modelling drivers' perceptual differences in evaluating travel cost by a probability distribution of perceived route costs $U_r\ (r \in R_k)$, with

$$U_r = u_r + e_r \qquad (r \in R_k;\ k = 1,2,\ldots,W)$$

where $E[U_r]=u_r$, and where $\mathbf{e} = \{e_r : r \in R_k;\ k = 1,2,\ldots,W\}$ follows some given joint probability distribution. The choice fractions are then given by

$$p_r(\mathbf{u}) = \Pr(u_r + e_r \le u_s + e_s\ ,\ \forall s \in R_k\ ,\ s \ne r) \qquad (r \in R_k;\ k = 1,2,\ldots,W). \quad (1.3)$$

For example, underlying the logit model is the assumption that the e_r are independent Gumbel random variables. More plausibly, it is often assumed that the route cost perceptual errors are formed from link cost perceptual errors (thereby implicitly taking account of correlations between overlapping routes). For example, if the link errors are independent Normal random variables, then **e** will follow a multivariate Normal distribution, as underlies the probit choice model (Sheffi, 1985).

It will later prove useful to also define corresponding absolute flow variables, rather than flow rates, with the inter-zonal demands denoted by the vector $\tilde{\mathbf{q}}$, with elements in the discrete (integer) units of "vehicles" or "drivers". These demands relate to a particular period of the day, of duration τ hours. That is to say, $\mathbf{q} = \tau^{-1}\tilde{\mathbf{q}}$. The demand-feasible absolute route and link flows are denoted respectively $\tilde{\Omega}_1 = \{\tilde{\mathbf{f}} \in Z_+^N : \mathbf{\Gamma}^T\tilde{\mathbf{f}} = \tilde{\mathbf{q}}\}$ and $\tilde{\Omega}_2 = \{\tilde{\mathbf{v}} \in Z_+^A : \tilde{\mathbf{v}} = \mathbf{\Delta}\tilde{\mathbf{f}}$ where $\tilde{\mathbf{f}} \in \tilde{\Omega}_1\}$, where Z_+^N is the N-dimensional space of non-negative integers. Capitalised versions of $\mathbf{v}, \mathbf{f}, \tilde{\mathbf{v}}$ and $\tilde{\mathbf{f}}$—namely $\mathbf{V}, \mathbf{F}, \tilde{\mathbf{V}}$ and $\tilde{\mathbf{F}}$—will denote vector random variables of the relevant flow/cost measures.

2. EQUILIBRIUM CONDITIONS UNDER STOCHASTIC FLOWS

The first step in the development of the proposed new model is the generalisation of the SUE model to permit random variation in the route and link flows, while taking account of the effect of flow variability on the variation in actual costs. The first question to answer is: what do we mean by "equilibrium" conditions in this more general setting? In conventional (deterministic flow) problems, the concept of an economic equilibrium is used to define a "self-consistency" condition between demand (some behavioural rule describing the demand for alternative routes depending on the travel costs) and supply (generalised travel costs of the alternatives as a function of the aggregate numbers using each alternative, represented by the link performance functions). For example, underlying the SUE model (1.1), the hypothesis is that **f** is in equilibrium if the route proportions given by the route choice model $\mathbf{p}(\cdot)$ at costs $\mathbf{c}(\mathbf{f})$ are consistent with flows **f**.

Once we permit the flows instead to be stochastic, a natural measure to equilibrate is then some kind of flow probability distribution, for which there are a number of choices. We shall choose to consider the absolute stochastic link flow vector $\tilde{\mathbf{V}}$, and its probability distribution $\left\{\Pr(\tilde{\mathbf{V}}=\tilde{\mathbf{v}}) : \tilde{\mathbf{v}} \in \tilde{\Omega}_2\right\}$. This has the advantage of dimensionality over route-based formulations, since in typical real-life networks, there are many more possible routes than there are links.

Suppose first, then, that the underlying probability distribution of $\tilde{\mathbf{V}}$ were known. Suppose that drivers base their choices on a finite collection of the actual link costs experienced in their trips on days in the past, in such a way that the flows $\left\{\tilde{\mathbf{V}}^{(1)},\tilde{\mathbf{V}}^{(2)},\ldots,\tilde{\mathbf{V}}^{(m)}\right\}$ on the m days concerned can be assumed to represent a sample of independent, identically distributed (i.i.d.) random variables drawn from the distribution of $\tilde{\mathbf{V}}$, with corresponding link costs $\left\{\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}}^{(1)}),\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}}^{(2)}),\ldots,\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}}^{(m)})\right\}$. Now introduce the vector random variable $\mathbf{Y}=\frac{1}{m}\sum_{j=1}^{m}\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}}^{(j)})$ of dimension A to denote the driver population-average estimate of actual link costs based on this sample of days. Let Ψ_m denote the space of demand-feasible $\mathbf{Y}$:

$$\Psi_m=\left\{\mathbf{y}\ :\ \mathbf{y}=\frac{1}{m}\sum_{j=1}^{m}\mathbf{t}(\tau^{-1}\tilde{\mathbf{v}}^{(j)})\ \text{ and }\ \tilde{\mathbf{v}}^{(j)}\in\Omega_2\ \ (j=1,2,\ldots,m)\right\}. \tag{2.1}$$

Then, by the i.i.d. property,

$$\Pr(\mathbf{Y}=\mathbf{y})=\sum_{\substack{(\tilde{\mathbf{v}}^{(1)},\tilde{\mathbf{v}}^{(2)},\ldots,\tilde{\mathbf{v}}^{(m)})\ \in\tilde{\Omega}_2{}^m \\ \text{such that } \frac{1}{m}\sum_{i=1}^{m}\mathbf{t}(\tau^{-1}\tilde{\mathbf{v}}^{(i)})=\mathbf{y}}}\ \prod_{j=1}^{m}\ \Pr(\tilde{\mathbf{V}}=\tilde{\mathbf{v}}^{(j)}) \qquad (\mathbf{y}\in\Psi_m)\ . \tag{2.2}$$

On the other hand, suppose that we did not know the underlying distribution of $\tilde{\mathbf{V}}$. Then,

$$\Pr(\tilde{\mathbf{V}}=\tilde{\mathbf{v}})=\sum_{\tilde{\mathbf{f}}\in\tilde{\Omega}_1 \text{ such that } \Delta\tilde{\mathbf{f}}=\tilde{\mathbf{v}}}\Pr(\tilde{\mathbf{F}}=\tilde{\mathbf{f}}) \qquad (\tilde{\mathbf{v}}\in\tilde{\Omega}_2) \tag{2.3}$$

and then by standard laws of conditional probabilities we may write:

$$\Pr(\tilde{\mathbf{V}}=\tilde{\mathbf{v}})=\sum_{\tilde{\mathbf{f}}\in\tilde{\Omega}_1 \text{ such that } \Delta\tilde{\mathbf{f}}=\tilde{\mathbf{v}}}\ \sum_{\mathbf{y}\in\Psi_{\mathbf{m}}}\Pr(\tilde{\mathbf{F}}=\tilde{\mathbf{f}}\mid\mathbf{Y}=\mathbf{y})\ \Pr(\mathbf{Y}=\mathbf{y}) \qquad (\tilde{\mathbf{v}}\in\tilde{\Omega}_2)\ . \tag{2.4}$$

Now, define the following partitions by inter-zonal movement:

$$\tilde{\mathbf{F}}=\begin{pmatrix}\tilde{\mathbf{F}}_{[1]}\\ \tilde{\mathbf{F}}_{[2]}\\ \vdots\\ \tilde{\mathbf{F}}_{[W]}\end{pmatrix} \quad\text{and}\quad \mathbf{p}(\mathbf{u})=\begin{pmatrix}\mathbf{p}_{[1]}(\mathbf{u})\\ \mathbf{p}_{[2]}(\mathbf{u})\\ \vdots\\ \mathbf{p}_{[W]}(\mathbf{u})\end{pmatrix}. \tag{2.5}$$

Suppose that, conditionally on $\mathbf{Y}=\mathbf{y}$, for each inter-zonal movement $k=1,2,\ldots,W$ independently, each of the $\tilde{q}_k$ drivers independently chooses between the available routes with probabilities $\mathbf{p}_{[k]}(\boldsymbol{\Delta}^{T}\mathbf{y})$, where

$\boldsymbol{\Delta}^{\mathrm{T}}\mathbf{y}$ is the vector of corresponding route costs. Here, $\mathbf{p}(\mathbf{u})$ is the route choice model given a special and rather different interpretation to that in conventional SUE analyses, as a disaggregate probability rather than an aggregate choice fraction (recall the discussion in the introduction section). Then:

$$\tilde{\mathbf{F}}_{[k]} \mid \mathbf{Y} = \mathbf{y} \sim \text{ Multinomial } (\tilde{q}_k, \mathbf{p}_{[k]}(\boldsymbol{\Delta}^{\mathrm{T}}\mathbf{y})) \qquad (\text{independently for } k = 1,2,\ldots,W)\,. \tag{2.6}$$

That is to say, the conditional probabilities in (2.4), relating to the distribution of $\tilde{\mathbf{F}} \mid \mathbf{Y}$, are known values based on the model assumptions. (They are multinomial probabilities since, conditional on **Y**, the fixed number of drivers are independently making choices according to fixed probabilities.) On the other hand, the unconditional distribution of **Y** may be related to the probability distribution of $\tilde{\mathbf{V}}$ by (2.2). Substituting (2.2) into (2.4) therefore gives a consistency (equilibrium) condition for the probability distribution of $\tilde{\mathbf{V}}$.

Therefore, (2.2)/(2.4) together give an equilibrium condition for any given value of m. We shall focus specifically on what happens to these conditions as $m \to \infty$. Letting the number of experiences become large is intuitively consistent with the assumption underlying conventional network equilibrium theory, that the drivers represent a population of well-informed, experienced travellers.

Lemma 1 Suppose that drivers form estimates of actual costs from a random sample $\{\mathbf{T}^{(1)},\mathbf{T}^{(2)},\ldots,\mathbf{T}^{(m)}\}$ of the link costs from their previous travel experiences, where m is given, $\mathbf{T}^{(j)} = \mathbf{t}(\tau^{-1}\mathbf{V}^{(j)})$ $(j = 1,2,\ldots,m)$, and $\{\tilde{\mathbf{V}}^{(1)},\tilde{\mathbf{V}}^{(2)},\ldots,\tilde{\mathbf{V}}^{(m)}\}$ is a sample of independent, identically distributed, demand-feasible link flow vectors. Suppose further that the functions $t_a(\tau^{-1}\tilde{\mathbf{v}})$ (a=1,2,…,A) are bounded for $\mathbf{v} \in \tilde{\Omega}_2$. Then as $m \to \infty$, $\operatorname{var}\left(\frac{1}{m}\sum_{j=1}^{m} T_a^{(j)}\right) \to 0$ (a=1,2,…,A).

Proof The result is a trivial consequence of assuming $\{\tilde{\mathbf{V}}^{(1)},\tilde{\mathbf{V}}^{(2)},\ldots,\tilde{\mathbf{V}}^{(m)}\}$ to be i.i.d., since then $\{\mathbf{T}^{(1)},\mathbf{T}^{(2)},\ldots,\mathbf{T}^{(m)}\}$ are too, and so (for each a=1,2,…,A) are $\{T_a^{(1)},T_a^{(2)},\ldots,T_a^{(m)}\}$, being drawn from a common marginal distribution with variance $\sigma_a^{\,2}$, say. Hence, $\operatorname{var}\left(\frac{1}{m}\sum_{j=1}^{m} T_a^{(j)}\right) = \dfrac{\sigma_a^{\,2}}{m}$ (a=1,2,…,A). Since $\sigma_a^{\,2} = \operatorname{var}(t_a(\tau^{-1}\tilde{\mathbf{V}}))$, where $\tilde{\mathbf{V}}$ follows the common distribution of the sampled flows, then $\sigma_a^{\,2} < \infty$ by virtue of $\tilde{\mathbf{V}}$ being a bounded random variable by the demand-feasibility constraints, and $t_a(\cdot)$ by hypothesis being a bounded function. Letting $m \to \infty$, the required result follows.

