On Wavelet Multiresolution Approximations of Random Processes

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Abstract

Some convergence issues concerning wavelet multiresolution approximation of random processes are investigated together with the properties of the stochastic coefficients associated with the multiresolution decomposition of second-order processes. Existing convergence results derived for orthonormal wavelet multiresolution approximations are extended to more general non-orthogonal multiresolution approximations. The mean and covariance functions of the stochastic expansion coefficients of second order processes are derived explicitly and it is shown that for white noise processes the variance of the coefficients is invariant across the scale. Simulation results illustrate the theoretical findings.

1 Introduction

It is often desirable to represent random processes as a weighted sum of basis functions. An example of such a representation is the wavelet multiresolution decomposition [7, 3, 9, 2] in which the random process is expressed in terms of dilates and translates of a scaling and a wavelet function.

The convergence of such representations has been addressed for several types of stochastic processes in the context of orthonormal wavelet multiresolution approximations implemented using compactly supported wavelets [6]. Explicit expressions for the approximation errors for deterministic and random signals have also been established [1]. In this paper it is shown that the convergence results can easily be extended to more general non-orthogonal classes of wavelet multiresolution decompositions.

The main focus of previous work however was the study of the expansion coefficients. The coefficients of the approximation define, at each resolution level, a discrete stochastic process that can be characterised in terms of mean, correlation and covariance functions. In this context the correlation structure of the wavelet coefficients of Brownian motion [4, 10] and more general random processes [8, 5] has been studied assuming orthogonal wavelets with compact support.

In this study, the stochastic properties of wavelet coefficients of second-order random processes i.e. the mean and covariance functions are derived in the context of general non-orthogonal wavelet multiresolution decompositions. It is shown that the coefficients associated with a wide sense stationary process also constitute wide sense stationary processes at every resolution level. As a special case, the second order properties of the coefficients of white noise processes are investigated. It is shown that the variance of the white noise coefficients is invariant across the scale. If the decomposition is orthonormal it is inferred that the coefficients define a discrete white noise process with the same variance as the continuous-time counterpart.

2 Wavelet Multiresolution Approximations of Random Processes

It is well known [1] that many random processes do not have sample paths in $L^2(\mathbb{R})$, that is, the signals are not square integrable over the real line, and therefore the wavelet approximation framework does not apply in such cases.

However, stochastic processes which are square integrable with probability one over the real line, i.e. *finite energy stochastic processes*, and the sample functions of stationary and nonstationary random processes with *finite (mean) power*, which are square integrable with probability one over every finite interval, can be represented meaningfully as multiresolution wavelet series. In this context the issue of convergence will be addressed for more general non-orthogonal wavelet bases.

2.1 Finite Energy Stochastic Processes

A measurable random process with finite energy $x(t,\omega)$ satisfies

$$\int_{-\infty}^{\infty} \mathcal{E}\{|x(t,\omega)|^2\}dt = \int_{-\infty}^{\infty} R(t,t)dt < \infty \tag{1}$$

where $\mathcal{E}\{\cdot\}$ and $R(\cdot)$ denote the expected value and covariance function respectively. From Fubini's theorem it follows that $x(t,\omega)$ defined on $\mathbb{R} \times \Omega$ is square integrable with probability one i.e. almost every realisation $x(t,\omega)$ with $\omega \in \Omega$ will be a square integrable function in $L^2(\mathbb{R})$.

Previous studies [6] have shown that such random processes can be represented meaningfully in terms of orthonormal wavelet series. In what follows, it will be shown that the convergence results still hold if the orthogonality constraints are dropped.

The main difference is that for non-orthogonal MRA's (Multiresolution approximations) the basis functions used for analysis are not identical with the basis functions used for synthesis. Each wavelet and scaling basis function used in the reconstruction corresponds to a dual wavelet and scaling basis function used in the analysis. The roles of the dual bases are normally interchangeable, each pair defining a MRA.

The following lemma considers the convergence with probability one of a non-orthogonal MRA of a finite energy random process, extending the applicability of the original lemma due to Genossar (1991).

