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Enhanced Hybrid Positioning in Wireless Networks II: AoA-RSS

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***Abstract*—In order to achieve higher location estimation accuracy through utilizing all the available information, in this paper we propose a hybrid localization system. We use the angle of arrival (AoA) measurement with the inherent received signal strength (RSS) information to develop an AoA-RSS linear least squares (LLS) location estimator. To accurately predict the performance of the LLS estimator, a closed form expression for the mean square error (MSE) is also derived. Furthermore, the information present in the covariance of the incoming signals is utilized and a novel weighted linear least squares (WLLS) method is proposed. It is shown via simulation that the theoretical MSE accurately predicts the performance of the LLS estimator. It is also shown via simulation that the WLLS algorithm exhibits better performance than the LLS algorithm.**

I. INTRODUCTION

Accurate wireless node localization has been the focus of many researchers in the past decade. Wireless nodes could be located using the global positioning system (GPS) however due to the high cost in terms of power consumption and price of the GPS chip and in many scenarios the unavailability of direct line of sight (LoS) to the satellites, GPS positioning is not always favored. Hence alternative methods to obtain node location have been developed. Most localization systems first estimate the distance between the target node (TN) and a node with known location called an anchor node (AN). One of the simplest and cheapest techniques to determine the distance between two nodes is the received signal strength (RSS) technique. RSS range estimation does not require any additional hardware on the nodes. Distance between nodes can also be determined using the angle of arrival (AoA) at the TN from two or more ANs [1]. The AoA could be estimated by using an array of antennas as in [2] or a rotating beam of radiation [3], and using techniques such as Multiple Signal Classification (MUSIC) [4] or estimation of signal parameters via rotational invariance techniques (ESPIRT) [5]. As for the RSS distance estimation, various techniques have been proposed to enhance its performance [6]. Once the distance information is made available, it can then be used to estimate the location coordinates. Moreover the localization scheme can be either iterative [7] or non-iterative [8], or it can be co-operative [9] or non-cooperative [10], [11].

In this paper, instead of emphasizing either AoA or RSS individually, we use all the available information to develop a hybrid AoA-RSS LLS approach. We study the performance of LLS estimator for hybrid AoA-RSS systems and develop a theoretical mean square error (MSE) equation. We also propose a WLLS algorithm that shows better performance than the LLS algorithm.

The rest of the paper is organized as follows. In section II, basic RSS, AoA and the hybrid localization systems are reviewed . Section III deals with the derivation of the theoretical MSE expression. WLLS algorithm is proposed in section IV. In Section V we compare the results of LLS with WLLS algorithm and section VI concludes the paper.

II. RSS, AoA AND HYBRID LOCALIZATION

For future use, we define the following notations $\mathbb{R}^n$ represents a set of n dimensional real numbers, $(.)^T$ is the transpose operator, $Tr(M)$ is the trace of matrix M, $E_x(.)$ refers to the expectation with respect to the random variable x.

1) ***RSS.*** If only RSS measurement is present then at least 3 or 4 ANs are required for 2d or 3d localization respectively. Each distance estimate can be represented as the radius of a circle, the center of which is the ANs location. Hence we have a number of circles depending upon the number of ANs. The point of intersection of all these circles is the position of the TN.

We assume that our network consists of N ANs with locations $\boldsymbol{\theta}_i = [x_i, y_i]^T \left(\boldsymbol{\theta}_i \in \mathbb{R}^2\right)$ for $i = 1, ..., N$. Where the distance d_i between the TN and the i^{th} AN, is related to the path-loss at the i^{th} AN, $\mathscr{L}_i$, and the path loss exponent (PLE), α, as [12]

$$\mathscr{L}_i = \mathscr{L}_0 + 10\alpha \log_{10} d_i + w_i, \tag{1}$$

where $\mathscr{L}_0$ being the path-loss at the reference distance d_0 ($d_0 < d_i$, and is normally taken as 1 m) and w_i is a zero-mean Gaussian random variable representing the log-normal shadowing effect, i.e. $w_i \sim \left(\mathcal{N}\left(0, \sigma_i^2\right)\right)$. The path-loss is the dB difference between the transmit power and the received power and is given by $\mathscr{L}_i = 10\log_{10} P - 10\log_{10} P_i$ where P is the transmit power at the TN and P_i is the received power at the i^{th} AN. The distance d_i is given by $d_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}$. The observed path loss (in dB) from d_0 to d_i, $z_i = \mathscr{L}_i - \mathscr{L}_0$, can be expressed as

$$z_i = \gamma\alpha \ln d_i + w_i, \tag{2}$$

for $\gamma = \frac{10}{\ln 10}$.