The result means that for each a=1,2,…,A, as m increases the probability distribution of the random variable $Y_a = \frac{1}{m}\sum_{j=1}^{m} t_a(\tau^{-1}\tilde{\mathbf{V}}^{(j)})$ becomes increasingly centred on $E[t_a(\tau^{-1}\tilde{\mathbf{V}})]$ ('centred' since Y_a, being a mean of i.i.d. variables, will clearly be asymptotically normally distributed). This result leads finally to the equilibrium conditions, following one further piece of additional notation. In particular, let $\boldsymbol{\psi}$ denote the column vector of probabilities $\left\{\Pr(\tilde{\mathbf{V}} = \tilde{\mathbf{v}}) : \tilde{\mathbf{v}} \in \tilde{\Omega}_2\right\}$, with dimension equal to the cardinality $|\tilde{\Omega}_2|$ of $\tilde{\Omega}_2$. This distribution is related to the route flow probability distribution $\boldsymbol{\varsigma}$ (a column vector of probabilities $\left\{\Pr(\tilde{\mathbf{F}} = \tilde{\mathbf{f}}) : \tilde{\mathbf{f}} \in \tilde{\Omega}_1\right\}$ of dimension $|\tilde{\Omega}_1|$), by $\boldsymbol{\psi} = \boldsymbol{\Pi\varsigma}$, where $\boldsymbol{\Pi}$ is a $|\tilde{\Omega}_2| \times |\tilde{\Omega}_1|$ matrix with elements

$$\Pi_{ij} = \begin{cases} 1 & \text{if the route flow } \tilde{\mathbf{f}} \text{ referred to by } \varsigma_j \text{ "corresponds" to the link flow } \tilde{\mathbf{v}} \\ & \text{referred to by } \psi_i\text{, in the sense that } \tilde{\mathbf{v}} = \boldsymbol{\Delta}\,\tilde{\mathbf{f}} \\ 0 & \text{otherwise} \end{cases} \quad . \qquad (2.7)$$

That is to say, $\boldsymbol{\psi} = \boldsymbol{\Pi\varsigma}$ is a vector representation of (2.3).

Theorem (Asymptotic equilibrium condition) Suppose that the hypotheses of Lemma 1 are met, and suppose that subsequently, conditionally on the driver population-average estimated actual link costs **y**, for each inter-zonal movement k=1,2,…,W independently, each of the $\tilde{q}_k$ drivers independently chooses between the available routes with probabilities $\mathbf{p}_{[k]}(\boldsymbol{\Delta}^T\mathbf{y})$. Then asymptotically, as $m \to \infty$, the link flow probability distribution $\boldsymbol{\psi}$ satisfies the equilibrium condition:

$$\boldsymbol{\psi} = \boldsymbol{\Pi}\,\boldsymbol{\varsigma}(\boldsymbol{\psi}) \qquad (2.8)$$

where $\boldsymbol{\varsigma}(\boldsymbol{\psi})$ is a vector of dimension $|\tilde{\Omega}_1|$ with elements the probabilities

$$\Pr(\,\tilde{\mathbf{F}} = \tilde{\mathbf{f}} \;\big|\; \mathbf{Y} = E_{\tilde{\mathbf{V}}}[\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}})] \quad \text{where } \tilde{\mathbf{V}} \sim \boldsymbol{\psi}\,) \qquad (\tilde{\mathbf{f}} \in \tilde{\Omega}_1) \qquad (2.9)$$

where $\tilde{\mathbf{V}} \sim \boldsymbol{\psi}$ denotes that $\tilde{\mathbf{V}}$ has a given probability distribution $\boldsymbol{\psi}$, and where the distribution of $\tilde{\mathbf{F}} \mid \mathbf{Y}$ is given by (2.6) based on the partition (2.5).

Proof Now, $\Pr(\tilde{\mathbf{F}} = \tilde{\mathbf{f}}) = \sum_{\mathbf{y} \in \Psi_m} \Pr\left(\tilde{\mathbf{F}} = \tilde{\mathbf{f}} \mid \mathbf{Y} = \mathbf{y}\right) \Pr\left(\mathbf{Y} = \mathbf{y}\right)$ $(\tilde{\mathbf{f}} \in \tilde{\Omega}_1)$. Lemma 1 tells us that as $m \to \infty$, an increasingly good approximation to the probability distribution of **Y** will be that distribution which puts all the probability mass at its long-run expected value, yielding:

$$\Pr(\tilde{\mathbf{F}} = \tilde{\mathbf{f}}) \approx \Pr(\, \tilde{\mathbf{F}} = \tilde{\mathbf{f}} \mid \mathbf{Y} = \mathrm{E}_{\tilde{\mathbf{V}}}[\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}}) \text{ where } \tilde{\mathbf{V}} \sim \boldsymbol{\psi}]\,) \quad . \tag{2.10}$$

The equilibrium conditions are obtained by noting that the left-hand side of this expression also defines the elements of $\boldsymbol{\varsigma}$, with this latter vector related to $\boldsymbol{\psi}$ by $\boldsymbol{\psi} = \boldsymbol{\Pi}\boldsymbol{\varsigma}$.

A number of crucial statistical assumptions underlie this equilibrium condition, notably that the flows $\left\{\tilde{\mathbf{V}}^{(1)}, \tilde{\mathbf{V}}^{(2)}, \ldots, \tilde{\mathbf{V}}^{(m)}\right\}$ which generate drivers' estimates of actual cost (a) are identically distributed; (b) are statistically independent; and (c) represent a large sample (i.e. m is large). Systematic, seasonal changes in travel demand (e.g. due to school semesters/vacations) or activities specific to particular days of the week (e.g. days of major shopping activity) may bring assumption (a) into question. Stochastic variation in the demand **q** may, however, be incorporated by an extension of the proposed model, discussed elsewhere (Watling, 1998). Assumption (b) might be questioned from the viewpoint of internal model consistency, in the sense that the dependence of $\tilde{\mathbf{V}}$ on $\left\{\tilde{\mathbf{V}}^{(1)}, \tilde{\mathbf{V}}^{(2)}, \ldots, \tilde{\mathbf{V}}^{(m)}\right\}$ in the equilibrium definition might be said to infer—assuming the sampled experiences are time-ordered —that $\tilde{\mathbf{V}}^{(j)}$ depends on $\left\{\tilde{\mathbf{V}}^{(1)}, \tilde{\mathbf{V}}^{(2)}, \ldots, \tilde{\mathbf{V}}^{(j-1)}\right\}$ for $1 \leq j \leq m$. This feature certainly distinguishes it from the stochastic process approach reviewed in section 1, where the equilibrium flows are auto-correlated across time epochs. Finally, assumption (c) is critical for the multinomial distribution to be a valid approximation for the unconditional flows.

3. GENERALISED SUE OF ORDER n

In the previous section, equilibrium conditions on the probability distribution of the network link flow vector were presented. However, solving conditions (2.8)/(2.9) for $\boldsymbol{\psi}$ is unlikely to be feasible for networks of a size of practical interest. One particular difficulty is the dimension of the problem, as determined by the number of possible (demand-feasible) integer link flow states. In the present section, a related family of fixed point conditions is deduced, which essentially allows us to approximate the true equilibrium distribution "to an abitrary order". For low order approximations, the dimension of the approximate conditions is an extremely small fraction of the dimension of the underlying problem.

For any positive integer k, define the following sub-space of A-dimensional non-negative integers:

$$M(k)=\left\{ \mathbf{m} : \mathbf{m} \in Z^A, 0 \le m_a \le k \ \ (a=1,2,\ldots,A) \ \text{ and } \sum_{a=1}^{A} m_a = k \right\} . \tag{3.1}$$

Now consider the vector random variable $\mathbf{V}$ of dimension A, which contains the network link flow rates. For given k (k=1,2,…), the order k moments of $\mathbf{V}$ are given by the expectations:

$$\theta(\mathbf{m},k)=\begin{cases} E\left[\prod_{a=1}^{A} V_a^{\,m_a}\right] & \text{for } k=1 \\ E\left[\prod_{a=1}^{A} \left(V_a - E[V_a]\right)^{m_a}\right] & \text{for } k=2,3,\ldots \end{cases} \qquad \left(\mathbf{m} \in M(k)\right) . \tag{3.2}$$

Furthermore, define the following as column vectors of the appropriate dimension:

$$\mathbf{\Theta}(k)=\left\{\theta(\mathbf{m},k) : \mathbf{m} \in M(k)\right\} \qquad (k=1,2,3,\ldots) . \tag{3.3}$$

Thus, for example, the order 1 moments are the link flow means, contained in the vector $\mathbf{\Theta}(1)$, and the order 2 moments are the link flow variances and covariances, contained in the vector $\mathbf{\Theta}(2)$.

These moments will now be used to make an approximation to the expected link cost vector that is part of (2.9). This will be achieved by first making an n^{th} order Taylor series approximation to the cost-flow function for link a, $t_a(\mathbf{v})$, in the neighbourhood of $\mathbf{v}=\mathbf{v}'$ (for some given $\mathbf{v}'$):

$$t_a(\mathbf{v}) \approx t_a(\mathbf{v}') + \sum_{k=1}^{n} \sum_{\mathbf{m} \in M(k)} \frac{1}{m_1!m_2!\ldots m_A!} \left. \frac{\partial^k t_a}{\partial v_1^{\,m_1} \partial v_2^{\,m_2} \ldots \partial v_A^{\,m_A}} \right|_{\mathbf{v}=\mathbf{v}'} \prod_{a=1}^{A} \left(v_a - v'_a\right)^{m_a} . \tag{3.4}$$

Substituting the vector random variable $\mathbf{V}$ for $\mathbf{v}$, the expectation $E[\mathbf{V}]$ for $\mathbf{v}'$, and then taking expectations throughout with respect to $\mathbf{V}$ yields:

$$E_{\mathbf{V}}\left[t_a(\mathbf{V})\right] \approx t_a\left(E[\mathbf{V}]\right) + \sum_{k=1}^{n} \sum_{\mathbf{m} \in M(k)} \frac{1}{m_1!m_2!\ldots m_A!} \left. \frac{\partial^k t_a}{\partial v_1^{\,m_1} \partial v_2^{\,m_2} \ldots \partial v_A^{\,m_A}} \right|_{\mathbf{v}=E[\mathbf{V}]} E\left[\prod_{a=1}^{A} \left(V_a - E[V_a]\right)^{m_a}\right]. \tag{3.5}$$

Since for any $a=1,2,\ldots,A$, $E\left[V_a - E[V_a]\right]=0$, the terms in the summation over k in (3.5) corresponding to k = 1 are zero, since exactly one $m_a=1$ for each $\mathbf{m}$ corresponding to k = 1. Simplifying and using notation (3.2)/(3.3), then (3.5) may be written

$$E_{\mathbf{V}}\left[t_a(\mathbf{V})\right] \approx t_a\left(\mathbf{\Theta}(1)\right) + \sum_{k=2}^{n} \sum_{\mathbf{m} \in M(k)} \frac{1}{m_1!m_2!\ldots m_A!} \left. \frac{\partial^k t_a}{\partial v_1^{\,m_1} \partial v_2^{\,m_2} \ldots \partial v_A^{\,m_A}} \right|_{\mathbf{v}=\mathbf{\Theta}(1)} \theta(\mathbf{m},k) . \tag{3.6}$$

Hence, for a given positive integer n, expression (3.6) represents an approximation to the expected link costs, as a function of the moments of $\mathbf{V}$ of order n and below. It is noted in passing that the general technique for approximating the expectation of a non-linear function of a random variable (by the expectation of that function's Taylor series expansion about the variable's mean) is sometimes called the "method of statistical differentials" (e.g. Ben-Akiva and Lerman, 1985, pp 140-142). Such an approximation has the following trivially-proven but significant property—namely, that by taking a sufficiently large n, any vector polynomial cost function may be approximated to an arbitrary accuracy:

Lemma 2 Suppose that $t(\mathbf{v})$ is an order r vector polynomial:

$$t(\mathbf{v}) = \sum_{i=0}^{I} \alpha_i \prod_{a=1}^{A} v_a^{\,r_{ai}} \quad \text{where } \sum_{a=1}^{A} r_{ai} \le r \;\; (i = 1,2...,I) \tag{3.7}$$

where the powers $r_{ai}\ (a = 1,2,...,A; i = 1,2,...,I)$ are given non-negative integers, and the coefficients $\alpha_i\ (i = 1,2,\ldots,I)$ are given real-valued numbers. Then if $n \ge r$, (3.6) gives the exact expectations (i.e. no approximation is involved).

