Monograph:

Gomma, H.W. and Owens, D.H. (2000) Generalized Predictive Control: Step Responses and Stability Thoery. Research Report. ACSE Research Report 773 . Department of Automatic Control and Systems Engineering

RESEARCH REPORT No. 773

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July 2000

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Generalized Predictive Control: step responses and stability theory

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Abstract

This paper presents new results that use step response data to produce sufficient conditions to guarantee the stability of the closed loop system in Generalized Predictive Control (GPC). The analysis produces easily checked conditions that provide considerable insight into the effect of parameters such as prediction horizons and control weightings on stability characteristics.

1.0 Introduction

Stability is not only important in the design of dynamic systems, but also in the design of all control systems. Although most people have an intuitive feeling as to what stability means, the concept is very subtle and rigorous definitions are necessary. In this paper, the stability of Generalized Predictive Control (GPC) is considered by obtaining the sufficient conditions for the roots of the closed-loop characteristic equation to lie inside the unit circle.

In spite of the fact that extensive research has been conducted into the GPC technique, there is no clear theory to guarantee the closed loop stability in terms of GPC tuning parameters. Since the first appearance of the GPC algorithm in 1987 few researchers have tackled this problem. The first trial was made by Clarke et al (1987), where the stability problem was approximated, under certain conditions, to the state space LQ controller. Later, using results in state space theory of Kwon and Pearson (1975), some stability results were presented (Clarke and Scattolini 1991 and Mosca and Zhang 1992). However, the most significant contribution can be ascribed to De Nicolao and Scattolini (1994), who introduced a clear representation for the closed loop system in terms of the impulse response coefficients. Later, Yoon (1994) used this representation and introduced some results under certain conditions for the stability of GPC. On the other hand, Zhang (1998) used the same principle with a different approach and referred to it: as an explicit closed loop description.

Despite the above, the stability issue and the mystery behind the GPC and how it works still need further research and analysis. This paper will deal with the GPC stability issue and try to introduce new results and explain in more detail many relative aspects which have been avoided by most researchers. Moreover, the results will reveal some facts which have been left without clear explanation.

2.0 Modelling of the system

The GPC approach is applicable to both single-input/single-output (SISO) and multi-input/multi-output (MIMO) systems. In general, non-linear models can frequently be linearised around a particular operating point and described by

$$A(z^{-1})y(t) = B(z^{-1})u(t-1) + C(z^{-1})\zeta(t) \qquad (1)$$

where $y(t)$ is the output. $u(t)$ is the control sequence, $\zeta(t)$ is the zero mean white noise. A, B and C are polynomials in the backward shift operator (z^{-1}).

$$A(z^{-1}) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots\cdots + a_{na} z^{-na}$$
$$B(z^{-1}) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots\cdots + b_{nb} z^{-nb}$$
$$C(z^{-1}) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots\cdots + c_{nc} z^{-nc}$$

This model is known as a CARMA (Controlled Auto-Regressive and Moving-Average) model. In industrial applications where the disturbances are non-stationary, an integral action is more appropriate (Clarke et al, 1987). This will lead to automatic steady state reference setpoint tracking despite the presence of unmodelled disturbances

$$A(z^{-1})y(t) = B(z^{-1})u(t-1) + C(z^{-1})\frac{\zeta(t)}{\Delta} \qquad (2)$$

where Δ is the differencing operator $1 - z^{-1}$. This model is known as CARIMA model (Controlled Auto-Regressive Integrated Moving-Average). For simplicity the C polynomial is chosen to be 1 or C^{-1} is truncated and absorbed into the A and B polynomials.

3.0 The optimal prediction

The main idea of GPC is to find a control sequence to minimise the multistage cost function of the form:

$$J(N_1, N_2, N_u) = E\left\{ \sum_{j=N_1}^{N_2} \delta(j)[\hat{y}(t+j\,|\,t) - w(t+j)]^2 + \sum_{j=1}^{N_u} \lambda(j)[\Delta u(t+j-1)]^2 \right\} \qquad (3)$$

where $\hat{y}(t+j|t)$ is the *j-step* ahead prediction of the system on data up to time *t*, $w(t+j)$ is the future reference trajectory, *E{.}* denotes the expectation operator and has been used to indicate that the control values chosen are calculated conditioned upon the data available up to and including time *t* and presuming the stochastic disturbance model. N_1, N_2 *and* N_u are the minimum costing horizon, maximum costing horizon and control horizon; $\delta(j)$ and $\lambda(j)$ are weighting function to penalise the error and the control sequence respectively.

To minimise the above function the future values of the output $\hat{y}(t+j)$ should be obtained by performing long division of 1 by $\Delta A(z^{-1})$. In fact, for long control horizon, an alternative method such as the recursion of the following *Diophantine* equation can be used:

$$1 = E_j(z^{-1})\tilde{A}(z^{-1}) + z^{-j}F_j(z^{-1}) \tag{4}$$

where $\tilde{A}(z^{-1}) = A(z^{-1})\Delta$. For a unique solution the degree of the polynomials E_j and F_j should be equal to $j-1$ and n_a respectively. From Equation (4), it is clear by dividing 1 by $\tilde{A}(z^{-1})$, the polynomial E_j is the quotient and the remainder is the factorisation of $z^{-j}F_j$. By multiplying each side in Equation (4) by $\Delta E_j(z^{-1})z^j$, it is easy to see that the prediction output could be written as (see: Clarke, 1987)

$$\hat{y}(t+j\,|\,t) = F_j(z^{-1})y(t) + G_j(z^{-1})\Delta u(t+j-1) \tag{5}$$

where $G_j(z^{-1}) = E_j(z^{-1})B(z^{-1})$.

