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CONSISTENT FORMULATION OF NETWORK EQUILIBRIUM WITH STOCHASTIC FLOWS

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ABSTRACT

Traffic flows in real-life transportation systems vary on a daily basis. According to traffic flow theory, such variability should induce a similar variability in travel times, but this “internal consistency” is generally not captured by existing network equilibrium models. We present an internally-consistent network equilibrium approach, which considers two potential sources of flow variability: i) daily variation in route choice and ii) daily variation in origin-destination demand. We particularly aspire to a flexible formulation that permits alternative statistical assumptions, which allows the best fit to be made to observed variability data in particular applications. Joint probability distributions of route—and therefore link—flows are derived under several assumptions concerning stochastic driver behavior. A stochastic network equilibrium model with stochastic demands and route choices is formulated as a fixed point problem. We explore limiting cases which allow an equivalent convex optimization problem to be defined, and finally apply this method to a real-life network of Kanazawa City, Japan.

Keywords: network equilibrium, stochastic demand, route choice, consistency, variability.

1. Introduction, Review & General Framework

In recent years, topics under the broad heading of “network reliability” have received an increasing share of research attention. A considerable body of work now exists on explanatory models that relate traveler behavior (especially route choice) to variation in service levels offered by the available alternatives (e.g. travel time variation), or to the inconvenient consequences of that variation (e.g., arriving late at a destination). Mirchandani & Soroush (1987) proposed an extension to the well-known Stochastic User Equilibrium (SUE) model, in which the actual travel times are random in addition to travelers’ perceptions of them. To analyze the effect of traffic information, Arnott et al. (1991) introduced random capacity into network equilibrium, whereby informed users were aware of the variations while uninformed users based their routing decisions on long-term expectations. Chen et al. (2002) formulated the “capacity reliability” concept, considering the probability that a network can serve a given level of demand, given stochastic variations in the link capacities (subsequently, also with stochastic variation in the demands themselves). Lo & Tung (2003) and Lo et al. (2006) formulated a probabilistic user equilibrium model under link capacity variations.

More recently, Nie (2011) proposed a percentile user equilibrium model based on random variations in capacity. Yin & Ieda (2001), Yin et al. (2004) and Watling (2006) developed network equilibrium models under the assumption of exogenously-specified travel time distributions. Chen & Zhou (2010), assuming an exogenously-specified lognormal travel time distribution, proposed a model in which travelers aim to minimize their “mean-excess travel time”. Several authors have explicitly considered the impact of stochastic supply and demand on network equilibrium, including Shao et al (2006), Siu & Lo (2008), and the studies of adverse weather by Lam et al. (2008) and Sumalee et al. (2010). On a different, but related, theme of network robustness, Waller et al. (2001) and Waller & Ziliaskopoulos (2006) investigated how a planner’s uncertainty in the mean demand level affects errors in equilibrium traffic forecasts, and Zhang et al. (2011) introduced the concept of expected residual minimization into stochastic-flow network equilibrium.

Clearly there have been many developments to the array of tools available for the analysis of stochastic networks. The purpose of this paper is to highlight an issue that, to some degree, is common to any such method of analysis, namely that of the internal consistency between the assumptions made regarding stochastic variation of various components of the traffic system, and in particular how this may be resolved within the context of an equilibrium approach. One approach adopted in several reliability/robustness analyses is to view the variability as external to the equilibrium process, in that the approach generates random input data to which some conventional notion of equilibrium is applied. As argued in Clark & Watling (2005), such an approach seems less appropriate for studying network unreliability due to day-to-day variability, since it is unlikely that the travelers in the transport system will be able to equilibrate on a daily basis. In this paper, on the other hand, we consider what ‘equilibrium’ might mean in a daily varying system. While we could consider more complex model forms and sets of assumptions, our focus will be on a relatively ‘stripped down’ class of models, in which we return to the basic foundation of SUE, and explore how we might formulate it in a coherent way when we may have stochastic, day-to-day variation (a) in the OD demands and/or (b) in the route choices given any demands.

Consistently incorporating the resulting distributions of both route flow and route travel time into network equilibrium is a non-trivial problem. Consider, as a starting point, the conventional SUE model, in which route utility is a sum of a systematic part (typically, the mean travel time) and a random residual term. Although the random term introduces a stochastic element, the flows in the SUE model are regarded as deterministic. However, once the equilibrium route choice proportions

have been computed, a probability distribution of route flows between each origin-destination (OD) pair could be derived, *ex post facto*, as a multinomial distribution (Sheffi, 1985, p.281), which could then be combined to generate a probability distribution of link flows and thence link travel times. However, for a non-linear travel time function $t(x)$, this will induce an inconsistency; for example, since under a random flow X , it is the case that $E[t(X)] \neq t(E[X])$ (see Watling, 2002a), it follows that the mean travel times on which the flow distribution was predicated are not equal to the mean travel times that would arise from a post-analysis of the model.

One approach to addressing this inconsistency is to use Markov processes to model the uncertain, dynamic evolution of networks—see Watling & Cantarella (2013a, 2013b) for recent reviews of this literature. (The Markov process approach is compared, both theoretically and numerically, with network equilibrium models of the kind studied in the present paper in Watling, 2002b). We do not adopt this approach in the present paper, but rather present an extended formulation of the SUE model that is able to accommodate such variability. The general framework we present is both a synthesis and extension of several existing works in the literature.

A generic description of a network equilibrium mechanism with stochastic flows and travel times is presented in Fig. 1. Since stochastic travel behavior is the main contributor to stochastic network flow, we consider that route choice and/or demand (i.e. whether a traveler makes a trip) could be represented as stochastic variables. Allowing for the possibility of the modeler to represent either route choice or demand as a deterministic or stochastic entity, and given that the case in which both are deterministic is already handled through conventional network equilibrium approaches, there are four important classes of stochastic-flow network equilibrium problems which we shall address: i) stochastic route choice with deterministic demand, ii) deterministic route choice with stochastic demand, iii) stochastic route choice with stochastic demand (or “doubly-stochastic” demand and route choice), and iv) “compound” stochastic route choice with stochastic (or deterministic) demand.

Table 1 presents existing approaches that fall within our framework. Class i includes the model of Watling (2002a), who addressed the consistency problem between distributions of flow and travel time through a second-order approximation based on multinomial route flows, thus equilibrating the first and second order flow moments (means, variances, and covariances). Class iii includes the extension of this model to include binomial demand variation, as described in Watling (2002c). Also within Class iii, Nakayama and colleagues presented a similar modeling approach (though not requiring any approximation) assuming negative-binomially distributed demand and stochastic route choice (Nakayama & Takayama, 2006; Nakayama, 2007), whereas Clark & Watling (Appendix A, 2005) suggested an approach for consistently modeling Poisson variation.

The purpose of this paper is to first bring together, under a common theoretical framework, these previous approaches to consistent modeling of stochastic flows and travel times. In doing so, we aim to highlight a broader range of assumptions that could be adopted within this overall framework; these alternative assumptions are useful when fitting to observed data (some may fit better than others) or because they may have more attractive theoretical properties or be more conducive to efficient large-scale computation. Aside from the general theoretical framing of the problem, our specific technical contributions within this general framework are highlighted in Table 1. A general formulation is presented as a fixed point problem. We subsequently establish the existence of solutions to such models we consider. We also examine limiting or approximate cases, which are appealing as they may be formulated as tractable optimization problems, which is both useful for solution and can be used to establish uniqueness. We conclude by presenting an application of such an approach to the real-life road network of Kanazawa city in Japan.

In summary, then, the key contributions of the paper are:

- i. To synthesize, and propose new possibilities to, previous studies on the problem of stochastic network equilibrium with stochastic flows, as shown in Table 1.
- ii. To prove the existence of network equilibrium with stochastic flows in all presented models.
- iii. To develop establish limiting models, and to formulate these as convex optimization problems, and hence to prove the uniqueness of solution of such problems.
- iv. To illustrate the approach with a numerical application to a realistic network.

2. General Formulation of Network Equilibrium with Stochastic Flows

In this paper, we aim to present a generalized framework for modeling stochastic network equilibrium with stochastic route and link flows. All of the models have four common, and internally consistent, elements. Firstly, the stochastic route flows give rise to stochastic link flows, simply by a linear transformation from the route flow vector random variable to the link flow vector random variable. Secondly, the stochastic link flows give rise to stochastic link travel times, through a transformation of the link flow variables implied by the link performance functions (i.e. those functions that relate given link flows to given levels of congested link travel times). Thirdly, the stochastic link travel times give rise to stochastic route travel times, simply by a further linear transformation of the relevant vector random variables. Fourthly, expectations (means) of the stochastic route travel times are fed into a Random Utility Model (RUM), which is used to describe the relative desirability of the route alternatives.

The way that these four common and consistent elements are then utilized varies between the different types of model within our framework. The different types of model vary in the way in which they ‘generate’ the stochastic route flows. Within our framework, we suggest there are four important classes to identify:

Class i: OD demand is deterministic, whereas travelers are assumed to make probabilistic decisions in their choice of route. That is to say, the RUM provides a probability of an individual traveler selecting a route in any particular trip, and so random variation in the route flows arises due to the given population of travelers playing out these random choices of route over repeated ‘trials’, which may be assumed to represent the particular days on which they make journeys.

Class ii: OD demand is stochastic, whereas travelers are assumed to make deterministic decisions in their choice of route. In this case, the RUM provides a fixed proportion of the aggregate OD demand that selects each route. Variations in the route flows now only arise due to the fact that these fixed proportions are applied to randomly-varying OD demands.

Class iii: OD demand is stochastic, and additionally—conditional on the decision to travel—travelers are assumed to make probabilistic decisions in their choice of route. This is an extension of Class i, with the RUM providing a conditional probability of an individual traveler selecting a route, given that the decision to travel has been made. Random variation in the route flows thus arises due to two sources: (a) the total demand is randomly varying, and (b) given the demand realized on any one occasion (e.g. day), the population that have chosen to travel on that day play out a random choice of route.

Class iv: As extensions of Class i and Class iii, now we also suppose that the route choice probabilities vary about some mean probability level (the RUM giving these mean levels). This class in-

cludes three distinct sources of random variation: (a) in OD demand, (b) in the route choice probabilities (about the mean probability from the RUM), and (c) in the traveler route choices given the realized probabilities. We shall refer to (b) and (c) together as compound stochastic route choice.

Before proceeding further, we shall now introduce some basic common concepts and notation. Throughout this paper, route choice is assumed to be described by random utility discrete choice models. A utility function is supposed to generate the deterministic (or systematic) utility, where throughout we assume that the deterministic utility is a continuous function of the mean route travel time¹. We also suppose that the travel time $t_a(x_a)$ on the a -th link depends only on the flow x_a on the a -th link, and not additionally on the flows on other links, i.e. so-called ‘separable’ link travel time functions. Furthermore, we assume that $t_a(x_a)$ is a polynomial of order n .

The notation throughout this paper is summarized in Table 2. When \mathbf{Y} is given, then the mean link travel time, which satisfies consistency condition 4 in Fig. 1, may be calculated as $E[t_a(\mathbf{X}_a)] = E[t_a(\sum_{i=1}^I \sum_{j=1}^J \delta_{a,ij} Y_{ij})] (\neq t_a(E[\mathbf{X}_a]))$, for non-linear $t_a(\cdot)$. Efficient methods for calculating the mean travel time from the random link flow distribution are developed in Lo & Tung (2003), Nakayama (2007), and Ng & Waller (2010).

Though stochastic, our modeling framework comprises standard elements of a ‘demand-side’ (which incorporates the distributions of OD demand levels and route choice) and a ‘supply-side’ (which links the flow and travel time distributions). From the ‘demand-side’, then, the joint probability distribution of the route flow vector \mathbf{Y} is identified if we are given (a) the OD demand distribution for each OD movement, and (b) the (random utility) model which maps the travelers’ perceptions of deterministic utilities onto route choice probabilities, conditional on the demand. The process of connecting these distributions together in a consistent way follows the steps illustrated in Fig. 1.