The expectations (3.6) represent the effect of a given stochastic flow distribution on the "supply-side" of our network problem. We now turn attention to the process by which drivers choose routes conditional on their perceived costs. The equilibrium conditions (2.8)/(2.9) are based on a multinomial route flow allocation for each inter-zonal movement, according to (2.5)/(2.6). Now, since the absolute link flows $\tilde{\mathbf{V}}$ are related to the link flow rates by $\mathbf{V} = \tau^{-1}\tilde{\mathbf{V}}$, it follows that the order n moments of $\tilde{\mathbf{V}}$ are $\tau^n\, \boldsymbol{\Theta}(n) \left(= \{\tau^n \theta(\mathbf{m}, n) : \mathbf{m} \in M(n)\}\right)$. In turn, as the link flows $\tilde{\mathbf{V}}$ are a linear combination of the route flows $\tilde{\mathbf{F}}$, then the order n moments of $\tilde{\mathbf{V}}$ are a linear combination of the order n moments of $\tilde{\mathbf{F}}$, since:

$$E\left[\prod_{a=1}^{A}\left(\tilde{V}_a - E\left[\tilde{V}_a\right]\right)^{m_a}\right] = E\left[\prod_{a=1}^{A}\left(\sum_{r=1}^{N}\Delta_{ar}\tilde{F}_r - E\left[\sum_{r=1}^{N}\Delta_{ar}\tilde{F}_r\right]\right)^{m_a}\right] = E\left[\prod_{a=1}^{A}\left(\sum_{r=1}^{N}\Delta_{ar}\left(\tilde{F}_r - E\left[\tilde{F}_r\right]\right)\right)^{m_a}\right] \tag{3.8}$$

and multiplying out the product over a gives a linear combination of n^{th} order moments of $\tilde{\mathbf{F}}$, where the coefficients are a function of the path-link incidence matrix $\boldsymbol{\Delta}$. Now, according to (2.6), we shall suppose that route flows on different inter-zonal movements are statistically independent, and so we need not consider such cross-moments. Therefore, based on the partition (2.5), we suppose that for each $k = 1,2,\ldots,W$, the n^{th} order moments of $\tilde{\mathbf{F}}_{[k]}$ are collected (in a similar way to those of $\mathbf{V}$) in a vector $\boldsymbol{\varpi}_{[k]}(n)$ of dimension $(N_k)^n$, with these vectors $\boldsymbol{\varpi}_{[k]}(n)\ (k = 1,2,\ldots,W)$ themselves being components of a partition of the vector $\boldsymbol{\varpi}(n)$ of dimension $\sum_{k=1}^{W}(N_k)^n$. The linear relationship between moments of the link flow rates $\tau^n\,\boldsymbol{\Theta}(n)$, and moments of the absolute route flows $\boldsymbol{\varpi}(n)$, is then denoted:

$$\tau^n\,\boldsymbol{\Theta}(n)=\widehat{\boldsymbol{\Delta}}(n)\,\boldsymbol{\varpi}(n) \qquad (n=1,2,...) \tag{3.9}$$

where (for each $n=1,2,\ldots$) $\widehat{\boldsymbol{\Delta}}(n)$ is an $A^n\times\sum_{k=1}^{W}(N_k)^n$ constant matrix that is a known function of n and the path-link incidence matrix $\boldsymbol{\Delta}$.

Now, if we know the choice probabilities $\mathbf{p}_{[k]}(\boldsymbol{\Delta}^T\mathbf{y})$ in (2.6), then for given n we can compute the moments $\boldsymbol{\varpi}_{[k]}(j)$ $(k=1,2,\ldots,W;\ j=1,2,\ldots,n)$ by known properties of the multinomial distribution, and hence from (3.8) can compute the moments $\boldsymbol{\Theta}(j)$ $(j=1,2,...,n)$. Then the only unknown in $\mathbf{p}_{[k]}(\boldsymbol{\Delta}^{\perp}\mathbf{y})$ is $\mathbf{y}$, which in equilibrium, by (2.9), is given by $\mathbf{y}=E_{\tilde{\mathbf{V}}}\left[\mathbf{t}(\tau^{-1}\tilde{\mathbf{V}})\right]=E_{\mathbf{V}}\left[\mathbf{t}(\mathbf{V})\right]$, and in (3.6) we have shown how this expectation may in turn be approximated by the moments $\boldsymbol{\Theta}(j)$ $(j=1,2,...,n)$. This leads us to the following definition.

Definition (Generalised SUE of order n) For a given positive integer n, consider a network with n times differentiable cost-flow functions $\mathbf{t}(\mathbf{v})$. The collection of link flow moments $(\boldsymbol{\Theta}(1),\boldsymbol{\Theta}(2),...,\boldsymbol{\Theta}(n))$ given by (3.3) is termed a Generalised Stochastic User Equilibrium of order n (and denoted GSUE(n)) if and only if $\boldsymbol{\Theta}(j)=\tau^{-j}\,\widehat{\boldsymbol{\Delta}}(j)\,\boldsymbol{\varpi}(j)$ $(j=1,2,\ldots,n)$, where $\widehat{\boldsymbol{\Delta}}(j)$ is a coefficient matrix defined in (3.9), and where—with $\boldsymbol{\varpi}_{[k]}(j)$ $(k=1,2,\ldots,W)$ representing the components of a partition of the vector $\boldsymbol{\varpi}(j)$—the vector $\boldsymbol{\varpi}_{[k]}(j)$ contains the j^{th} order moments of the multinomial route flow vector $\tilde{\mathbf{F}}_{[k]}$ with parameters $\tilde{q}_k$ (the inter-zonal demand) and probabilities $\mathbf{p}_{[k]}(\boldsymbol{\Delta}^{\perp}\mathbf{y})$, where $\mathbf{y}=E_{\mathbf{V}}\left[\mathbf{t}(\mathbf{V})\right]$ and is approximated by the moments $(\boldsymbol{\Theta}(1),\boldsymbol{\Theta}(2),...,\boldsymbol{\Theta}(n))$ according to (3.6).

Effectively, the only difference between the GSUE(n) condition and the moments of an equilibrium probability distribution satisfying (2.8)/(2.9), is that the expected costs upon which choices are based are in the former case approximated using only the first n moments of the distribution, rather than the full distribution. The first point to note is that one member of the GSUE family is a familiar model.

Lemma 3 The link flow vector $\hat{\mathbf{v}}$ is an SUE if and only if $\hat{\mathbf{v}}$ is a GSUE(1).

Proof Suppose that $\hat{\mathbf{v}}$ is a GSUE(1). Then from (3.6) with n=1, $E_{\mathbf{V}}\left[t_a(\mathbf{V})\right] \approx t_a(\hat{\mathbf{v}})$, the multinomial probabilities of the GSUE model are then $\mathbf{p}_{[k]}(\boldsymbol{\Delta}^T\mathbf{t}(\hat{\mathbf{v}}))$, and the first order moments of the route flows for inter-zonal movement k $(k = 1,2,\ldots,W)$ are $E\left[\tilde{\mathbf{F}}_{[k]}\right] = \tilde{q}_k\,\mathbf{p}_{[k]}(\boldsymbol{\Delta}^T\mathbf{t}(\hat{\mathbf{v}}))$, a partition of $E\left[\tilde{\mathbf{F}}\right]$. Now, $E\left[\tilde{\mathbf{V}}\right] = E\left[\boldsymbol{\Delta}\,\tilde{\mathbf{F}}\right] = \boldsymbol{\Delta}\,E\left[\tilde{\mathbf{F}}\right]$ (that is to say, in terms of (3.9), $\widehat{\boldsymbol{\Delta}}(1) = \boldsymbol{\Delta}$). With notation defined earlier (see (1.1)/(1.2)), we may combine these results as $E\left[\tilde{\mathbf{V}}\right] = \boldsymbol{\Delta}.\mathrm{diag}(\boldsymbol{\Gamma}\tilde{\mathbf{q}}).\mathbf{p}(\boldsymbol{\Delta}^T\mathbf{t}(\hat{\mathbf{v}}))$. But by (3.9), $\hat{\mathbf{v}} = E[\mathbf{V}] = \tau^{-1}E\left[\tilde{\mathbf{V}}\right]$, and we obtain the SUE condition (1.2) on $\hat{\mathbf{v}}$ by noting that $\mathrm{diag}(\boldsymbol{\Gamma}\mathbf{q}) = \tau^{-1}\mathrm{diag}(\boldsymbol{\Gamma}\tilde{\mathbf{q}})$. The "only if" part of the proof arises by reversing this logical argument.

A corollary to Lemmas 2 and 3 is that if $\mathbf{t}(\mathbf{v})$ is linear in $\mathbf{v}$, then for any $n \geq 1$ the mean GSUE(n) flows $\boldsymbol{\Theta}(1)$ will be SUE flows (see also Cascetta, 1989); in practice, however, $\mathbf{t}(\mathbf{v})$ will invariably be non-linear. It is notable that Lemma 3 makes no reference to the magnitude of the demand flows, and so is not a limiting, large population relationship (in contrast with Davis & Nihan, 1993). A limiting relationship does, however, exist for general GSUE(n) models, as established in the results below.

Lemma 4 Suppose the m-vector random variable **X** follows a multinomial distribution with parameters q (the fixed sum of the elements of **X**) and **p** (the probabilities of the m alternatives). Then for any non-negative integers n_i $(i = 1,2,\ldots,m)$ with $\sum_{i=1}^{m} n_i = n \geq 2$, $\lim_{q\to\infty} \dfrac{E\left[\prod_{i=1}^{m}\left(X_i - E[X_i]\right)^{n_i}\right]}{q^n} = 0$.

Proof Expanding, $E\left[\prod_{i=1}^{m}\left(X_i - \mu_i\right)^{n_i}\right] = \sum_{\substack{(r_1,r_2,\ldots,r_m)\\ \text{such that } 0\leq r_i\leq n_i}} E\left[\prod_{i=1}^{m} X_i^{r_i}\right]\prod_{i=1}^{m}\left(-\mu_i\right)^{n_i - r_i}$. Then, with $\mu_i = np_i$, $n = \sum_{i=1}^{m} n_i$ and $r = \sum_{i=1}^{m} r_i$, we obtain $E\left[\prod_{i=1}^{m}\left(X_i - \mu_i\right)^{n_i}\right] = \sum_{\substack{(r_1,r_2,\ldots,r_m)\\ \text{such that } 0\leq r_i\leq n_i}} E\left[\prod_{i=1}^{m} X_i^{r_i}\right](-1)^{n-r}q^{n-r}\prod_{i=1}^{m} p_i^{n_i - r_i}$. But from the multinomial moment generating function $\phi(\mathbf{t}) = \left(p_1 + \sum_{j=2}^{m} p_j \exp(t_j)\right)^q$ (see, e.g., Stuart & Ord, 1987), it is straightforward to show that $E\left[\prod_{i=1}^{m} X_i^{r_i}\right] = q^r\prod_{i=1}^{m} p_i^{r_i} + o(q^r)$, where $o(q^r)$ in general denotes an arbitrary function for which $\lim_{q\to\infty}\dfrac{o(q^r)}{q^r} = 0$. Hence:

$$E\left[\prod_{i=1}^{m}\left(X_i-\mu_i\right)^{n_i}\right]=\sum_{\substack{(r_1,r_2,\ldots,r_m)\\ \text{such that } 0\le r_i\le n_i}}\left[(-1)^{n-r}q^n\prod_{i=1}^{m}p_i^{\,n_i}+o(q^n)\right]$$

and the lemma is proven by noting that the summation over the first term is a summation over 2^n terms which are identical except that half have a negative and half a positive sign, and so cancel.

Theorem (Large Sample Approximation Theorem) For fixed, non-zero demand rates $\mathbf{q}$ considered over a period of duration τ, in the limit as $\tau\to\infty$, if $\left(\mathbf{\Theta}(1),\mathbf{\Theta}(2),\ldots,\mathbf{\Theta}(n)\right)$ is a GSUE(n), then all components of $\mathbf{\Theta}(j)$ $(j=2,3,\ldots,n)$ approach zero, and $\mathbf{\Theta}(1)$ is a SUE.