From (2), we can obtain the noisy distance equation as

$$\hat{d}_i = d_i \exp\left(\frac{w_i}{\gamma\alpha}\right) \tag{3}$$

Clearly (3) is non-linear and can be solved using iterative techniques, however a least squares (LS) can also be be used by first linearizing (3) [6]. From (3) we obtain

$$\hat{d}_i^2 \approx \left[(x-x_i)^2+(y-y_i)^2\right]\exp\left(\frac{2w_i}{\gamma\alpha}\right), \quad (4)$$

then a reference AN is selected and its distance equation is subtracted from (4) for $i=1,...,N$ $(i \neq r)$. Let $\hat{d}_r$ represent the reference distance of this reference AN.

$$\hat{d}_r^2 \approx \left[(x-x_r)^2+(y-y_r)^2\right]\exp\left(\frac{2w_r}{\gamma\alpha}\right). \quad (5)$$

The reference AN can be randomly chosen or a special criterion can be developed to choose the reference distance as in [6]. From (3) and (5) we get

$$\begin{aligned}&(x_i-x_r)\,x+(y_i-y_r)\,y= \\ &0.5\left[\frac{1}{\beta_r}\left(d_r\exp\left(\frac{w_r}{\gamma\alpha}\right)\right)^2-\frac{1}{\beta_i}\left(d_1\exp\left(\frac{w_1}{\gamma\alpha}\right)\right)^2-\kappa_r+\kappa_i\right],\end{aligned} \quad (6)$$

where $\kappa_r = x_r^2+y_r^2$ and $\kappa_i = x_i^2+y_i^2$. (6) can be rewritten in matrix form as

$$\mathbf{A}_r\mathbf{u}_r = 0.5\hat{\mathbf{b}}_r. \quad (7)$$

where,

$$\mathbf{A}_r = \begin{bmatrix} x_1-x_r & y_1-y_r \\ x_2-x_r & y_2-y_r \\ \vdots & \vdots \\ x_N-x_r & y_N-y_r \end{bmatrix} \in \mathbb{R}^{N\times 2}, \quad \mathbf{u}_r = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$$\mathbf{b}_r = \begin{bmatrix} \frac{1}{\beta_r}\left(d_r\exp\left(\frac{w_r}{\gamma\alpha}\right)\right)^2-\frac{1}{\beta_1}\left(d_1\exp\left(\frac{w_1}{\gamma\alpha}\right)\right)^2-\kappa_r+\kappa_1 \\ \frac{1}{\beta_r}\left(d_r\exp\left(\frac{w_r}{\gamma\alpha}\right)\right)^2-\frac{1}{\beta_2}\left(d_2\exp\left(\frac{w_2}{\gamma\alpha}\right)\right)^2-\kappa_r+\kappa_2 \\ \vdots \\ \frac{1}{\beta_r}\left(d_r\exp\left(\frac{w_r}{\gamma\alpha}\right)\right)^2-\frac{1}{\beta_N}\left(d_N\exp\left(\frac{w_N}{\gamma\alpha}\right)\right)^2-\kappa_r+\kappa_N \end{bmatrix},$$

and $\mathbf{b}_r \in \mathbb{R}^{N\times 1}$

Moore-Penrose pseudo inverse is taken on both sides of (7) to obtain the solution

$$\hat{\mathbf{u}}_r = 0.5\mathbf{A}_r^{\dagger}\hat{\mathbf{b}}_r. \quad (8)$$

where

$$\mathbf{A}_r^{\dagger} = \left(\mathbf{A}_r^T\mathbf{A}_r\right)^{-1}\mathbf{A}_r^T$$

2) *Angle of Arrival.* If angle estimates are only available then we need only 2 and 3 ANs for 2d and 3d localization respectively. Each AN forms a line on which the AN and TN are situated. Hence we get a number of lines depending upon the number of ANs. The point of intersection of these lines is the estimated position of the TN. The AoA system generally shows good results but the estimation error increases significantly as the target moves away from the ANs.