From the ‘supply-side’, the link flow probability distribution is consistently determined from the transformation $\mathbf{X} = \Delta \mathbf{Y}$. The mean link travel time under such a stochastic flow model is then given by $E_{T_a}[T_a] = E_{X_a}[t_a(X_a)]$. The details of how to construct this function are described later (Section 3.6), but are not important for the present section; the important issue for our present discussion is that such a mean travel time function exists, which captures the effect of the stochastic flow variation. The mean route travel time is then also a function of \mathbf{p} , since: the mean route travel time depends on the mean link travel times, the mean link travel times depend on the distribution of link flows, the distribution of link flows depends on the distribution of route flows, and the distribution of route flows depends on \mathbf{p} (as well as other parameters). Under the models we consider, the deterministic utility v_{ij} —as a continuous function of the mean route travel time—is then also a continuous function of \mathbf{p} . The deterministic utility $v_{ij}(\mathbf{p})$ is therefore defined for all $\mathbf{p} \in \tilde{\Omega}$, through this construction process. If $f(\cdot)$ denotes the utility function which transforms the mean route travel time into the deterministic utility, then we are saying that: $v_{ij}(\mathbf{p}) = f(\mu_{ij})$ where $\mu_{ij} = \sum_a \delta_{a,ij} E_{X_a}[t_a(X_a)]$ where $\mathbf{X} = \Delta \mathbf{Y}$ and where the distribution of \mathbf{Y} is parameterized by \mathbf{p} .

The vector-valued function $\phi(\mathbf{p})$ is used to denote the RUM evaluated at deterministic utilities $v_{ij}(\mathbf{p})$ when the input route choice is given by \mathbf{p} , i.e.:

¹ This function may be non-linear so as to capture, for example, different attitude of travellers to risk. As an alternative, a natural extension of the formulation presented would be to suppose that the utility function depends on other elements of the random distribution of travel times, such as higher order moments. Our reason not to present such an extended formulation above is that we felt the additional complication distracted from the main thrust of the present paper.

$$\phi_{ij}(\mathbf{p}) = \Pr \left[\nu_{ij}(\mathbf{p}) + \varepsilon_{ij} > \max_{j' \in \tilde{J}_{-j}^i} \{ \nu_{ij'}(\mathbf{p}) + \varepsilon_{ij'} \} \right]. \quad (1)$$

What is exactly meant by \mathbf{p} and therefore by the term ‘input route choice’ depends on the particular type of model adopted. In the four classes of model above, the interpretations are for the four classes i–iv above: i) the probabilities of a randomly-selected traveler choosing the alternative routes; ii) the deterministic proportions of travelers choosing the alternative routes; iii) the conditional probabilities of a randomly-selected traveler choosing the alternative routes, given that a decision to travel has been made; and iv) the means of the conditional probabilities to choose the alternative routes as described in i) or iii). The word ‘input’ is used to indicate that the relevant probabilities/proportions/conditional-probabilities/mean-probabilities are assumed to be given as a ‘known’, and so the $\nu_{ij}(\mathbf{p})$ are then constructed from the mean route travel times, that is the expectations of the route travel time random variables. These latter random variables are constructed from the link travel time random variables, which are in turn constructed from the link flow random variables, which are in turn constructed from the route flow random variables. Each of the different classes has a different way of generating these last, route flow random variables, from the input \mathbf{p} and the other assumptions/inputs of the model. This last step will be the focus of Section 3, which follows.

By constructing $\nu_{ij}(\mathbf{p})$ in this way, and then applying Eq. (1), the resulting RUM provides in $\phi(\mathbf{p})$ an output route choice—that is to say, for any given class of model, it provides an interpretation of route choice as an output which is presumed to coincide with the interpretation of route choice relevant to \mathbf{p} . Requiring consistency of the input and output route choice is then the common mechanism for requiring network equilibrium in all the cases considered. That is to say, in all our presented models of network equilibrium with stochastic flows, we enforce the fixed point condition:

$$\mathbf{p} = \phi(\mathbf{p}) \quad (\mathbf{p} \in \tilde{\Omega}). \quad (2)$$

We remark that in all of the classes considered, if the link travel time functions are linear in the link flows, then Eq. (2) defines a conventional Stochastic User Equilibrium condition (Sheffi, 1985). In practice, such linearity is of course highly unlikely to hold, and so the quite different notions of equilibrium that emerge from the different classes of stochastic flow model merit their own particular investigation.

3. Flow Distributions on Stochastic Network Equilibrium

In the present section we explore particular model specifications that fall within the framework described in Section 2. In order to implement the approach described, the key question we need to ultimately address in each case is: in what way does the distribution of route flows \mathbf{Y} depend on the choices from the random utility model \mathbf{p} , as defined in Section 2? This is the element that differs according to the model specification. Once this element is derived, we are then able to derive $\nu_{ij}(\mathbf{p})$ and $\phi_{ij}(\mathbf{p})$, and hence have a well-defined problem (2) to solve. In fact, since in all specifications we shall consider, we suppose that given \mathbf{p} , any two routes serving different OD movements are statistically independent², then since the route flow vector $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_I)^T$ our primary task is to de-

² Clearly in equilibrium OD movements must be related, due to their interactions on a common network, which our approach captures of course. Statistical independence here is more concerned with the question of whether,

duce the dependence on \mathbf{p} of the distribution of route travel times \mathbf{Y}_i for each OD movement.

3.1 Class i models: Deterministic OD demand and stochastic route choice

As described earlier, Class i models are characterized by a situation in which the OD demand is fixed, but random variation occurs in route flows due to travelers making randomly varying preferences. In the case of a single OD movement connected by two routes, each traveler conducts a Bernoulli trial, leading to Binomial variation in the total route flows; for more than two routes, the variation is multinomial, and if we assume OD movements to be statistically independent, then the route flows are independent across OD movement and multinomially distributed within an OD movement. The only issue that makes this non-trivial is the fact that the choice probabilities are not fixed a priori, but depend on mean travel times which themselves depend on flows, giving rise to an equilibrium condition. This problem is described in detail in Watling (2002a), where it was proposed to approximate the equilibrium condition through a second order approximation, meaning that flow means, variances and covariances are equilibrated. However, such an approximation is not necessary, and following the logic presented in Section 2, we may specify an exact fixed point condition (2) for such a case based on equilibrating the individual choice probabilities \mathbf{p} . As we remarked in the opening to Section 3, the key element we require is then the probability distribution of route flows

Thus, we suppose that a route is randomly selected by each driver in any given scenario (e.g. day). If the drivers are assumed homogenous, each driver traveling on the i -th OD movement chooses the j -th route with the same probability p_{ij} . Suppose further that each driver selects a route independently of any other driver. Then the joint probability distribution of route flows \mathbf{Y}_i for the i -th OD movement is multinomial with parameters n_i and \mathbf{p}_i , where n_i is the given OD demand as described in Table 2. Inserted into Eqs. (1) and (2), this allows a fixed point condition to be defined on \mathbf{p} as individual choice probabilities.

3.2 Class ii models: Stochastic OD demand and deterministic route choice

In contrast with Class i models, where the variation is all due to variations in individuals' route choice preferences, Class ii models suppose all the variation to be due to variability in the OD demand levels. In this case, the RUM gives rise to a \mathbf{p} which is assumed to represent the fixed proportion of the OD demand choosing a particular route.

The first question, then, is: what are sensible candidate distributions for modeling stochastic OD demand? In practice, the choice of distribution should be resolved through empirical evidence, and our proposal is that this is achieved by considering the ratio of the observed OD demand variance to the mean, and matching this to the known theoretical properties of the underlying discrete and non-negative models:

- binomial distribution if the variance of the demand is less than its mean;
- Poisson distribution if the variance and mean of the demand are (approximately) equal; and
- beta-binomial or negative binomial distribution if the variance of the demand exceeds its mean.

These candidate distributions offer the modeler a range of possibilities to use for any particular case-study, which can be chosen on the basis of the observed variability in the actual demand data.

before making a trip on a particular day, a traveler's choices might depend on the pre-trip choices of other travelers on that day. This may occur, for example, if OD demand is correlated due to seasonal, special event or weather-related factors. A natural extension of the approaches presented would be to suppose some correlation between OD movements which might occur.

This includes the possibility that different distributions may be suitable for different OD movements on the same network.

Of course, these are not the only candidate distributions, but in the following paragraphs we provide a ‘deductive’ justification for suggesting them. Such a deductive justification may also prove useful in cases where it is not feasible to obtain empirical evidence of the mean and variance in OD demand; in such cases, a suitable distribution might be hypothesized, and the impact of different levels of assumed variance investigated as a form of sensitivity analysis. Our deductive justification is based on the concept of latent drivers. Each latent driver randomly decides whether to make a trip. The realized demand is the number of latent drivers who actually make trips. The random travels of latent drivers lead to stochasticity in the demand. We assume that the latent drivers are homogeneous and mutually independent. Then, it follows that the demand is independent between OD pairs. Since the latent drivers are assumed homogenous, they share a common trip probability π . Whether or not a driver makes a trip constitutes a Bernoulli trial, when the trip probability π is given and fixed, while drivers’ decisions are independent. Therefore, the demand generated by ν_i latent drivers follows a binomial distribution, namely $N_i \sim B[\nu_i, \pi]$. Thus we have a justification for the binomial as a candidate distribution.

The variance of a $B[\nu_i, \pi]$ variable, namely $\nu_i \pi(1-\pi)$, is smaller than its mean, $\nu_i \pi$. It is well-known that as π approaches 0, the variance of the demand converges to the mean, and the distribution tends to the Poisson, an approximation of the binomial distribution (Stuart & Ord, 1994). Thus, the Poisson might be justified as a candidate distribution by virtue of the fact that it approximates the binomial variation described above. However, it may only fit a limited number of situations; for example, with Poisson-distributed demand, the coefficient of variation may be too small in comparison with observed data, especially if the mean demand is high.

In real traffic networks, the trip probability may itself vary over different decisions of whether to make a trip, for example due to variations in the activity patterns that motivate trip-making. For such a situation, we may consider the case of a randomly-distributed trip probability following a beta distribution on the interval $[0, 1]$; conditional on the realized probability, drivers make a Bernoulli trial of whether to travel. The resulting OD demand probability distribution is a compound of the binomial and beta distributions, known as the beta-binomial distribution (Johnson et al., 1993). The mean and variance of a beta-binomial random variable, $BB[\nu_i, \bar{\pi}_i, \gamma_i]$, are $\nu_i \bar{\pi}_i$ and $[(\nu_i + \gamma_i)/(\gamma_i + 1)]\nu_i \bar{\pi}_i(1 - \bar{\pi}_i)$, respectively, meaning that provided the number of latent drivers ν_i is sufficiently large, a much greater variance may arise than with binomial or Poisson distributed demands. Indeed, the variance of a beta-binomially distributed demand may exceed its mean, rendering it suitable for cases with relatively large variation. The negative binomial distribution similarly permits larger variance, as illustrated in Nakayama (2007) and Nakayama & Takayama (2006).

Having chosen a suitable distribution to represent OD demand variation, the second issue is how this is integrated with the route choice element, which in Class ii models is assumed deterministic, with the fixed route choice proportions p_{ij} provided by the RUM.

The first possibility we consider is that of binomially distributed OD demand. The situation is visualized as follows: Consider that the latent drivers travelling on the i -th OD movement are divided into as many groups as there are routes available for that movement. Suppose, for this given i , there is a proportion p_{ij} of latent drivers in the j -th group. When a latent driver in the j -th group makes a trip, he always takes the j -th route. In other words, each latent driver has planned his route before making a trip and merely decides at random whether to travel. The number of latent drivers traveling on the i -th OD movement and in the j -th group are denoted ν_i and $p_{ij} \nu_i$, respectively. Since the probability of whether to make a trip is fixed, the homogenous population of drivers share a

common π . Under these conditions, the route flows within each group are binomially distributed, $Y_{ij} \sim B[p_{ij} v_i, \pi]^3$. In addition, the Y_{ij} are mutually independent, because randomness of Y_{ij} results only from random travel decisions of latent drivers in the j -th group of the i -th OD movement.

The binomial distribution has the property of partial reproducibility; that is, $N_1 + N_2 \sim B[v_1 + v_2, \pi]$ when $N_1 \sim B[v_1, \pi]$ and $N_2 \sim B[v_2, \pi]$. Therefore,

$$\sum_{j=1}^{J_i} Y_{ij} = N_i \sim \mathbb{B}\left[\sum_{j=1}^{J_i} p_{ij} v_i, \pi\right] = \mathbb{B}[v_i, \pi] \quad (3)$$

The mean and variance of $Y_{ij} \sim B[p_{ij} v_i, \pi]$ are $\pi p_{ij} v_i = \mu_{ij}$ and $\pi(1-\pi)p_{ij} v_i = (1-\pi)\mu_{ij} = \sigma_{ij}^2$ respectively. Clearly, $E[N_i] = \tilde{\mu}_i = \sum_{j=1}^{J_i} \mu_{ij}$ and $\text{Var}[N_i] = \tilde{\sigma}_i^2 = \sum_{j=1}^{J_i} \sigma_{ij}^2 = (1-\pi)\tilde{\mu}_i$. Furthermore, $m_a = \sum_{i=1}^I \sum_{j=1}^{J_i} \delta_{a,ij} \mu_{ij} = \pi \sum_{i=1}^I v_i \sum_{j=1}^{J_i} \delta_{a,ij} p_{ij}$ and $s_a^2 = \text{Var}[X_a] = \pi(1-\pi) \sum_{i=1}^I v_i \sum_{j=1}^{J_i} \delta_{a,ij} p_{ij} = (1-\pi)m_a$. Therefore,

$$X_a \sim \mathbb{B}\left[\sum_{i=1}^I v_i \sum_{j=1}^{J_i} \delta_{a,ij} p_{ij}, \pi\right] = \mathbb{B}\left[\frac{m_a}{\pi}, \pi\right]. \quad (4)$$

Thus, the variances of the demand, link flow and route flow are all $(1-\pi)$ multiples of their means.