Proof Denote the components of the movement k route flow moment vector $\boldsymbol{\varpi}_{[k]}(j)$ (defined above (3.9)) by $\varpi_{i[k]}(j)$. Then by Lemma 4, for each $j\ge 2$, $(\tilde{q}_k)^{-j}\varpi_{i[k]}(j)\to 0$ as $\tilde{q}_k\to\infty$ where $\tilde{q}_k=q_k\tau$ (the absolute demand). Hence for $j\ge 2$, since all $q_k>0$ by hypothesis, then for fixed q_k, replacing $\tilde{q}_k$ by $q_k\tau$, then $\tau^{-j}\varpi_{i[k]}(j)\to 0$ as $\tau\to\infty$. Hence, for $j\ge 2$ all the components of $\mathbf{\Theta}(j)=\tau^{-j}\,\hat{\mathbf{\Delta}}(j)\,\boldsymbol{\varpi}(j)$ tend to zero as $\tau\to\infty$. In that case (in the limit), from (3.6), $E_{\mathbf{V}}[\mathbf{t}(\mathbf{V})]=\mathbf{\Theta}(1)$. The same logic used in the proof of Lemma 3 may then be applied to establish that $\mathbf{\Theta}(1)$ is a SUE.

The Theorem above constitutes a “large sample” result, since as $\tau\to\infty$ for fixed $\mathbf{q}$, the absolute number of travellers on each inter-zonal movement will also approach infinity. It is therefore the absolute demand that is critical here—it is not sufficient that the demand rates be “large”, if the time interval is short. It is interesting to note that the predominant methodological development in equilibrium approaches over the last ten years has been the attempt to model simultaneously a number of shorter time periods. The move to such “dynamic equilibrium” approaches therefore weakens the justification for SUE as an approximation to mean flows.

4. EXISTENCE OF SOLUTIONS AND THE SECOND ORDER MODEL, GSUE(2)

In the remainder of the paper, the focus will be on one particular member of the GSUE(n) family, namely the GSUE(2) model. This gives a major advantage in terms of allowing more transparent notation, as introduced in the Lemma below. Moreover, it would reasonably be expected that in most practical cases, a local quadratic approximation to the link cost-flow functions would be satisfactory.

Lemma 5 Consider a network with twice-differentiable link cost-flow functions **t**(**v**), choice probability model **p**(**u**) and demands **q**. Then the mean A-vector $\boldsymbol{\mu}$ and $A \times A$ covariance matrix $\boldsymbol{\Sigma}$, as a pair $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, is a GSUE(2) if and only if:

$$\begin{cases} \boldsymbol{\mu} = \Delta \cdot \operatorname{diag}(\boldsymbol{\Gamma}\mathbf{q}) \,.\, \mathbf{p}(\Delta^{T}\breve{\mathbf{t}}(\mu, \boldsymbol{\Sigma})) & \text{(4.1a)} \\ \boldsymbol{\Sigma} = \tau^{-1} \, \boldsymbol{\Delta} \,.\, \boldsymbol{\Psi}(\mathbf{q}, \mathbf{p}(\Delta^{T}\breve{\mathbf{t}}(\mu, \boldsymbol{\Sigma}))) \,.\, \boldsymbol{\Delta}^{T} & \text{(4.1b)} \end{cases}$$

where $\breve{\mathbf{t}}(\mu, \boldsymbol{\Sigma})$ is an A-vector with elements

$$\breve{t}_a(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = t_a(\boldsymbol{\mu}) + \tfrac{1}{2} \| \mathbf{H}_a(\boldsymbol{\mu}), \boldsymbol{\Sigma} \| \qquad (a = 1, 2, \ldots, A) \tag{4.2}$$

where $\mathbf{H}_a(\mathbf{v})$ is the $A \times A$ Hessian matrix of $t_a(\mathbf{v})$, where the scalar product of two $m \times n$ matrices is

$$\| \mathbf{X}, \mathbf{Y} \| = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij} \qquad (\mathbf{X}, \mathbf{Y} \in \Re^{m} \times \Re^{n}) \tag{4.3}$$

and where $\boldsymbol{\Psi}(\mathbf{q}, \mathbf{p})$ is a function whose result is an $N \times N$ block diagonal matrix, with blocks (based on the partition (2.5) of **p**) the matrices of dimension $N_k \times N_k$:

$$\boldsymbol{\Psi}_{[k]}(q_k, \mathbf{p}_{[k]}) = q_k \left(\operatorname{diag}(\mathbf{p}_{[k]}) - \mathbf{p}_{[k]} \mathbf{p}_{[k]}^{T} \right) \qquad (k = 1, 2, \ldots, W) \,. \tag{4.4}$$

Proof With $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ replacing $(\boldsymbol{\Theta}(1), \boldsymbol{\Theta}(2))$ in (3.3), and (4.2) corresponding to (3.6), the mean of the absolute route flows $\tilde{\mathbf{F}}$ (from (3.7)) is $\varpi(1) = \operatorname{diag}(\boldsymbol{\Gamma}\tilde{\mathbf{q}}).\mathbf{p}(\Delta^{T}\breve{\mathbf{t}}(\mu, \boldsymbol{\Sigma}))$, noting the use of $\tilde{\mathbf{q}}$, not $\mathbf{q}$. The second order moments of each $\tilde{\mathbf{F}}_{[k]}$ (represented in (3.7) as a vector $\boldsymbol{\varpi}_{[k]}(2)$) are contained in the matrix $\boldsymbol{\Psi}_{[k]}(\tilde{q}_k, \mathbf{p}_{[k]}(\Delta^{T}\breve{\mathbf{t}}(\mu, \boldsymbol{\Sigma})))$ where $\boldsymbol{\Psi}_{[k]}(\cdot)$ is given by (4.4)—again noting the use of $\tilde{q}_k$ rather than q_k. Similarly, in (3.7), $\hat{\boldsymbol{\Delta}}(1) = \boldsymbol{\Delta}$, and $\hat{\boldsymbol{\Delta}}(2)\boldsymbol{\varpi}(2)$ corresponds to $\boldsymbol{\Delta\Psi\Delta}^{T}$ in the new notation. The conditions (4.1) are finally obtained by replacing $\tilde{\mathbf{q}}$ by $\mathbf{q}\tau$.

The existence of GSUE(2) solutions may then be established, using techniques that mirror those used for conventional deterministic user equilibrium (Smith, 1979) and SUE (Cantarella & Cascetta, 1995).

Existence Theorem Consider a network with demand rates **q** over time period τ, with cost functions **t**(**v**) that for all $\mathbf{v} \in \Omega_2$ are continuous, twice differentiable, and have all second derivatives continuous. Suppose further that the route choice function **p**(**u**) is continuous for all $\mathbf{u} \in \Re_{+}^{N}$, this being a choice probability function in the sense that

$$\sum_{r \in R_k} p_r(\mathbf{u}) = 1 \qquad p_r(\mathbf{u}) \geq 0 \;\; (r \in R_k) \qquad (k = 1, 2, \ldots, W; \text{ at all } \mathbf{u} \in \Re_{+}^{N}) \,. \tag{4.5}$$

Then at least one GSUE(2) exists.

Proof Since we assume without loss of generality that all $q_k > 0$ (see section 2), then the inverse matrix $\left(\mathrm{diag}(\mathbf{\Gamma q})\right)^{-1}$ exists. Now consider the mapping **G**(**f**) of the N-vector **f** given by

$$\mathbf{G}(\mathbf{f}) = \mathrm{diag}(\mathbf{\Gamma q}) \,.\, \mathbf{p}\left(\Delta^{\mathrm{T}} \breve{\mathbf{t}}\left(\mathbf{\Delta f}, \tau^{-1} \mathbf{\Delta}\, \mathbf{\Psi}(\mathbf{q}, (\mathrm{diag}(\mathbf{\Gamma q}))^{-1} \mathbf{f})\, \Delta^{\mathrm{T}}\right)\right) \qquad (\mathbf{f} \in \Omega_1) \tag{4.6}$$

where the elements of $\breve{\mathbf{t}}(\cdot,\cdot)$ are given by (4.2), and $\mathbf{\Psi}(\cdot,\cdot)$ is given by (4.4). By construction, under (4.5), $\mathrm{diag}(\mathbf{\Gamma q}).\mathbf{p}(\mathbf{u})$ maps to the feasible route flow rate space Ω_1 for any $\mathbf{u} \in \Re_+^N$. That is to say, $\mathbf{G} : \mathbf{\Omega}_1 \to \mathbf{\Omega}_1$. As noted previously by Smith (1979), Ω_1 is a closed, bounded, convex subset of $\Re^N$.

Now under the hypotheses on $\mathbf{t}(\cdot)$, the modified cost functions $\breve{\mathbf{t}}(\boldsymbol{\mu}, \mathbf{\Sigma})$ given by (4.2) are continuous in $(\boldsymbol{\mu}, \mathbf{\Sigma})$. Since $\mathbf{\Psi}$ given by (4.4) is clearly continuous, and $\mathbf{p}(\cdot)$ is continuous by hypothesis, it follows that $\mathbf{G}(\cdot)$ given by (4.6) is continuous, being a continuous composition of continuous functions. Hence, $\mathbf{G}(\cdot)$ satisfies all the conditions of Brouwer's fixed point theorem (e.g. Baiocchi and Capelo, 1984), establishing that there exists at least one $\mathbf{f}^* \in \Omega_1$ such that $\mathbf{f}^* = \mathbf{G}(\mathbf{f}^*)$.

For one such fixed point $\mathbf{f}^*$ of **G**, let:

$$\boldsymbol{\mu}^* = \mathbf{\Delta f}^* \qquad\qquad \mathbf{\Sigma}^* = \tau^{-1} \mathbf{\Delta}\, \mathbf{\Psi}(\mathbf{q}, (\mathrm{diag}(\mathbf{\Gamma q}))^{-1} \mathbf{f}^*)\, \Delta^{\perp}\ . \tag{4.7}$$

Now,

$$\mathbf{f}^* = \mathbf{G}(\mathbf{f}^*) = \mathrm{diag}(\mathbf{\Gamma q})\, \mathbf{p}\left(\Delta^{\mathrm{T}} \breve{\mathbf{t}}(\mathbf{\Delta f}^*, \tau^{-1} \mathbf{\Delta}\, \mathbf{\Psi}(\mathbf{q}, (\mathrm{diag}(\mathbf{\Gamma q}))^{-1} \mathbf{f}^*) \Delta^{\mathrm{T}})\right) \tag{4.8}$$

and substituting for the arguments of $\breve{\mathbf{t}}(\cdot,\cdot)$ in (4.8) from (4.7), we obtain

$$\mathbf{f}^* = \mathrm{diag}(\mathbf{\Gamma q})\, \mathbf{p}\left(\Delta^{\mathrm{T}} \breve{\mathbf{t}}(\mu^*, \mathbf{\Sigma}^*)\right). \tag{4.9}$$

Substituting then for $\mathbf{f}^*$ in (4.7) from (4.9), we obtain the GSUE(2) conditions (4.1). That is to say, we have shown that a pair $(\mu^*, \mathbf{\Sigma}^*)$ exists that satisfies the GSUE(2) conditions.

In fact, the result above may be extended to the case of the general GSUE(n) model $(n \geq 2)$, where existence of solutions is guaranteed under continuity of **p**, **t** and all derivatives of the n times differentiable function **t** of order n and below. The proof is again based on applying Brouwer's fixed point theorem to a function of the mean route flow rates **f**, of the form

$$\mathbf{G}(\mathbf{f}) = \mathrm{diag}(\mathbf{\Gamma q})\, \mathbf{p}\left(\mathbf{\Delta}^{\mathrm{T}} \breve{\mathbf{t}}(\mathbf{\Delta f}, \{\tau^{-j} \widehat{\mathbf{\Delta}}^{(j)} \mathbf{H}^{(j)}(\tilde{\mathbf{q}}, (\mathrm{diag}(\mathbf{\Gamma q}))^{-1} \mathbf{f}) :\ j = 2,3,\ldots,n\})\right) \tag{4.10}$$

where $\mathbf{H}^{(j)}(\tilde{\mathbf{q}},\boldsymbol{\beta})$ is a vector of j^{th} order moments (of the absolute route flows) of W independent Multinomial $(\tilde{q}_k,\boldsymbol{\beta}_{[k]})$ variables, and the coefficient matrices $\hat{\boldsymbol{\Delta}}^{(j)}$ $(j=2,3,\ldots,n)$ are given by (3.7).

5. UNIQUENESS OF GSUE(2) SOLUTIONS

An important question for any forecasting model is whether it will produce a single unique output for given fixed input data. A special case of the GSUE(2) model is considered here, as defined below.