Lemma 2.1 *Let $(\phi(t), \tilde{\phi}(t))$ and $(\psi(t), \tilde{\psi}(t))$ be pairs of dual scaling and wavelet functions corresponding to a multiresolution analysis, not-necessarily orthogonal, in $L^2(\mathbb{R})$. For any measurable, finite energy random process $x(t,\omega)$ defined on $\mathbb{R} \times \Omega$ the following hold:*

- *The coefficients*

$$\begin{aligned} c_{j,k} &= \langle x(t), \tilde{\phi}_{j,k}(t)\rangle = \int_{-\infty}^{\infty} x(t)\tilde{\phi}_{j,k}(t)dt \\ d_{j,k} &= \langle x(t), \tilde{\psi}_{j,k}(t)\rangle = \int_{-\infty}^{\infty} x(t)\tilde{\psi}_{j,k}(t)dt \end{aligned} \tag{2}$$

are well defined for any $j, k \in \mathbb{Z}$

- *The approximation*

$$x_j(t) = \sum_{k \in \mathbb{Z}} c_{j,k}\phi_{j,k}(t) = \sum_{l=-\infty}^{j} \sum_{k \in \mathbb{Z}} d_{j,k}\psi_{j,k}(t) \tag{3}$$

is well defined for any $j \in \mathbb{Z}$ *and converges with probability one to* $x(t)$ *that is*

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} |x(t) - x_j(t)|^2 dt = 0 \tag{4}$$

with probability one.

Proof: Using a similar approach to that in the original proof due to Genossar (1991), since each realisation of $x(t)$ is square integrable with probability one it follows that with probability one, the coefficients of the approximation can be calculated as an inner product $\langle x, \tilde{\phi}_{j,k}(t)\rangle$ and $\langle x, \tilde{\psi}_{j,k}(t)\rangle$ involving the dual scaling $\tilde{\phi}_{j,k}(t)$ and wavelet $\tilde{\psi}_{j,k}(t)$ functions respectively. The approximation $x_j(t)$, which is the projection onto the scaling subspace V_j expanded in terms of scaling and wavelet functions respectively, converges to $x(t)$ in the $L^2(\mathbb{R})$ norm according to the wavelet theory.

The theorem regarding the convergence of the approximation in the quadratic mean [6] can also be modified accordingly

Theorem 2.1 *Let* $(\phi(t), \tilde{\phi}(t))$ *and* $(\psi(t), \tilde{\psi}(t))$ *be pairs of dual scaling and wavelet functions corresponding to a multiresolution analysis, not necessarily orthogonal, in* $L^2(\mathbb{R})$*. For any measurable, finite energy random process* $x(t, \omega)$ *defined on* $\mathbb{R} \times \Omega$ *with associated covariance function* $R(t, s)$ *satisfying* $\int_{-\infty}^{\infty} R(t, t)dt < \infty$ *the approximation*

$$x_j(t) = \sum_{k \in \mathbb{Z}} c_{j,k}\phi_{j,k}(t) = \sum_{l=-\infty}^{j} \sum_{k \in \mathbb{Z}} d_{j,k}\psi_{j,k}(t) \tag{5}$$

converges to $x(t, \omega)$ *in the* $L^2(\mathbb{R} \times \Omega)$ *norm, that is,*

$$\lim_{j \to \infty} \int_{-\infty}^{\infty} \mathcal{E}\left\{|x(t) - x_j(t)|^2\right\} dt = 0 \tag{6}$$

Proof: Similar to Genossar (1991), define $e_j(t)$, the approximation error at level j,

$$e_j(t) = x(t) - x_j(t) = x(t) - \sum_{k \in \mathbb{Z}} c_{j,k}\phi_{j,k}(t) \tag{7}$$