A second possibility considered is that of Poisson-distributed OD demands. One (but not the only) justification for such an assumption would be as an approximation of the binomial model above, since $B[v_i, \pi] \rightarrow \Pi[\tilde{\mu}_i]$ as π approaches 0, and we have $Y_{ij} \sim \Pi[\mu_{ij}]$, where $\tilde{\mu}_i = \pi v_i$ and $\mu_{ij} = p_{ij} \tilde{\mu}_i$. The Poisson distribution has the property of strong reproducibility (Stuart & Ord, 1994, p.395). Therefore, $X_a \sim \Pi[m_a] = \Pi[\sum_{i=1}^I \sum_{j=1}^{J_i} \delta_{a,ij} \mu_{ij}]$. Since the variance and mean of a Poisson distribution are equal, we have $\sigma_{ij}^2 = \mu_{ij}$ and $s_a^2 = m_a$.

A third possibility we consider is that the OD demand follows a beta-binomial distribution. This distribution has one more parameter than the binomial distribution, and permits a more flexible behavior of the demand. Assuming common $\bar{\pi}$ and that $(v_i + \gamma_i)/(\gamma_i + 1)$ is constant, the variance of the demand is proportional to its mean, because the mean and variance of $BB[v_i, \bar{\pi}_i, \gamma_i]$ are $v_i \bar{\pi}_i$ and $[(v_i + \gamma_i)/(\gamma_i + 1)]v_i \bar{\pi}_i(1 - \bar{\pi}_i)$, respectively. Setting $(v_i + \gamma_i)/(\gamma_i + 1) = \eta$, we obtain $\gamma_i = (v_i - \eta)/(\eta - 1)$. These are applicable when γ_i increases as v_i increases or when γ_i is sufficiently larger than v_i . In many cases, v_i is sufficiently large, and $v_i/\gamma_i \approx \eta - 1$. In the former case, $Y_{ij} \sim BB[p_{ij} v_i, \bar{\pi}, (v_i - \eta)/(\eta - 1)]$ yields $\sigma_{ij}^2 = \eta(1 - \bar{\pi})\mu_{ij}$. Similarly, we obtain $s_a^2 = \eta(1 - \bar{\pi})m_a$.

It is not always appropriate to set $(v_i + \gamma_i)/(\gamma_i + 1) = \eta$. In general, the variance of $BB[v_i, \bar{\pi}, \gamma_i]$ is

$$\tilde{\sigma}_i^2 = \frac{v_i + \gamma_i}{\gamma_i + 1} v_i \bar{\pi}_i(1 - \bar{\pi}_i) = \tilde{\mu}_i(\tilde{\mu}_i + \gamma_i \bar{\pi}_i) \frac{1 - \bar{\pi}_i}{\bar{\pi}_i(\gamma_i + 1)}. \quad (5)$$

due to $\tilde{\mu}_i = v_i \bar{\pi}_i$. The above implies that, as the mean $\tilde{\mu}_i$ increases, the variance $\tilde{\sigma}_i^2$ increases by second order in the mean. As mentioned above, the beta-binomial distribution has an extra parameter, which imparts greater flexibility to the demand. Therefore, the beta-binomial distribution admits both linear and quadratic relationships between the mean and variance of the flow.

The fourth possibility we consider is the possibility for the OD demand to follow a negative binomial distribution. Like the binomial distribution, the negative binomial distribution has the property of partial reproducibility. The mean and variance of $Y_{ij} (\sim NB[p_{ij} \alpha_i, \beta])$ are $p_{ij} \alpha_i \beta = \mu_{ij}$ and $p_{ij} \alpha_i \beta(1 + \beta) = (1 + \beta)\mu_{ij}$, respectively. Clearly, $E[N_i] = \tilde{\mu}_i = \sum_{j=1}^{J_i} \mu_{ij}$ and $\text{Var}[N_i] = \tilde{\sigma}_i^2 = \sum_{j=1}^{J_i} \sigma_{ij}^2 = (1 + \beta)\tilde{\mu}_i$. Furthermore, $m_a = \sum_{i=1}^I \sum_{j=1}^{J_i} \delta_{a,ij} \mu_{ij} = \beta \sum_{i=1}^I \alpha_i \sum_{j=1}^{J_i} \delta_{a,ij} p_{ij}$, and $s_a^2 = \text{Var}[X_a] = \beta(1 + \beta) \sum_{i=1}^I$

³ Since we cannot ensure in the subsequent equilibrium process that $p_{ij} v_i$ is necessarily a natural number, such a binomial distribution may not be well-defined. We thus make a pragmatic approximation by adopting the gamma function as an extension of the factorial function, since $\Gamma(y + 1) = y!$ when y is a natural number. In this way the binomial probability of a $B[n, \pi]$ variable when n is non-integer is presumed to be $\Gamma(n + 1) \pi^y (1 - \pi)^{n - y} / [\Gamma(y + 1) \Gamma(n - y + 1)]$.

$$\alpha_i \sum_{j=1}^{J_i} \delta_{a,ij} p_{ij} = (1+\beta) m_a.$$

If the OD demand follows any of the binomial, Poisson, beta-binomial or negative binomial distributions and route choice is deterministic, the variances of the demand, link flow, and route flow are all constant multiples of their means, and are defined as follows:

$$\sigma_{ij,i'j'} = \begin{cases} \rho \mu_{ij} = \sigma_{ij}^2 & \text{if } i = i' \text{ and } j = j' \\ 0 & \text{otherwise (} i \neq i' \text{ or } j \neq j') \end{cases} \quad (6)$$

$$\tilde{\sigma}_i^2 = \rho \tilde{\mu}_i \quad (7)$$

$$s_a^2 = \rho m_a, \quad (8)$$

where

$$\rho = \begin{cases} 1 - \pi & \text{if } N_i \sim \text{Bin}[v_i, \pi] \\ 1 & \text{if } N_i \sim \text{Poi}[\tilde{\mu}_i] \\ 1 + \beta & \text{if } N_i \sim \text{BetaBin}[\alpha_i, \beta] \\ \eta(1 - \bar{\pi}) & \text{if } N_i \sim \text{NegBin}\left[v_i, \bar{\pi}, \frac{v_i - \eta}{\eta - 1}\right] \end{cases} \quad (9)$$

Thus, from the collection of expressions above, we are able to fully characterize the route flow distributions for each of the four candidate choices of OD demand distribution (as well as information on the link flow distributions, useful for computing the expected link travel times). This is then used in Eq. (1) to solve Eq. (2) in \mathbf{p} , where \mathbf{p} in this class denotes the equilibrium proportions of flow on the alternative routes.

3.3 Class iii models: Stochastic OD demand and stochastic route choice

In Class iii models there are two components of random variation in the route flows, namely OD demand variation and random variation in route choice conditional on the OD demand. This is an extension of Class i, with route choice conditional on the demand described by a multinomial distribution. Three consistent formulations of such a class have already been described in detail in the literature, and so are simply summarized here for completeness (the source papers may be consulted for further details):

- Watling (2002c) compounded binomial OD demand variation with a multinomial distribution for conditional route choice, resulting in a multinomial distribution for the unconditional route flows.
- Clark & Watling (2005, Appendix A) compounded Poisson OD demand with multinomial conditional route choice, whereby the resulting (unconditional) route flows follow independent Poisson distributions.
- Nakayama & Takayama (2006) and Nakayama (2007) compounded negative-binomially distributed demand with multinomial conditional route choice, with the resulting route flows then following a negative multinomial distribution.

The route flow distributions are used to solve Eq. (2) given Eq. (1), this time with the equilibrium process acting on \mathbf{p} as the conditional probabilities to choose the alternative routes, given that a decision to travel has been made.

3.4 Class iv models: Compound stochastic route choice

Class iv models extend those in either Class i (deterministic demand) or Class iii (stochastic

demand) by supposing two contributory sources to variation in route flows aside from whether the OD demand varies: the route choice probabilities, and the traveler route choices given the realized route probabilities. In Classes i and iii, the multinomial route choice model assumes that all drivers choose routes with fixed and common probabilities, whereas in class iv we suppose the route choice probabilities themselves to be randomly distributed, according to a Dirichlet distribution (a generalization of the beta distribution). Conditional on the realized probabilities, each driver randomly selects a route. The distribution of route choice probabilities is given by $\mathbf{P}_i \sim \Delta[\bar{\mathbf{p}}_i, r_i]$. In this case, the route choice input, \mathbf{p} , introduced in Section 2 is the vector of mean route choice probabilities. This route choice is referred to as compound stochastic, since it combines variation in the route choice probabilities with variation in the choice itself.

If the OD demand is fixed (i.e. we aim to generalize Class i), the resultant route flows are a compound of multinomial and Dirichlet distributions, known as the Dirichlet-compound multinomial distribution. The joint probability $f_{\mathbf{Y}_i}(\mathbf{y}_i)$ is $[n_i! \Gamma(r_i) / \Gamma(n_i + r_i)] \prod_{j=1}^{J_i} [\Gamma(y_{ij} + r_i p_{ij}) / y_{ij}! \Gamma(r_i p_{ij})]$. The mean, variance, and covariance of this distribution are as follows (Mosimann, 1962):

$$\begin{aligned} \mu_{ij} &= E[Y_{ij}] = n_i p_{ij} & (10) \\ \sigma_{ij,i'j'} &= \text{Cov}[Y_{ij}, Y_{i'j'}] = \begin{cases} \frac{n_i + r_i}{1 + r_i} n_i p_{ij} (1 - p_{ij}) = \text{Var}[Y_{ij}] & \text{if } i = i' \text{ and } j = j' \\ -\frac{n_i + r_i}{1 + r_i} n_i p_{ij} p_{i'j'} & \text{if } i = i' \text{ and } j \neq j' \\ 0 & \text{otherwise (} i \neq i') \end{cases} & (11) \end{aligned}$$

The parameter r_i can be interpreted as a variance scale parameter. According to Eq. (11), the variance of route flows enlarges as r_i increases. We shall call this compound stochastic route choice ‘Dirichlet-compound multinomial route choice.’

If the OD demand is stochastic (i.e. we aim to generalize Class iii), then it is natural to consider the Dirichlet-compound multinomial distribution for conditional route choice when the demand is assumed to follow a beta-binomial distribution. In order to do so, we now introduce a hypothetical link, where y_{i0} is the number of no-travel latent drivers; that is, $y_{i0} = \nu_i - n_i$. Let $p_{i0} = 1 - \bar{\pi}_i$, and p_{i0} is the probability of no-travel because $\bar{\pi}_i$ is the mean trip probability as mentioned before. Set $r_i = \gamma_i \bar{\pi}_i$. As a compound of beta-binomial distribution⁴ and Dirichlet-compound multinomial distribution, we then obtain the following for the unconditional route flows:

$$\begin{aligned} f_{\mathbf{Y}_i}(\mathbf{y}_i) &= f_{\mathbf{Y}_i | n_i}(\mathbf{y}_i) g_{N_i}(n_i) \\ &= \frac{\nu_i! \Gamma(\gamma_i)}{\Gamma(\nu_i + \gamma_i)} \prod_{j=0}^{J_i} \frac{\Gamma(y_{ij} + \gamma_i p_{ij})}{y_{ij}! \Gamma(\gamma_i p_{ij})} \end{aligned} \quad (12)$$

The above is the probability function of the Dirichlet-compound multinomial distribution. Thus, the route flows between a single OD pair follow the Dirichlet-compound multinomial distribution. The link flows are given by the sum of Dirichlet-compound multinomial distributed route flows. The sum of Dirichlet-compound multinomial distributed variables does not necessarily follow the Dirichlet-compound multinomial, because it does not have the property of reproducibility or partial

⁴ The probability function of beta-binomially distribution, $\text{BB}[\nu_i, \bar{\pi}_i, \gamma_i]$, is as follows: $g_{N_i}(n_i) = \{\nu_i! \text{B}[n_i + \gamma_i \bar{\pi}_i, \nu_i - n_i + \gamma_i(1 - \bar{\pi}_i)]\} / \{(\nu_i - n_i)! n_i! \text{B}[\gamma_i \bar{\pi}_i, \gamma_i(1 - \bar{\pi}_i)]\}$ where $\text{B}(x, y)$ is the beta function, and $\text{B}(x, y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$.

reproducibility. It is, therefore, difficult to derive a clear form of link distribution. However, the consistent mean travel time can directly be calculated from the route flow distribution (e.g., Nakayama, 2007).