Corollary 1 Consider a network with twice-differentiable costs $\mathbf{t}(\cdot)$, choice probability model $\mathbf{p}(\cdot)$, and demands **q**, and suppose additionally that the Jacobian of **t**(**v**) is diagonal for all demand-feasible $\mathbf{v}\in\Omega_2$ (that is to say, $t_a(\mathbf{v})=f_a(v_a)$ for some function $f_a(\cdot)$, for $a=1,2,\ldots,A$). Then the active GSUE(2) conditions are those on $\boldsymbol{\mu}$ and the diagonal elements of $\boldsymbol{\Sigma}$ (denoted by the A-vector of link flow variances $\boldsymbol{\phi}$). These active conditions may be written:

$$\boldsymbol{\mu}=\boldsymbol{\rho}(\boldsymbol{\mu},\boldsymbol{\phi})\ \mathbf{q} \tag{5.1a}$$

$$\phi_a=\tau^{-1}\left(\mu_a-\sum_{k=1}^{W}(\rho_{ak}(\boldsymbol{\mu},\boldsymbol{\phi}))^2\ q_k\right)\qquad (a=1,2,\ldots,A) \tag{5.1b}$$

where $\boldsymbol{\rho}(\boldsymbol{\mu},\boldsymbol{\phi})$ is an $A\times W$ matrix that depends on $(\boldsymbol{\mu},\boldsymbol{\phi})$ (with elements $\rho_{ak}(\boldsymbol{\mu},\boldsymbol{\phi})$ representing the proportion of the demand flow on movement k that uses link a), given by

$$\rho(\mu,\phi)=\boldsymbol{\Delta}\ \mathrm{diag}(\mathbf{p}(\Delta^T\breve{\mathbf{t}}(\mu,\mathrm{diag}(\phi))))\,\boldsymbol{\Gamma}\quad . \tag{5.2}$$

Proof As the Jacobian of **t**(**v**) is diagonal, we may replace $\breve{\mathbf{t}}(\boldsymbol{\mu},\boldsymbol{\Sigma})$ in (4.1)/(4.2) with $\breve{\mathbf{t}}(\boldsymbol{\mu},\mathrm{diag}(\boldsymbol{\phi}))$. Then, (5.1a) is established by noting that (dropping arguments) $\boldsymbol{\mu}=\boldsymbol{\Delta}\ \mathrm{diag}(\boldsymbol{\Gamma}\mathbf{q})\ \mathbf{p}=\boldsymbol{\Delta}\ \mathrm{diag}(\mathbf{p})\ \boldsymbol{\Gamma}\mathbf{q}=\boldsymbol{\rho}\ \mathbf{q}$. Decomposing the diagonal terms (variances) of (4.1b) into a sum of terms over the different inter-zonal movements:

$$\phi_a=\tau^{-1}\sum_{k=1}^{W}q_k\left(\sum_{r\in R_k}\Delta_{ar}\,p_r-\sum_{r\in R_k}\sum_{s\in R_k}\Delta_{ar}\Delta_{as}\,p_r\,p_s\right)=\tau^{-1}\sum_{k=1}^{W}q_k\sum_{r\in R_k}\Delta_{ar}\,p_r\left(1-\sum_{s\in R_k}\Delta_{as}\,p_s\right). \tag{5.3}$$

Since if $\boldsymbol{\rho}=\boldsymbol{\Delta}\,\mathbf{p}$, then $\rho_{ak}=\sum_{r\in R_k}\Delta_{ar}\,p_r$ and the result is proven by writing expression (5.3) as:

$$\phi_a=\tau^{-1}\sum_{k=1}^{W}q_k\sum_{r\in R_k}\boldsymbol{\Delta}_{ar}\,p_r\,(1-\rho_{ak})=\tau^{-1}\sum_{k=1}^{W}q_k(1-\rho_{ak})\sum_{r\in R_k}\boldsymbol{\Delta}_{ar}\,p_r=\tau^{-1}\sum_{k=1}^{W}q_k(1-\rho_{ak})\rho_{ak}=\tau^{-1}\left(\mu_a-\sum_{k=1}^{W}q_k\rho_{ak}{}^2\right)$$

Note that having solved (5.1), the GSUE(2) off-diagonal terms (covariances) in $\mathbf{\Sigma}$ are given by:

$$\mathbf{\Sigma} = \tau^{-1}\,\mathbf{\Delta}\,.\,\mathbf{\Psi}(\mathbf{q},\mathbf{p}(\Delta^{T}\breve{\mathbf{t}}(\mu,\mathrm{diag}(\phi))))\,.\,\mathbf{\Delta}^{T} \quad . \tag{5.4}$$

In corollary 1, we replace $\breve{\mathbf{t}}(\boldsymbol{\mu},\mathbf{\Sigma})$ with $\breve{\mathbf{t}}(\boldsymbol{\mu},\mathrm{diag}(\boldsymbol{\phi}))$ since on the supply-side (i.e. link performance side) of our traffic assignment problem, the cross-link flow covariances have no effect. However, this is not the same as assuming that these covariances are zero, and indeed from the demand-side (i.e. from the choices of drivers between alternative routes) there will in general be link flow correlations between different links through (5.4), due to the fact that link flows are formed from route flows.

The conditions (5.1) are rather attractive in their simplicity, and their form motivates us to consider a problem related to the GSUE(2) problem. In particular, we shall consider a fixed point problem cast in the form of an $A\times W$ matrix of disaggregated link flows $\boldsymbol{\beta}$, with elements β_{ak} denoting the flow on inter-zonal movement k that uses link a (a=1,2,…,A; k=1,2,…,W):

$$\beta = \mathbf{\Delta}\ \mathrm{diag}\big(\mathbf{p}(\Delta^{T}\vec{\mathbf{t}}(\beta))\big)\mathbf{\Gamma}\ \mathrm{diag}(\mathbf{q}) \qquad\qquad (\boldsymbol{\beta}\in\Omega_3) \tag{5.5}$$

where the convex set of demand-feasible disaggregated link flows $\Omega_3 \subseteq \Re^{A}\times\Re^{W}$ is given by

$$\Omega_3 = \{\ \boldsymbol{\beta} : \beta_{ak} = \sum_{r\in R_k} f_r\,\Delta_{ar}\ \ (a=1,2,\ldots,A;\ k=1,2,\ldots,W)\ \text{where}\ \mathbf{f}\in\Omega_1\ \} \tag{5.6}$$

and where $\vec{\mathbf{t}}(\boldsymbol{\beta})$ is given by

$$\vec{\mathbf{t}}(\boldsymbol{\beta}) = \breve{\mathbf{t}}(\mathbf{g}(\boldsymbol{\beta}),\mathrm{diag}(\mathbf{h}(\boldsymbol{\beta}))) \tag{5.7}$$

where $\breve{\mathbf{t}}(\boldsymbol{\mu},\mathbf{\Sigma})$ is given by (4.2), and the elements of $\mathbf{g}(\boldsymbol{\beta})$ and $\mathbf{h}(\boldsymbol{\beta})$ are (recalling that all $q_k > 0$):

$$g_a(\boldsymbol{\beta}) = \sum_{k=1}^{W}\beta_{ak} \qquad\qquad h_a(\boldsymbol{\beta}) = \tau^{-1}\sum_{k=1}^{W}\left(\beta_{ak} - \frac{\beta_{ak}^{\ 2}}{q_k}\right) \qquad\qquad (a=1,2,\ldots,A). \tag{5.8}$$

This problem will be of particular interest due to the relationships established in the two Lemmas below between fixed points of (5.5) and those of the GSUE(2) problem (5.1).

Lemma 6 Suppose the hypotheses of Corollary 1 are met, and that the vector pair $(\boldsymbol{\mu}^*,\boldsymbol{\phi}^*)$ satisfies the active GSUE(2) conditions (5.1). Then $\boldsymbol{\beta}^*$ given by

$$\boldsymbol{\beta}^* = \boldsymbol{\rho}(\boldsymbol{\mu}^*,\boldsymbol{\phi}^*)\ \mathrm{diag}(\mathbf{q}) \tag{5.9}$$

where $\boldsymbol{\rho}(\boldsymbol{\mu},\boldsymbol{\phi})$ is given by (5.2), is a solution to the fixed point problem (5.5).

Proof Consider $\boldsymbol{\beta}^*$ given by (5.9), then in view of (5.8) (and, for the last equality, (5.1a)):

$$\mathbf{g}(\boldsymbol{\beta}^*) = \boldsymbol{\beta}^*\,\mathbf{1}_W = \boldsymbol{\rho}(\boldsymbol{\mu}^*,\boldsymbol{\phi}^*)\ \mathrm{diag}(\mathbf{q})\ \mathbf{1}_W = \boldsymbol{\rho}(\boldsymbol{\mu}^*,\boldsymbol{\phi}^*)\ \mathbf{q} = \boldsymbol{\mu}^*\ . \tag{5.10}$$

Similarly, since by (5.9), $\beta^*_{ak} = \rho_{ak}(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*)\, q_k \quad (a = 1,2,\ldots, A;\ k = 1,2,\ldots,W)$, then (5.8) implies:

$$h_a(\boldsymbol{\beta}^*) = \tau^{-1} \sum_{k=1}^{W} \left(\beta^*_{ak} - \frac{(\beta^*_{ak})^2}{q_k} \right) = \tau^{-1} \sum_{k=1}^{W} \left(\rho_{ak}(\mu^*, \phi^*)\, q_k - (\rho_{ak}(\mu^*, \phi^*))^2\, q_k \right) \tag{5.11}$$

and by the GSUE(2) conditions (5.1) we obtain

$$h_a(\boldsymbol{\beta}^*) = \tau^{-1} \left(\mu^*_a - \sum_{k=1}^{W} (\rho_{ak}(\mu^*, \phi^*))^2 q_k \right) = \phi^*_a \ . \tag{5.12}$$

Hence, $\ddot{\mathbf{t}}(\boldsymbol{\beta}^*) = \breve{\mathbf{t}}(\mathbf{g}(\boldsymbol{\beta}^*), \mathrm{diag}(\mathbf{h}(\boldsymbol{\beta}^*))) = \breve{\mathbf{t}}(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*)$. But then this $\boldsymbol{\beta}^*$ is a solution of (5.5), since:

$$\begin{aligned} \boldsymbol{\Delta}\mathrm{diag}(\mathbf{p}(\boldsymbol{\Delta}^{\mathrm{T}} \ddot{\mathbf{t}}(\beta^*)))\boldsymbol{\Gamma}\mathrm{diag}(\mathbf{q}) &= \boldsymbol{\Delta}\mathrm{diag}(\mathbf{p}(\boldsymbol{\Delta}^{\mathrm{T}} \breve{\mathbf{t}}(\mu^*, \mathrm{diag}(\phi^*))))\boldsymbol{\Gamma}\mathrm{diag}(\mathbf{q}) \\ &= \boldsymbol{\rho}(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*)\ \mathrm{diag}(\mathbf{q}) = \boldsymbol{\beta}^* \ . \end{aligned}$$

Lemma 7 Suppose the hypotheses of Corollary 1 hold, and that $(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*) \neq (\boldsymbol{\mu}^{\oplus}, \boldsymbol{\phi}^{\oplus})$ are two solutions to the active GSUE(2) conditions (5.1). Suppose that $\boldsymbol{\beta}^*$ and $\boldsymbol{\beta}^{\oplus}$ denote the corresponding solutions to (5.5), constructed (by Lemma 6) through (5.9). Then $\boldsymbol{\beta}^* \neq \boldsymbol{\beta}^{\oplus}$.