For simplicity, $N_1 = 1, N_2 = N_u = N$, $\delta(j) = 1$ and $\lambda(j) = \lambda$. From Equation (5), the optimal output predictions could be stated as:

$$\begin{aligned} \hat{y}(t+1|t) &= G_1\Delta u(t) + F_1 y(t) \\ \hat{y}(t+2|t) &= G_2\Delta u(t+1) + F_2 y(t) \\ &\vdots \\ \hat{y}(t+N|t) &= G_N\Delta u(t+N-1) + F_N y(t) \end{aligned} \tag{6}$$

which can be written as:

$$\begin{aligned} \mathbf{y} &= \mathbf{Gu} + \mathbf{F}(z^{-1})y(t) + \mathbf{G'}(z^{-1})\Delta u(t-1) \\ \mathbf{y} &= \mathbf{Gu} + \mathbf{f} \end{aligned} \tag{7}$$

where **f** is the free response and **Gu** is the forced response where

$$\begin{aligned} \mathbf{y} &= [\hat{y}(t+1\,|\,t) \quad \hat{y}(t+2\,|\,t) \quad \cdots \quad \hat{y}(t+N\,|\,t)]^T \\ \mathbf{u} &= [\Delta u(t) \quad \Delta u(t+1) \quad \cdots \quad \Delta u(t+N-1)]^T \end{aligned} \tag{8}$$

$$G=\begin{bmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ g_{N-1} & g_{N-2} & \cdots & g_0 \end{bmatrix}, \mathbf{G}'(z^{-1})=\begin{bmatrix} (G_1(z^{-1})-g_0)z \\ (G_2(z^{-1})-g_0-g_1z^{-1})z^2 \\ \cdot \\ (G_N(z^{-1})-g_0-g_1z^{-1}-\cdots-g_{N-1}z^{-(N-1)})z^N \end{bmatrix}$$

$$F(z^{-1})=\begin{bmatrix} F_1(z^{-1}) \\ F_2(z^{-1}) \\ \vdots \\ F_N(z^{-1}) \end{bmatrix}$$

Now the cost function in Equation (3) can be written as:

$$\boldsymbol{J} = (\mathbf{Gu}+\mathbf{f}-\mathbf{w})^{\mathrm{T}}(\mathbf{Gu}+\mathbf{f}-\mathbf{w})+\lambda\,\mathbf{u}^{\mathrm{T}}\mathbf{u} \tag{9}$$

where

$$\mathbf{w} = [w(t+1) \quad w(t+2) \quad \cdots \quad w(t+N)]^T \tag{10}$$

To minimise the $\boldsymbol{J}$, assuming that there are no constraints in the future so:

$$\mathbf{u} = -(\mathbf{G}^{\mathrm{T}}\mathbf{G}+\lambda)^{-1}\mathbf{G}^{\mathrm{T}}(\mathbf{f}-\mathbf{w}) \tag{11}$$

The first element, $\Delta u(t)$, of the matrix **u**, will be applied to the system and will be repeated at every sampling period. In non-adaptive design with a time invariant model, this leads to a time invariant controller. In general, to reduce the computation needed, it is assumed that the control signals will be constant after the control horizon. Many algorithms have been presented to minimise the computation effort by using the Neural networks (Quero et al 1990) and the reaction curve for process modelling (Camacho et al 1995). The stability results of the above control law are summarised in the following sections.

4.0 Stability with finite maximum prediction horizon

The first theorem presents a description of the closed loop system in terms of the step response coefficients, which will be the key element in this paper. However, the main objective of the theorem is to study a popular case in GPC where the control horizon (N_u) is set to one. Under this condition, the control law will be unique and can be evaluated with no matrix inversion.

First, it should be mentioned that the step response is one of the simplest mechanisms which can be used to predict the output of a process. In this paper this mechanism will be used intensively through the following results. The predicted output is related to the input by the equation

$$\hat{y}(t+k \mid t) = \sum_{i=1}^{\infty} g_j \Delta u(t+k-j) \tag{12}$$

where g_i are the sampled output values for the step input and Δ is the differencing operator ($\Delta = 1 - z^{-1}$) with z^{-1} the backward shift operator.

Theorem 1

Assume that the open loop system is stable and $N_u = 1$, $N_1 = 1$, $N_2 = N$ and $\lambda(i) = \lambda$. Assume also that the step response coefficients $\{g_i\}_{i=1,2,..}$ satisfy

$$0 \le g_1 \le g_2 \le \cdots g_k \le \cdots \le g_\infty, \quad \sum_{i=1}^{N} g_i (g_{i+1} - g_i) \ge \lambda \tag{13}$$

and that they satisfy the inequality

$$\sum_{i=1}^{N} g_i^2 + 2\lambda > \sum_{i=1}^{N} g_i (g_\infty - g_i) \tag{14}$$

Then the closed loop system is stable.

Proof

By using Equations (11) and (12), the control law can be written as

$$\Delta u(t) = -\frac{\sum_{i=1}^{N} g_i \left(f(t+i) - w(t+i)\right)}{\left(\sum_{i=1}^{N} g_i^2 + \lambda\right)} \tag{15}$$

but from the above

$$\begin{aligned}\sum_{i=1}^{N} g_i \left(f(t+i)\right) &= g_1 f(t+1) + g_2 f(t+2) + \cdots g_N f(t+N) \\ &= g_1 \sum_{i=2}^{\infty} g_i \Delta u(t+1-i) + g_2 \sum_{i=3}^{\infty} g_i \Delta u(t+2-i) + \cdots g_N \sum_{i=N+1}^{\infty} g_i \Delta u(t+N-i)\end{aligned} \tag{16}$$

From the control law it is clear that

$$\Delta u(t) = \frac{\sum_{i=1}^{N} g_i \omega(t+i)}{(\sum_{i=1}^{N} g_i^2 + \lambda) + g_1 \sum_{i=2}^{\infty} g_i z^{1-i} + g_2 \sum_{i=3}^{\infty} g_i z^{2-i} + \cdots g_N \sum_{i=N+1}^{\infty} g_i z^{N-i}} \qquad (17)$$

For simplicity, let the reference be constant over the prediction horizon N i.e. $\omega(t+1) = \omega(t+2) = \cdots \omega(t+N) = \omega$. By substituting Equation (17) into the system model