Once derived, the appropriate route flow distribution may be used in Eq. (1) to solve Eq. (2) in \mathbf{p} , where the equilibrium process now defines a self-consistent \mathbf{p} as the mean route choice probabilities.

3.5 Approximate Flow Distributions

3.5.1 Multivariate normal distributed flows

If the demand is sufficiently large and the number of routes is limited, the central limit theorem dictates that the route flows \mathbf{Y}_i approximate the multivariate normal distribution. We formally establish this result in Appendix A. For practical applications, it is important to examine how large the demand is required for the normal approximation to be reasonable, and how accurate it may be as an approximation to some other model such as those presented in sections 3.1–3.4. Such a question has been studied well. For example, according to the textbook of Hald (1952), as a rule-of-thumb it is suggested that the normal approximation to the binomial distribution will likely be adequate when $n \geq 36$, though the required level of accuracy may differ between contexts.

Assuming that the just-described conditions hold, the vector of all route flows follows a multivariate normal distribution, $\mathbf{Y} \sim \mathcal{N}[\boldsymbol{\mu}, \boldsymbol{\Sigma}]$. Because $\mathbf{X} = \boldsymbol{\Delta} \mathbf{Y}$, then \mathbf{X} also follows a multivariate normal distribution, $N[\mathbf{m}, \mathbf{S}]$, whose mean vector \mathbf{m} and variance-covariance matrix \mathbf{S} , are given by

$$\mathbf{m} = \boldsymbol{\Delta} \boldsymbol{\mu} \quad (13)$$

$$\mathbf{S} = \boldsymbol{\Delta} \boldsymbol{\Sigma} \boldsymbol{\Delta}^T, \quad (14)$$

where we have exploited the property of $N[\boldsymbol{\mu}, \boldsymbol{\Sigma}]$ (Stuart & Ord, 1994, p.512). One practical difficulty that may arise in some cases is that $\boldsymbol{\Delta} \boldsymbol{\Sigma} \boldsymbol{\Delta}^T$ may be singular if it contains some ‘redundant’ link variables, however these may be discarded as follows: If the a -th and $(a+1)$ -th links directly connect to the $(a+2)$ -th link, the start node of the $(a+2)$ -th link, which is also the end node of the a -th and $(a+1)$ -th links, is neither the origin nor the destination (the latter is described by $X_{a+2} = X_{a+1} + X_a$), then the variance-covariance matrix of X_a , X_{a+1} , and X_{a+2} cannot be defined. To remedy this problem, one of the three links should be abbreviated, because the link flow can be computed from the remaining two link flows. Otherwise, the component column (or row) vectors of $\boldsymbol{\Delta} \boldsymbol{\Sigma} \boldsymbol{\Delta}^T$ are not linearly independent. Consequently, $\boldsymbol{\Delta} \boldsymbol{\Sigma} \boldsymbol{\Delta}^T$ is non-invertible and the joint probability density function of \mathbf{X} is undefined.

The route flow is normally distributed when each latent demand is sufficiently large. If Eqs. (6)–(8) in Class ii are satisfied, the route flows approximately follow the independent normal distribution given by

$$Y_{ij} \sim \mathcal{N}[\mu_{ij}, \rho \mu_{ij}]. \quad (15)$$

From Eqs. (13) and (14) we then have for the corresponding link flows that:

$$\mathbf{X} \sim \mathcal{N}[\mathbf{m}, \mathbf{S}] = \mathcal{N}[\boldsymbol{\Delta} \boldsymbol{\mu}, \rho \boldsymbol{\Delta} \text{diag}(\boldsymbol{\mu}) \boldsymbol{\Delta}^T]. \quad (16)$$

3.5.2 Poisson distributed flows in large-scale networks

The probability density function of a normal distribution is, of course, symmetric and allows negative values. An alternative that does not possess such properties is the Poisson, and so in this section we explore the possibility to adopt this as an approximating model. As is well-known, the binomial distribution $B[n, p]$ is approximated by the Poisson distribution $\Pi[n p]$ if $n p$ is finite and p

is sufficiently small. Therefore, it may be justified to assume that any route flow with low probability of being selected is approximately Poisson distributed. The accuracy of the Poisson approximation has already been studied well: the maximum difference between the binomial and limiting Poisson probabilities ($B[n, p]$ and $\Pi[n p]$) is given as $\mu^2 e^{-\mu} |1 - \mu/2|/n + O(n^{-2})$ where $\mu = np$ (Stuart & Ord, 1994, p.171). Thus, the accuracy depends on n and p .

In practice, then, in which situations might we justify an assumption that p is “sufficiently small” to apply the Poisson approximation? One possibility could be a case, such as a large grid network, where there are many similar route possibilities, since in such a case the selection probabilities of most routes should be small but non-zero because they are assigned by the RUM. However, the assumption that all route choice probabilities are sufficiently small would be difficult to justify for all OD movements in many real life cases, given the evidence that drivers tend to select from a relatively small number of routes, and given the existence of major highways which are likely to be dominant in their proportional allocation of demand.

A second, and perhaps more plausible, case is an argument that does not rely on limiting distributions. In particular, the assumption of uncorrelated Poisson route flows may be justified in an alternative theoretical way, following Clark & Watling (2005): if the OD demand flows are Poisson distributed, and the route flows conditional on the demands are multinomial, then the unconditional route flows are exactly Poisson and uncorrelated. It then follows that link flows, though correlated, have marginal distributions which are also Poisson. The Poisson condition arises as we are effectively seeing route choice as sampling from a time-homogeneous Poisson process, and the uncorrelated property arises as we can think of stochastic demand being as if we add an additional ‘no-travel’ hypothetical route for each OD movement, with no conservation bound then required on the sum of route flows in the extended network (conservation of total route flows by OD movement being the reason for the negative correlation in the multinomial model of conditional route choice). This result does not require the OD demands to be large, the network to be large, nor the route choice probabilities to be small, as it is not a limiting result. The strongest assumption it makes is that OD demands follow a Poisson distribution, and the reasonableness of this assumption should be verified with actual data, on a case-by-case basis.

Since the Poisson approximation offers a simple mathematical treatment of traffic assignment, it is therefore worthy of consideration, though of course we should keep in mind its potentially limited applicability. Certainly it has been a model of some interest to transportation researchers, e.g. van Zuylen & Willumsen (1980).

In the cases when we may assume p_{ij} sufficiently small that the Poisson approximation to the Binomial is valid, we may then have that Y_{ij} approximately follows a Poisson $\Pi[n_i p_{ij}] (= \Pi[\mu_{ij}])$ distribution. In multinomial route choice, the route flows are not independent between a given OD pair. In this case, the covariance of flows on the j -th and j' -th routes between the i -th OD pair is $-n_i p_{ij} p_{ij'}$. However, the covariance can be assumed as 0 because $n_i p_{ij}$ is finite and $p_{ij'}$ is sufficiently small. In other words, each route flow can be assumed to follow an independent Poisson distribution. Then $X_a \sim \Pi[m_a] = \Pi[\sum_{i=1}^I \sum_{j=1}^{J_i} \delta_{a,ij} \mu_{ij}]$, as described in Section 3.2.

3.6 Existence of Equilibrium Solutions

In the present section we consider existence of solutions to the equilibrium model (2) for each of the models defined in the previous section.

Now, clearly the set $\tilde{\Omega}$ as defined in Table 2 is compact, and as given by Eq. (1), $\phi(\mathbf{p}) \in \tilde{\Omega}$. In addition, if $v_{ij}(\mathbf{p})$ is continuous, $\phi(\mathbf{p})$ is also continuous from Eq. (1). If $\phi(\mathbf{p})$ is continuous, the fixed point problem has at least one solution, according to Brouwer’s fixed point theorem (e.g., Ortega &

Rheinboldt, 1970). Let us assume that the travel time functions $t_a(x_a)$ are continuous and strictly increasing on $x_a \geq 0$. Existence of solutions then hinges on the continuity of the mean link travel time in \mathbf{p} , which in turn depends on the specific link flow probability function $f_{X_a}(x_a)$ and on the travel time function $t_a(x_a)$.

In the case that each route flow is binomially distributed—which applies to (the marginal distributions of) the multinomial flows in Class i, the binomial case in Class ii, and the multinomial case in Class iii—the mean link travel time is given by:

$$E[T_a] = E[t_a(X_a)] = \sum_{y_{11}=0}^{n_1} \sum_{y_{12}=0}^{n_1} \cdots \sum_{y_{1i}=0}^{n_1} \delta_{a,ij} t_a \left(\sum_{i=1}^I y_{ia} \right) \prod_{i=1}^I \frac{n_i!}{(n_i - y_{ia})! y_{ia}!} p_{ia}^{y_{ia}} (1 - p_{ia})^{n_i - y_{ia}}, \quad (17)$$

where $y_{ia} = \sum_{j=1}^I \delta_{a,ij} y_{ij}$ and $p_{ia} = \sum_{j=1}^I \delta_{a,ij} p_{ij}$. Because n_i is given and fixed, the above is a function of \mathbf{p} , namely, $\bar{t}_a(\mathbf{p})$. Because a finite sum of continuous functions is itself continuous, implying that the route mean travel time is continuous, $\bar{t}_a(\mathbf{p})$ is continuous w.r.t. \mathbf{p} . By the same reasoning, the mean travel time is continuous w.r.t. \mathbf{p} if the route flows are beta-binomially distributed.

It is more complicated to examine the continuity of mean travel time with the normally distributed flow. Normally distributed flows can become negative, albeit with small probability. We assume that $t_a(x_a) = t_a(0) = \tau_a$ on $x_a < 0$ ⁵. The mean link travel time is given by the indefinite integral of $(1/\sqrt{2\pi}s_a) \int_{-\infty}^{\infty} t_a(x) \exp[-(x - m_a)^2/2s_a^2] dx$, and is a function of m_a and s_a . Unlike the definite integral, $\int_{-\infty}^{\infty} t_a(x) f_{X_a}(x) dx$ is not necessarily continuous even though both $f_{X_a}(x_a)$ and $t_a(x_a)$ are continuous.

If $\{\xi_i\}$ ($i = 1, 2, \dots$) is a sequence of continuous functions, and if the ξ_i uniformly converge to ξ , then ξ is continuous (see Theorem 7.12 in Rudin (1976)). If $|\{\xi_i(m)\}| \leq \varpi_i$ and if $\sum \varpi_i$ converges, then $\sum \xi_i(m)$ is uniformly convergent (Theorem 7.10 in Rudin (1976)). Now set $\xi_i^a(m_a, s_a) = \int_0^i t_a(x) f_{X_a}(x) dx$. Defined as a definite integral, $\xi_i^a(m_a, s_a)$ is seen to be continuous w.r.t. m_a and s_a when $X_a \sim N[m_a, s_a^2]$. Let $\varpi_i^a = \int_0^i \varpi^a(x) dx$, where

$$\varpi^a(x) = \begin{cases} \frac{t_a(x)}{\sqrt{2\pi} s_{\downarrow}} \exp \left[-\frac{1}{2} \left| \frac{x - m_{\uparrow}}{s_{\downarrow}} \right|^2 \right] & \text{if } x \leq m_{\uparrow} - 2s_{\downarrow} \text{ and } x \geq m_{\uparrow} + 2s_{\downarrow} \\ \frac{t_a(m_{\uparrow})}{\sqrt{2\pi} s_{\downarrow}} & \text{otherwise} \end{cases}. \quad (18)$$

Clearly, since $e^{-0.5x^2} < e^{-0.5x}$ when $|x| > 1$, $t_a(x) f_{X_a}(x) \leq \varpi^a(x)$ on a real field. Applying integration by parts, and assuming that a mean travel time exists, we obtain $\int_{m_{\uparrow} + 2s_{\downarrow}}^{\infty} \varpi^a(x) dx = t_a(m_{\uparrow} + 2s_{\downarrow}) / \sqrt{2\pi e^2} - e^{-\infty} t_a(\infty) = t_a(m_{\uparrow} + 2s_{\downarrow}) / \sqrt{2\pi e^2}$, because $\lim_{x \rightarrow \infty} t_a(x)/e^x = 0$ since $t_a(x_a)$ is a polynomial function. Similarly, we can confirm that the value of $\int_{-\infty}^{m_{\uparrow} - 2s_{\downarrow}} \varpi^a(x) dx$ is finite. Being a definite integral of a finite function, $\int_{m_{\uparrow} - 2s_{\downarrow}}^{m_{\uparrow} + 2s_{\downarrow}} \varpi^a(x) dx$ is guaranteed finite-valued. According to the above theorems, the finiteness of $\int_{-\infty}^{\infty} \varpi^a(x) dx$ implies (uniform) convergence of $\int_{-\infty}^{\infty} t_a(x) f_{X_a}(x) dx$. Thus, $\int_{-\infty}^{\infty} t_a(x) f_{X_a}(x) dx$ is continuous w.r.t. m_a and s_a . As mentioned in Section 3.5.1, the mean and variance of approximately normal-distributed flow is equal to those of the original flows. Therefore, m_a and s_a are given by $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ of the original distributions. The mean and variance of route flows are determined by \mathbf{p} if the type of route flow distribution is given, as described in Sections 3.1–3.4. Accordingly, the mean travel time of normally distributed link flow is considered as a function of \mathbf{p} , namely, $\bar{t}_a(\mathbf{p})$. In conclusion, we have confirmed that the function of mean link travel time is continuous w.r.t. \mathbf{p} in the case of normally distributed route flows.