Proof From (5.10) and (5.12) in the proof of Lemma 6, it follows that $\mathbf{g}(\boldsymbol{\beta}^*) = \boldsymbol{\mu}^*$, $\mathbf{h}(\boldsymbol{\beta}^*) = \boldsymbol{\phi}^*$, $\mathbf{g}(\boldsymbol{\beta}^{\oplus}) = \boldsymbol{\mu}^{\oplus}$, $\mathbf{h}(\boldsymbol{\beta}^{\oplus}) = \boldsymbol{\phi}^{\oplus}$. Hence,

$$\breve{\mathbf{t}}(\mathbf{g}(\boldsymbol{\beta}^*), \mathrm{diag}(\mathbf{h}(\boldsymbol{\beta}^*))) = \breve{\mathbf{t}}(\boldsymbol{\mu}^*, \mathrm{diag}(\boldsymbol{\phi}^*)) \ . \tag{5.13}$$

Now, let us suppose conversely that $\boldsymbol{\beta}^* = \boldsymbol{\beta}^{\oplus}$, and aim for a contradiction. Then,

$$\breve{\mathbf{t}}(\mathbf{g}(\boldsymbol{\beta}^*), \mathrm{diag}(\mathbf{h}(\boldsymbol{\beta}^*))) = \breve{\mathbf{t}}(\mathbf{g}(\boldsymbol{\beta}^{\oplus}), \mathrm{diag}(\mathbf{h}(\boldsymbol{\beta}^{\oplus}))) = \breve{\mathbf{t}}(\boldsymbol{\mu}^{\oplus}, \mathrm{diag}(\boldsymbol{\phi}^{\oplus})) \ . \tag{5.14}$$

Then, (5.13) and (5.14) together imply that $\breve{\mathbf{t}}(\boldsymbol{\mu}^*, \mathrm{diag}(\boldsymbol{\phi}^*)) = \breve{\mathbf{t}}(\boldsymbol{\mu}^{\oplus}, \mathrm{diag}(\boldsymbol{\phi}^{\oplus}))$. Hence, by (5.2), $\boldsymbol{\rho}(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*) = \boldsymbol{\rho}(\boldsymbol{\mu}^{\oplus}, \boldsymbol{\phi}^{\oplus})$. Hence, by substitution for $\boldsymbol{\rho}$ in the right hand sides of (5.1), $(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*) = (\boldsymbol{\mu}^{\oplus}, \boldsymbol{\phi}^{\oplus})$, contradicting the original hypothesis, and thus establishing the lemma.

The fixed point problem (5.5) effectively works by embedding part of the probabilistic choice mechanism (i.e. multinomial route split conditional on predicted costs) within a modified kind of expected link cost function (5.7)/(5.8). Turning attention, then, to the choice process, we introduce some additional terminology, motivated by that of Cantarella and Cascetta (1995). The route demand map is a function $\boldsymbol{\Phi}(\mathbf{u})$ $(\boldsymbol{\Phi}: \Re^N_+ \to \Omega_1 \subseteq \Re^N_+)$ relating a vector of route flows to given route costs $\mathbf{u}$:

$$\boldsymbol{\Phi}(\mathbf{u}) = \mathrm{diag}(\boldsymbol{\Gamma}\mathbf{q})\, \mathbf{p}(\mathbf{u}) \tag{5.15}$$

and the link demand map $\boldsymbol{\zeta}: \Re^A_+ \to \Omega_2 \subseteq \Re^A_+$ relates a vector of link flows $\boldsymbol{\zeta}(\mathbf{y})$ to given link costs $\mathbf{y}$:

$$\boldsymbol{\zeta}(\mathbf{y}) = \Delta \mathrm{diag}(\boldsymbol{\Gamma}\mathbf{q})\mathbf{p}(\Delta^{\mathrm{T}}\mathbf{y}). \tag{5.16}$$

Denoting (for positive integers m and n) $\Re_{+}^{m}(n) = \{\mathbf{X} \in \Re_{+}^{m} \times \Re_{+}^{n} : \mathbf{X} = \mathbf{x}\ \mathbf{1}_{n}^{\mathbf{T}} \text{ and } \mathbf{x} \in \Re_{+}^{m}\}$, where $\mathbf{1}_{n}$ is an n-vector of 1's, we define finally the disaggregated link demand map $\boldsymbol{\lambda} : \Re_{+}^{A}(W) \to \boldsymbol{\Omega}_{3}$ as the relationship between the matrix of disaggregated link flows and the matrix of disaggregated link costs $\mathbf{Y}$, the latter formed from W identical copies of an A-vector of common link costs $\mathbf{y}$ (i.e. $\mathbf{Y} = \mathbf{y}\ \mathbf{1}_{W}^{\mathbf{T}}$):

$$\boldsymbol{\lambda}(\mathbf{Y}) = \boldsymbol{\lambda}(\mathbf{y}\,\mathbf{1}_{W}^{\mathbf{T}}) = \Delta\ \mathrm{diag}(\mathbf{p}(\Delta^{\mathrm{T}}\mathbf{y}))\,\boldsymbol{\Gamma}\ \mathrm{diag}(\mathbf{q}) \qquad (\mathbf{Y} \in \Re_{+}^{A}(W)). \tag{5.17}$$

In addition, adopting notation (4.3), a real-valued vector mapping $\boldsymbol{\psi}(.)\,(\boldsymbol{\psi} : X \to Y)$ will be termed monotonically increasing over X if and only if

$$\left\|\boldsymbol{\psi}(\mathbf{x}') - \boldsymbol{\psi}(\mathbf{x}''),\ \mathbf{x}' - \mathbf{x}''\right\| > 0 \qquad \forall\ \mathbf{x}', \mathbf{x}'' \in X \ \ (\mathbf{x}' \neq \mathbf{x}'')$$

and monotonically non-increasing if the > is replaced by $\leq$. By a slight abuse, the same notation will be used both when $\mathbf{x}', \mathbf{x}''$ and $\boldsymbol{\psi} : X \to Y$ are matrices ($X, Y \in \Re^{m} \times \Re^{n}$), and vectors $(X, Y \in \Re^{m})$.

Lemma 8 Suppose the route demand map $\boldsymbol{\Phi}$ in (5.15) is monotonically non-increasing over $\Re_{+}^{N}$. Then the disaggregated link demand map $\boldsymbol{\lambda}$ in (5.17) is monotonically non-increasing over $\Re_{+}^{A}(W)$.

Proof Now under (5.17), for $\mathbf{Y} = \mathbf{y}\ \mathbf{1}_{W}^{\mathbf{T}} \in \Re_{+}^{A}(W)$,

$$\begin{aligned}
\boldsymbol{\lambda}(\mathbf{Y})\ \mathbf{1}_{W} = \boldsymbol{\lambda}(\mathbf{y}\,\mathbf{1}_{W}^{\mathbf{T}})\ \mathbf{1}_{W} &= \Delta \mathrm{diag}(\mathbf{p}(\Delta^{\mathrm{T}}\mathbf{y}))\boldsymbol{\Gamma}\mathrm{diag}(\mathbf{q})\ \mathbf{1}_{W} \\
&= \Delta \mathrm{diag}(\mathbf{p}(\Delta^{\mathrm{T}}\mathbf{y}))\boldsymbol{\Gamma}\mathbf{q} && (\text{since } \mathrm{diag}(\mathbf{q})\ \mathbf{1}_{W} = \mathbf{q}) \\
&= \Delta \mathrm{diag}(\boldsymbol{\Gamma}\mathbf{q})\mathbf{p}(\Delta^{\mathrm{T}}\mathbf{y}) && (\text{for n-vectors } \mathbf{a} \text{ and } \mathbf{b},\ \mathrm{diag}(\mathbf{a})\mathbf{b} = \mathrm{diag}(\mathbf{b})\mathbf{a}) \\
&= \boldsymbol{\zeta}(\mathbf{y}) && (\text{given by (5.16)}).
\end{aligned} \tag{5.18}$$

Then,

$$\begin{aligned}
\left\|\boldsymbol{\lambda}(\mathbf{Y}') - \boldsymbol{\lambda}(\mathbf{Y}''), \mathbf{Y}' - \mathbf{Y}''\right\| &= \sum_{a=1}^{A}\sum_{k=1}^{W}\left(\lambda_{ak}(\mathbf{Y}') - \lambda_{ak}(\mathbf{Y}'')\right)\left(y_a' - y_a''\right) \\
&= \sum_{a=1}^{A}\left(y_a' - y_a''\right)\left(\sum_{k=1}^{W}\lambda_{ak}(\mathbf{Y}') - \sum_{k=1}^{W}\lambda_{ak}(\mathbf{Y}'')\right) \\
&= \left\|\boldsymbol{\zeta}(\mathbf{y}') - \boldsymbol{\zeta}(\mathbf{y}''), \mathbf{y}' - \mathbf{y}''\right\|
\end{aligned} \tag{5.19}$$

by (5.18). Hence, if the link demand map $\boldsymbol{\zeta}(.)$ is monotonically non-increasing over $\Re_{+}^{A}$, then $\boldsymbol{\lambda}(.)$ will be monotonically non-increasing over $\Re_{+}^{A}(W)$. But by hypothesis, the route demand map is monotonically non-increasing over $\Re_{+}^{N}$ and—as proven in Cantarella & Cascetta (1995, p. 314)—this is a sufficient condition for the link demand map to be monotonically non-increasing over $\Re_{+}^{A}$.

The relevance of Lemma 8 derives from the fact that it is satisfied by common forms of choice probability model $\mathbf{p}(\mathbf{u})$ used in practice. Following Daganzo (1979), we define $\mathbf{p}(\mathbf{u})$ to be a regular random utility model if it is of the form (1.3) and the probability distribution of perceptual differences $\mathbf{e}$ is independent of measured utility/cost. The logit model satisfies this condition, as does the probit provided the covariance matrix is constant. If $\mathbf{p}(\mathbf{u})$ is a regular random utility model, then the route demand map (5.15) is monotonically non-increasing over $\Re_{+}^{N}$ (Cantarella & Casetta, 1995, p 315).

Having dealt with the choice-side assumptions of our problem, we turn attention to the modified expected cost functions (5.7), and introduce a corresponding monotonicity-like condition. In particular, for any $\boldsymbol{\beta} \in \Omega_3$, the monotone complement of $\boldsymbol{\beta}$ is defined as the set $M_{\boldsymbol{\beta}} \subseteq \Omega_3$ given by:

$$M_{\boldsymbol{\beta}} = \left\{\boldsymbol{\beta}^* \in \Omega_3 : \left\| \vec{\mathbf{t}}(\boldsymbol{\beta}) - \vec{\mathbf{t}}(\boldsymbol{\beta}^*),\, \boldsymbol{\beta}\mathbf{1}_W - \boldsymbol{\beta}^*\mathbf{1}_W \right\| > 0 \right\} \qquad (\boldsymbol{\beta} \in \Omega_3)\,. \tag{5.20}$$

Note that (5.20) is not in fact a monotonicity condition over some subset of Ω_3, since $\boldsymbol{\beta}\mathbf{1}_W$ appears rather than the functional argument $\boldsymbol{\beta}$. The intepretation of $M_{\boldsymbol{\beta}}$ shall be explored shortly, below. Before that, however, a general condition is established on the "proximity" of GSUE(2) solutions.

Theorem (Proximity Theorem) Consider a problem satisfying the hypotheses of Corollary 1, with a route demand map (5.15) that is monotonically non-increasing over $\Re_{+}^{N}$. Suppose that GSUE(2) solutions exist, and that $(\boldsymbol{\mu}^*, \boldsymbol{\phi}^*)$ is one such solution, inducing a corresponding $\boldsymbol{\beta}^*$ through (5.9). Then no other GSUE(2) solution exists that has an induced $\boldsymbol{\beta}$ in the monotone complement $M_{\boldsymbol{\beta}^*}$ of $\boldsymbol{\beta}^*$.