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t) \qquad (18)$$

this leads to

$$y(t) = \frac{B \sum_{i=1}^{N} g_i \omega}{A\Delta \left\{ (\sum_{i=1}^{N} g_i^2 + \lambda) + g_1 \sum_{i=2}^{\infty} g_i z^{1-i} + g_2 \sum_{i=3}^{\infty} g_i z^{2-i} + \cdots g_N \sum_{i=N+1}^{\infty} g_i z^{N-i} \right\}} \qquad (19)$$

where the characteristic polynomial can be written in the following form

$$H = A\Delta \left\{ (\sum_{i=1}^{N} g_i^2 + \lambda) + g_1 \sum_{i=2}^{\infty} g_i z^{1-i} + g_2 \sum_{i=3}^{\infty} g_i z^{2-i} + \cdots g_N \sum_{i=N+1}^{\infty} g_i z^{N-i} \right\}$$

$$\frac{H}{A} = \Delta \left\{ (\sum_{i=1}^{N} g_i^2 + \lambda) + (g_1 g_2 + g_2 g_3 + \cdots g_N g_{N+1}) z^{-1} + (g_1 g_3 + g_2 g_4 + \cdots g_N g_{N+2}) z^{-2} \cdots \right\}$$

$$\frac{H}{A} = (\sum_{i=1}^{N} g_i^2 + \lambda) + \left(\sum_{i=1}^{N} g_i (g_{i+1} - g_i) - \lambda \right) z^{-1} + \sum_{i=1}^{N} g_i (g_{i+2} - g_{i+1}) z^{-2} + \cdots\cdots \qquad (20)$$

From Rouche's theorem (Spiegel, 1964), as A is stable, for all roots to lie in the unit circle it is sufficient that

$$\left| \sum_{i=1}^{N} g_i^2 + \lambda \right| > \left| \sum_{i=1}^{N} g_i (g_{i+1} - g_i) - \lambda \right| + \left| \sum_{i=1}^{N} g_i (g_{i+2} - g_{i+1}) \right| + \cdots\cdots \qquad (21)$$

By assumption (Equation (13)), all elements are positive and hence the inequality holds if

$$\left| \sum_{i=1}^{N} g_i^2 + \lambda \right| > \left| \sum_{i=1}^{N} g_i (g_\infty - g_i) - \lambda \right| \qquad (22)$$

or

$$\sum_{i=1}^{N} g_i^2 + 2\lambda > \sum_{i=1}^{N} g_i (g_\infty - g_i) \tag{23}$$

which completes the proof. □

Remark: From Equation (23) it can be seen that λ is a very influential parameter in the system stability, where increasing λ by a reasonable amount to satisfy the conditions stated above (Equations (13) and (23)) can lead to the stability of the closed loop system. Moreover, this theorem is essential in giving an interpretation for the effect of λ on the system performance/robustness. From Equation (15), it is clear that the gain of the control law increases as λ decreases, which in turn improves the performance and diminishes the robustness. However, it is still worth mentioning that, from Equation (13), the maximum value of λ which can be used in the theorem is bounded by

$$\begin{aligned} \lambda_{\max} = \sum_{i=1}^{N} g_i (g_{i+1} - g_i) &< \sum_{i=1}^{N} g_\infty (g_{i+1} - g_i) \\ &< g_\infty (g_{N+1} - g_1) \\ &< g_\infty (g_\infty - g_1) \end{aligned} \tag{24}$$

It is clear that increasing the maximum prediction horizon can allow larger values of λ to be used with guaranteed stability. However, the effect of an infinite maximum prediction horizon will be investigated in more detail, in the following section.

5.0 Stability with infinite maximum prediction horizon

It is clear that the previous results were looking at a special case where the maximum prediction horizon is finite. It is worth mentioning that since the use of an infinite costing horizon has been advocated in the context of GPC leading to GPC^∞ (Clarke et al 1987, Bitmead et al 1990 and Scokaert 1994); it has been criticized because it was not considered as a practical optimization problem. A few years later, Scokaert (1997) used the technique in the state-space controller (Muske and Rawlings (1993a, b)) as a practical implementation of GPC^∞. The first attempt to prove the stability under this condition was introduced by Clarke et al 1987 in which the state space GPC was approximated to state space LQ controller. Later, Scokaert et al (1994) introduced different approach by invoking the monotonicity of the receding horizon cost function, with respect to time. It is obvious that both analyses did not describe the representation of the closed loop system. Thus, in this section the effects of setting the prediction horizon to infinity will be considered and new stability result will be presented using the same approach which is used above.

Theorem 2

Assume the open loop system is stable and $N_u = 1$, $N_1 = 1$, $N_2 = N$ and $\lambda(i) = \lambda$. Assume also that the open-loop system's step response coefficients satisfy

$$0 < g_1 < g_2 < \cdots g_k < \cdots < g_\infty, \quad \sum_{i=1}^{N} g_i(g_{i+1} - g_i) > \lambda \tag{25}$$

and that $\sum_{i=1}^{\infty} g_i(g_\infty - g_i)$ converges. Then the closed loop system is stable for all sufficiently large values of *N*.

Proof

As in Theorem 1, for all roots to lie inside the unit circle it is sufficient that

$$\left|\sum_{i=1}^{N} g_i^2 + \lambda\right| > \left|\sum_{i=1}^{N} g_i(g_{i+1} - g_i) - \lambda\right| + \left|\sum_{i=1}^{N} g_i(g_{i+2} - g_{i+1})\right| + \cdots\cdots \tag{26}$$

By the above assumption (Equation 25), all elements are positive and hence the inequality holds if

$$\left|\sum_{i=1}^{N} g_i^2 + \lambda\right| > \left|\sum_{i=1}^{N} g_i(g_\infty - g_i) - \lambda\right| \tag{27}$$

or

$$\sum_{i=1}^{N} g_i^2 + 2\lambda > \sum_{i=1}^{N} g_i(g_\infty - g_i) \tag{28}$$

The left-hand-side is unbounded as $N \to \infty$ while the right-hand-side is finite by assumption. The inequality is achieved foe all large enough values of N. This completes the proof of the result. □

Remark: Theorem 2 has a substantial contribution to the case where $N \to \infty$, due to the explicit representation of the closed loop system in terms of the step response coefficients.

From the above, Theorems 1 and 2 have given sufficient conditions for the stability when the minimum of the prediction horizon is finite. In the following section a new case will be addressed, in which the minimum of the prediction horizon has large values.