⁵We assume that $t_a(x_a)$ is strictly increasing on $x_a \geq 0$, but may not be on $x_a < 0$. The continuity of $t_a(x_a)$ w.r.t. x_a is guaranteed on a real field.

The negative binomial and Poisson distributes are discrete, but, unlike the binomially distributed case, the mean travel time with negative-binomially or Poisson distributed flows is given by the infinite series of $E[T_a] = \sum_{x_a=0}^{\infty} \dots \sum_{x_a=0}^{\infty} t_a(x_a) f_{x_a}(x_a)$. Due to the infinite series, even if each $t_a(x_a) f_{x_a}(x_a)$ is continuous, $E[T_a]$ is not necessarily continuous, as described above. Although, for the limited space, the detail of the proof is omitted, uniform convergence of $E[T_a]$ may be shown, and, then the proof proceeds as for the above normally-distributed case.

From the above discussion, then, we infer that a solution exists to the fixed point problem (2) for the model specifications presented in Section 3.

4. Convex Optimization Problem of Network Equilibrium with Stochastic Flows

The general formulation of network equilibrium with stochastic flows is given by the fixed point problem (2) defined in Section 2. While there exist a variety of general-purpose algorithms for solving fixed point problems in transportation networks (e.g. Liu et al, 2009; Cantarella et al, 2013), which might be applied to problem (2), an attractive and efficient alternative for practical applications is a formulation as a convex optimization problem. Since some of the proposed models presented in Section 3 may be formulated as convex optimization problems, we discuss these below. We do not intend to suggest these as ‘recommended models’, as opposed to the more complex models which may not admit a convex optimization formulation, but rather we intend to point out efficient formulations that exist for at least a subset of the models in Section 2.

4.1 Formulation

An example of models that can be reformulated as convex optimization problems are those models that exist in Class ii, when we adopt the multinomial logit model, with a utility function that is linear in the route travel time (with scale parameter θ).

As mentioned in Table 2, $\bar{c}_a(m_a)$ is the function that calculates the mean travel time on the a-th link from the mean link flow. In all the models in Class ii, $s_a^2 = \rho m_a$ and $\sigma_{ij}^2 = \rho \mu_{ij}$ as mentioned in Eqs. (6) and (8), and mean link travel time is a function solely of m_a . For example, if the link flow is Poisson distributed and if $t_a(x_a) = \tau_a[1 + 0.15(x_a/\zeta_a)^4]$, the mean travel time function is $\bar{c}_a(m_a) = \tau_a[1 + 0.15(m_a^4 + 6m_a^3 + 7m_a^2 + m_a)/\zeta_a^4]$; the same principle can be applied to derive functions $\bar{c}_a(m_a)$ for the other (non-Poisson) models in Class ii.

The model with multinomial-logit-type deterministic route choice and stochastic demand can therefore be formulated as the following optimization problem:

$$\min . \zeta = \sum_{a=1}^A \int_0^{m_a} \bar{c}_a(w) dw + \frac{1}{\theta} \sum_{j=1}^{J_i} \mu_{ij} \ln \mu_{ij} . \quad (19)$$

s.t.

$$\tilde{\mu}_i = \sum_{j=1}^{J_i} \mu_{ij} \quad i = 1, 2, \dots, I \quad (20)$$

$$m_a = \sum_{i=1}^I \sum_{j=1}^{J_i} \delta_{a,ij} \mu_{ij} \quad a = 1, 2, \dots, A \quad (21)$$

$$\mu_{ij} \geq 0 \quad i = 1, 2, \dots, I, \quad j = 1, 2, \dots, J_i \quad (22)$$

The above is similar to Fisk’s formulation (Fisk, 1980), but uses mean link travel times and route

flows instead of deterministic link travel times and route flows. Therefore, we can confirm that the above problem solves the stochastic network equilibrium with deterministic logit-type route choice in the same manner of Fisk (1980), but in this case with stochastic demand.

4.2. Uniqueness of Equilibrium

4.2.1 Normally distributed flow case

Let us investigate uniqueness of the equilibrium in the optimization problem of Eqs. (19)–(22). Clearly, the feasible domain of this problem is convex. If the objective function of Eq. (25) is strictly convex, the optimization problem has a unique equilibrium. To demonstrate that the objective function is convex, it is sufficient to show that the Hessian matrix of the objective function is positive definite.

The second derivative of ζ in Eq. (19) is given by

$$\frac{\partial^2 \zeta}{\partial \mu_{ij} \partial \mu_{i'j'}} = \begin{cases} \frac{1}{\theta \mu_{ij}} + \sum_{a=1}^A \delta_{a,ij} \bar{c}'_a(m_a) & \text{if } i = i' \text{ and } j = j' \\ \sum_{a=1}^A \delta_{a,ij} \delta_{a,i'j'} \bar{c}'_a(m_a) & \text{otherwise} \end{cases}, \quad (23)$$

where $\bar{c}'_a(m_a) = d \bar{c}_a / dm_a$, which is mentioned in the next paragraph. The above gives the component of the Hessian matrix of ζ . Let $\nabla^2 \zeta$ denote the Hessian matrix of ζ . Then,

$$\mathbf{u}^T [\nabla^2 \zeta] \mathbf{u} = \frac{1}{\theta} \sum_{i=1}^I \sum_{j=1}^{J_i} \frac{u_{ij}^2}{\mu_{ij}} + \sum_{a=1}^A \bar{c}'_a(m_a) \left(\sum_{i=1}^I \sum_{j=1}^{J_i} \delta_{a,ij} u_{ij} \right)^2. \quad (24)$$

For $\mathbf{u} = (u_{11}, u_{12}, \dots, u_{IJ})^T \neq \mathbf{0}$, the first term on the right-hand side of the above equation is positive and the second term is non-negative if $\bar{c}'_a(m_a) > 0$. Therefore, $\nabla^2 \zeta$ is positive definite, and ζ is convex if $\bar{c}'_a(m_a) > 0$.

In Class ii, $X_a \sim N[m_a, \rho m_a]$ as mentioned in Eq. (16), if m_a is sufficiently large. Because $t_a(x_a)$ is polynomial, $\bar{c}_a(m_a) < \infty$ on $0 \leq m_a \leq m_\uparrow$. Furthermore, $\bar{c}_a(m_a)$ is continuous. The derivative of mean travel time function is given as

$$\bar{c}'_a(m_a) = \frac{d}{dm_a} E[t_a(X_a)] = \frac{d}{dm_a} \int_{-\infty}^{\infty} t_a(x) f_{X_a}(x) dx. \quad (25)$$

In the region of $[0, m_\uparrow] \times (-\infty, \infty)$, $\int_{-\infty}^{\infty} t_a(x) f_{X_a}(x) dx$ is uniformly convergent as mentioned in Section 3.6. Therefore, when $\int_{-\infty}^{\infty} \{\partial[t_a(x) f_{X_a}(x)] / \partial m_a\} dx$ is uniformly convergent, we can interchange integration and differentiation as follows (e.g. Theorem 9.42 in Rudin (1976)):

$$\frac{d}{dm_a} \int_{-\infty}^{\infty} t_a(x) f_{X_a}(x) dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial m_a} [t_a(x) f_{X_a}(x)] dx \quad (26)$$

Because of $X_a \sim N[m_a, \rho m_a]$,

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\partial}{\partial m_a} [t_a(x) f_{X_a}(x)] dx &= \int_{-\infty}^{\infty} t_a(x) \frac{\partial}{\partial m_a} \left\{ \frac{1}{\sqrt{2\pi\rho m_a}} \exp\left[-\frac{(x-m_a)^2}{2\rho m_a}\right] \right\} dx \\ &= \frac{1}{2\rho m_a^2} \int_{-\infty}^{\infty} t_a(x) [x^2 - (\rho + m_a)m_a] f_{X_a}(x) dx \end{aligned} \quad (27)$$

We can confirm that $\int_{-\infty}^{\infty} \{\partial[t_a(x) f_{X_a}(x)] / \partial m_a\} dx$ is uniformly convergent in the same manner of Section 3.6. Let $l_a(x) = x^2 - (\rho + m_a)m_a$, and, then

$$\begin{aligned} & \int_{-\infty}^{\infty} t_a(x) [x^2 - (\rho + m_a)m_a] f_{X_a}(x) dx \\ &= \int_{-\infty}^0 \tau_a 1_a(x) f_{X_a}(x) dx + \int_0^{\infty} t_a(\sqrt{1_a(x) + (\rho + m_a)m_a}) 1_a(x) f_{X_a}(x) dx \end{aligned} \quad (28)$$

Because $t_a(\cdot)$ is strictly increasing and $t_a(0) = \tau_a$, $t_a(\sqrt{1_a(x) + (\rho + m_a)m_a}) 1_a(x) \geq \tau_a 1_a(x)$. Therefore,

$$\int_{-\infty}^{\infty} t_a(x) [x^2 - (\rho + m_a)m_a] f_{X_a}(x) dx \geq \tau_a \int_{-\infty}^{\infty} 1_a(x) f_{X_a}(x) dx = 0 \quad (29)$$

since $\int_{-\infty}^{\infty} [x^2 - (\rho + m_a)m_a] f_{X_a}(x) dx = \int_{-\infty}^{\infty} x^2 f_{X_a}(x) dx - (\rho + m_a)m_a = s_a^2 + m_a^2 - (\rho + m_a)m_a = 0$ due to $s_a^2 = \rho m_a$. Thus, $\bar{c}'_a(m_a) > 0$ is confirmed, so $\bar{c}_a(m_a)$ is strictly increasing. Therefore, the optimization problem of Eqs. (19)–(22) is proved to have a unique equilibrium. Also, the problem is convex.

4.2.2 Poisson distributed flow case

When the link flow is Poisson-distributed, the mean travel time function is given as

$$\bar{c}_a(m_a) = \sum_{k=0}^{\infty} t_a(k) f_{X_a}(k). \quad (30)$$

In addition, the derived function of the mean travel time can be written as

$$\bar{c}'_a(m_a) = \sum_{k=0}^{\infty} t_a(k) \frac{d}{dm_a} f_{X_a}(k) = -t_a(0) e^{-m_a} + \sum_{k=1}^{\infty} t_a(k) \left[\frac{e^{-m_a} m_a^{k-1}}{(k-1)!} - \frac{e^{-m_a} m_a^k}{k!} \right]. \quad (31)$$

In terms of the Poisson distribution, the above becomes

$$\begin{aligned} \bar{c}'_a(m_a) &= -t_a(0) f_{X_a}(0) + \sum_{k=1}^{\infty} t_a(k) \{ f_{X_a}(k-1) - f_{X_a}(k) \} \\ &= \sum_{k=0}^{\infty} f_{X_a}(k) \{ t_a(k+1) - t_a(k) \} > 0 \end{aligned} \quad (32)$$

Note that $t_a(x_a)$ is strictly increasing as described above. Thus, $\bar{c}'_a(m_a) > 0$ when $m_a \geq 0$, whereby the mean travel time function $\bar{c}_a(m_a)$ is strictly increasing. Therefore, the uniqueness of the optimization problem is guaranteed.