Clearly, $e_j(t, \omega)$ is a random process defined on $\mathbb{R} \times \Omega$ hence the squared error at resolution j, defined as

$$\varepsilon_j = \|e_j(t)\|_2^2 = \int_{-\infty}^{\infty} |e_j(t)|^2 dt \tag{8}$$

is a random variable in Ω. From Lemma (2.1) it follows that almost any realisation $x_j(t)$ converges to $x(t)$ in $L^2(\mathbb{R})$. Hence

$$\lim_{j\to\infty} e_j(t) = 0 \tag{9}$$

with probability one. This means that

$$\lim_{j\to\infty} \varepsilon_j = 0 \tag{10}$$

with probability one. Regardless of whether the multiresolution is orthogonal or not, by definition, $x_j(t)$ is the orthogonal projection of $x(t)$ onto $V_j = ... + W_{j-2} + W_{j-1}$. This means that $e_j(t)$ is the projection of $x(t)$ onto the orthogonal complement of V_j in $L^2(\mathbb{R})$ such that $x(t) = x_j(t) \oplus e_j(t)$. Hence

$$\|x(t)\|_2^2 = \|e_j(t) + x_j(t)\|_2^2 = \|e_j(t)\|_2^2 + \|x_j(t)\|_2^2 \tag{11}$$

Consequently

$$\varepsilon_j = \|e_j(t)\|_2^2 \leq \|x(t)\|_2^2 \tag{12}$$

with probability one.

Since $R(t,t)$ is integrable

$$\mathcal{E}\left\{\|x(t)\|_2^2\right\} = \int_{-\infty}^{\infty} R(t,t)dt < \infty \tag{13}$$

Equations (10), (12), (13) state that the sequence of random variables $\{\varepsilon_j\}$ which converges to zero with probability one, is dominated by $\|x(t)\|_2^2$ with probability one and $\|x(t)\|_2^2$ has finite expectation. By the Dominated Convergence Theorem [11]

$$\lim_{j\to\infty} \mathcal{E}\{\epsilon_j\} = 0 \tag{14}$$

Equation (6) follows since the value of an absolutely convergent iterated integral is independent of the order of integration (Fubini's theorem).

2.2 Finite Power Random Processes

Random processes with finite (mean) power,

$$\int_I \mathcal{E}\{|x(t,\omega)|^2\}dt = \int_I R(t,t)dt < \infty \tag{15}$$

for any compact interval $I \in \mathbb{R}$, are square integrable over every finite interval. The wavelet representation over a finite interval of such processes is relevant in practice since real life processes are usually observed over a finite time interval.

Using Fubini's Theorem, it can be shown that with probability one the sample paths $x(t,\omega)$ of these random processes are square integrable over every finite interval.

The following lemma, and the subsequent theorem are generalisations of the original results derived in the context of orthogonal wavelet approximations [6].

Lemma 2.2 *If $(\phi(t), \tilde{\phi}(t))$ and $(\psi(t), \tilde{\psi}(t))$ are pairs of dual scaling and wavelet functions which define a multiresolution analysis not necessarily orthogonal, then for any finite (mean) power random process $x(t,\omega)$ defined over $L^2(\mathbb{R} \times \Omega)$, the following statements hold:*

- *The coefficients*

$$\begin{aligned} c_{j,k} &= \langle x(t), \tilde{\phi}_{j,k}(t)\rangle_I = \int_I x(t)\tilde{\phi}_{j,k}(t)dt \\ d_{j,k} &= \langle x(t), \tilde{\psi}_{j,k}(t)\rangle_I = \int_I x(t)\tilde{\psi}_{j,k}(t)dt \end{aligned} \tag{16}$$

  are well defined for any $j, k \in \mathbb{Z}$

- *The approximating series*

$$x_j(t) = \sum_{k\in\mathbb{Z}} c_{j,k}\, \phi_{j,k}(t) = \sum_{l=-\infty}^{j} \sum_{k\in\mathbb{Z}} d_{j,k}\, \psi_{j,k}(t) \tag{17}$$

  is well defined for all and $j, k \in \mathbb{Z}$

- *For any compact interval $I \in \mathbb{R}$,*

$$\lim_{j\to\infty} \int_I |x(t) - x_j(t)|^2 dt = 0 \tag{18}$$

  with probability one.

Proof: The integral used to calculate the coefficients is convergent (finite) for any compact interval, since $x(t,\omega)$ is square integrable over every compact I with probability one. It follows that the approximating sums are well defined and converge to $x(t)$ in the $L^2(\mathbb{R})$ norm.

The following theorem guarantees convergence in the mean of the wavelet approximation of a finite (mean) power random process.