6.0 Large values of minimum and maximum prediction horizon

In this section the idea of designing a controller with a very large minimum prediction horizon will be considered. This case was first presented by Yoon et al (1995); however, it cannot be considered as a precise result due to the approximations which were used in developing the proof. On the other hand, it did not present the effect of selecting λ greater than zero. Thus, the next theorem will shed some light on the effect of setting the minimum prediction horizon to a relatively large value.

Theorem 3

Suppose that the open-loop system is stable, $N_2 > N_1$, $N_u = 1$, $\lambda > 0$ and that $N_2 - N_1$ is kept constant. Suppose also that the step response coefficients have the property that

$$0 \le g_{N_1} \le \cdots \le g_{N_2} \le \cdots \le g_\infty \tag{29}$$

Then the closed loop system is stable for all large enough valued of N_1.

Proof

The control law can be represented as:

$$\Delta u(t) = -\frac{\sum_{i=N_1}^{N_2} g_i \left(f(t+i) - w(t+i)\right)}{M_1} \tag{30}$$

$$\text{where } M_1 = \sum_{i=N_1}^{N_2} g_i^2 + \lambda \tag{31}$$

Equation (30) can be written in the following form

$$\Delta u(t) + \frac{1}{M_1}[g_{N_1} f(t+N_1) + g_{N_1+1} f(t+N_1+1) + \cdots g_{N_2} f(t+N_2)] = \frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i \omega$$

$$\Delta u(t)\{1 + \frac{1}{M_1}[g_{N_1} \sum_{i=N_1+1}^{\infty} g_i z^{N_1-i} + g_{N_1+1} \sum_{i=N_1+2}^{\infty} g_i z^{N_1+1-i} + \cdots g_{N_2} \sum_{i=N_2+1}^{\infty} g_i z^{N_2-i}]\} = \frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i \omega \tag{32}$$

Substituting Equation (32) into Equation (18) leads to the following characteristic polynomial

$$\frac{H}{A}=1+\left[(\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i g_{i+1})-1\right]z^{-1}+\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i(g_{i+2}-g_{i+1})z^{-2}+\cdots \tag{33}$$

Similar to the above theorems, for stability, it is sufficient that

$$1>\left|(\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i g_{i+1})-1\right|+\left|\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i(g_{i+2}-g_{i+1})\right|\cdots \tag{34}$$

$$\text{but } \frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i g_{i+1}=\frac{\sum_{i=N_1}^{N_2} g_i g_{i+1}}{\sum_{i=N_1}^{N_2} g_i^2+\lambda}<1 \tag{35}$$

for N_1 large and N_2-N_1 constant.

From the assumption $g_{i+1}>g_i$, accordingly, Equation (34) reduces to

$$1>1-(\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i g_{i+1})+\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i(g_{i+2}-g_{i+1})\cdots \tag{36}$$

This is true if

$$0>-(\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i g_{i+1})+\frac{1}{M_1}\sum_{i=N_1}^{N_2} g_i(g_\infty-g_{i+1})\cdots \tag{37}$$

or

$$g_\infty\sum_{i=N_1}^{N_2} g_i>2\sum_{i=N_1}^{N_2} g_i g_{i+1} \tag{38}$$

which is true for large N_1 and N_2-N_1 constant. Therefore the system is stable and this completes the proof. □

Remark: The above theorem provides new useful information on the stability of the closed loop system in the presence of the previous conditions. The advantage of this theorem is that it has shown that selecting $\lambda>0$, for large values of N_1, still leads to stable closed loop system. On the other hand, it differs from the one developed by Yoon et al (1995) in which the control strategy of a stable open loop system was considered to tend towards a mean-level law. To reach this conclusion, Yoon et al (1995) ignored the coefficients of the higher order of the shift operator (z^{-1}) (see: Equation 33), they considered the differences between g_{i+k+1} and g_{i+k}, in the summation $\sum_{i=N_1}^{N_2} g_i(g_{i+k+1} - g_{i+k})$ goes to zero, when $N_1 \rightarrow \infty$, which means ignoring any residual errors could appear. Therefore to avoid this approximation and to prove the closed loop stability, in Theorem 3, Rouche's theorem has been used.

In general, Theorems 1, 2 and 3 support the idea of setting the control horizon N_u to one when controlling a stable system and they suggest the possibility of saving computation time and achieving acceptable performance.

7.0 The effect of prediction horizons on the closed loop poles' location

Apart from the study presented by Lim et al (1998), the authors have not seen anywhere in the literature an attempt to create a link between the GPC parameters and the pole locations of the closed-loop system. The proposed solution (Lim et al 1998) used the same approach as the one that was developed by Hang et al (1991) for minimum variance controller, which involved the use of bilinear transformation to restricts the closed loop poles to a certain area. However, the effect of the minimum/maximum prediction horizons (N_1 and N_2) on the pole location has not been investigated yet in the literature. The next theorem provides a very interesting result about the location of the closed loop poles, using new approach, which in turn will give an indication about the system performance.

Theorem 4

Suppose that the open loop system is stable, $\lambda = 0$ and that the closed loop polynomial is written as (see: Theorem 1)

$$\frac{H}{A} = \sum_{i=N_1}^{N_2} g_i^2 + \sum_{i=N_1}^{N_2} g_i(g_{i+1} - g_i)z^{-1} + \sum_{i=N_1}^{N_2} g_i(g_{i+2} - g_{i+1})z^{-2} + \cdots \tag{39}$$

or as

$$\xi(z) = a_0 + \sum_{j=1}^{\infty} a_j z^{-j} , \tag{40}$$

where

$$a_0 = \sum_{i=N_1}^{N_2} g_i^2 \text{ and } a_j = \sum_{i=N_1}^{N_2} g_i (g_{i+j} - g_{i+j-1})$$

Suppose that the open-loop system's step response has the monotonic property i.e. $0 \le g_k \le g_{k+1} \le \cdots \le g_\infty$ and that there exists a number r such that $(\frac{a_{j+1}}{a_j}) < r < 1$ for all j. Regarding N_1 as a variable and assuming that N_2 is also varied so that $N_2 - N_1$ remains constant, then for all $\varepsilon \in (r, 1)$ there exists $k^* \ge k$ such that any $N_1 \ge k^*$ and $\lambda = 0$ results in closed loop stability and all poles of the closed loop system lie inside a circle $|z| < \varepsilon$.