Keeping the same notation as for RSS, we have

$$\hat{\theta}_i \approx \arctan\left[\frac{(y-y_i)}{(x-x_i)}\right]+m_i, \quad (9)$$

where m_i represents the zero mean Gaussian noise in the estimate of the angle $\hat{\theta}_i$ of the TN with i^{th}AN, i.e. $m_i \sim \left(\mathcal{N}\left(0,\sigma_{m_i}^2\right)\right)$. (9) can be written in matrix form as

$$\mathbf{A}_a\mathbf{u}_a = \hat{\mathbf{b}}_a \quad (10)$$

where,

$$\mathbf{A}_a = \begin{bmatrix} \tan\hat{\theta}_1 & -1 \\ \tan\hat{\theta}_2 & -1 \\ \vdots & \vdots \\ \tan\hat{\theta}_N & -1 \end{bmatrix} \in \mathbb{R}^{N\times 2}, \quad \mathbf{u}_a = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$$\hat{\mathbf{b}}_a = \begin{bmatrix} x_1\tan\hat{\theta}_1-y_1 \\ x_2\tan\hat{\theta}_2-y_2 \\ \vdots \\ \vdots \\ x_N\tan\hat{\theta}_N-y_N \end{bmatrix} \in \mathbb{R}^{N\times 1}$$

For the standard LLS estimator the solution is given by

$$\hat{\mathbf{u}}_a = \mathbf{A}_a^{\dagger}\hat{\mathbf{b}}_a \quad (11)$$

3) *Hybrid(ToA/AoA).* If both RSS and angle estimates are available then localization can be achieved with only one AN. But in order to achieve better accuracy more ANs can be introduced to the system. In this case each AN forms a line, rather than a circle. At one end of the line the AN is situated with known position while at the opposite end the TN is situated for which the coordinates are to be estimated. If the slope (AoA) and the magnitude (RSS) of this line is available, then the TN coordinates can be easily determined. The error in this case also increases with the increase in distance between AN and TN.

From the RSS and AoA equations given by (3) and (9) respectively we obtain

$$\mathbf{A} = \begin{bmatrix} 1_1 & 0 \\ \vdots & \vdots \\ 1_N & 0 \\ 0 & 1_1 \\ \vdots & \vdots \\ 0 & 1_N \end{bmatrix} \in \mathbb{R}^{2N\times 2} \quad \mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{2\times 1}$$

$$\hat{\mathbf{b}} = \begin{bmatrix} \hat{\mathbf{b}}_x \\ \hat{\mathbf{b}}_y \end{bmatrix} \in \mathbb{R}^{2N\times 1}$$

$$\hat{\mathbf{b}}_x = \begin{bmatrix} x_1 + \hat{d}_1 \cos\hat{\theta}_1 \delta_1 \\ \vdots \\ x_N + \hat{d}_N \cos\hat{\theta}_N \delta_N \end{bmatrix}$$

$$\hat{\mathbf{b}}_y = \begin{bmatrix} x_1 + \hat{d}_1 \sin\hat{\theta}_1 \delta_1 \\ \vdots \\ x_N + \hat{d}_N \sin\hat{\theta}_N \delta_N \end{bmatrix},$$

where δ_i is the bias reducing constant and is given by

$$\delta_i = \exp\left(\frac{\sigma_{m_i}^2}{2} - \frac{\sigma_{w_i}^2}{2(\gamma\alpha)^2}\right).$$

The LLS solution for the hybrid system is given by

$$\hat{\mathbf{u}} = \mathbf{A}^{\dagger}\hat{\mathbf{b}}. \tag{12}$$

III. Theoretical MSE of LLS

In this section we derive the theoretical MSE expression for the least square estimator in (12). The MSE for the hybrid system is given by

$$\text{MSE}(\mathbf{u}) = Tr\left\{E\left[(\hat{\mathbf{u}} - \mathbf{u})(\hat{\mathbf{u}} - \mathbf{u})^T\right]\right\}, \tag{13}$$

where $\hat{\mathbf{u}}$ is the erroneous estimated location and $\mathbf{u}$ is the location with no error. Putting (12) in (13) we get