Theorem 2.2 *Let $(\phi(t), \tilde{\phi}(t))$ and $(\psi(t), \tilde{\psi}(t))$ be pairs of dual scaling and wavelet functions which define a multiresolution analysis not necessarily orthogonal. For any random process $x(t,\omega)$ defined over $L^2(\mathbb{R} \times \Omega)$ which satisfies the finite (mean) power requirement, the approximation*

$$x_j(t) = \sum_{k\in\mathbb{Z}} \langle x(t), \tilde{\phi}_{j,k}(t)\rangle_I \phi_{j,k}(t) = \sum_{l=-\infty}^{j} \sum_{k\in\mathbb{Z}} \langle x(t), \tilde{\psi}_{j,k}(t)\rangle_I\, \psi_{j,k}(t) \tag{19}$$

satisfies

$$\lim_{j\to\infty} \int_I \mathcal{E}\left\{|x(t) - x_j(t)|^2\right\} dt = 0 \tag{20}$$

for any compact $I \in \mathbb{R}$.

Proof: The truncation $x_0(t)$ of the random process over the interval I, such that $x_0(t,\omega) = x(t,\omega)$ when $t \in I$ and zero elsewhere, is a finite energy process hence the results of Theorem (2.1) hold. Thus,

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} \mathcal{E}\left\{ |x_0(t) - \sum_{k\in\mathbb{Z}} \langle x_0(t), \tilde{\phi}_{j,k}(t)\rangle \phi_{j,k}(t)|^2 \right\} dt = 0 \tag{21}$$

Because the integrand is everywhere nonnegative and $I \subset \mathbb{R}$, the previous expression is equivalent to

$$\lim_{j\to\infty} \int_{I} \mathcal{E}\left\{ |x_0(t) - \sum_{k\in\mathbb{Z}} \langle x_0(t), \tilde{\phi}_{j,k}(t)\rangle \phi_{j,k}(t)|^2 \right\} dt = 0 \tag{22}$$

Since $x_0(t,\omega) = x(t,\omega)$ for $t \in I$ and $\langle x_0(t), \tilde{\phi}_{j,k}(t)\rangle = \langle x(t), \tilde{\phi}_{j,k}(t)\rangle_I$ it follows that

$$\lim_{j\to\infty} \int_{I} \mathcal{E}\left\{ |x(t) - \sum_{k\in\mathbb{Z}} \langle x(t), \tilde{\phi}_{j,k}(t)\rangle_I \phi_{j,k}(t)|^2 \right\} dt = 0 \tag{23}$$

Note that if the analysis scaling and wavelet functions are compactly supported, only a finite number of nonzero scaling and wavelet function coefficients exist. It follows that at every resolution level only a finite number of synthesis basis functions, which may have infinite support, are practically used.

In contrast when the analysis functions are not compactly supported, the number of nonzero coefficients is theoretically infinite, although their values vanish rapidly outside the interval of interest. However, if the scaling and wavelet functions used in reconstruction have compact support, only the basis functions that cover the interval of interest are useful to approximate $x(t)$. Hence, at every resolution level, the expansion will include a finite number of scaling and wavelet functions.

2.3 Properties of Stochastic Wavelet Coefficients

The coefficients associated with a multiresolution decomposition of a random process are stochastic variables. If such a decomposition exists (is meaningful) the coefficient sequence at every resolution defines a discrete random process over the space of square summable sequences $l^2(\mathbb{Z} \times \Omega)$. The correlation properties of these coefficients are induced by the correlation structure of the continuous-time process.

Consider $x(t,\omega)$ a second-order random process ($\mathcal{E}\{|x(t,\omega)|^2\} < \infty$) with mean $\mu(t)$ correlation function $r(t,s)$ and covariance function $R(t,s)$.