5. Numerical Examples

5.1. Simple Network Case

A simple example is first considered, with a Class i model selected so as to illustrate some general features of the stochastic flow models presented (in the following section, we shall consider Class ii models). We consider four example networks, all serving a single OD movement joined by parallel links/routes, but with differing numbers of routes in each case (2, 4, 10 and 20). All networks consist of an even division between A-type routes and B-type routes; so the network with ten parallel routes has five A-type and five B-type routes. The travel time functions of the A-type and B-type routes are $t_A(x_A) = 10[1 + (x_A/100)^2]$ and $t_B = 20[1 + 0.25(x_B/100)^2]$, where t_A and t_B are the travel times and x_A and x_B are the flows of the A-type and B-type routes, respectively. The fixed OD demand levels in the four cases are proportional to the number of routes available, namely 300, 600, 1500 and 3000. The multinomial logit model is adopted for the random utility model, with scale

parameter $\theta = 0.5$ in all cases.

The mean travel times are given by $E[10\{1+(X_A/100)^2\}] = 10[1 + \{\text{Var}[X_A] + (E[X_A])^2\}/100^2]$ and $E[20\{1+0.25(X_B/100)^2\}] = 20[1 + 0.25\{\text{Var}[X_B] + (E[X_B])^2\}/100^2]$, where X_A and X_B are the random variables of flows of the A-type and B-type routes, respectively.

Since all four example networks are made up of an even split between the two kinds of routes, it is easy to ascertain that the same flow is carried on routes of the same type, regardless of the total number of routes. Therefore, it is sufficient to consider one pair of A-type and B-type route flows, x_A and x_B , in each example. Fig. 2(a) illustrates the mean flows on A-type routes in the case with multinomial (binomial) route choice (Class i model, Section 3.1), those with the Poisson approximation (Section 3.5.2), and for comparison the conventional SUE flows. We have not illustrated the mean of the approximate normally distributed route flow (Section 3.5.1), since it coincides with that of the underlying multinomial model. Fig. 2(a) indicates that the flows in the conventional SUE case are different from the others; this is due to the mean travel time functions including the term for the variance in flows, which impacts on the assignment of flows when equilibrating. The mean flows of the binomial (Class i) model approach those of the Poisson approximation as the number of routes (and OD demand) increases, as would be expected. In this simple example, the difference between binomially distributed and approximately Poisson distributed flows may not be that great.

Unlike conventional SUE models, the solution of network equilibrium with stochastic flows is influenced by the absolute levels of the flows involved, not simply the flow rates. We illustrate this by parameterizing the demands and capacities by $h > 0$, such that the demands in the 2-route, 4-route and 10-route networks are $300h$, $600h$, and $1500h$, respectively, and the capacities on both links are $100h$. Figure 2(b) illustrates the resulting equilibrium solutions for where $0.1 \leq h \leq 2$. The SUE flow on the A-type route in the 2-route network is $10[1 + (300hp)^2/(100h)^2] = 10(1 + 9p^2)$, where p is the probability of choosing the A-type route, and so the SUE flows are invariant to h . In the stochastic flow model, on the other hand, the mean A-type flows in the three networks are $10[1 + 3p(1-p)/h + 9p^2]$, $10[1 + 6p(1-p)/h + 9p^2]$ and $10[1 + 15p(1-p)/h + 9p^2]$, respectively, which depend on h . These expressions (as illustrated in the figure) demonstrate that the mean binomially distributed flows approach the SUE flow as $h \rightarrow \infty$, but that for finite h , the stochastic models in this study have different properties from SUE.

5.2. Kanazawa Road Network Case

In this subsection, we apply both a normal approximation model (Section 3.5.1) and a Class ii model (Section 3.2) to the real-life network of Kanazawa in Japan. The network consists of arterial roads interconnected by 140 nodes and 467 links. The time period considered is the morning peak hour, i.e. 7:00 a.m.–8:00 a.m., and the mean OD demands were derived from a person trip survey in the Kanazawa urban area. The travel time functions used are of the standard BPR-type, $t_a = \tau_a [1 + 0.15(x_a/\zeta_a)^4]$. We consider first the use of the (approximate) normally distributed flow model (as described in Section 3.5.1), before considering other more complex forms. For practical applications, we believe this to be a starting point, since the normal approximation is relatively simple to implement, and so is more convenient for policy-testing; alternative, more complex distributions may then be tested to consider to what extent their equilibria depart from those of the simple model. The parameter ρ in Eq. (15) of the normal approximation represents an index of dispersion (variance to mean ratio) for the OD flows. We did not have direct information on variation in OD flows, so instead set ρ to the average index of dispersion in flows obtained from a year of weekday hourly link flow data⁶ from 7:00 a.m.–8:00 a.m., recorded by traffic counters. This gave rise to a

⁶A better method that might be explored in the future may be to reconstruct variances in the OD flows from

value of $\rho = 42.0$. We assume the logit scale parameter $\theta = \infty$, i.e. the objective function of the optimization problem in Section 4.1 consists solely of the first term on the right-hand-side of Eq. (19). The resulting stochastic flow network equilibrium problem was numerically solved by the Frank-Wolfe algorithm.

The resulting equilibrium solution was compared with observed link flows. The correlation coefficient between the observed and mean equilibrium link flows was calculated as 0.914, and this shows an apparently reasonable correspondence between the two. Detailed results from the model are illustrated in Fig. 3. Fig. 3(a) illustrates the equilibrium standard deviations in link travel times, where a number of critical links may be observed which greatly influence variations in trip times. Figure 3(b) presents the coefficients of variation (CVs) of link flows in the network. Many links with high CVs are located near Route 8, one of the main national roads in Japan. Consequently, the demand of access/egress to/from Route 8 is large. However, the access/egress roads are of lower capacity than Route 8, and their travel time reliability is low. Thus the results are plausible and specific to the local details of the network, in spite of assuming a common index of dispersion across all OD movements.

The parameter ρ , we might expect, is highly influential in describing the variability of flows and travel times. To examine the influence of ρ on the variability, three other cases with $\rho = 10, 21$, and 84 are considered. Figure 3(c) shows the mean of CVs of link flows and travel times, respectively. We find from the figure that the mean CVs become higher as ρ increases. The means of CVs of link travel times are higher than those of link flows. The travel times are more variable than the link flows, because the roads are congested, and a slight increase in link flow is amplified in terms of travel time. However, in this case there is relatively little feedback effect from the level of travel time variation to the equilibrium mean link flows: the correlation coefficients between the observed and calculated mean link flows in the four cases with $\rho = 10, 21, 42$ and 84 are from 0.912 to 0.914, and there is no significant difference.

Now we explore the use of alternative, more complex models, and in this case we shall focus on those in Class ii (Section 3.2). Since the parameter ρ is greater than 1, the negative binomial distribution is a candidate. Fig. 4 shows the scatter plots between mean link flows of the negative binomially and approximately normally distributed cases. The correlation coefficient between them is 0.99994, and they are almost the same. Thus, the normal approximation is a fairly accurate approximation to the negative binomial Class ii model, in the case of the Kanazawa road network.

As discussed in Section 3.2, the mean of a $NB[\alpha, \beta]$ variable is $\alpha\beta$, and $\rho = 1 + \beta$. The mean of the a -th link flow is m_a , so the a -th link flow follows a $NB[m_a/\beta, \beta] = NB[m_a/(\rho - 1), \rho - 1]$ distribution. The mean of the observed link flows in the network was 879.2 (pcu/hr), and the maximum of the observed link flow was 3,585. Fig. 5 illustrates the probability function of a negative-binomially distributed link flow whose mean is 879.2, $NB[879.2/(\rho - 1), \rho - 1]$, with $\rho = 10, 21, 42$ and 84 . Fig. 5 also includes each probability density function of the approximate normal distribution. As we can see, the average link distribution with $\rho = 10$ is quite close to the normal distribution. The distribution becomes skewed and diverges from the normal distribution as ρ gets larger. Thus the accuracy of the normal approximation depends on the parameter, ρ . Fig. 6 shows the probability function of a negative-binomially distributed link flow whose mean is 1,600 and 3,200, with $\rho = 42$. Fig. 5(c) and Fig. 6 illustrate that the link distribution approaches the normal distribution as the mean enlarges. Thus, we confirm that the normal approximation is reasonable for the link flows in the Kanazawa road network.

variances in link counts, as these may be somewhat different; a method such as that developed by Hazelton (2000) might be explored for this purpose, if extended to the case of congested networks..

6. Conclusions

Most previous network equilibrium models, including SUE models, presuppose that network flows are deterministic. This study examined network equilibrium models with stochastic flows. Although several authors have considered such a problem, few have considered the issue of how to consistently formulate network equilibrium in such a case, such that the assumptions regarding stochastic variation are followed in all aspects of the equilibration process. In order to address this issue, we have set out a general framework and have proposed four classes of model that fit within this framework. Each class is based on a different use and interpretation of the RUM in this context, and each leads to a different (but internally consistent) representation of variability in route flows, link flows, link travel times and route travel times. We establish formulations of these models as fixed point problems, establish existence of solutions to the resulting fixed point problems, and propose approximation methods that may be applicable in some scenarios. We show that in a limited number of cases, it is possible to formulate these problems as a convex optimization problem, similarly to the optimization problem of logit-based SUE. We have applied some of the proposed models to the Kanazawa road network, where it was seen to give rise to plausible phenomena.

In future research, it should be considered how stochastic factors other than demand and route choice, such as random capacities, could be incorporated within such a general framework, while still ensuring consistency. Furthermore, we assumed independent drivers, and hence independent demand, but as the work of Duthie et al. (2011) implies, neglecting the correlation among demands may lead to a mis-estimation. Considering correlated demands and identifying the demand distributions from the actual day-to-day data are therefore also important areas for future work. In the past, estimating a mean OD demand matrix has proved sufficiently challenging in practice, but emerging data sources provide the potential for more precise tracking of daily OD demands, and hence the possibility in the future to obtain direct information on levels of OD demand variability.

In the paper, we prove uniqueness of optimization problems of the models in Class ii and with Poisson distributed flows in all classes in the present paper, but it would be useful in the future to extend these results so as to establish uniqueness for other models with stochastic flows. In Watling (2002a) this was achieved for some limited cases in terms of link flow moments; a fruitful area of work may be to consider extending this work for the wider class of models considered in the present paper. It would also be fruitful to explore consistent formulations under other behavioural mechanisms, especially those related to risk, in which travellers may respond to aspects of the route travel time distribution other than its mean.

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Appendix A: Justification of normal approximation

We consider the behavior of a single (latent) driver. Define the mean vector and variance-covariance matrix as follows:

$$\boldsymbol{\mu}_{ik} = \begin{pmatrix} E[Y_{i0k}] \\ E[Y_{i1k}] \\ \vdots \\ E[Y_{ij,k}] \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{ik} = \begin{pmatrix} \text{Var}[Y_{i0k}] & \text{Cov}[Y_{i0k}, Y_{i1k}] & \cdots & \text{Cov}[Y_{i0k}, Y_{ij,k}] \\ \text{Cov}[Y_{i1k}, Y_{i0k}] & \text{Var}[Y_{i1k}] & \cdots & \text{Cov}[Y_{i1k}, Y_{ij,k}] \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}[Y_{ij,k}, Y_{i0k}] & \text{Cov}[Y_{ij,k}, Y_{i1k}] & \cdots & \text{Var}[Y_{ij,k}] \end{pmatrix}.$$

Because each driver independently selects a route, $\boldsymbol{\mu}_i = \sum_{k=1}^{v_i} \boldsymbol{\mu}_{ik}$ and $\boldsymbol{\Sigma}_i = \sum_{k=1}^{v_i} \boldsymbol{\Sigma}_{ik}$. If the demand is fixed, we have $E[Y_{i0k}] = \text{Var}[Y_{i0k}] = \text{Cov}[Y_{i0k}, Y_{i1k}] = \dots = \text{Cov}[Y_{i0k}, Y_{ij,k}] = 0$.

