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**Proceedings Paper:**
Bogachev, LV, Derfel, G and Molchanov, SA (2015) Analysis of the archetypal functional equation in the non-critical case. In: Proceedings of the 10th AIMS Conference. 10TH AIMS International Conference, 07-11 Jul 2014, Madrid, Spain. American Institute of Mathematical Sciences , Wilmington, North Carolina, U.S.A. , pp. 132-141.

https://doi.org/10.3934/proc.2015.0132

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Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

# ANALYSIS OF THE ARCHETYPAL FUNCTIONAL EQUATION IN THE NON-CRITICAL CASE

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Abstract. We study the archetypal functional equation of the form $y(x) = \iint_{\mathbb{R}^2} y(a(x-b))\,\mu(\mathrm{d}a,\mathrm{d}b)$ $(x \in \mathbb{R})$, where $\mu$ is a probability measure on $\mathbb{R}^2$; equivalently, $y(x) = \mathbb{E}\{y(\alpha(x-\beta))\}$, where $\mathbb{E}$ is expectation with respect to the distribution $\mu$ of random coefficients $(\alpha,\beta)$. Existence of non-trivial (i.e. non-constant) bounded continuous solutions is governed by the value $K := \iint_{\mathbb{R}^2} \ln|a|\,\mu(\mathrm{d}a,\mathrm{d}b) = \mathbb{E}\{\ln|\alpha|\}$; namely, under mild technical conditions no such solutions exist whenever $K < 0$, whereas if $K > 0$ (and $\alpha > 0$) then there is a non-trivial solution constructed as the distribution function of a certain random series representing a self-similar measure associated with $(\alpha,\beta)$. Further results are obtained in the supercritical case $K > 0$, including existence, uniqueness and a maximum principle. The case with $\mathbb{P}(\alpha < 0) > 0$ is drastically different from that with $\alpha > 0$; in particular, we prove that a bounded solution $y(\cdot)$ possessing limits at $\pm\infty$ must be constant. The proofs employ martingale techniques applied to the martingale $y(X_n)$, where $(X_n)$ is an associated Markov chain with jumps of the form $x \rightsquigarrow \alpha(x-\beta)$.

## 1. Introduction.

### 1.1. The archetypal equation and main results.

This paper concerns the *archetypal* functional equation with rescaled argument [2, 8] of the form

$$y(x) = \iint_{\mathbb{R}^2} y(a(x-b))\,\mu(\mathrm{d}a,\mathrm{d}b), \qquad x \in \mathbb{R}, \tag{1}$$

where $\mu(\mathrm{d}a,\mathrm{d}b)$ is a probability measure on $\mathbb{R}^2$. Due to the normalization of the measure $\mu$ to unity, such an equation is *balanced* in that the total weighted contribution of the (scaled) solution $y(\cdot)$ on the right-hand side of (1) is matched by the non-scaled input on the left-hand side. The integral in (1) has the meaning

2010 *Mathematics Subject Classification.* Primary: 39B05; Secondary: 34K06, 39A22, 60G42, 60J05.

*Key words and phrases.* Functional & functional-differential equations, pantograph equation, Markov chain, harmonic function, martingale.

L.V.B. was partially supported by a Leverhulme Research Fellowship. G.D. received funding from the Israel Science Foundation, Grant 35/10. Support during various research visits provided by the Center for Advanced Studies in Mathematics (Ben-Gurion University) and also by the Center for Interdisciplinary Research (ZiF) and SFB 701 (Bielefeld University) is acknowledged.

of expectation with respect to a random vector $(\alpha, \beta)$ with distribution $\mathbb{P}\{(\alpha,\beta) \in \mathrm{d}a \times \mathrm{d}b\} = \mu(\mathrm{d}a, \mathrm{d}b)$; thus, equation (1) can be represented in the compact form

$$y(x) = \mathbb{E}\{y(\alpha(x-\beta))\}, \qquad x \in \mathbb{R}. \tag{2}$$

The equation (1)–(2) is a rich source of various equations specified by a suitable choice of the measure $\mu$, which has motivated its name "archetypal" [2]. Examples include many well-known classes of equations with rescaling, such as: equations in convolutions, e.g. the *Choquet–Deny equation* $y = y \star \sigma$ [5]; equations for *Hutchinson's self-similar measures* [13], e.g. $y(x) = \frac{1}{2}\,y(a(x+1)) + \frac{1}{2}\,y(a(x-1))$ $(a > 1)$ arising in the *Bernoulli convolutions problem* [21]; *two-scale* (*refinement*) *equations*[1] of the form $z(x) = a\sum_{i=1}^{\ell} p_i\, z(a(x-b_i))$ with $z(x) := y'(x)$ [7, 9], exemplified by *Schilling's equation* $z(x) = a\big(\frac{1}{4}z(ax-1)+\frac{1}{2}z(ax)+\frac{1}{4}z(ax+1)\big)$ describing spatially chaotic structures in amorphous materials [11, 19]; etc. Furthermore, as was observed by Derfel [8], the archetypal equation (1)–(2) also contains some important functional-differential classes, including the (*balanced*) *pantograph equation*[2] [1, 2, 8]

$$y'(x) + y(x) = \sum\nolimits_i p_i\, y(a_i x), \qquad a_i, p_i > 0, \quad \sum\nolimits_i p_i = 1, \tag{3}$$

and *Rvachev's equation*[3] $z'(x) = 2\big(z(2x+1) - z(2x-1)\big)$ [18]. See an extensive review of examples and applications of the archetypal equation (1)–(2) in Bogachev *et al.* [2], together with further references therein.