Proof

Let $a_0 = \sum_{i=N_1}^{N_2} g_i^2$, $a_j = \sum_{i=N_1}^{N_2} g_i (g_{i+j} - g_{i+j-1})$. As $a_0 > 0$, $a_j > 0$ and $(\frac{a_{j+1}}{a_j}) < r$, then a sufficient condition for no zeros in $|z| \ge \varepsilon$, is that

$$a_0 > \sum_{j=1}^{\infty} a_j \varepsilon^{-j} \tag{41}$$

which holds if

$$a_0 > a_1 \sum_{j=1}^{\infty} r^{j-1} \varepsilon^{-j} \qquad \text{as } a_j = (\frac{a_j}{a_{j-1}}) \cdots (\frac{a_2}{a_1}) a_1 \tag{42}$$

$$\text{i.e. } a_0 > \varepsilon^{-1} \frac{a_1}{1 - \frac{r}{\varepsilon}} = \frac{a_1}{\varepsilon - r} \tag{43}$$

From the assumption $r < \varepsilon < 1$ and the above equation can be written in terms of step response coefficients as

$$(\varepsilon - r) \sum_{i=N_1}^{N_2} g_i^{\,2} > \sum_{i=N_1}^{N_2} g_i (g_{i+1} - g_i) \tag{44}$$

which is true if $g_\infty < \infty$, $N_2 - N_1$=constant and very large ($N_1 \to \infty$). Thus the closed loop system is stable and there are no zeros lying inside the circle $|z| \geq \varepsilon$, which completes the proof. □

Remark: From Theorem 4, it can be deduced that in the presence of the above conditions, the use of large values of N_1 and N_2 will move the closed poles to lie inside a circle with radius ε (less than one) which indicates faster response. The radius of this circle depends on the values of N_1 and N_2, which means for ε close to r, it is clear that N_1 could be large. However, for the poles to lie inside a bigger circle with radius ε (closer to one) which means slower response, smaller values of N_1 can be used to validate Equation (44). This suggests that increasing N_1 and N_2 moves the poles to lie inside a smaller circle closer to r and vice versa.

8.0 Stability with control horizon greater than or equal to one ($N_u \geq 1$)

The reader's attention is now brought to a common case in GPC, where the control horizon is chosen to have values greater than ($N_u \geq 1$). Although the increase in the value of N_u increases the amount of computation required, it is still preferable in most applications, as it provides better performance with high optimality to the GPC (see: Chapter 5). The following theorems present new sufficient conditions for stability with a special case of $N_u \geq 1$. Although the stability of this case has been tackled before (Clark et al 1987, Clarke and Mohtadi 1989, Kwon and Byun 1989), none of them have dealt with the problem through an explicit representation of the closed loop system. The stability was proved by observing that the predictive scheme in question tends to the steady-state LQ controller for which there is a stability guarantee or by using the monotonicity of the optimal cost function such as (Rawlings et al 1993 and Yoon et al 1995). It should be mentioned that the main difficulty of finding a clear expression for the closed loop system is the presence of the square matrix ($N_u \times N_u$) inversion. Thus, the following theorem will try to approach this problem through certain conditions.

Theorem 5

If $N_1 = 1$, $(N_2 - N_1 + 1) = N_u = N$ and $\lambda = 0$, then the closed loop is stable and the control strategy tends to a deadbeat law.

Proof

As the weighting function λ is selected to be equal to zero, the GPC control law can be written in the following form:

$$\mathbf{u} = -\underbrace{\left(\mathbf{G}^{\mathrm{T}}\mathbf{G}\right)^{-1}\mathbf{G}^{\mathrm{T}}}_{\mathbf{P}}(\mathbf{f} - \mathbf{w}) \tag{45}$$

As **G** is a square matrix then for all **G**:

$$\begin{aligned} \mathbf{P} &= (\mathbf{G}^{\mathrm{T}}\mathbf{G})^{-1}\mathbf{G}^{\mathrm{T}} \\ &= \mathbf{G}^{-1}(\mathbf{G}^{\mathrm{T}})^{-1}\mathbf{G}^{\mathrm{T}} = \mathbf{G}^{-1} \end{aligned} \tag{46}$$

where the matrix **G** is a square matrix consists of the plant's step responses (g_i) and it has the following form

$$\mathbf{G} = \begin{bmatrix} g_1 & 0 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & g_{N-1} & \cdots & g_1 \end{bmatrix}_{N\times N}$$

Thus, the control law can be given in the following form

$$g_1^N \Delta u(t) = -g_1^{N-1}\left(f(t+1) - w(t+1)\right) \tag{47}$$

Let the reference trajectory be constant over the prediction horizon, Equation (47) can be written as

$$g_1 \Delta u(t) = -\sum_{i=2}^{\infty} g_i \Delta u(t+1-i) + \omega \tag{48}$$

which can be written

$$\Delta u(t) = \frac{\omega}{g_1 + \sum_{i=2}^{\infty} g_i z^{1-i}} \tag{49}$$

Recalling the system transfer function

$$y(t) = \frac{B(z^{-1})}{A(z^{-1})} u(t) \tag{50}$$

Consequently, substituting Equation (49) into (50) leads to the following closed loop system

$$y(t) = \frac{B\omega}{A\Delta\left(g_1 + \sum_{i=2}^{\infty} g_i z^{1-i}\right)} \tag{51}$$

From the above, the characteristic polynomial can be written as

$$\begin{aligned} \frac{H}{A} &= \Delta\left\{g_1 + \sum_{i=2}^{\infty} g_i z^{1-i}\right\} = \Delta\{g_1 + g_2 z^{-1} + g_3 z^{-2} \cdots\} \\ \frac{H}{A} &= g_1 + (g_2 - g_1)z^{-1} + (g_3 - g_2)z^{-2} \cdots \end{aligned} \tag{52}$$

If the step response can be written as

$$y(t) = \sum_{i=1}^{\infty} g_i \Delta u(t-i) \tag{53}$$

Thus the relation between the step responses and plant transfer function can be written as

$$\frac{B(z^{-1})}{A(z^{-1})} = g_1 z^{-1} + (g_2 - g_1)z^{-2} + (g_3 - g_2)z^{-3} \cdots \tag{54}$$