Assume that the mean vector and variance-covariance matrix of \mathbf{Y}_{ik} are finite and positive. Although the Y_{ijk} are independent, they need not follow an identical distribution because each driver autonomously decides his behavior. The above behavior is general, and permits multinomial or Dirichlet-compound multinomial route choice and binomially, Poisson, negative- or beta-binomially distributed demand. Thus, it is more relaxed than the preceding subsections. Since Y_{ijk} is binary, $\Pr[|Y_{i0k}| \leq 1, |Y_{i1k}| \leq 1 \dots |Y_{ij,k}| \leq 1] = 1$, and the random vector \mathbf{Y}_{ik} is uniformly bounded. In the following proof, we introduce for convenience the weighted composite route choice variable, Z_{ik} . Let $Z_{ik} = \boldsymbol{\kappa}_i^T \mathbf{Y}_{ik}$, where $\boldsymbol{\kappa}_i = (\kappa_0, \kappa_1 \dots \kappa_{j_i})^T$. Using Z_{ik} , the vector \mathbf{Y}_{ik} can be treated as a scalar. Then $E[Z_{ik}] = E[\boldsymbol{\kappa}_i^T \mathbf{Y}_{ik}] = \boldsymbol{\kappa}_i^T \boldsymbol{\mu}_{ik}$ and $\text{Var}[Z_{ik}] = \text{Var}[\boldsymbol{\kappa}_i^T \mathbf{Y}_{ik}] = \boldsymbol{\kappa}_i^T \boldsymbol{\Sigma}_{ik} \boldsymbol{\kappa}_i$. Since \mathbf{Y}_{ik} is uniformly bounded, Z_{ik} is also uniformly bounded when $\boldsymbol{\kappa}$ is in the finite sphere. Clearly, $\text{Var}[\sum_{k=1}^{v_i} Z_{ik}] \rightarrow \infty$ as $v_i \rightarrow \infty$ ($\boldsymbol{\kappa} \neq \mathbf{0}$) because $\text{Var}[Z_{ik}] > 0$. If the standardized random variables $\mathbf{V} = (V_0, V_1 \dots V_n)^T$ are independent and uniformly bounded, then $(1/\sqrt{n}) \sum_{k=0}^n V_k \rightarrow N[0, 1]$ as $n \rightarrow \infty$ (Loève, 1977, p.289). As $v_i \rightarrow \infty$,

$$\frac{\sum_{k=1}^{v_i} Z_{ik} - \sum_{k=1}^{v_i} E[Z_{ik}]}{\sqrt{\sum_{k=1}^{v_i} \text{Var}[Z_{ik}]}} = \frac{\sum_{k=1}^{v_i} Z_{ik} - \boldsymbol{\kappa}_i^T \boldsymbol{\mu}_i}{\sqrt{\boldsymbol{\kappa}_i^T \boldsymbol{\Sigma}_i \boldsymbol{\kappa}_i}} \rightarrow \mathfrak{N}[0, 1]$$

because $\boldsymbol{\mu}_i = \sum_{k=1}^{v_i} \boldsymbol{\mu}_{ik}$ and $\boldsymbol{\Sigma}_i = \sum_{k=1}^{v_i} \boldsymbol{\Sigma}_{ik}$. Therefore, $\sum_{k=1}^{v_i} Z_{ik} \rightarrow \mathfrak{N}[\boldsymbol{\kappa}_i^T \boldsymbol{\mu}_i, \boldsymbol{\kappa}_i^T \boldsymbol{\Sigma}_i \boldsymbol{\kappa}_i]$.

According to the Cramér-Wold device (Billingsley, 1995, p380), a necessary and sufficient condition for $\mathbf{Y}_{ik} \rightarrow \mathbf{Z}_i$ is that $\sum_{j=0}^{j_i} \kappa_j Y_{ijk} \rightarrow \sum_{j=0}^{j_i} \kappa_j Z_{ij}$ for each $\boldsymbol{\kappa} = (\kappa_0, \kappa_1 \dots \kappa_{j_i})^T$ in the finite $(j_i + 1)$ -dimensional sphere, where the random vector $\mathbf{Z}_i = (Z_{i0}, Z_{i1} \dots Z_{ij_i})^T$. Therefore, as $n_i \rightarrow \infty$, we have

$$\mathbf{Y}_i = \sum_{k=1}^{v_i} \mathbf{Y}_{ik} \rightarrow \mathfrak{N}[\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i].$$

Thus, the distribution of route flows between the same OD pair approaches multivariate normal when the demand is sufficiently large.

Table 1: Consistent formulations of stochastic queueing networks with stochastic flows: Model classes considered

class	Class i	Class ii	Class iii	Class iv						
route choice	stochastic (multinomial)	deterministic			stochastic (multinomial)	compound stochastic (Dirichlet-compound multinomial)				
demand	deterministic	stochastic			stochastic	stochastic				
resultant route flow	multinomial	binomial	Poisson	beta- binomial	negative binomial	binomial	Poisson	negative binomial	deterministic	beta-binomial
vector of decision variables,	route choice probabilities	route choice proportions			conditional route choice probabilities	mean route choice probabilities	Dirichlet-compound multinomial			
reference	Watling (2002a)	this study	this study	this study	this study	Watling (2002c), Clark & Watling (Appendix A, 2005), Nakayama (2006), Nakayama & Takayama (2007)	this study	this study		

Table 2 (a) Notation 1

i	origin-destination (OD) pair (i, j)
I	total number of OD pairs
j	route ($j = 1, 2, \dots, J_i$)
J_i	number of routes between the OD pair
J_i^*	set of routes between the OD pair, excluding the j -th route
a	(single) link ($a = 1, 2, \dots, A$)
A	total number of links
$a_{i,j}$	link-route incidence variable ($a_{i,j} = 1$ if the a -th link is on the j -th route, and 0 otherwise)
	link-route incidence matrix $= [a_{i,j}]$
N_i	random variable of the demand between the OD pair
\bar{n}_i	mean of the demand between the OD pair $= E[N_i]$
σ_i^2	variance of the demand between the OD pair $= Var[N_i]$
n_i	fixed demand of driver traveling between the i -th OD pair or realized value of the demand between the OD pair
\tilde{N}_i	random variable of the demand between the OD pair
$g_{N_i}(n_i)$	probability function of N_i
S_i	latent demand between the OD pair
p_{ij}	depending on the specific model, the probability of choosing the route between the i -th OD pair for a randomly-selected traveler, or the proportion of demand choosing the route, where clearly $\sum_{j \in J_i} p_{ij} = 1$.
p_i	vector of route choice probabilities/proportions for the i -th OD pair $= (p_{i1}, p_{i2}, \dots, p_{iJ_i})^T$
p	vector of all route choice probabilities/proportions $= (p_1, p_2, \dots, p_I)^T$
P_i	random vector of route choice probabilities between the i -th OD pair when the probabilities themselves are random (specific to Section 3.4, Class IV models).
\mathcal{P}_i	set of p satisfying $\sum_{j \in J_i} p_{ij} = 1$ for any i and $p_{ij} \geq 0$ for any i and j .
$(\cdot)_{ij}$	function that generates either the probability of choosing the j -th route between the i -th OD pair or the proportion choosing the route
$(\cdot)_i$	vector-valued function that generates the route choice probabilities/proportions, whose components are $(\cdot)_{ij}$
λ	common probability of making a trip (trip probability)
Y_{ijk}	random variable that decides whether the k -th driver traveling between the i -th OD pair chooses the j -th route ($Y_{ijk} = 1$ if the k -th driver takes the j -th route, and 0 otherwise). Clearly $\sum_{j \in J_i} Y_{ijk} = 1$, and $\sum_{k \in K_i} Y_{ijk} = Y_{ij}$.
Y_{ik}	random vector of $Y_{ijk} = (Y_{i0k}, Y_{i1k}, \dots, Y_{iJ_i k})^T$
Y_{ij}	random variable of route flow on the j -th route between the i -th OD pair.
y_{ij}	realized value of Y_{ij}
Y_i	random vector of route flows between the OD pair $= (Y_{i1}, Y_{i2}, \dots, Y_{iJ_i})^T$
Y	random vector of all route flows $= (Y_1, Y_2, \dots, Y_I)^T$

Table 2 (b) Notation 2

y_{ij}	mean flow on the j -th route between the i -th O D pair = $E[Y_{ij}]$
y_i	vector of mean route flows between the i -th O D pair = $(y_{i1}, y_{i2}, \dots, y_{ij})^T$
	vector of all mean route flows = $(y_1, y_2, \dots, y_i)^T$
y_{ij}^2	variance of flow on the j -th route between the i -th O D pair = $Var[Y_{ij}]$
$y_{ij,i'j'}^2$	covariance between the flow on the j -th route between the i -th O D pair and flow on the j' -th route between the i' -th O D pair = $Cov[Y_{ij}, Y_{i'j'}]$
y_i	variance-covariance matrix of route flows between the i -th O D pair
y	variance-covariance matrix of all route flows
$f_{Y_i}(y_i)$	joint probability function of Y_i
$f_{Y_i N_i}(y_i)$	joint probability function of Y_i conditional on $N_i = n_i$.
X_a	random variable of flow on the a -th link
x_a	realized value of X_a
X	random vector of all link flows = $(X_1, X_2, \dots, X_A)^T$
m_a	mean flow on the a -th link = $E[X_a]$
m	mean vector of link flows = $(m_1, m_2, \dots, m_A)^T$
m	finite value that is greater than the maximum of mean link flows ($\max[m_a a = 1, 2, \dots, A] < m < \infty$).
s_a^2	variance of flow on the a -th link
S	variance-covariance matrix of link flows
s_i	positive value that is less than the minimum of standard deviations of link flows ($0 < s_i < \min[s_a a = 1, 2, \dots, A]$). Each random link flow is assumed to have a positive variance.
$f_{X_a}(x_a)$	probability function of X_a
T_a	random variable of travel time on the a -th link
$t_a(x_a)$	travel time function on the a -th link
$\bar{t}_a(p)$	function that calculates the mean travel time on the a -th link from the route choice probabilities
$\bar{c}_a(m_a)$	function that calculates the mean travel time on the a -th link from the mean link flow m_a ; $\bar{c}_a(m_a) = \bar{c}_a(m_a(p)) = \bar{t}_a(p)$, because m_a is determined by p .
τ_a	free-flow travel time on the a -th link
μ_a	capacity on the a -th link
$u_{ij}(p)$	utility function of the j -th route between the i -th O D pair.
\bar{t}_{ij}	mean travel time on the j -th route between the i -th O D pair
ϵ_{ij}	error term of the j -th route between the i -th O D pair
	positive parameter in the route choice model
$E[X]$	mean of random variable X = $E_X[X]$
$Var[X]$	variance of random variable X of
$Cov[X, Y]$	covariance of the two random variables X and Y
$Pr[\cdot]$	probability calculation operator

Table 2(c) Notation 3

$[\cdot, \cdot]$	normal distribution of mean vector and variance-covariance matrix and \cdot , respectively
$! [\cdot]$	Poisson distribution of mean
$" [n, \cdot]$	binomial distribution with parameter n and \cdot
$" [\cdot, \#]$	negative binomial distribution with parameter \cdot and $\#$
$" [\cdot, \cdot, r]$	beta-binomial distribution with parameters \cdot , \cdot , and r
$\# [p, r]$	Dirichlet distribution with parameter vector p and parameter r
0	null vector
t	transpose of vector or matrix
$\% (x)$	gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$
$\max \{ \}$	maximum-value operator
$\min \{ \}$ \min	minimum-value operator
$\text{diag} (\cdot)$ diag	diagonal matrix containing diagonal components

Random variables are expressed in capital letters and vectors or matrices are expressed in bold font