Observing that any function $y(x) \equiv \mathrm{const}$ satisfies equations (1)–(2), it is natural to investigate if there are any *non-trivial* (i.e. non-constant) bounded continuous solutions. Such a question naturally arises in the context of functional and functional-differential equations with rescaling, where the possible existence of bounded solutions (e.g. periodic, almost periodic, compactly supported, etc.) is of major interest in physical and other applications (see e.g. [4, 18, 19, 23]).

Investigation of the archetypal equation (1)–(2), with a focus on bounded continuous solutions (abbreviated below as *b.c.-solutions*), was initiated by Derfel [8] (in the case $\alpha > 0$) who showed that the problem crucially depends on the value

$$K := \iint_{\mathbb{R}^2} \ln|a|\, \mu(\mathrm{d}a, \mathrm{d}b) = \mathbb{E}\{\ln|\alpha|\}. \tag{4}$$

More precisely, if $K < 0$ (*subcritical case*) then, under some mild technical conditions on the measure $\mu$, there are no b.c.-solutions other than constants,[4] whereas if $K > 0$ (*supercritical case*) then a non-trivial b.c.-solution does exist.

However, the *critical case* $K = 0$ was left open in [8]. Some recent progress was due to Bogachev *et al.* [1] who settled the problem for the balanced pantograph equation (3) by showing that if $K = \sum_i p_i \ln a_i = 0$ then there are no non-trivial b.c.-solutions of (3). Recently (see [2]) we proved the same result for a general equation (1)–(2) in the critical case subject to an *a priori* condition of uniform continuity of $y(\cdot)$, which is satisfied for a large class of examples including (3).

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[1] Compactly supported continuous solutions of such equations play a crucial role in wavelet theory [6, 23], and also in subdivision schemes and curve design [4, 9].

[2] Pantograph equation $y'(x) = c_0\, y(x) + c_1 y(\alpha x)$ dating back to Ockendon and Tayler [16] arises in diverse areas, e.g. number theory, astrophysics, radioactive decay, queues and risk theory, population dynamics, medicine, quantum theory, stochastic games, etc.; for general results and further bibliography on the pantograph equation, see [1, 2, 3, 10, 14, 15].

[3] Its compactly supported solutions are instrumental in approximation theory [18].

[4] A similar result was obtained earlier (via a different method) by Steinmetz and Volkmann [22] for a special case of equation (2), $y(x) = p\,y(px-1) + q\,y(qx+1)$ $(p, q > 0,\ p+q=1)$.

The focus of the present work is on the non-critical case $K \neq 0$, especially when $K > 0$ with $\alpha$ possibly taking negative values, aiming to obtain further results including existence, uniqueness and a maximum principle. Under a slightly weaker moment condition on $\beta$ as compared to [8] we establish the dichotomy of non-existence *vs.* existence of non-trivial b.c.-solutions in the subcritical ($K < 0$) and supercritical ($K > 0$) regimes, respectively.

Let us stress though that in contrast to the subcritical case which is insensitive to the sign of $\alpha$, for $K > 0$ we are only able to produce a non-trivial solution under the assumption that $\alpha > 0$ almost surely (a.s.). Such a solution is constructed, with the help of results by Grintsevichyus [12], as the distribution function $F_\Upsilon(x) = \mathbb{P}(\Upsilon \leq x)$ of the random series $\Upsilon = \sum_{n=1}^{\infty} \beta_n \prod_{i=1}^{n-1} \alpha_i^{-1}$ representing a self-similar measure associated with $(\alpha, \beta)$, where $\{(\alpha_n, \beta_n)\}_{n\geq 1}$ are independent identically distributed (i.i.d.) random pairs with distribution $\mu$ each. This solution is unique (up to linear transformations) in the class of functions with finite limits at $\pm\infty$ (Theorem 4.3(a)), but the uniqueness in the class of b.c.-solutions may fail to be true: we will present an example of such a solution $y(\cdot)$ oscillating at $+\infty$ (see Remark 4.2).

In the case $K > 0$ with $\mathbb{P}(\alpha < 0) > 0$, the function $F_\Upsilon(\cdot)$ (which is still well defined) is no longer a solution to the equation (1)–(2); e.g. if $\alpha < 0$ a.s. then $y = F_\Upsilon(x)$ satisfies another functional equation, $y(x) = 1 - \mathbb{E}\{y(\alpha(x-\beta))\}$ (cf. [12, Eq. (5)]). Thus, the problem of existence remains largely open here. More to the point, this case is completely different from the purely positive case, $\alpha > 0$ (a.s.); for instance, a b.c.-solution $y(\cdot)$ with limits at $\pm\infty$ must be constant (Theorem 4.3(b)). This follows from Theorem 4.2 stating that the limits superior at $\pm\infty$ coincide (the same is true for the limits inferior). Heuristically, this is a manifestation of "mixing" in (2) for (large) positive and negative arguments of $y(\cdot)$ due to possible negative values of $\alpha$. Note that Theorem 4.2 is proved with the help of the maximum principle of Theorem 4.1, which is of interest in its own right.