$$\text{MSE}(\mathbf{u}) = \mathbf{A}^{\dagger}\mathbf{C}(\mathbf{u})\mathbf{A}^{\dagger^T}. \tag{14}$$

where $\mathbf{C}(\mathbf{u}) = E\left[\left(\hat{\mathbf{b}} - \mathbf{b}\right)\left(\hat{\mathbf{b}} - \mathbf{b}\right)^T\right]$, for $\mathbf{b}$ representing the noise free observation. The covariance $\mathbf{C}(\mathbf{u})$ can be partitioned into separate matrices as follows

$$\mathbf{C}(\mathbf{u}) = \begin{bmatrix} \mathbf{C}(x) & \mathbf{C}(xy) \\ \mathbf{C}(xy) & \mathbf{C}(y) \end{bmatrix} \tag{15}$$

$$\mathbf{C}(x) = E\left[\left(\hat{\mathbf{b}}_x - \mathbf{b}_x\right)\left(\hat{\mathbf{b}}_x - \mathbf{b}_x\right)^T\right] \in \mathbb{R}^{N\times N}. \tag{16}$$

For the diagonal terms i.e. $i = j$,

$$\mathbf{C}(x)_{ii} = \frac{d_i^2}{2}\exp\left(\frac{\sigma_{w_i}^2}{(\gamma\alpha)^2} + \sigma_{m_i}^2\right) + \frac{d_i^2}{2}\cos(2\theta_i)\exp\left(\frac{\sigma_{w_i}^2}{(\gamma\alpha)^2} - \sigma_{m_i}^2\right) - \left(d_i\cos(\theta_i)\right)^2. \tag{17}$$

On the other hand, for the non-diagonal terms i.e. $i \neq j$, we have

$$\mathbf{C}(x)_{ij} = 0. \tag{18}$$

$$\mathbf{C}(y) = E\left[\left(\hat{\mathbf{b}}_y - \mathbf{b}_y\right)\left(\hat{\mathbf{b}}_y - \mathbf{b}_y\right)^T\right] \in \mathbb{R}^{N\times N}. \tag{19}$$

For the diagonal terms i.e. $i = j$,

$$\mathbf{C}(y)_{ii} = \frac{d_i^2}{2}\exp\left(\frac{\sigma_{w_i}^2}{(\gamma\alpha)^2} + \sigma_{m_i}^2\right) - \frac{d_i^2}{2}\cos(2\theta_i)\exp\left(\frac{\sigma_{w_i}^2}{(\gamma\alpha)^2} - \sigma_{m_i}^2\right) - \left(d_i\sin(\theta_i)\right)^2. \tag{20}$$

On the other hand, for the non-diagonal terms i.e. $i \neq j$, we have

$$\mathbf{C}(y)_{ij} = 0. \tag{21}$$

Similarly,

$$\mathbf{C}(xy) = E\left[\left(\hat{\mathbf{b}}_x - \mathbf{b}_x\right)\left(\hat{\mathbf{b}}_y - \mathbf{b}_y\right)^T\right] \in \mathbb{R}^{N\times N}, \tag{22}$$

for the diagonal terms i.e. $i = j$,

$$\mathbf{C}(xy)_{ii} = d_i^2\cos\theta_i\sin\theta_i\exp\left(\frac{\sigma_{w_i}^2}{(\gamma\alpha)^2} - \sigma_{m_i}^2\right) - d_i^2\cos\theta_i\sin\theta_i. \tag{23}$$

On the other hand, for the non-diagonal terms i.e. $i \neq j$, we have

$$\mathbf{C}(xy)_{ij} = 0. \tag{24}$$

Proof: Appendix. ■

IV. Weighted Linear Least Squares Algorithm

The location estimates obtained from (12) shows a high error because in (12) the information about the link quality is not utilized. If this information is available at hand, we can use it to improve the performance of the system. Thus in this section we propose a weighted linear least squares (WLLS) algorithm by exploiting the covariance matrix. The covariance matrix $\mathbf{C}(\mathbf{u})$ gives us the information about the link quality. Thus the links with large noise variance are given less weights and vice versa. The WLLS solution is obtained by minimizing the cost function.

$$\varepsilon_{WLLS}(\hat{\mathbf{u}}) = (\mathbf{b} - \mathbf{A}\hat{\mathbf{u}})^T\mathbf{C}(\mathbf{u})^{-1}(\mathbf{b} - \mathbf{A}\hat{\mathbf{u}}) \tag{25}$$

where $\mathbf{C}(\mathbf{u})^{-1}$ is the inverse of the covariance matrix defined in (15).