If $x(t,\omega)$ is measurable and

$$\int_{-\infty}^{\infty} \mathcal{E}\{|x(t,\omega)|\}|\tilde{\phi}(t)|dt < \infty \tag{24}$$

is convergent the following integrals

$$\int_{-\infty}^{\infty} x(t,\omega)\tilde{\phi}(t)dt < \infty \tag{25}$$

$$\int_{-\infty}^{\infty} x(t,\omega)\tilde{\psi}(t)dt < \infty \tag{26}$$

are well defined so the coefficients

$$c_{j,k} = \int_{-\infty}^{\infty} x(t,\omega)\tilde{\phi}_{j,k}(t)dt < \infty$$
$$d_{j,k} = \int_{-\infty}^{\infty} x(t,\omega)\tilde{\psi}_{j,k}(t)dt < \infty \tag{27}$$

are finite.

At each resolution level j, the integrals in (27) define two discrete stochastic processes $c_j(k,\omega)$ and $d_j(k,\omega)$ respectively, with $k, j \in \mathbb{Z}$, which can be characterised in terms of mean, correlation and covariance functions.

The mean of the random scaling and wavelet coefficients in (27) can be calculated at each resolution as follows

$$\mathcal{E}\{c_{j,k}\} = \mathcal{E}\left\{\int_{-\infty}^{\infty} x(t)\tilde{\phi}_{j,k}(t)dt\right\} = \int_{-\infty}^{\infty} \mathcal{E}\{x(t)\}\tilde{\phi}_{j,k}(t)dt = 2^{-j/2}\int_{-\infty}^{\infty} \mu(t)\tilde{\phi}(t)dt$$
$$\mathcal{E}\{d_{j,k}\} = \mathcal{E}\left\{\int_{-\infty}^{\infty} x(t)\tilde{\psi}_{j,k}(t)dt\right\} = \int_{-\infty}^{\infty} \mathcal{E}\{x(t)\}\tilde{\psi}_{j,k}(t)dt = 2^{-j/2}\int_{-\infty}^{\infty} \mu(t)\tilde{\psi}(t)dt \tag{28}$$

where Fubini's Theorem has been applied to change the order of integration. In particular, assuming $\mu(t) = \mu$ is constant if follows that $\mathcal{E}\{c_{j,k}\} = 2^{-j/2}\mu\|\phi\|_1 = 2^{-j/2}\mu$ (usually $\|\phi\|_1 = 1$) and $\mathcal{E}\{d_{j,k}\} = 0$

A far as the second-order properties are concerned it is always possible to assume that $x(t,\omega)$ is zero-mean with little loss of generality. In this case the correlation and covariance functions coincide.

The covariance function $R_{c_j}(l,m)$ of $c_{j,k}$ where $j,k,l,m \in \mathbb{Z}$ can be calculated as follows

$$\begin{aligned} R_{c_j}(l,m) = \mathcal{E}\{c_{j,l}\overline{c_{j,m}}\} &= \mathcal{E}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\phi}_{j,l}(u)\tilde{\phi}_{j,m}(v)x(u)x(v)dudv\right\} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\phi}_{j,l}(u)\tilde{\phi}_{j,m}(v)\mathcal{E}\{x(u)x(v)\}dudv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\phi}_{j,l}(u)\tilde{\phi}_{j,m}(v)R(u,v)dudv \end{aligned} \tag{29}$$

Note that equation (29) can be used to calculate the correlation structure of the coefficients across the scale by simply changing the analysis functions accordingly.

If the random process $x(t,\omega)$ is assumed to be wide-sense stationary, that is, the covariance function $R(t,s)$ is a function of only the time difference

$$R(t,s) = R(t-s) \tag{30}$$

after a change of variable equation (29) can be written as

$$\begin{aligned} R_{c_j}(l,m) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\phi}_{j,l}(\tau+v)\tilde{\phi}_{j,m}(v)R(\tau)d\tau dv \\ &= \int_{-\infty}^{\infty} R(\tau)d\tau \int_{-\infty}^{\infty} \tilde{\phi}_{j,l}(\tau+v)\tilde{\phi}_{j,m}(v)dv \\ &= \int_{-\infty}^{\infty} R(\tau)\gamma_{\tilde{\phi}_{j,l},\tilde{\phi}_{j,m}}(\tau)d\tau \end{aligned} \tag{31}$$

where $\tau = u - v$ and $\gamma_{\tilde{\phi}_{j,l},\tilde{\phi}_{j,m}}(\tau)$ is the correlation function between $\tilde{\phi}_{j,l}$ and $\tilde{\phi}_{j,m}$. Since $\gamma_{\tilde{\phi}_{j,l},\tilde{\phi}_{j,m}}(\tau)$ depends only of the difference $(l-m)$ (the functions $\tilde{\phi}_{j,l}(t)$ can be obtained from $\tilde{\phi}_{j,m}(t)$ by a translation with $(l-m)$) it follows that $R_{c_j}(l,m)$ depends only on $(l-k)$ and therefore the discrete-time process $c_j(k,\omega)$ is also a wide-sense stationary process.