This analysis is complemented by uniqueness results in the class of absolutely continuous (a.c.) solutions (using the Fourier transform methods); here, boundedness is not assumed *a priori*. Again, we demonstrate a striking difference between the cases $\alpha > 0$ (a.s.) and $\mathbb{P}(\alpha < 0) > 0$ (see Theorems 4.4 and 4.5, respectively).

Throughout the paper, it is assumed that

$$\text{(i) } \mathbb{P}(\alpha \neq 0) = 1; \quad \text{(ii) } \mathbb{P}(|\alpha| \neq 1) > 0; \quad \text{(iii) } \forall c \in \mathbb{R},\ \mathbb{P}(\alpha(c-\beta) = c) < 1. \tag{5}$$

Note that the remaining degenerate cases are treated in full detail in [2].

The rest of the paper is organized as follows. We start in §2 by introducing an associated Markov chain $(X_n)$ with jumps of the form $x \rightsquigarrow \alpha(x-\beta)$, and also extend the iterated equation $y(x) = \mathbb{E}_x\{y(X_n)\}$ to its "optional stopping" analog $y(x) = \mathbb{E}_x\{y(X_\tau)\}$, where $\tau$ is a (random) stopping time and $\mathbb{E}_x$ stands for the expectation subject to the initial condition $X_0 = x$. Suitable iterations of such a kind will be instrumental. In §3 we prove a stronger version of the dichotomy between the cases $K < 0$ and $K > 0$ (the latter subject to $\alpha > 0$). Finally, §4 contains further discussion of the supercritical case, as briefly indicated above.

## 2. **Preliminaries.**

2.1. **Associated Markov chain and harmonic functions.** The archetypal equation (2) admits an important interpretation via an associated Markov chain $(X_n)$ on $\mathbb{R}$ determined by the recursion

$$X_n = \alpha_n(X_{n-1} - \beta_n) \quad (n \in \mathbb{N}), \qquad X_0 = x \in \mathbb{R}, \tag{6}$$

where $\{(\alpha_n, \beta_n)\}_{n\ge1}$ are i.i.d. random pairs with the same distribution as a generic copy $(\alpha, \beta)$. Transition operator $T$ of the Markov chain (6) is given by

$$Tf(x) := \mathbb{E}_x\{f(X_1)\} \equiv \mathbb{E}\{f(\alpha(x-\beta))\}, \tag{7}$$

where the index $x$ indicates the initial condition $X_0 = x$. A function $f(\cdot)$ is called *$T$-harmonic* (or simply *harmonic*) if $Tf = f$ (cf. [17, p. 40]); hence, according to (7) solutions of equation (2) are equivalently described as harmonic functions.

2.2. **Iterations and stopping times.** Equation (2) can be expressed as $y(x) = \mathbb{E}_x\{y(X_1)\}$, and by iteration $y(x) = \mathbb{E}_x\{y(X_n)\}$ $(n \in \mathbb{N})$. Explicitly,

$$X_n = A_n x - D_n, \qquad n \ge 0, \tag{8}$$

$$A_n := \prod_{i=1}^{n} \alpha_i \quad (A_0 := 1), \qquad D_n := \sum_{i=1}^{n} \beta_i \prod_{j=i}^{n} \alpha_j \quad (D_0 := 0). \tag{9}$$

For $n \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, let $\mathcal{F}_n := \sigma\{X_i,\, i \le n\}$ be the $\sigma$-algebra generated by events $\{X_i \in B\}$ (with Borel sets $B \in \mathcal{B}(\mathbb{R})$); the increasing sequence $(\mathcal{F}_n)_{n\ge0}$ is referred to as the (natural) filtration of $(X_n)$. A random variable $\tau$ with values in $\mathbb{N} \cup \{+\infty\}$ is called a *stopping time* with respect to filtration $(\mathcal{F}_n)$ if it is adapted to $(\mathcal{F}_n)$ (i.e. $\{\tau = n\} \in \mathcal{F}_n$, $n \in \mathbb{N}_0$) and $\tau < \infty$ a.s. We shall systematically use the following simple fact. (Note that the continuity of $y(\cdot)$ is not required.)

**Lemma 2.1.** *Let $\tau$ be a stopping time with respect to filtration $\mathcal{F}_n^{\alpha} := \sigma\{\alpha_1, \dots, \alpha_n\} \subset \mathcal{F}_n$, $n \in \mathbb{N}_0$. If $y(\cdot)$ is a bounded $T$-harmonic function then*

$$y(x) = \mathbb{E}_x\{y(X_\tau)\}, \qquad x \in \mathbb{R}. \tag{10}$$

*Proof.* Clearly, $\tau$ is adapted to the filtration $\mathcal{F}_n^{\alpha,\beta} := \sigma\{(\alpha_i, \beta_i),\, i \le n\} \equiv \mathcal{F}_n$. Using (6) it is easy to check that $\mathbb{E}\{y(X_n)\,|\,\mathcal{F}_{n-1}\} = y(X_{n-1})$ (a.s.), and hence $(y(X_n))$ is a *martingale* [17, p. 43, Proposition 1.8]. Since $y(\cdot)$ is bounded, formula (10) now readily follows by Doob's Optional Stopping Theorem [20, pp. 485–486, Theorem 1 and Corollary]. □

3. **The subcritical ($K < 0$) and supercritical ($K > 0$) cases.** In the case $\alpha \ne 0$ a.s., formula (8) can be rewritten in the form (cf. (8), (9))

$$X_n = A_n(x - B_n), \qquad n \ge 0, \tag{11}$$

$$A_n := \prod_{i=1}^{n} \alpha_i \quad (A_0 := 1), \qquad B_n := D_n A_n^{-1} = \sum_{i=1}^{n} \beta_i A_{i-1}^{-1} \quad (B_0 := 0). \tag{12}$$

The following important result is due to Grintsevichyus [12, pp. 164–165].