The elements of (15) are dependent on the real values of distance and angles which are not available. Thus we use the estimated values to get the estimated covariance matrix $\mathbf{C}(\hat{\mathbf{u}})^{-1}$. The WLLS solution is obtained as follows,

$$\hat{\mathbf{u}}_{WLLS} = \mathbf{A}^{\pm}\hat{\mathbf{b}}^{\pm} \tag{26}$$

where
$\mathbf{A}^{\pm} = \left[\mathbf{A}^T\mathbf{C}(\hat{\mathbf{u}})\mathbf{A}\right]^{-1}\mathbf{A}^T$ and $\hat{\mathbf{b}}^{\pm} = \mathbf{C}(\hat{\mathbf{u}})\hat{\mathbf{b}}$.

V. SIMULATION RESULTS

In this section, performance of the system, obtained by Monte Carlo simulation is compared with the theoretical MSE derived in section III. Furthermore it is shown through simulation that the WLLS algorithm produces better results than LLS. System performance is also analyzed by changing different variable like PLE, angle noise variance and distance noise variance. All simulations are run independently η number of times.

In Fig. 1, theoretical MSE is compared with Avg. RMSE obtained by Monte Carlo simulation. Four ANs are taken at the corners of a $40\text{m} \times 40\text{m}$ network and five targets are taken at random locations. The PLE is kept fixed at 2.5 for all links, while noise variance in angle and distance estimates is increased gradually. From Fig. 1 it can be seen that the theoretical MSE accurately predicts the system performance.

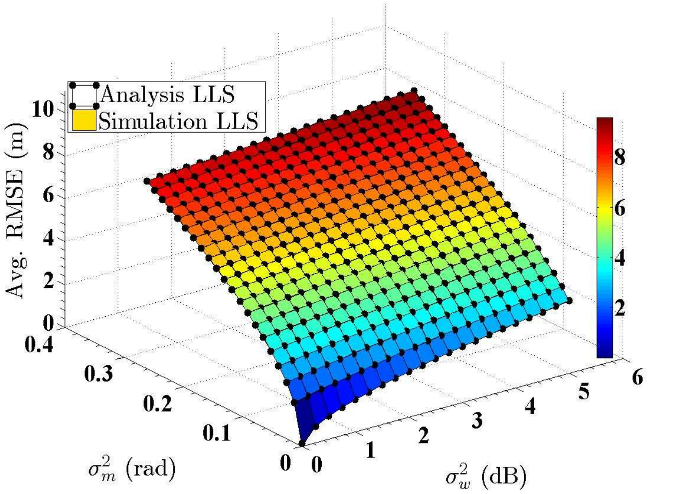

Figure 1. RMSE for analytical LLS and simulation LLS. $\alpha = 2.5$, $N = 4$, $\eta = 2000$.

Fig. 2 compares the performance of LLS with WLLS algorithm. Again we take four ANs at the corners of the network and 5 TNs are taken at random locations, PLE is kept at 2.5 and RMSE is plotted against shadowing and angle noise variance. Fig. 2 shows that WLLS outperforms LLS.

In Fig. 3, RMSE is plotted against shadowing noise variance and PLE. Performance is analyzed by increasing shadowing noise variance from 0 to 6 dB and PLE from 2 to 4.5, while keeping angle noise variance fixed at 4^o. Fig. 3 shows that as the PLE increases, performance of the system slightly improves for both LLS and WLLS, which is the expected result. Also expected is the better performance of WLLS, which is justified by the figure.

VI. CONCLUSION

In this paper, a LLS model for hybrid localization was reviewed. The system performance was analyzed and an analytical MSE equation was derived. Performance was improved by introducing a WLLS algorithm which exploits the covariance matrix. Via simulation it was shown that the WLLS algorithm shows better performance than the LLS algorithm. In future work a combined PLE/location estimator will be developed using AoA-RSS model.