Proof Suppose the hypotheses of the theorem hold, yet conversely there exists a second GSUE(2) solution—denoted $(\boldsymbol{\mu}^{\oplus}, \boldsymbol{\phi}^{\oplus})$ and inducing $\boldsymbol{\beta}^{\oplus}$ through (5.9)—such that $\boldsymbol{\beta}^{\oplus} \in M_{\boldsymbol{\beta}^*}$. By Lemma 7, $\boldsymbol{\beta}^* \neq \boldsymbol{\beta}^{\oplus}$. Now by (5.17) we can write the fixed point problem (5.5) as:

$$\boldsymbol{\beta} = \boldsymbol{\lambda}(\vec{\mathbf{t}}(\boldsymbol{\beta})\; \mathbf{1}_W^{\mathbf{T}}) \equiv \boldsymbol{\lambda}(\boldsymbol{\psi}(\boldsymbol{\beta})) \quad \text{where} \quad \boldsymbol{\psi}(\boldsymbol{\beta}) = \vec{\mathbf{t}}(\boldsymbol{\beta})\; \mathbf{1}_W^{\mathbf{T}}\,. \tag{5.21}$$

Denoting $\boldsymbol{\psi}^* = \boldsymbol{\psi}(\boldsymbol{\beta}^*)$ and $\boldsymbol{\psi}^{\oplus} = \boldsymbol{\psi}(\boldsymbol{\beta}^{\oplus})$, then as $\boldsymbol{\beta}^*$ and $\boldsymbol{\beta}^{\oplus}$ are fixed points of (5.21), we have also that $\boldsymbol{\beta}^* = \boldsymbol{\lambda}(\boldsymbol{\psi}^*)$ and $\boldsymbol{\beta}^{\oplus} = \boldsymbol{\lambda}(\boldsymbol{\psi}^{\oplus})$. Then, on the one hand,

$$\left\| \boldsymbol{\beta}^* - \boldsymbol{\beta}^{\oplus}, \boldsymbol{\psi}^* - \boldsymbol{\psi}^{\oplus} \right\| = \left\| \boldsymbol{\lambda}(\boldsymbol{\psi}^*) - \boldsymbol{\lambda}(\boldsymbol{\psi}^{\oplus}), \boldsymbol{\psi}^* - \boldsymbol{\psi}^{\oplus} \right\| \leq 0 \tag{5.22}$$

by virtue of the fact that under the hypotheses of the theorem, $\boldsymbol{\lambda}(.)$ is monotonically non-increasing (using Lemma 8). On the other hand,

$$\begin{aligned}
\left\| \boldsymbol{\beta}^* - \boldsymbol{\beta}^{\oplus}, \boldsymbol{\psi}^* - \boldsymbol{\psi}^{\oplus} \right\| &= \sum_{a=1}^{A} \sum_{k=1}^{W} (\beta^*_{ak} - \beta^{\oplus}_{ak})(\psi^*_{ak} - \psi^{\oplus}_{ak}) \\
&= \sum_{a=1}^{A} \sum_{k=1}^{W} (\beta^*_{ak} - \beta^{\oplus}_{ak})\ (\vec{t}_a(\boldsymbol{\beta}^*) - \vec{t}_a(\boldsymbol{\beta}^{\oplus})) \\
&= \sum_{a=1}^{A} (\vec{t}_a(\boldsymbol{\beta}^*) - \vec{t}_a(\boldsymbol{\beta}^{\oplus})) \left(\sum_{k=1}^{W} \boldsymbol{\beta}^*_{ak} - \sum_{k=1}^{W} \boldsymbol{\beta}^{\oplus}_{ak} \right) \\
&= \left\| \vec{\mathbf{t}}(\boldsymbol{\beta}^*) - \vec{\mathbf{t}}(\boldsymbol{\beta}^{\oplus}),\ \boldsymbol{\beta}^* \mathbf{1}_W - \boldsymbol{\beta}^{\oplus} \mathbf{1}_W \right\| \quad > 0 \qquad (5.23)
\end{aligned}$$

where the last line holds since $\boldsymbol{\beta}^{\oplus} \in M_{\boldsymbol{\beta}^*}$, by definition (5.20). Expressions (5.22) and (5.23) taken together give a contradiction, thus establishing the theorem.

The value of the Proximity Theorem therefore rests on the usefulness of the monotone complement concept. This is demonstrated in the two corollaries below, the first addressing the case of networks with a single inter-zonal movement, the second addressing the general case of multiple movements.

Corollary 2 Consider a problem satisfying the hypotheses of the Proximity Theorem, with a single inter-zonal movement with demand q_1. Suppose $\mathbf{t}(\mathbf{v})$ is three times continuously differentiable for all $\mathbf{v} \in \Omega_2$, with the Jacobian of $\mathbf{t}(\mathbf{v})$ diagonal for all $\mathbf{v} \in \Omega_2$. Denoting the derivatives $\frac{\partial t_a}{\partial v_a}, \frac{\partial^2 t_a}{\partial {v_a}^2}, \frac{\partial^3 t_a}{\partial {v_a}^3}$ as a function of $\mathbf{v}$ by respectively $t'_a(\mathbf{v}),\ t''_a(\mathbf{v}),$ and $t'''_a(\mathbf{v})$, suppose further that

$$t'_a(\mathbf{v}) > 0 \quad \forall\, \mathbf{v} \in \Omega_2 \ \text{ where } v_a > 0 \quad (a = 1,2,\ldots, A) \qquad (5.24)$$

and that for each link $a = 1,2,\ldots, A$, either

$$t''_a(\mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \Omega_2 \qquad (5.25)$$

or the following three conditions hold:

$$t''_a(\mathbf{v}) > 0 \quad \forall\, \mathbf{v} \in \Omega_2 \ \text{ where } v_a > 0 \qquad (5.26)$$

$$t'''_a(\mathbf{v}) \ge 0 \quad \forall\, \mathbf{v} \in \Omega_2 \qquad (5.27)$$

$$\frac{v_a\ t''_a(\mathbf{v})}{t'_a(\mathbf{v})} \le q_1 \tau \quad \forall\, \mathbf{v} \in \Omega_2\ . \qquad (5.28)$$

Then if a GSUE(2) solution exists, it is unique.

Proof In the single inter-zonal movement case, $\boldsymbol{\beta}$ and $\mathbf{v}$ are synonymous. Now, since the Jacobian of $\mathbf{t}(\mathbf{v})$ is diagonal,

$$\overleftrightarrow{t}_a(\mathbf{v}) = t_a(\mathbf{v}) + \tfrac{1}{2} t''_a(\mathbf{v})\,\tau^{-1}\left(v_a - \frac{{v_a}^2}{q_1} \right). \tag{5.29}$$

Since by hypothesis $t_a(\mathbf{v})$ depends only on v_a, then so does $\overleftrightarrow{t}_a(\mathbf{v})$, and we therefore aim to show that $\overleftrightarrow{t}_a$ is an increasing function of v_a. On a link for which (5.24) and (5.25) hold, this can be seen to be true by inspection of (5.29). Turning attention, then, to a link on which (5.26)-(5.28) hold, since $t_a(\cdot)$ is three times differentiable, we obtain from (5.29):

$$\begin{aligned}\frac{\partial \overleftrightarrow{t}_a}{\partial v_a} &= t'_a(\mathbf{v}) + \frac{1}{2\tau} t''_a(\mathbf{v})\left(1 - \frac{2v_a}{q_1}\right) + \frac{1}{2\tau} t'''_a(\mathbf{v})\left(v_a - \frac{{v_a}^2}{q_1}\right) \\ &= \left(t'_a(\mathbf{v}) - \frac{v_a t''_a(\mathbf{v})}{q_1 \tau} \right) + \frac{1}{2\tau} t''_a(\mathbf{v}) + \frac{1}{2\tau} t'''_a(\mathbf{v})\left(v_a - \frac{{v_a}^2}{q_1}\right).\end{aligned} \tag{5.30}$$

The first bracketed term in (5.30) is non-negative under (5.28), the second term is positive by (5.26) provided $v_a > 0$, and the third term is non-negative by (5.27) for demand-feasible v_a since $0 \le v_a \le q_1$. Therefore the overall derivative is the sum of two non-negative and one positive term, and so is positive for $v_a > 0$. Since therefore $\overleftrightarrow{t}_a$ is increasing in, and depends only on, $v_a = \beta_{a1}$, it follows that for any $\boldsymbol{\beta} \in \Omega_3$, the monotone complement is maximal: i.e. $M_{\boldsymbol{\beta}} = \Omega_3 - \{\boldsymbol{\beta}\}$. Hence, by the Proximity Theorem, only one GSUE(2) solution can exist.

The relevance of conditions (5.24)-(5.28) may be illustrated by considering the common "power-law" family of cost-flow relationships:

$$t_a(\mathbf{v}) = \alpha_a + \gamma_a {v_a}^{n_a} \qquad (\alpha_a \ge 0;\ \gamma_a > 0;\ a = 1,2,\ldots, A) \tag{5.31}$$

in which case some simple analysis shows (5.24)-(5.28) to be satisfied provided that for each link $a = 1,2,\ldots, A$, either $n_a = 1$ or $2 \le n_a \le 1 + q_1\tau$. That is to say, the conditions are satisfied provided that the absolute number of travellers $q_1\tau$ is not too small (in practice, we would certainly expect $n_a \le 10$).

Turning attention to multiple inter-zonal movement problems, we first establish the following bound.

Lemma 9 Suppose $\mathbf{t}(\mathbf{v})$ is twice differentiable with a diagonal Jacobian, and suppose further that $t''_a(\mathbf{v}) \geq 0$ at all $\mathbf{v} \in \Omega_2$. Then:

$$t_a(\boldsymbol{\mu}) \;\leq\; \vec{t}_a(\boldsymbol{\beta}) \;\leq\; t_a(\boldsymbol{\mu}) + \frac{\mu_a}{2\tau}\left(1 - \frac{\mu_a}{\sum_{k=1}^{W} q_k}\right) t''_a(\boldsymbol{\mu}) \qquad (a = 1,2,\ldots, A;\ \boldsymbol{\mu} = \boldsymbol{\beta}\mathbf{1}_W;\ \boldsymbol{\beta} \in \Omega_3).$$

Proof The lower bound on $\vec{t}_a(\boldsymbol{\beta})$ is clear by inspection of (4.2)/(5.7), since $t''_a(\cdot) \geq 0$ by hypothesis, and we turn attention to the upper bound. By (5.7)/(5.8) with $\boldsymbol{\mu} = \boldsymbol{\beta}\mathbf{1}_W$,

$$\vec{t}_a(\boldsymbol{\beta}) = t_a(\boldsymbol{\mu}) + \frac{1}{2\tau}\left(\mu_a - \sum_{k=1}^{W} \frac{\beta_{ak}{}^2}{q_k}\right) t''_a(\boldsymbol{\mu}) \ . \tag{5.32}$$

Since $t''_a(\cdot) \geq 0$ by hypothesis, then (5.32) is clearly maximised (with respect to $\boldsymbol{\beta}$) for fixed $\boldsymbol{\mu}$ by the solution to minimisation problem (5.33):

$$\text{minimise } \sum_{k=1}^{W} \frac{\beta_{ak}{}^2}{q_k} \ \text{ subject to } \sum_{k=1}^{W} \beta_{ak} = \mu_a \ \text{ and } \ 0 \leq \beta_{ak} \leq q_k \ (k = 1,2,\ldots,W). \tag{5.33}$$

Removing the equality constraint by substitution of $\beta_{aW} = \mu_a - \sum_{k=1}^{W-1} \beta_{ak}$ in the objective function yields a modified objective function g. Now, neglecting the inequality constraints, the unique unconstrained minimum of g occurs when simultaneously:

$$0 = \frac{\partial g}{\partial \beta_{am}} = \frac{2\beta_{am}}{q_m} - \frac{2\left(\mu_a - \sum_{k=1}^{W-1} \beta_{ak}\right)}{q_W} \qquad (m = 1,2,\ldots,W-1) \tag{5.34}$$

and since the second term is a common constant in all $W-1$ of these conditions, they together imply:

$$\frac{\beta_{ak}}{q_k} = \frac{\beta_{am}}{q_m} \quad (k = 1,2,\ldots,W-1;\ \ m = 1,2,\ldots,W-1) \ . \tag{5.35}$$

With β_{ak} substituted in (5.34) from (5.35), we obtain after some rearrangement the minimum point:

$$\beta^*_{am} = \frac{q_m}{\sum_{k=1}^{W} q_k} \, \mu_a \quad (m = 1,2,\ldots,W) \, . \tag{5.36}$$

By inspection, the unconstrained minimum (5.36) also satisfies the constraints of the constrained problem (5.33), and so minimises (5.33) too. The upper bound of the Lemma is obtained by substitution of (5.36) into (5.32), and we are finished.

The bounds in Lemma 9 then make it possible to establish the following result.