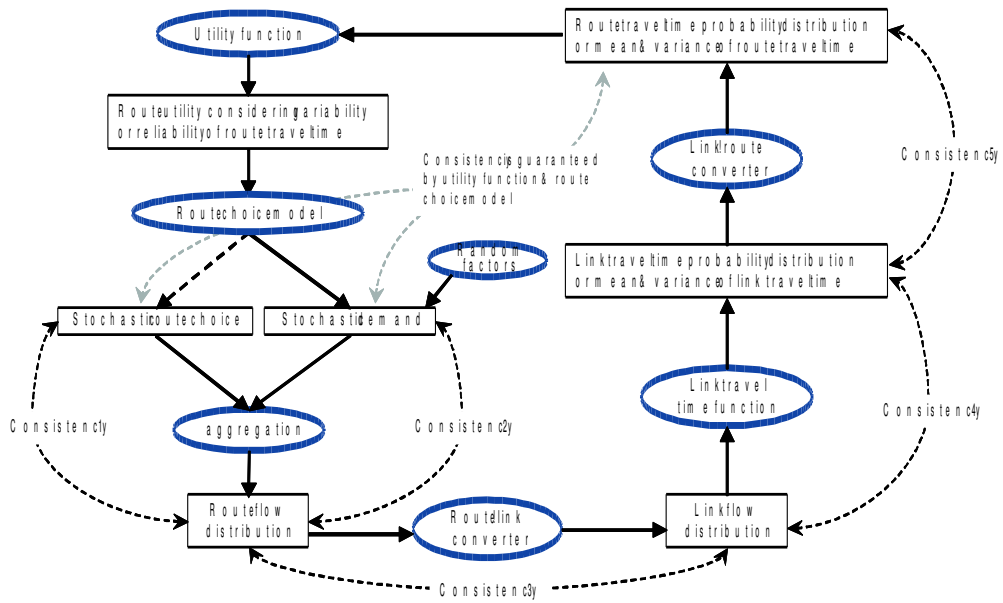
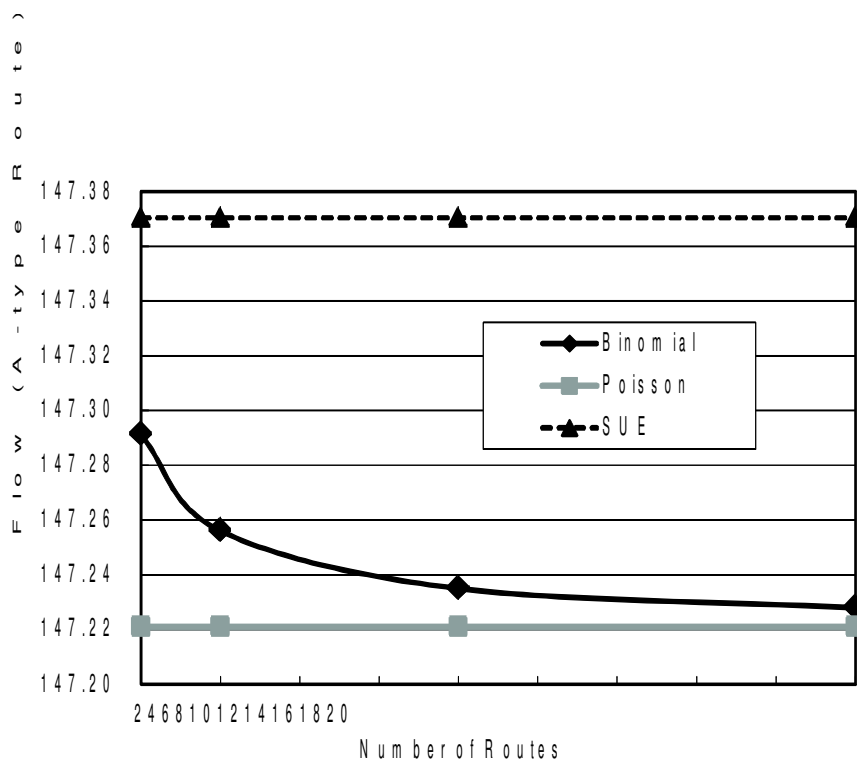
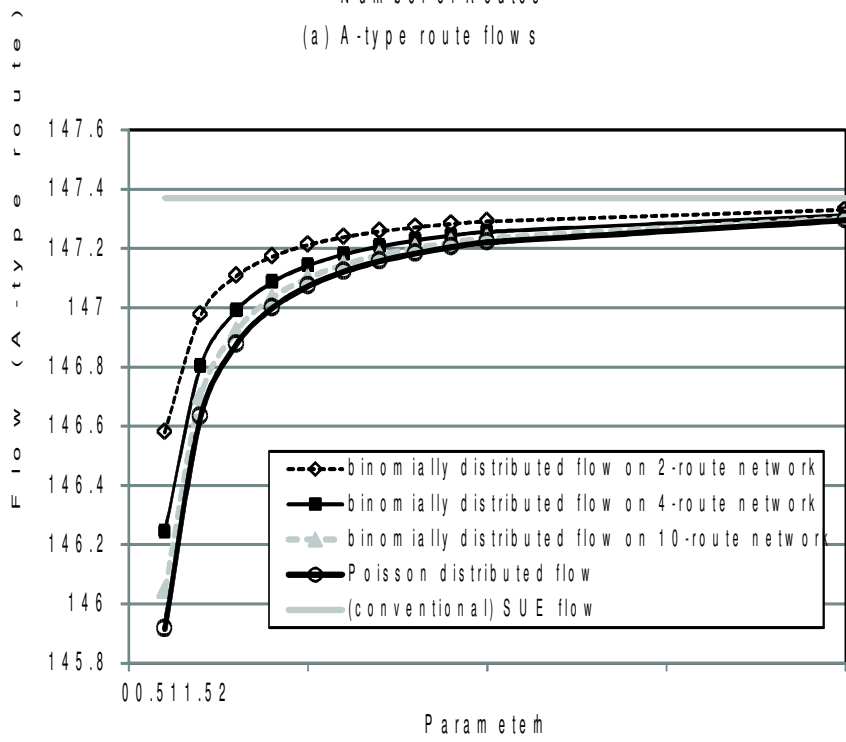


Fig. 1: Schematic of network equilibrium mechanism with stochastic flows and travel times

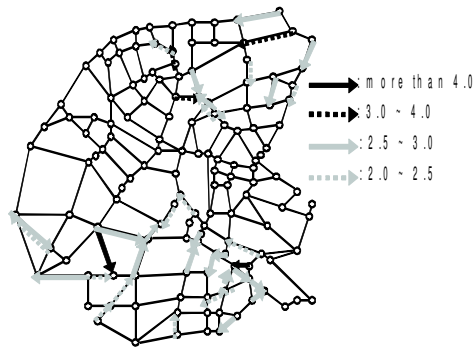


(a) A-type route flows

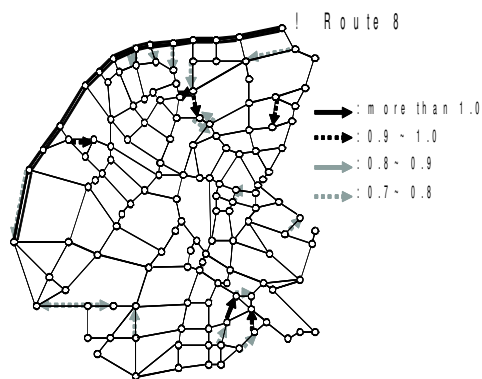


(b) A-type route flows for differenth

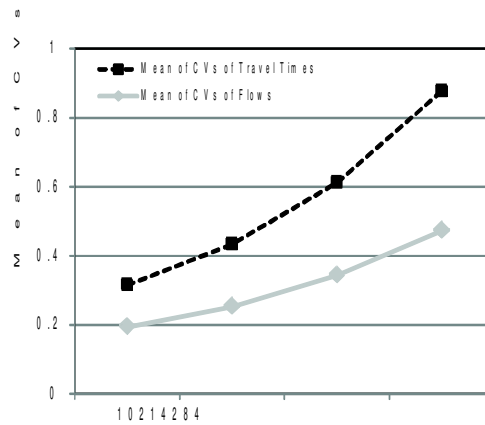
Fig. 2: Results in the four examples



(a) SDs of link travel times with $w=42$



(b) CVs of link travel times with $w=42$



(c) Mean of CVs of flows and travel times among links with $w=10, 21, 42$, and 84 , respectively

Fig. 3: Results of application to the Kanazawa road network during peak morning traveling time (7:00 a.m.~8:00 a.m.)

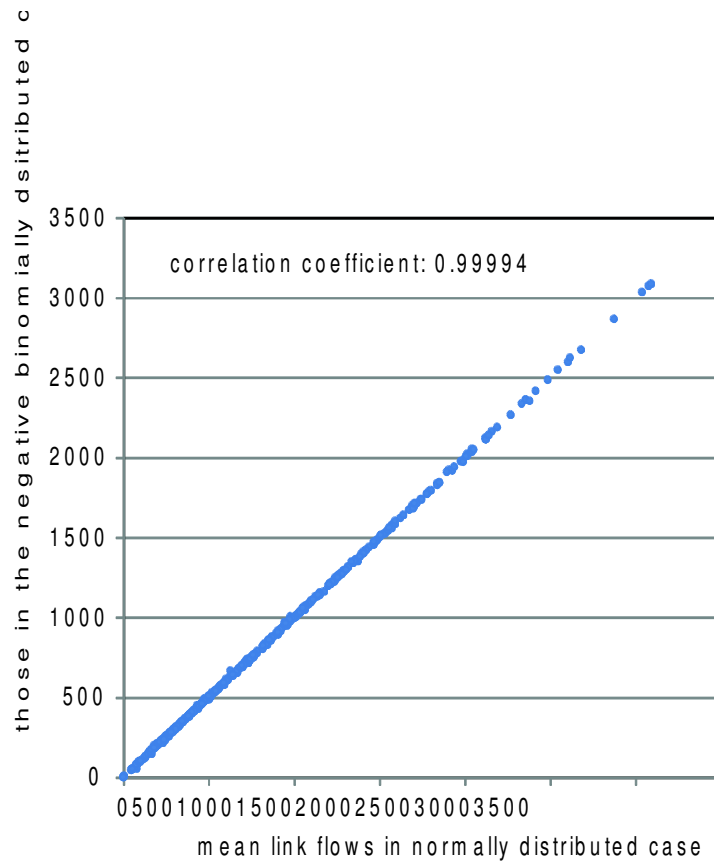


Fig. 4: Scatter plots between mean link flows of negative binomially and normally distributed cases

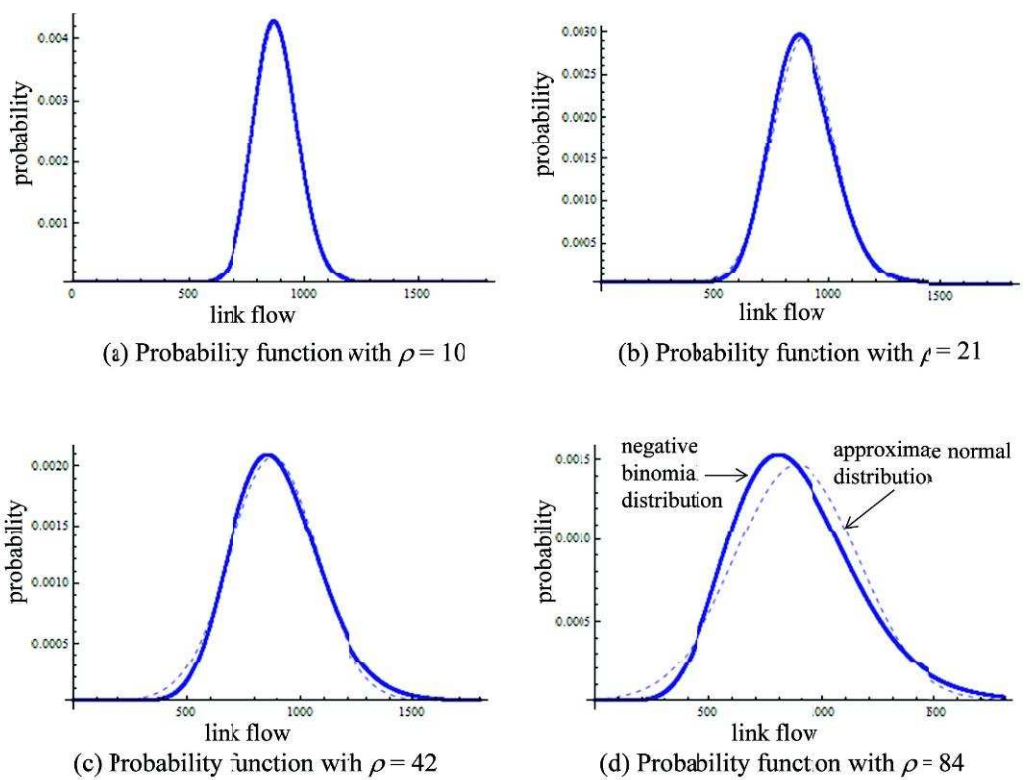


Fig. 5: Probability function of average link flow distribution with $\rho = 10, 21, 42,$ and 84

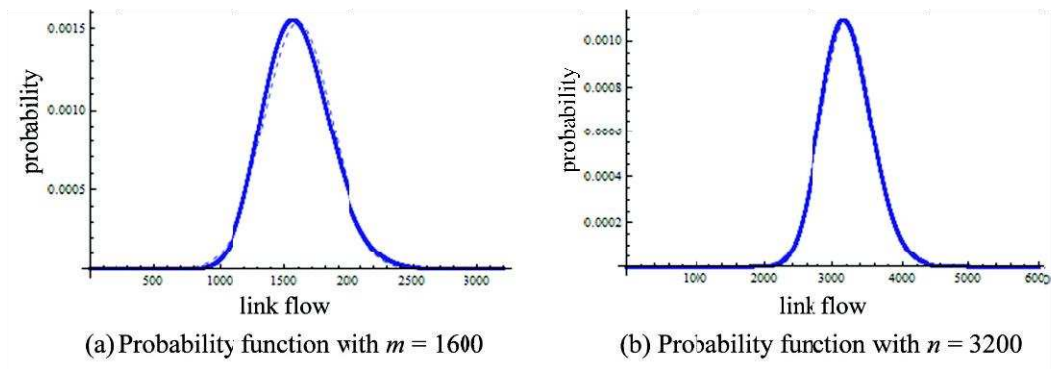


Fig. 6: Probability function of link flow distribution whose mean is 1,600 and 3,200 with $\sigma = 42$