**Lemma 3.1.** *Let assumption (5) be in force, and also assume that*

$$0 < \mathbb{E}\{\ln|\alpha|\} < \infty, \qquad \mathbb{E}\{\ln\max(|\beta|, 1)\} < \infty. \tag{13}$$

*Then the random series*

$$\Upsilon := \beta_1 + \beta_2\,\alpha_1^{-1} + \beta_3\,\alpha_1^{-1}\alpha_2^{-1} + \dots = \sum_{n=1}^{\infty} \beta_n A_{n-1}^{-1} \tag{14}$$

*converges a.s., and its distribution function $F_\Upsilon(x) := \mathbb{P}(\Upsilon \le x)$ is continuous on $\mathbb{R}$.*

**Remark 3.1.** The results in [12] entail that $F_\Upsilon(\cdot)$ is either a.c. or singularly continuous; a purely discrete case (with a single atom!) arises if $\alpha(c - \beta) = c$ (a.s.).

Recall that the parameter $K$ is defined in (4). The next two results (for $K<0$ and $K>0$, respectively) were obtained by Derfel [8] in the case $\alpha>0$ (a.s.) under a more stringent condition $\mathbb{E}\{|\beta|\}<\infty$; but his proofs essentially remain valid in a more general situation as described below.

### 3.1. **The subcritical case.**

**Theorem 3.2** ($K<0$)**.** *Assume that the second integrability condition in* (13) *is fulfilled, but the first one is replaced by* $-\infty<\mathbb{E}\{\ln|\alpha|\}<0$*. Then any b.c.-solution of the archetypal equation* (2) *is constant.*

*Proof.* Applying Lemma 2.1 with $\tau\equiv n\in\mathbb{N}$, we obtain (see (8), (9))

$$y(x)=\mathbb{E}\{y(A_nx-D_n)\},\qquad x\in\mathbb{R}. \tag{15}$$

Setting $D_n^\circ:=\sum_{i=1}^n\beta_iA_i=\alpha_1(\beta_1+\beta_2\alpha_2+\dots+\beta_n\alpha_2\cdots\alpha_n)$ (cf. (9)), observe that the pair $(A_n,D_n)$ has the same distribution as $(A_n,D_n^\circ)$, which is evident by reversing the numbering $(\alpha_i,\beta_i)\mapsto(\alpha_{n-i+1},\beta_{n-i+1})$ $(i=1,\dots,n)$. Hence, equation (15) can be rewritten as

$$y(x)=\mathbb{E}\{y(A_nx-D_n^\circ)\},\qquad x\in\mathbb{R}. \tag{16}$$

Due to Lemma 3.1 (with $\alpha_i^{-1}$ in place of $\alpha_i$), $D_n^\circ$ converges a.s. as $n\to\infty$, say $D_n^\circ\to\Upsilon^\circ$ (cf. (14)). On the other hand, $A_n\to0$ a.s., since $\mathbb{E}\{\ln|\alpha|\}<0$ and, by the strong low of large numbers, $\ln|A_n|=\sum_{i=1}^n\ln|\alpha_i|\to-\infty$ (a.s.). As a result, for each $x\in\mathbb{R}$ we have $A_nx-D_n^\circ\to-\Upsilon^\circ$ (a.s.). Since $y(\cdot)$ is bounded and continuous, one can apply Lebesgue's dominated convergence theorem [20, p. 187, Theorem 3] and pass to the limit in (16), yielding $y(x)=\mathbb{E}\{y(-\Upsilon^\circ)\}$; since the right-hand side does not depend on $x$, it follows that $y(x)\equiv\text{const}$. □

### 3.2. **Canonical solution in the supercritical case with $\alpha>0$.**

The next theorem provides a non-trivial b.c.-solution to the archetypal equation (2) in the case of *positive* $\alpha$. Recall that $\Upsilon$ is the random series (14) and $F_\Upsilon(x)$ is its distribution function (see Lemma 3.1).