But, from Equation (52), the closed loop characteristic polynomial can be written as

$$\frac{H}{A} = g_1 + (g_2 - g_1)z^{-1} + (g_3 - g_2)z^{-2} \cdots \tag{55}$$

By comparing Equations (54) and (55), it can be found that

$$\frac{H(z^{-1})z^{-1}}{A(z^{-1})} = \frac{B(z^{-1})}{A(z^{-1})} \tag{56}$$

Therefore it is clear that

$$H(z^{-1})z^{-1} = B(z^{-1}) \tag{57}$$

By recalling the closed loop transfer function Equation (51),

$$y(t) = \frac{B}{A\Delta\left(g_1 + \sum_{i=2}^{\infty} g_i z^{1-i}\right)}\omega(t) \tag{58}$$

and substituting from Equation (57)

$$y(t) = \frac{1}{z}\omega(t) \tag{59}$$

It is clear that the closed loop poles have moved to the origin and results in a stable dead beat control which completes the proof. □

Remark: The above theorem is believed to be of interest for GPC. It can be seen that in the presence of the above conditions, the control law results in a (stable) deadbeat control. Moreover, it provides a unique approach, which can guarantee the stability of an unstable open loop system which was discussed before by Yoon et al (1995) using the result of Rawlings et al (1993). The key element of those proofs was the observation of the monotonically decreasing optimal cost function. Neither of them sustained their proofs with a closed loop representation.

Note: Due to the zero-pole cancellation it should be mentioned that in real design a mismatch could occur which in turn will not result in a deadbeat closed loop controller. That is why new conditions which can guarantee stability, in this and similar circumstances should be considered.

Theorem 6

Assume that the open loop system is stable and has step response coefficients satisfying a convexity condition of the form

$$g_1 > (g_2 - g_1) > (g_3 - g_2) \cdots > 0 \tag{60}$$

Then if $N_1 = 1$, $N_u = (N_2 - N_1 + 1) = N$ and $\lambda = 0$ the closed loop system is stable.

Proof

Following the proof of Theorem 5, from Equation (55), the characteristic polynomial can be written as:

$$\frac{H}{A} = g_1 + (g_2 - g_1)z^{-1} + (g_3 - g_2)z^{-2} \cdots \tag{61}$$

Thus, from the sufficient condition in Zhang et al (1998) (see the Appendix), the closed loop system will be stable if:

$$g_1 > (g_2 - g_1) > (g_3 - g_2) \cdots > 0 \tag{62}$$

Which completes the proof. □

Remark: It is important to note that many systems such as the gas turbines (Gomma, 1999) have a convex step response. Accordingly, the significant of the above theorems (Theorems 5 and 6) that they have a great importance in GPC stability as they show new sufficient condition where the system can be stable when $(N_2 - N_1 + 1) = N_u = N \geq 1$. In addition, they are applicable to many engines such as gas turbines (Gomma 1999) and many other systems. Furthermore, it should be noticed that when the above theorems are applied to systems with time delay (d), the minimum prediction horizon (N_1) can be selected such as $N_1 = d + 1$.

9.0 Illustrative examples

The previous sections have provided a range of stability tests for GPC. This section considers the application of the ideas to some simple illustrative examples.

Note: in each figure of this section, the dotted line and solid lines are the set point and output, respectively.

Example 1

Consider the following second order system which can be written as

$$\frac{B(z^{-1})}{A(z^{-1})} = \frac{0.0132z^{-1} + 0.0115z^{-2}}{1 - 1.6457z^{-1} + 0.6703z^{-2}} \tag{63}$$

The open loop poles and zeros are located as following:

Open loop zeros	*Open loop poles*
$z = -0.8333$	$p_1 = 0.7405,\ p_2 = 0.9052$ and

the step response coefficients can be written as

$g_1 = 0.0132,\ g_2 = 0.0463,\ g_3 = 0.921, \cdots g_\infty = 1$

It should noted that in the above the step response coefficients achieve the conditions which have been stated in Theorem 1. The values (1, 8, 1) are chosen for the horizons

(N_1, N_2, N_u) and the simulation results are shown in Figures 1 (a and b) for two different values of the weighting function $\lambda = 0$ and $\lambda = 0.8$.

Figure 1 (a) indicates that decreasing λ i.e. decreasing the weighting on the control signal leads to better performance (attributed to the increase of the control gain). Also, as hinted in Theorem 6.1, using control horizon equal to unity with systems with monotonically increasing step response, yields closed loop stability. Similarly, Figure 1 (b) shows, as mentioned in Theorem 1, that the increase of the weighing function λ by a reasonable amount (see: Equation 6.21) is still maintains system stability.

Example 2

In this case the main objective is to show the effect of choosing large values of maximum prediction (Theorem 2) on the system stability. The process under control is a continuous process sampled with a time constant of approximately 1.7 secs and a sampling time of 0.1 sec giving the transfer function

$$\frac{B(z^{-1})}{A(z^{-1})} = \frac{0.0125z^{-1} + 0.0104z^{-2}}{1 - 1.5483z^{-1} + 0.5712z^{-2}} \qquad (64)$$

The open loop poles and zeros are located as following:

Open loop zeros	*Open loop poles*
$z = -0.832$	$p_1 = 0.6065,\ p_2 = 0.9418$

and the *step response coefficients* can be written as

$g_1 = 0.0125,\ g_2 = 0.0423,\ g_3 = 0.0812, \cdots g_\infty = 1$

This data satisfies the conditions of the theorem. In order to examine the effect of large maximum prediction horizon, the values $(1, 70, 1)$ are chosen for the horizons (N_1, N_2, N_u). Figures 2 (a and b) show the simulation values of $\lambda = 0$ and $\lambda = 2$. It is clear that both controllers are able to stabilise the system, with very similar responses. This happens because the increase of λ has a very minor effect on the system performance when large values of N_2 are used.