The covariance function $R_{d_j}(l,m)$ of $d_{j,k}$ can be calculated in a similar manner. In particular the variance of the scaling and wavelet coefficients at every scale is

$$R_{c_j}(k,k) = \int_{-\infty}^{\infty} R(\tau)\gamma_{\tilde{\phi}_{j,k},\tilde{\phi}_{j,k}}(\tau)d\tau$$
$$R_{d_j}(k,k) = \int_{-\infty}^{\infty} R(\tau)\gamma_{\tilde{\psi}_{j,k},\tilde{\psi}_{j,k}}(\tau)d\tau \tag{32}$$

where $\gamma_{\tilde{\phi}_{j,k},\tilde{\phi}_{j,k}}$ and $\gamma_{\tilde{\psi}_{j,k},\tilde{\psi}_{j,k}}$ are the autocorrelation functions of $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$ which are independent of k.

A special case of a wide-sense stationary process, which is extremely useful in practical applications, is the *white noise* process. By definition, the spectral-density function of a white noise process is given by

$$S(\nu) = S_0 \tag{33}$$

for all ν where $S(\nu)$ is defined as

$$S(\nu) = \int_{-\infty}^{\infty} e^{-i2\pi\nu\tau}R(\tau)d\tau \tag{34}$$

Although in this case

$$\int_{-\infty}^{\infty} S(\nu)d\nu = R(0) = \infty \tag{35}$$

indicates that a white noise is not a second order process, heuristically, equation (35) suggests that $R(\tau)$ must be given by

$$R(\tau) = \delta(\tau)S_0 \tag{36}$$

where $\delta(\tau)$ is the Dirac δ function. In practice [11], one deals with random processes $x(t,\omega)$ that are approximately white noise and for which the integral

$$\int_{-\infty}^{\infty} f(t)x(t)dt < \infty \tag{37}$$

is convergent for every square integrable function f.

In what follows some specific properties of wavelet representations of white noise will be derived. If $x(t,\omega)$ is white noise then the covariance function $R(\tau) = S_0\delta(\tau)$ can be substituted in equation (37) to give

$$R_{c_j}(l-m) = \int_{-\infty}^{\infty} S_0\delta(\tau)\gamma_{\tilde{\phi}_{j,l},\tilde{\phi}_{j,m}}(\tau)d\tau$$
$$R_{d_j}(l-m) = \int_{-\infty}^{\infty} S_0\delta(\tau)\gamma_{\tilde{\psi}_{j,l},\tilde{\psi}_{j,m}}(\tau)d\tau \tag{38}$$

Using the 'sifting' property of the delta function

$$\int_{-\infty}^{\infty} \delta(\tau) f(\tau) d\tau = f(0) \tag{39}$$

the integrals in (38) can be evaluated as

$$R_{c_j}(l-m) = S_0 \gamma_{\tilde{\phi}_{j,l},\tilde{\phi}_{j,m}}(0)$$
$$R_{d_j}(l-m) = S_0 \gamma_{\tilde{\psi}_{j,l},\tilde{\psi}_{j,m}}(0) \tag{40}$$

In particular the variances of the scaling and wavelet coefficients are

$$R_{c_j}(0) = S_0 \gamma_{\tilde{\phi}_{j,k},\tilde{\phi}_{j,k}}(0) = S_0 \|\tilde{\phi}_{j,k}\|_2^2$$
$$R_{d_j}(0) = S_0 \gamma_{\tilde{\psi}_{j,k},\tilde{\psi}_{j,k}}(0) = S_0 \|\tilde{\psi}_{j,k}\|_2^2 \tag{41}$$