**Theorem 3.3** ($K>0$)**.** *Suppose that assumption* (5) *is satisfied, along with conditions* (13)*, and also assume that* $\alpha>0$ *a.s. Then* $y=F_\Upsilon(x)$ *is a b.c.-solution of the archetypal equation* (2)*.*

*Proof.* Thanks to Lemma 3.1 we only have to verify that $F_\Upsilon(x)$ satisfies (2). Observe from (14) that $\Upsilon=\beta_1+\alpha_1^{-1}\widetilde{\Upsilon}$, where $\widetilde{\Upsilon}$ is independent of $(\alpha_1,\beta_1)$ and has the same distribution as $\Upsilon$. Hence, we obtain (using that $\alpha_1>0$ a.s.)

$$\begin{aligned}F_\Upsilon(x)&=\mathbb{P}(\beta_1+\alpha_1^{-1}\widetilde{\Upsilon}\le x)=\mathbb{P}(\widetilde{\Upsilon}\le\alpha_1(x-\beta_1))\\&=\mathbb{E}\big\{\mathbb{P}\big(\widetilde{\Upsilon}\le\alpha_1(x-\beta_1)\,|\,\alpha_1,\beta_1\big)\big\}=\mathbb{E}\{F_\Upsilon(\alpha_1(x-\beta_1))\},\end{aligned}$$

that is, the function $y=F_\Upsilon(x)$ satisfies equation (2). □

We will refer to $y=F_\Upsilon(x)$ as the *canonical solution* of equation (2).

**Remark 3.2.** For some concrete equations with $\alpha\equiv\text{const}>1$, b.c.-solutions different from the canonical one may be constructed (see Remark 4.2).

**Remark 3.3.** To the best of our knowledge, no non-trivial b.c.-solutions of equation (2) are known if $\mathbb{P}(\alpha<0)>0$ except in the special case $|\alpha|\equiv1$ (see [2, Theorem 2.2(b-ii)]).

## 4. Further results in the supercritical case.

4.1. **Bounds coming from infinity.** The next result is akin to the maximum principle for the usual harmonic functions. The continuity of $y(\cdot)$ is not presumed.

**Theorem 4.1** (Maximum Principle)**.** *Suppose that assumption (5) is satisfied, along with conditions (13). Let $y(\cdot)$ be a bounded solution of (2), and denote*

$$m^{\pm} := \liminf_{x\to\pm\infty} y(x), \qquad M^{\pm} := \limsup_{x\to\pm\infty} y(x), \tag{17}$$

*where the same $+$ or $-$ sign should be chosen on both sides of each equality. Then*

$$m \le y(x) \le M, \qquad x \in \mathbb{R}, \tag{18}$$

*where $m := \min\{m^+, m^-\}$, $M := \max\{M^+, M^-\}$.*

*Proof.* Applying Lemma 2.1 with $\tau \equiv n \in \mathbb{N}$, for any $x \in \mathbb{R}$ we obtain

$$y(x) = \mathbb{E}\{y(A_n(x - B_n))\}, \tag{19}$$

where $A_n = \prod_{i=1}^n \alpha_i$ and $B_n = \sum_{i=1}^n \beta_i A_{i-1}^{-1}$ (see (11), (12)). By Lemma 3.1, the limiting random variable $\Upsilon = \lim_{n\to\infty} B_n$ is continuous, hence $\lim_{n\to\infty}(x - B_n) = x - \Upsilon \ne 0$ (a.s.). Combined with $|A_n| \to \infty$ a.s. (which follows by the strong law of large numbers due to the first moment condition in (13), cf. the proof of Theorem 3.2), this implies that $|A_n(x - B_n)| \to \infty$ (a.s.). Hence, Fatou's lemma [20, p. 187, Theorem 2] applied to equation (19) yields

$$y(x) \le \mathbb{E}\left\{\limsup_{n\to\infty} y(A_n(x - B_n))\right\} \le \max\{M^+, M^-\} = M,$$

which proves the upper bound in (18). The lower bound follows similarly. □

The case where $\alpha$ may take on negative values has an interesting general property as follows (note that conditions (13) are not needed here).

**Theorem 4.2.** *Suppose that $q := \mathbb{P}(\alpha < 0) > 0$, and let $y(x)$ be a bounded solution of (2). Then, in the notation (17), we have*

$$m^- = m^+, \qquad M^- = M^+. \tag{20}$$

*Proof.* By Fatou's lemma applied to equation (2) we get

$$M^+ = \limsup_{x\to+\infty} y(x) \le \mathbb{E}\left\{\limsup_{x\to+\infty} y(\alpha(x - \beta))\right\} \le M^+(1 - q) + M^- q. \tag{21}$$

Since $q > 0$, (21) implies that $M^+ \le M^-$. By symmetry, the opposite inequality is also true, hence $M^- = M^+$. The first equality in (20) is proved similarly. □

4.2. **Uniqueness for solutions with limits at infinity.** We can now prove the following uniqueness result (again, the continuity of solutions is not presumed). Note that the cases $\alpha > 0$ (a.s.) and $\mathbb{P}(\alpha < 0) > 0$ are drastically different.

**Theorem 4.3.** *Let assumption (5) be in force, along with conditions (13). Let $y(\cdot)$ be a bounded solution of (2) such that the limits $L^{\pm} := \lim_{x\to\pm\infty} y(x)$ exist.*

(a) *Suppose that $\mathbb{P}(\alpha > 0) = 1$. Then $y(\cdot)$ coincides, up to an affine transformation, with the canonical solution $F_\Upsilon(\cdot)$ (see Theorem 3.3); specifically,*

$$y(x) = (L^+ - L^-)\, F_\Upsilon(x) + L^-, \qquad x \in \mathbb{R}. \tag{22}$$

*In particular, $y(\cdot)$ must be everywhere continuous.*

(b) *If $\mathbb{P}(\alpha < 0) > 0$ then $y(x) \equiv \text{const}$.*

*Proof.* (a) Denote the right-hand side of (22) by $y_*(x)$. By linearity of (2) and according to Theorem 3.3, $y_*(x)$ satisfies equation (2), and it has the same limits $L^\pm$ at $\pm\infty$ as the solution $y(x)$. Hence, $y(x) - y_*(x)$ is also a solution, with zero limits at $\pm\infty$. But Theorem 4.1 then implies that $y(x) - y_*(x) \equiv 0$.