Example 3

Consider the full gas turbine model with inner loop controller P=0.3 (Gomma 1999) which can be written in the following form

$$\frac{B(z^{-1})}{A(z^{-1})}=\frac{(0.1488-0.09971z^{-1}-0.0195z^{-2})z^{-1}}{1-0.6583z^{-1}-0.2977z^{-2}-0.0268z^{-3}} \qquad (65)$$

where the open loop poles and zeros are located as following

Open loop zeros

$z_1=0.8281, z_2=-0.1582$

Open loop poles

$p_{1,2}=-0.1645\pm j0.0103, p_3=0.9874$

and *the step response coefficients* can be written as

$g_1=0.1488,\ g_2=0.1471,\ g_3=0.1708,\cdots g_\infty=1.7311$

From the above, it is clear that the open loop system is minimum phase stable system. Let $N_2-N_1+1=N_u=N$, then, by assigning the following values (0, 6) to the parameters (λ, N), Theorem (5) indicates that this will result in closed loop stability. Furthermore, the closed loop poles should move to the location of the open loop numerator, which is found to be as:

Closed loop poles

$p_{c1}=0$

Figure 3 shows the step response for the closed loop system. As expected from Theorem (5), the system behaves as dead beat control with a sampling time $T_s=0.1$sec.

10.0 Conclusion

In this paper various stability results for GPC algorithms are proved using step response data and analyses of the closed loop characteristic polynomial. In contrast to most existing theorems, which relied on state space representation or the monotonicity decrease of the receding cost function, this paper introduced new theorems using an explicit representation for the closed loop system. These results cover many sufficient conditions for stability which can be considered as design guidelines for plants with monotonic or convex step responses.

The analyses are based on representing the closed loop system in terms of the step response coefficients. In addition a combination between this technique and the Lemma developed by Zhang (1998), led to a very interesting theorem which has shown a condition where the GPC can lead to dead-beat control law which can stabilize unstable systems. Furthermore, the effect of prediction horizons in pole location has been studied for the first time. Moreover, some examples have been given to support the above results. At the end, it should be mentioned that the complexity of developing a general stability theorem for GPC algorithm could be attributed to the complexity of the square inversion matrix which involves all system's parameters in a nonlinear way.

Appendix:

The following condition was based on Jury's table, which was presented in a form to be similar to the Routh-Hurwitz table for the stability in the left half of the *s*-plane (see: Jury 1964). It can be proven for the following polynomial

$$F(z^{-1}) = 1 + f_1 z^{-1} + f_2 z^{-2} + \cdots f_n z^{-n} \tag{6.61}$$

that if

$$1 > f_1 > f_2 \cdots > f_n > 0 \tag{6.62}$$

then $f(z^{-1})$ has no zeros when $|z| \geq 1$, i.e. $f(z^{-1})$ is stable. Later, Zhang and Xi (1998) have shown that by using Hurwitz's Theorem in complex analysis, given

$$L(z^{-1}) = 1 + L_1 z^{-1} + L_2 z^{-2} + \cdots \tag{6.63}$$

if

$$1 > L_1 > L_n > \cdots > 0 \tag{6.64}$$

then $L(z^{-1})$ will never equal to zero when $|z| \geq 1$, i.e. $L(z^{-1})$ is stable. It is clear that more stability results can be obtained by the using the above conditions.
These conditions are given in the following theorems.

References

Bitmead, R.R., Gevers, M., and Wertz, V., "Adaptive optimal control", Prenctice Hall International, 1990.

Camacho, E.F. and Bodons, C. "Model predictive control in the process industry: Advances in industrial control", Springer-Verlag, 1995.

Clarke, D.W. and Scattolini, R., "Constrained receding horizon predictive control", Proc. IEE-D, Vol. 138, No.4, pp.347-354, 1991.

Clarke, D.W., Mohtadi C., and Tuffs P.S., "Generalized predictive control-part I and II", Automatica, Vol. 23, No.2, pp.137-160, 1987.

De Nicolao., G and Scattolini, R., "Advances in model predictive control, chapter stability and output terminal constraints in predictive control", Oxford University Press, 1994.

Gomma H.W., "Robust and predictive control of 1.5 MW gas turbine engine". PhD Thesis, Exeter University, UK.

Kwon, W.H. and Pearson A.E., "On the stabilisation of a discrete constant linear system", IEEE Transactions on Automatic Control, Vol. 20, No. 6, pp. 800-801, 1975.

Lim K.W., Ho, W.K., Lee, T.H., Ling, K.V. and Xu, W., "Generalized predictive controller with pole restriction", IEE Proc. Control Theory Appl., Vol. 145, No. 2, pp. 219-225, 1998.

Mosca, E. and Zhang, J., "Stable receding of predictive control", Automatica, Vol.28, No.6, pp.1229-1233, 1992.

Muske, K. R., and Rawlings, J. B., "Model predictive control with linear models," AIChE Journal, Vol.39, No.2, pp.262-287, 1993 (b).

Muske, K.R., and Rawlings, J. B., "Linear model predictive control of unstable process", Journal Proceedings of Control, Vol.3, pp.85-95, 1993 (a).

Quero, J.M., and Camacho, E.F., 'Neural generalized predictive controllers', Porc. Of the IEEE International Conference on System Engineering, Pittsburgh, 1990.

Scokaert, P. O. M., "Infinite horizon generalized predictive control", International Journal of control, Vol. 66, No. 1, pp.161-175, 1997.

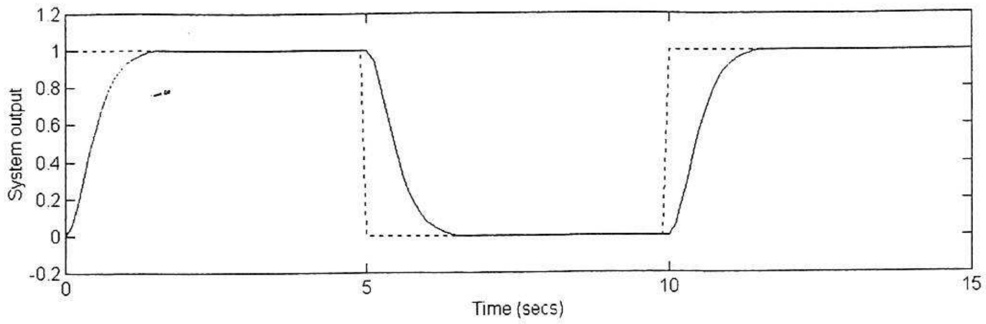
Scokaert, P.O.M "Constrained predictive control", D.Phil thesis, Department of Engineering Science, Oxford University, 1994.