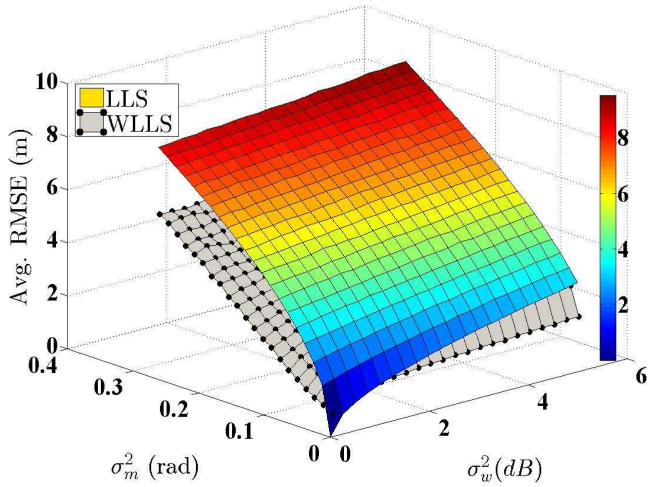

Figure 2. Performance comparison between LLS and WLLS. $\alpha = 2.5$, $N = 4$, $\eta = 2000$.

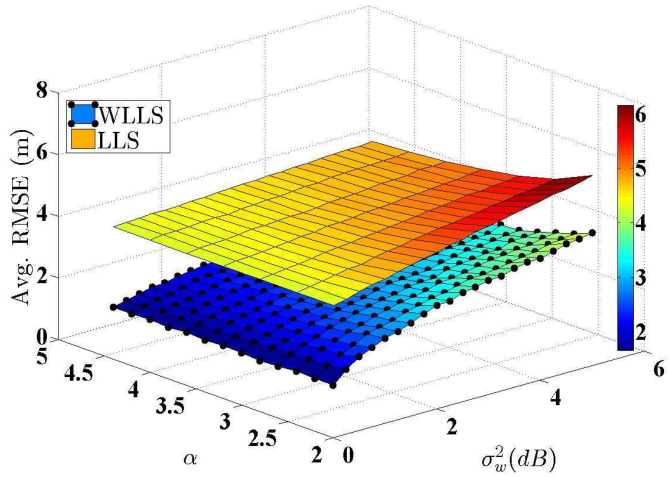

Figure 3. Performance comparison between LLS and WLLS for a different PLE's. $N = 4$, $\sigma_m = 2^o$, $\eta = 3000$.

APPENDIX

Proof of (17) and (18): For diagonal terms i.e. $i = j$, we have from (16).

$$\mathbf{C}\left(x\right)_{ii} = E_{w,m}\left[\left(\hat{\mathbf{b}}_x - \mathbf{b}_x\right)^2\right]_{ii}. \tag{27}$$

Putting values of $\hat{\mathbf{b}}_x$ and $\mathbf{b}_x$ in (27)

$$\begin{aligned}\mathbf{C}\left(x\right)_{ii} &= E_{w_i,m_i}\left[\left(d_i \exp\left(\tfrac{w_i}{\gamma\alpha}\right)\cos\left(\theta_i + m_i\right)\delta_i - d_i\cos\theta_i\right)^2\right],\\ &= E_{w_i,m_i}\Bigg[\left\{\left(d_i \exp\left(\frac{w_i}{\gamma\alpha}\right)\right)^2 \cos^2(\theta_i + m_i)\right\}\delta_i^2 + \left(d_i\cos\theta_i\right)^2\\ &\quad -2\delta_i\left(d_i\cos\theta_i\right)\left(d_i\exp\left(\frac{w_i}{\gamma\alpha}\right)\right)\cos\left(\theta_i + m_i\right)\Bigg],\end{aligned} \tag{28}$$

$$= E_{w,m}\Bigg[d_i^2 \exp\Big(\frac{2w_i}{\gamma\alpha}\Big) \times \Big\{0.5 + 0.5\Big(\cos(2\theta_i)\cos(2m_i) - \sin(2\theta_i)\sin(2m_i)\Big)\Big\}\delta_i^2 + \Big(d_i\cos\theta_i\Big)^2 - 2\delta_i\Big(d_i^2\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\Big) \times \Big(\cos^2(\theta_i)\cos(m_i) - \cos(\theta_i)\sin(\theta_i)\sin(m_i)\Big)\Bigg]. \quad (29)$$