(b) Theorem 4.2 implies that $L^- = L^+ =: L$, hence by the bound (18) of Theorem 4.1 we have $L \le y(x) \le L$, i.e. $y(x) \equiv L = \text{const}$. □

**Remark 4.1.** In the case $\mathbb{P}(\alpha < 0) > 0$, Theorem 4.3(b) holds true if just one of the limits $L^\pm$ is assumed (due to (20), the other limit exists automatically).

**Remark 4.2.** Kato and McLeod [15, p. 923, Theorem 9(iii)] showed *inter alia* that the pantograph equation $y'(x) + y(x) = y(\alpha x)$ with $\alpha = \text{const} > 1$ has a family of $C^\infty$-solutions on the half-line $x \in [0, \infty)$ such that $y(x) = \mathrm{g}(\ln x / \ln \alpha) + O(x^{-\theta})$ as $x \to +\infty$, where $\mathrm{g}(\cdot)$ is any 1-periodic function, Hölder continuous with exponent $0 < \theta \le 1$. Noting from the equation that $y'(0) = 0$, such solutions can be extended to the entire line $\mathbb{R}$ by defining $y(x) := y(0)$ for all $x < 0$. It is known (see [2, 8]) that $y(\cdot)$ automatically satisfies the archetypal equation (2) (with the same $\alpha > 1$ and exponentially distributed $\beta$), thus furnishing an example of (a family of) bounded continuous (even smooth) solutions that do not have limit at $+\infty$.

4.3. **Uniqueness via Fourier transform.** Here, we obtain uniqueness results in the class of a.c. solutions with integrable derivative. In what follows, abbreviation "a.e." stands for "almost everywhere" (with respect to Lebesgue measure on $\mathbb{R}$). Note that boundedness of solutions is not presumed. It is convenient to state and prove these results separately for positive and negative $\alpha$ (see Theorems 4.4 and 4.5, respectively). Recall that $\Upsilon$ is the random series (14).

**Theorem 4.4.** *Let assumption (5) be satisfied, together with conditions (13).*

(a) *Let $\alpha > 0$ a.s., and assume that a solution $y(\cdot)$ of equation (2) is a.e. differentiable, with $z(x) := y'(x) \in L^1(\mathbb{R})$. Then $z(\cdot)$ is determined uniquely (a.e.) up to a multiplicative factor, with Fourier transform given by*

$$\hat{z}(s) = c_1\, \mathbb{E}\{\mathrm{e}^{\mathrm{i}s\Upsilon}\} \qquad (s \in \mathbb{R}), \qquad c_1 := \hat{z}(0) \in \mathbb{R}. \tag{23}$$

(b) *If $y(\cdot)$ is also a.c. then it coincides, up to an affine transformation, with the canonical solution $F_\Upsilon(\cdot)$ (see Theorem 3.3), i.e. there are $c_0, c_1 \in \mathbb{R}$ such that*

$$y(x) = c_0 + c_1 F_\Upsilon(x), \qquad x \in \mathbb{R}. \tag{24}$$

*Proof.* (a) Differentiation of (2) shows that $z(x) := y'(x)$ satisfies a.e. the equation

$$z(x) = \mathbb{E}\{\alpha z(\alpha(x - \beta))\}. \tag{25}$$

Let $\hat{z}(s) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx} z(x)\, \mathrm{d}x$ be the Fourier transform of the function $z \in L^1(\mathbb{R})$, hence $\hat{z}(\cdot)$ is bounded and continuous on $\mathbb{R}$, with the sup-norm $\|\hat{z}\| \le \int_{\mathbb{R}} |z(x)|\, \mathrm{d}x < \infty$. Multiplying (25) by $\mathrm{e}^{\mathrm{i}sx}$ and integrating over $x \in \mathbb{R}$, we see, using Fubini's theorem and the substitution $t = \alpha(x - \beta)$, that $\hat{z}(\cdot)$ satisfies the equation

$$\hat{z}(s) = \mathbb{E}\{\mathrm{e}^{\mathrm{i}s\beta}\, \hat{z}(\alpha^{-1} s)\}, \qquad s \in \mathbb{R}. \tag{26}$$

Iterating (26) $n \ge 1$ times we get (see the notation (12))

$$\hat{z}(s) = \mathbb{E}\big\{\mathrm{e}^{\mathrm{i}sB_n}\, \hat{z}(A_n^{-1} s)\big\}, \qquad s \in \mathbb{R}. \tag{27}$$

Note that $\mathbb{E}\{\ln|\alpha^{-1}|\} \in (-\infty, 0)$, hence $A_n^{-1} \to 0$ a.s. (see the proof of Theorem 3.2); besides, $B_n \to \Upsilon$ a.s. by Lemma 3.1. Thus, passing to the limit in (27) (by dominated convergence) and recalling that $\hat{z}(\cdot)$ is continuous, we obtain (23).