This result motivates the following proposition:

Proposition 2.1 *The variance $R_{c_j}(0)$ and $R_{d_j}(0)$ of the discrete-time random processes $c_j(k,\omega)$ and $d_j(k,\omega)$ defined by white noise integrals involving the dual scaling and wavelet basis functions $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$ of a multiresolution approximation are scale (j) invariant.*

Proof: Since the dual scaling and wavelet functions $\tilde{\phi}$ and $\tilde{\psi}$ generate a multiresolution decomposition it follows that $\|\tilde{\phi}_{j,k}\|_2^2 = \|\tilde{\phi}\|_2^2$ and similarly $\|\tilde{\psi}_{j,k}\|_2^2 = \|\tilde{\psi}\|_2^2$ for all $j,k \in \mathbb{Z}$ hence the proposition is proven.

If white noise is superimposed over a bandlimited deterministic signal, this property could be used as a practical means to detect the resolution (wavelet) subspaces in the wavelet multiresolution decomposition, which mainly account for the noise.

In particular, for orthonormal multiresolution decompositions $\|\phi\|_2 = \|\tilde{\phi}\|_2 = 1$ and $\|\psi\|_2 = \|\tilde{\psi}\|_2 = 1$. Hence $R_{c_j}(0) = S_0 = R(0)$ and $R_{d_j}(0) = S_0 = R(0)$ i.e. the transformation preserves the variance. Moreover, since in this case $R_{c_j}(l-m) = \delta_{l,m} S_0$ and $R_{c_j}(l-m) = \delta_{l,m} S_0$, where $\delta_{j,k}$ is the Kronecker symbol defined on $\mathbb{Z} \times \mathbb{Z}$, it follows that c_j and d_j define at every scale a discrete white noise process.

3 Simulation Results

The theoretical results presented in the previous sections are illustrated here by means of a simple experiment. A white noise process with mean $\mu = 10$ and unit variance was represented as a orthonormal multiresolution approximation. The mean and variance of the resulting coefficients were calculated and compared with the theoretical results.

The white noise signal consisted of 2^{16} data samples. The projections of the signal over 9 wavelet subspaces W_j, $j = -1, \ldots, -9$ and 10 scaling subspaces V_j, $j = 0, -1, \ldots, -9$ were calculated using a pair of $6th$ order Daubechies filters. The mean and variance of the scaling and wavelet coefficients corresponding to each subspace, listed in Table 3, match very well with the theoretical findings.

Scale	$\mathcal{E}\{c_{j,k}\}$	$\mathcal{E}\{d_{j,k}\}$	$R_{c_j}(0)$	$R_{d_j}(0)$
j=0	10.000 $=\mu$	-	1.007	-
j=-1	14.142 $=2^{1/2}\mu$	0.001	1.005	1.003
j=-2	20.000 $=2^{1}\mu$	0.013	1.010	1.003
j=-3	28.284 $=2^{3/2}\mu$	0.010	0.991	1.029
j=-4	40.000 $=2^{4}\mu$	-0.005	0.982	1.001
j=-5	56.568 $=2^{5/2}\mu$	0.008	0.946	1.018
j=-6	80.000 $=2^{6}\mu$	-0.030	0.910	0.981
j=-7	113.137$=2^{7/2}\mu$	0.044	0.917	0.904
j=-8	160.000$=2^{8}\mu$	0.080	0.936	0.895
j=-9	226.274$=2^{9/2}\mu$	-0.023	0.880	0.997

Table 3

4 Summary

The convergence of general, non-orthogonal wavelet multiresolution approximations of random processes has been investigated. The convergence results, introduced by Genossar *et al* (1991) for orthonormal wavelet bases with compact support, have been extended to less restrictive multiresolution decompositions.

The second order properties of the stochastic coefficients associated with a wavelet multiresolution decomposition of a second order random processes, namely the mean and covariance function, have been derived as functions of the mean and covariance function of the original second order random process. It has been shown that the variance of the coefficients of a white noise process is invariant across the scale. In particular, in case the MRA is orthonormal the corresponding coefficients at every scale are also white and have the same variance as the original white noise process. Simulation results were provided to illustrate the theoretical findings.

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