Equ. (29) is obtained from (28) by using the identity $\cos^2(t) = 0.5 + 0.5\cos(2t)$. Also using the expectation $E_{m_i}[\sin(m_i)] = 0$ and $E_{m_i}[\sin(2m_i)] = 0$, (32) is obtained.

$$= \Bigg\{\frac{d_i^2}{2}E_{w_i}\Big[\exp\Big(\frac{2w_i}{\gamma\alpha}\Big)\Big] + \frac{d_i^2}{2}E_{w_i}\Big[\exp\Big(\frac{2w_i}{\gamma\alpha}\Big)\Big]\cos 2\theta_i \times E_{m_i}\Big[\cos(2m_i)\Big]\Bigg\}\delta_i^2 + \Big(d_i\cos(\theta_i)\Big)^2 - 2\delta_i\Big(d_i\cos(\theta_i)\Big)^2 \times E_{w_i}\Big[\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\Big]E_{m_i}\Big[\cos(m_i)\Big]. \quad (32)$$

Finally, using expectations

$$E[\cos(m_i)] = \exp\Big(-\frac{\sigma_{m_i}^2}{2}\Big),\ E\Big[\exp\Big(\frac{2w_i}{\gamma\alpha}\Big)\Big] = \exp\Bigg(\frac{2\sigma_{w_i}^2}{(\gamma\alpha)^2}\Bigg), \quad (33)$$

$$E[\cos(2m_i)] = \exp\big(-2\sigma_{m_i}^2\big),\ E\Big[\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\Big] = \exp\Bigg(\frac{\sigma_{w_i}^2}{2(\gamma\alpha)^2}\Bigg), \quad (34)$$

we conclude the proof by obtaining (17).

For non-diagonal terms i.e. $i \neq j$, we have from (16)

$$\mathbf{C}(x)_{ij} = E_{w_{ij},m_{ij}}\Bigg[\Big(d_i\exp\Big(\tfrac{w_i}{\gamma\alpha}\Big)\cos(\theta_i + m_i)\delta_i - d_i\cos\theta_i\Big) \times \Big(d_j\exp\Big(\frac{w_j}{\gamma\alpha}\Big)\cos(\theta_j + m_j)\delta_j - d_j\cos\theta_j\Big)\Bigg],$$

which yields (30) and then (31) given at the top of next page.

The derivation of $\mathbf{C}(y)$ is similar to the $\mathbf{C}(x)$ other than that the x coordinates are replaced by y.

Proof of (23) and (24).: From (22) we have

$$\mathbf{C}(x,y)_{ii} = E_{w_i,m_i}\Bigg[\Big(d_i\exp\Big(\tfrac{w_i}{\gamma\alpha}\Big)\cos(\theta_i + m_i)\delta_i - d_i\cos\theta_i\Big) \times \Big(d_i\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\sin(\theta_i + m_i)\delta_i - d_i\sin\theta_i\Big)\Bigg],$$

$$= E_{w_i,m_i}\Bigg[\Big(d_i\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\Big)^2\cos(\theta_i + m_i)\sin(\theta_i + m_i)\delta_i^2 - d_i\Big(d_i\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\Big)\cos(\theta_i + m_i)\sin\theta_i\delta_i - d_i\Big(d_i\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\Big)\sin(\theta_i + m_i)\cos\theta_i\delta_i + d_i^2\cos\theta_i\sin\theta_i\Bigg]. \quad (35)$$

Expanding (35), and then using double angle identity $\cos^2 t - \sin^2 t = \cos(2t)$, and also $E[\sin(m_i)] = 0$, we obtain

$$= E_{w_i,m_i}\Bigg[\Big\{d_i^2\exp\Big(\frac{2w_i}{\gamma\alpha}\Big)\cos(\theta_i)\sin(\theta_i)\cos(2m_i)\Big\}\delta_i^2 - \Big\{d_i^2\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\cos(\theta_i)\cos(m_i)\sin(\theta_i)\Big\}\delta_i - \Big\{d_i^2\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\sin(\theta_i)\cos(m_i)\cos(\theta_i)\Big\}\delta_i + d_i^2\cos\theta_i\sin\theta_i\Bigg]. \quad (36)$$

Finally, using expectation (33) and (34) we conclude the proof by obtaining (23) from equation (36).