Spiegel, M. R., "Complex variables", McGraw-Hill, 1964.

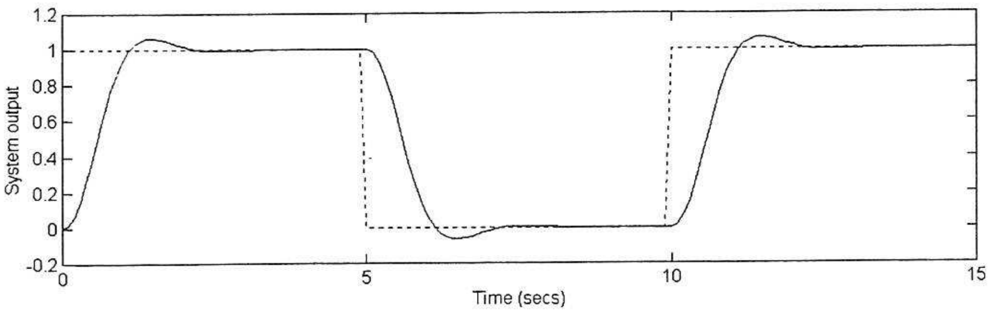
Yoon T.W. and Clarke, D.W., "A reformulation of receding-horizon predictive control", International Journal System Science, Vol. 26, No. 7, pp. 1383-1400, 1995.

Yoon T.W., "Robust adaptive predictive control", D.Phil. thesis, Department of Engineering Science, Oxford University, U.K., 1994.

Zhang, J., and Xi, Y., "Some Stability results', International Journal of control", Vol.70, No.5, pp.831-840, 1998.

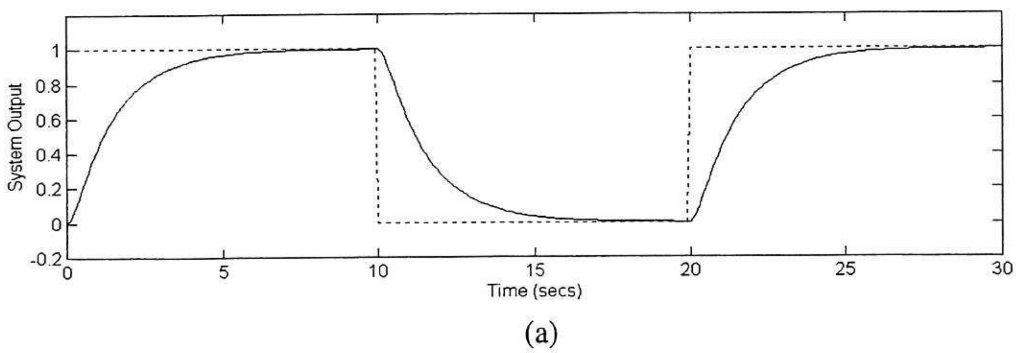


(a)

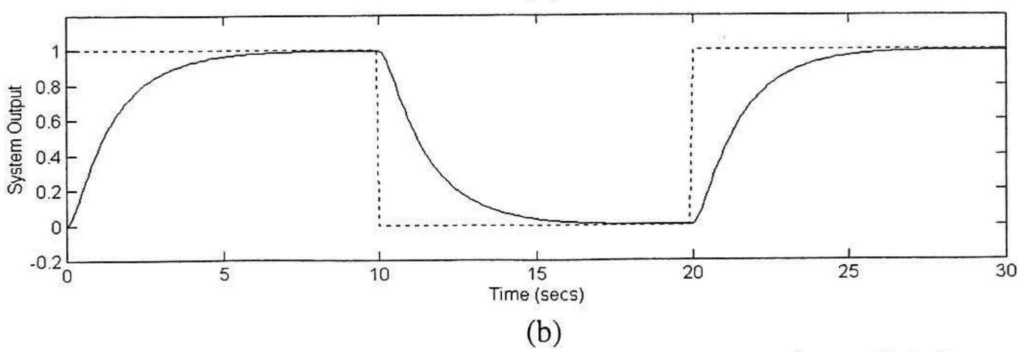


(b)

Figure 1: Stabilisation effects of tuning parameters (a) (N_1, N_2, N_u, λ)=(1, 8, 1, 0); (b) (N_1, N_2, N_u, λ)=(1, 8, 1, 0.8)



(a)



(b)

Figure 2: Stabilisation effects of tuning parameters (a) (N_1, N_2, N_u, λ)=(1, 70, 1, 0); (b) (N_1, N_2, N_u, λ)=(1, 70, 1, 1.2)

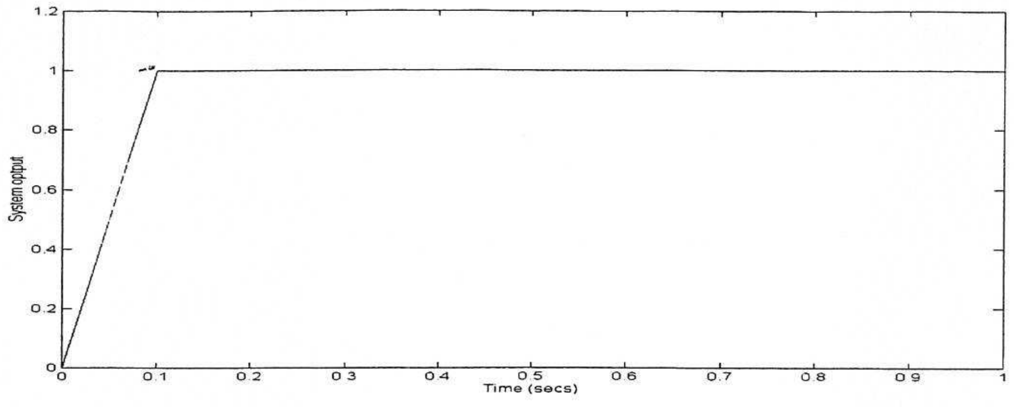


Figure 3: Step response of full free turbine model