(b) To identify $z(\cdot)$ from its Fourier transform (23), it is convenient to integrate both parts of equation (23) against a suitable class of test functions. Consider the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions $\varphi(x)$ with finite support and such that their Fourier transform $\widehat{\varphi}(s) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx}\varphi(x)\,\mathrm{d}x$ is integrable; by the inversion formula, $\varphi(x) = (2\pi)^{-1}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}sx}\,\widehat{\varphi}(s)\,\mathrm{d}s$. With this at hand, we can write

$$\int_{\mathbb{R}} \hat{z}(s)\,\widehat{\varphi}(s)\,\mathrm{d}s = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx}\,\widehat{\varphi}(s)\,\mathrm{d}s \right) z(x)\,\mathrm{d}x = 2\pi \int_{\mathbb{R}} \varphi(-x)\,z(x)\,\mathrm{d}x. \tag{28}$$

Similarly,

$$\begin{aligned} \int_{\mathbb{R}} \mathbb{E}\{\mathrm{e}^{\mathrm{i}s\Upsilon}\}\,\widehat{\varphi}(s)\,\mathrm{d}s &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx}\,\mathrm{d}F_{\Upsilon}(x) \right) \widehat{\varphi}(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}sx}\,\widehat{\varphi}(s)\,\mathrm{d}s \right) \mathrm{d}F(x) = 2\pi \int_{\mathbb{R}} \varphi(-x)\,\mathrm{d}F_{\Upsilon}(x). \end{aligned} \tag{29}$$

Thus, thanks to equation (23), from (28) and (29) we obtain

$$\int_{\mathbb{R}} \varphi(-x)\,z(x)\,\mathrm{d}x = c_1 \int_{\mathbb{R}} \varphi(-x)\,\mathrm{d}F_{\Upsilon}(x), \qquad \varphi \in \mathcal{S}(\mathbb{R}). \tag{30}$$

Since $\mathcal{S}(\mathbb{R})$ is dense in both $L^1(\mathbb{R};\, z(x)\,\mathrm{d}x)$ and $L^1(\mathbb{R}; \mathrm{d}F_{\Upsilon}(x))$, equation (30) extends to indicator functions of any intervals, yielding (by the continuity of $F_{\Upsilon}(\cdot)$)

$$y(x) - y(0) = \int_0^x z(u)\,\mathrm{d}u = c_1\{F_{\Upsilon}(x) - F_{\Upsilon}(0)\}, \qquad x \in \mathbb{R},$$

which is reduced to (24) by setting $c_0 := y(0) - c_1 F_{\Upsilon}(0)$. □

**Remark 4.3.** The result of Theorem 4.4 was obtained by Daubechies and Lagarias [7, p. 1392, Theorem 2.1(b)] in a particular case with $\alpha \equiv \mathrm{const} > 1$ and discrete $\beta$.

**Remark 4.4.** Uniqueness (up to a multiplicative factor) of b.c.-solutions of equation (26) was proved by Grintsevichyus [12, p. 165, Proposition l].

**Example 4.1.** De Rham's function (see [7, pp. 1403–1405] is a continuous (but nowhere differentiable) even solution of the difference equation

$$\phi(x) = \phi(3x) + \tfrac{1}{3}\big(\phi(3x+1) + \phi(3x-1)\big) + \tfrac{2}{3}\big(\phi(3x+2) + \phi(3x-2)\big).$$

Then $y(x) := \int_0^x \phi(u)\,\mathrm{d}u$ is an odd function of class $C^1(\mathbb{R})$ satisfying

$$y(x) = \tfrac{1}{3}\,y(3x) + \tfrac{1}{9}\big(y(3x+1) + y(3x-1)\big) + \tfrac{2}{9}\big(y(3x+2) + y(3x-2)\big),$$

which is an archetypal equation with $\alpha \equiv 3$ and $\beta$ taking values $0, -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$ with probabilities $\frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{2}{9}$, respectively. Now, according to Theorem 4.4 the solution $y(\cdot)$ is an affine version of the distribution function $F_{\Upsilon}(\cdot)$, the latter thus being automatically a.c. and, moreover, in $C^1(\mathbb{R})$; in turn, it follows that de Rham's function $\phi(\cdot)$ is proportional to the probability density of $\Upsilon$ (see (14)).

A counterpart of Theorem 4.4 for $\alpha$ with possible negative values is strikingly different (cf. Theorem 4.3).

**Theorem 4.5.** *Let $q := \mathbb{P}(\alpha < 0) > 0$, and let a solution $y(\cdot)$ be a.e. differentiable, with $y' \in L^1(\mathbb{R})$. Then $y' = 0$ a.e. If in addition $y(\cdot)$ is a.c. then $y \equiv \mathrm{const}$.*

*Proof.* The random time $\tau_- := \inf\{n \geq 1\colon A_n < 0\}$ is adapted to the filtration $\mathcal{F}_n^\alpha$ and has geometric distribution, $\mathbb{P}(\tau_- = n) = (1-q)^{n-1}q$ $(n \geq 1)$. Hence, $\tau_- < \infty$ a.s. and $\mathbb{E}\{\tau_-\} = q^{-1} < \infty$. Applying Lemma 2.1, we obtain the equation

$$y(x) = \mathbb{E}\{y(\tilde{\alpha}(x - \tilde{\beta})\}, \qquad x \in \mathbb{R}, \tag{31}$$

where $\tilde{\alpha} := A_{\tau_-} < 0$, $\tilde{\beta} := B_{\tau_-}$ (cf. (11), (12)).