Corollary 3 Suppose the hypotheses of the Proximity Theorem hold, and that $\mathbf{t}(\mathbf{v})$ is differentiable with a diagonal Jacobian that has positive entries at all $\mathbf{v} \in \Omega_2$, and that all derivatives of order 2 and above exist and are non-negative at all $\mathbf{v} \in \Omega_2$. Then for any GSUE(2) solution inducing $\boldsymbol{\beta}$, the monotone complement M_{β} is contained within $\overline{M}_{\beta}$ given by:

$$\overline{M}_{\beta} = \left\{ \boldsymbol{\beta}^* \in \Omega_3 : \boldsymbol{\mu}^* = \boldsymbol{\beta}^* \mathbf{1}_W \text{ and } \mu_a - \mu_a^* > \sup_{\mathbf{v}\in\Omega_2} \left\{ \frac{v_a t_a''(\mathbf{v})}{t_a'(\mathbf{v})\, 2\tau} \left(1 - \frac{v_a}{\sum_{k=1}^{W} q_k} \right) \right\} \ (a = 1,2,\dots, A) \right\}. \quad (5.37)$$

Proof Now, as noted in (5.23),

$$\left\| \vec{\mathbf{t}}(\boldsymbol{\beta}^*) - \vec{\mathbf{t}}(\boldsymbol{\beta}^{\oplus}),\ \boldsymbol{\beta}^* \mathbf{1}_W - \boldsymbol{\beta}^{\oplus} \mathbf{1}_W \right\| = \sum_{a=1}^{A} (\vec{t}_a(\boldsymbol{\beta}^*) - \vec{t}_a(\boldsymbol{\beta}^{\oplus})) \left(\sum_{k=1}^{W} \beta^*_{ak} - \sum_{k=1}^{W} \beta^{\oplus}_{ak} \right) \quad (5.38)$$

and so to ensure the left-hand-side is positive, we may impose the stronger condition that all terms in the summation over a in the right-hand-side are positive. It is this stronger condition that we will show for the monotone complement. Now, from Lemma 9, using the lower bound for $\vec{t}_a(\boldsymbol{\beta}^{\oplus})$ and the upper bound for $\vec{t}_a(\boldsymbol{\beta})$, and defining $\boldsymbol{\mu}^{\oplus} = \boldsymbol{\beta}^{\oplus} \mathbf{1}_W$ and $\boldsymbol{\mu} = \boldsymbol{\beta} \mathbf{1}_W$, we have:

$$\mu_a^{\oplus} > \mu_a \quad \Rightarrow \quad \vec{t}_a(\boldsymbol{\beta}^{\oplus}) - \vec{t}_a(\boldsymbol{\beta}) > t_a(\boldsymbol{\mu}^{\oplus}) - t_a(\boldsymbol{\mu}) - \frac{\mu_a}{2\tau} t_a''(\boldsymbol{\mu}) + \frac{{\mu_a}^2}{2\tau \sum_{k=1}^{W} q_k} t_a''(\boldsymbol{\mu}) \ . \quad (5.39)$$

If $t_a''(\boldsymbol{\mu}) = 0$ then we are finished, since by hypothesis $t_a(\boldsymbol{\mu})$ is dependent only on, and is increasing in, μ_a. So let us suppose instead that $t_a''(\boldsymbol{\mu}) > 0$ (since by hypothesis it is non-negative). By a Taylor series expansion in the neighbourhood of $\boldsymbol{\mu}^{\oplus}$:

$$t_a(\boldsymbol{\mu}^{\oplus}) = t_a(\boldsymbol{\mu}) + t_a'(\boldsymbol{\mu})\ (\mu_a^{\oplus} - \mu_a) + \sum_{i=2}^{\infty} t_a^{(i)}(\boldsymbol{\mu})\ \frac{(\mu_a^{\oplus} - \mu_a)^i}{i!} \ .$$

Since by hypothesis $t_a^{(i)}(\cdot) = \dfrac{\partial^i t_a}{\partial {v_a}^i}$ is everywhere non-negative for $i = 3,4,\dots$, and strictly positive for $i = 1,2$, then $t_a(\boldsymbol{\mu}^{\oplus}) > t_a(\boldsymbol{\mu}) + t_a'(\boldsymbol{\mu})\ (\mu_a^{\oplus} - \mu_a)$ when $\mu_a^{\oplus} > \mu_a$. Hence, by (5.39),

$$\mu_a^{\oplus} > \mu_a \quad \Rightarrow \quad \vec{t}_a(\boldsymbol{\beta}^{\oplus}) - \vec{t}_a(\boldsymbol{\beta}) > t_a'(\boldsymbol{\mu})\ (\mu_a^{\oplus} - \mu_a) - \frac{\mu_a}{2\tau} t_a''(\boldsymbol{\mu}) + \frac{{\mu_a}^2}{2\tau \sum_{k=1}^{W} q_k} t_a''(\boldsymbol{\mu}) \quad (5.40)$$

and the right hand side may be seen to be positive for any $\boldsymbol{\mu} = \boldsymbol{\beta} \mathbf{1}_W$ when $\boldsymbol{\beta} \in \overline{M}_{\beta^{\oplus}}$ given by (5.37).

The value of this result is again most easily illustrated by reference to the power law family (5.31). In that case, $\frac{v_a t''_a(\mathbf{v})}{t'_a(\mathbf{v})2\tau}\left(1-\frac{v_a}{\sum_{k=1}^{W} q_k}\right) \le \frac{v_a t''_a(\mathbf{v})}{t'_a(\mathbf{v})2\tau} = \frac{n_a - 1}{2\tau}$ for any $\boldsymbol{\mu} \in \Omega_2$. Thus, for example, if $n_a \le 5$ and $\tau \ge 0.5$, then for a given GSUE(2) solution $(\boldsymbol{\mu},\boldsymbol{\phi})$ there cannot be a second GSUE(2) solution $(\boldsymbol{\mu}^{\oplus},\boldsymbol{\phi}^{\oplus})$ at a distance greater than $\mu_a^{\oplus} - \mu_a > 4$. Since typically mean flow rates will be in the order of hundreds or thousands of vehicles per hour, the result establishes that if multiple solutions do exist, they will (in relative terms) have very similar mean flow rates. At smaller distances, the possibility cannot be ruled out that some of the terms in the summation over a in (5.38) will be negative, yet even in such instances there will still be many cases where the overall summation is positive, and the proximity theorem would again apply. (That is to say, the condition in Corollary 3 is rather more stringent than is necessary).

6. EXAMPLE

As an illustration of the GSUE(n) model, a simple example is presented, consisting of two parallel links/routes with cost-flow performance functions $t_1(\mathbf{v}) = \left(\frac{v_1}{10}\right)^4$ and $t_2(\mathbf{v}) = 10$, serving a single inter-zonal movement with demand $q = 20$ vehicles/hour. Suppose further that the choice probability function has the logit form $p_1(\mathbf{u}) = (1+\exp(0.5(u_1 - u_2)))^{-1}$. Now, since $\mathbf{t}(\mathbf{v})$ has a diagonal Jacobian, all cross-link moments in the GSUE(n) fixed point conditions will be inactive, since they will not appear in (3.6). Indeed, since $t_2(.)$ is constant, and derivatives of $t_1(.)$ of higher order than 4 are zero, only moments of V_1 of order 4 and below will be active. Now, since $V_1 = F_1 = \tau^{-1}\tilde{F}_1$, we require (on the choice side) expressions for the first four moments of $\tilde{F}_1 \sim \text{Binomial}(\tilde{q}, p_1)$, given by (Stuart & Ord, 1987, p. 76):

$$\varpi_1(1) = \tilde{q}p_1;\ \ \varpi_1(2) = \tilde{q}p_1(1-p_1);\ \ \varpi_1(3) = \varpi_1(2)\,(1-2p_1);\ \ \varpi_1(4) = \varpi_1(2)\left(1+3\left(1-\frac{2}{\tilde{q}}\right)\varpi_1(2)\right).$$

Using an initial fine grid search, and then a finer grid search on areas near to a solution, solutions to the GSUE(n) fixed point conditions were determined numerically, to three decimal place accuracy, for each of $n = 1,2,3,4$, and for a number of given values of τ. In this simple example, this may be achieved by a univariate grid search on p_1 for $0 \le p_1 \le 1$. For each value of τ and n tested, exactly one GSUE(n) solution was found to exist. The resulting GSUE(n) mean flow rates are plotted in **Figure 1**. Now, by Lemma 2, as the $t_a(\mathbf{v})$ are polynomials of order ≤ 4, the GSUE(n) mean flow for any n>4 coincides with that for n=4. Moreover, Lemma 2 implies that for $n \geq 4$ the approximation error in (3.6) is zero: the GSUE(4) solution gives the exact first four moments of the equilibrium distribution ψ given by (2.8)/(2.9). Thus, the GSUE(4) mean flow illustrated may be regarded as the 'target' (true mean flows corresponding to (2.8)/(2.9)), and the other curves as approximations to it.

The GSUE(1) (i.e. SUE) solution, which is invariant to τ, is at a link 1 flow of approximately 16.29 vehicles/hour. It can be seen that GSUE(n) $(n \geq 2)$ mean flows and SUE flows differ, indicating that the GSUE(n) is indeed a model in its own right. Furthermore, consistent with the Large Sample Approximation Theorem (see section 3), the difference between the two decays as $\tau \to \infty$. Considering the GSUE(2) prediction, as well as being different from the SUE solution, in this example the GSUE(2) mean flow is closer than SUE to the true mean flow of (2.8)/(2.9) (i.e. the GSUE(4) solution). In fact the GSUE(2) model slightly overcompensates, and the GSUE(3) model corrects the mean flow back in the opposite direction. The fact that the GSUE(3) mean flow is greater than the GSUE(2) one is to be expected by the nature of the example: Since in all cases the equilibrium choice probability $p_1 > 0.5$, then the binomial link/route flow distribution for link 1 is negatively skewed. Since the third derivative of $t_1(.)$ is positive, then at the GSUE(2) flows, the GSUE(3) model would predict a lower link 1 expected cost than the GSUE(2) model, thus (in the GSUE(3) equilibration process) attracting an increased link 1 mean flow. A similar argument can be made for the comparison with the GSUE(4) solution, based on the nature of the fourth moment as a measure of kurtosis.

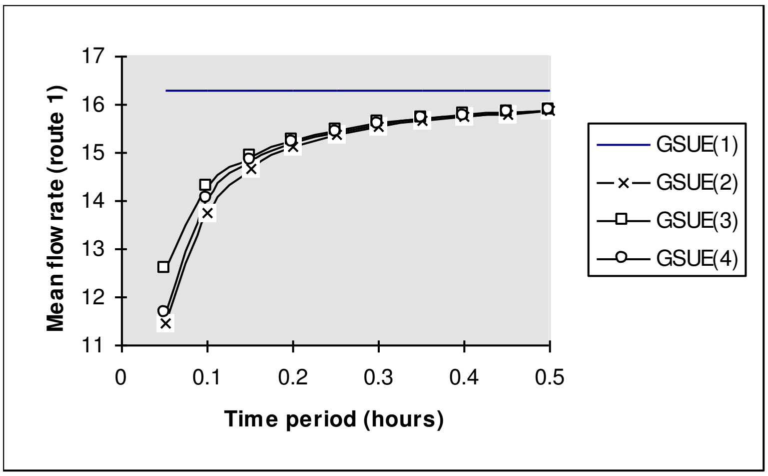

Figure 1: GSUE(n) mean flow rate on route 1 (n=1,2,3,4) as a function of time period duration τ

7. CONCLUSION

A new class of equilibrium models has been presented, capable of characterising long-run behaviour when network flow and cost variables are stochastic. Although these models may be applied as part of a conventional network assessment, their primary application is envisaged as being in cases that are beyond conventional SUE and UE models. These cases are generally where variability is an issue, either in the input data, in the output data (model predictions), in the representation of a 'responsive' policy, or in the assumed behavioural response. Potential applications include: the modelling of day-to-day variability in the inter-zonal trip demand matrix (Watling, 1998); the estimation of network reliability measures, including the impact of variable link capacities (Watling, 1999); and the impact of unreliability on travel behaviour, such as the risk of late arrival relative to some desired arrival time.

A key practical requirement for such models is the existence of an efficient solution algorithm for large networks. This aspect is explored in a companion paper (Watling, 2001), in which results are also reported of applications to realistic networks. On the technical side, promising areas for extension of the underlying models are: (i) elastic demand, including departure time choice, where the greater variance in flows over the inelastic case might be expected to induce a greater discrepancy with conventional deterministic flow models; and (ii) within-day dynamics, where—due to the potentially small absolute number of travellers per inter-zonal movement per departure time period—the large sample justification, often quoted in support of SUE as an approximation to mean stochastic flows, has much less credibility.

ACKNOWLEDGEMENTS

The support of an Advanced Fellowship from the UK Engineering and Physical Sciences Research Council is gratefully acknowledged, as are the critical insights of Martin Hazelton, and the constructive criticism of various earlier drafts by Mike Maher, Mike Bell, Dirck Van Vliet and three anonymous referees.

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