For non-diagonal terms i.e. $i \neq j$, we have from (22).

$$\mathbf{C}(x)_{ij} = E_{w_{ij},m_{ij}}\Bigg[\Big(d_i\exp\Big(\frac{w_i}{\gamma\alpha}\Big)\cos(\theta_i + m_i)\delta_i - d_i\cos\theta_i\Big) \times \Big(d_j\exp\Big(\frac{w_j}{\gamma\alpha}\Big)\sin(\theta_j + m_j)\delta_j - d_j\sin\theta_j\Big)\Bigg],$$

which yields (37) and then (38) given at the top of next page.

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$$\mathbf{C}(x)_{ij} = E_{w_{ij},m_{ij}}\Bigg[\Bigg\{\Bigg(d_i d_j \exp\left(\frac{w_i}{\gamma\alpha}\right)\exp\left(\frac{w_j}{\gamma\alpha}\right)\Bigg)\left(\cos(\theta_i)\cos(m_i)+\sin(\theta_i)\sin(m_i)\right)\left(\cos(\theta_j)\cos(m_j)+\sin(\theta_j)\sin(m_j)\right)\delta_{ij}\Bigg\}$$
$$\Bigg\{\Bigg(d_i d_j \exp\left(\frac{w_i}{\gamma\alpha}\right)\Bigg)\left(\cos(\theta_i)\cos(\theta_j)\cos(m_i)+\sin(\theta_i)\cos(\theta_j)\sin(m_i)\right)\delta_i\Bigg\}$$
$$-\Bigg\{\Bigg(d_i d_j \exp\left(\frac{w_j}{\gamma\alpha}\right)\Bigg)\left(\cos(\theta_i)\cos(\theta_j)\cos(m_j)+\sin(\theta_i)\cos(\theta_j)\sin(m_j)\right)\delta_j\Bigg\}+d_i d_j\cos(\theta_i)\cos(\theta_j)\Bigg], \quad (30)$$

where $\delta_{ij} = \delta_i \delta_j$.
Taking expectations in (30), we obtain

$$\mathbf{C}(x)_{ij} = d_i d_j \cos(\theta_i)\cos(\theta_j) - d_i d_j \cos(\theta_i)\cos(\theta_j) - d_i d_j \cos(\theta_i)\cos(\theta_j) + d_i d_j \cos(\theta_i)\cos(\theta_j) = 0. \quad (31)$$

which proves (18).

$$\mathbf{C}(xy)_{ij} = E_{w_{ij},m_{ij}}\Bigg[\Bigg\{\Bigg(d_i d_j \exp\left(\frac{w_i}{\gamma\alpha}\right)\exp\left(\frac{w_j}{\gamma\alpha}\right)\Bigg)\left(\cos(\theta_i)\cos(m_i)-\sin(\theta_i)\sin(m_i)\right)\left(\sin(\theta_j)\cos(m_j)+\cos(\theta_j)\sin(m_j)\right)\delta_{ij}\Bigg\}$$
$$-\Bigg\{\Bigg(d_i d_j \exp\left(\frac{w_i}{\gamma\alpha}\right)\Bigg)\left(\cos(\theta_i)\sin(\theta_j)\cos(m_i)+\sin(\theta_i)\sin(\theta_j)\sin(m_i)\right)\delta_i\Bigg\}$$
$$-\Bigg\{\Bigg(d_i d_j \exp\left(\frac{w_j}{\gamma\alpha}\right)\Bigg)\left(\cos(\theta_i)\sin(\theta_j)\cos(m_j)+\cos(\theta_i)\cos(\theta_j)\sin(m_j)\right)\delta_j\Bigg\}+d_i d_j\cos(\theta_i)\sin(\theta_j)\Bigg]. \quad (37)$$

Taking expectations in (37), we obtain

$$\mathbf{C}(xy)_{ij} = d_i d_j \cos(\theta_i)\sin(\theta_j) - d_i d_j \cos(\theta_i)\sin(\theta_j) - d_i d_j \cos(\theta_i)\sin(\theta_j) + d_i d_j \cos(\theta_i)\sin(\theta_j) = 0. \quad (38)$$

which proves (24).

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