Let us first verify that $\tilde{\alpha}$, $\tilde{\beta}$ satisfy the moment conditions (13). Indeed, noting that $\ln|\tilde{\alpha}| = \sum_{i=1}^{\tau_-} \ln|\alpha_i|$ and $\mathbb{E}\{\tau_-\} = q^{-1} < \infty$, by Wald's identity [20, p. 488, Theorem 3] we obtain, using the first condition in (13),

$$\mathbb{E}\{\ln|\tilde{\alpha}|\} = \mathbb{E}\{\tau_-\} \cdot \mathbb{E}\{\ln|\alpha|\} \in (0, \infty). \tag{32}$$

Recalling (12) and denoting $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, we have

$$|\tilde{\beta}| \leq \sum_{i=1}^{\tau_-} \frac{|\beta_i|}{|A_{i-1}|} \leq \prod_{i=1}^{\tau_-}(|\beta_i| \vee 1) \cdot \sum_{i=1}^{\tau_-} \frac{1}{|A_{i-1}|} \prod_{i=1}^{\tau_-}(|\beta_i| \vee 1) \cdot \tau_- \prod_{i=1}^{\tau_-} \frac{1}{|\alpha_i| \wedge 1}.$$

The right-hand side is not less than 1, hence the same bound holds for $|\tilde{\beta}| \vee 1$ and

$$\ln(|\tilde{\beta}| \vee 1) \leq \sum_{i=1}^{\tau_-} \ln(|\beta_i| \vee 1) + \ln(\tau_-) - \sum_{i=1}^{\tau_-} \ln(|\alpha_i| \wedge 1). \tag{33}$$

Again applying Wald's identity and using conditions (13), we get from (33)

$$\mathbb{E}\{\ln(|\tilde{\beta}| \vee 1)\} \leq \mathbb{E}\{\tau_-\} \cdot \Big(\mathbb{E}\{\ln(|\beta| \vee 1)\} + 1 - \mathbb{E}\{\ln(|\alpha| \wedge 1)\}\Big) < \infty.$$

Now we can apply to (31) the method used in the proof of Theorem 4.4. More specifically, a differentiated version of (31), for $z(x) := y'(x)$, reads (cf. (25))

$$z(x) = \mathbb{E}\{\tilde{\alpha}\, z(\tilde{\alpha}(x - \tilde{\beta}))\} \qquad \text{(a.e.)}.$$

However, here $\tilde{\alpha} < 0$ (a.s.), so the Fourier transform $\hat{z}(s)$ now satisfies (cf. (26))

$$\hat{z}(s) = -\,\mathbb{E}\{\mathrm{e}^{\mathrm{i}s\tilde{\beta}}\, \hat{z}(\tilde{\alpha}^{-1}s)\}, \qquad s \in \mathbb{R}.$$

Iterating as before, we obtain for each $n \in \mathbb{N}$

$$\hat{z}(s) = (-1)^n\, \mathbb{E}\{\mathrm{e}^{\mathrm{i}s\widetilde{\Upsilon}_n}\, \hat{z}(\tilde{A}_n^{-1}s)\}, \qquad s \in \mathbb{R}, \tag{34}$$

where due to (32) we have a.s. $\tilde{A}_n^{-1} = \prod_{i=1}^n \tilde{\alpha}_i^{-1} \to 0$, $\widetilde{\Upsilon}_n = \sum_{i=1}^n \tilde{\beta}_i \tilde{A}_{i-1}^{-1} \to \widetilde{\Upsilon}$. Hence, the expectation in (34) converges to $\hat{z}(0)\,\mathbb{E}\{\mathrm{e}^{\mathrm{i}s\widetilde{\Upsilon}}\}$; however, due to the sign alternation the limit of (34) does not exist unless $\hat{z}(0) = 0$, in which case $\hat{z}(s) = 0$ for all $s \in \mathbb{R}$. By the uniqueness theorem for the Fourier transform, this implies that $z(x) = y'(x) \equiv 0$ a.e. Finally, if $y(\cdot)$ is a.c. then it follows that $y(x) \equiv \text{const}$. □

**Remark 4.5.** The last statement (i.e. under the a.c.-condition) of each of Theorems 4.4 and 4.5 can be easily deduced by Theorem 4.3. Indeed, since the derivative $y'(\cdot)$ is a.c. and in $L^1(\mathbb{R})$, by the Newton–Leibniz formula we have

$$y(x) = y(0) + \int_0^x y'(u)\,\mathrm{d}u \to y(0) + \int_0^{\pm\infty} y'(u)\,\mathrm{d}u \qquad (x \to \pm\infty).$$

Thus, the limits of $y(x)$ at $\pm\infty$ exist, and the rest immediately follows from Theorem 4.3. However, the uniqueness results for the derivative $y'$, contained in Theorems 4.4 and 4.5, cannot be obtained along these lines.

**Acknowledgments.** The authors are grateful to John Ockendon and Anatoly Vershik for stimulating discussions.

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Received xxxx 20xx; revised xxxx 20xx.

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