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Determination of the ambient temperature in transient heat conduction

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Abstract

The restoration of the space- or time-dependent ambient temperature entering a third-kind convective Robin boundary condition in transient heat conduction is investigated. The temperature inside the solution domain together with the ambient temperature are determined from additional boundary measurements. In both cases of the space- or time-dependent unknown ambient temperature the inverse problems are linear and ill-posed. Least-squares penalised variational formulations are proposed and new formulae for the gradients are derived. Numerical results obtained using the conjugate gradient method combined with a boundary element direct solver are presented and discussed.

Keywords: Heat equation, ambient temperature, boundary element method, conjugate gradient method, inverse problem

1 Introduction

Ambient temperature refers to the temperature which surrounds a heating or cooling object under investigation and its knowledge is very important for safe and efficient performance of heat transfer equipment, e.g. thermal flow sensors, [17]. If convection occurs only on a "hostile" part of the boundary of the heat conductor which is inaccessible to practical measurements, then, in principle, the ambient temperature could be determined by solving an ill-posed inverse heat conduction problem using the Cauchy data measurements of both the temperature and the heat flux on the remaining "friendly" part of the boundary. However, in many physical situations, e.g. high pressures, high temperatures hostile environments, the measurements of the surface (boundary) temperature and the heat flux can experience practical difficulties and in some cases the relationship between these quantities is unattainable, see e.g. [1, 3, 4]. Therefore, in order to prevent this experimental difficulty, in the mathematical formulation of Section 2 we allow for the convection Robin boundary condition of the third kind (on the boundary of the solution domain there is convective heat transfer with the environment), as given by Newton's law of cooling or heating, to be prescribed over the whole boundary. Then, we study the inverse problems of restoring the ambient temperature from additional terminal, point or integral boundary temperature measurements (observations).

Further, in our study the unknown ambient temperature is allowed to vary with space or time. Therefore, a more realistic model can be proposed for the heat transfer in building enclosures, e.g. glazed surfaces, where the ambient temperature can vary spatially, or with time, depending on the local air patterns, e.g. type of flow, external weather conditions, etc., [16].

The plan of the paper is as follows. In Section 2 we formulate the inverse problems for the determination of a space-dependent (Problem I) or time-dependent (Problem II) ambient temperature and recall the available existence and uniqueness results in the classical sense. Section 3 is devoted to defining the weak solutions of the direct and adjoint Robin problems and recalling their unique solvability. The symmetric Galerkin formulation of the boundary element method (BEM) given in [2] for the Dirichlet and Neumann direct problems is extended in Section 4 to the Robin problem for the transient heat equation. Furthermore, in our inverse problems, all the unknowns and additional observations are at the boundary and the discretization of the boundary only is the essence of the BEM. Therefore, it seems more natural and appropriate to use the BEM instead of the domain discretization methods such as the finite element or finite difference methods. Sections 5 and 6 are devoted to developing the least-squares variational methods for solving the inverse problems I and II, respectively. In each of these sections we present the numerical results for several benchmark test examples of interest obtained using the iterative conjugate gradient method (CGM) combined with the BEM direct solver. In all cases, numerical stability and good accuracy are achieved provided that the iterative process is stopped according to the discrepancy principle. Finally, Section 7 presents the summary, conclusions and future work.

2 Mathematical formulation

Let $\Omega \in \mathbb{R}^d$ be a bounded domain and denote its boundary by Γ. In the cylinder $Q := \Omega \times (0, T]$, where $T > 0$, with the lateral surface area $S = \Gamma \times (0, T]$, consider the following inverse problems ([8] and [9]). Throughout the paper, u denotes the temperature, f the ambient temperature, a the initial temperature, g the heat source, and σ the heat transfer coefficient.

Inverse Problem I. Find a pair of functions $\{u(x,t), f(\xi)\}$ such that

$$u_t - \Delta u = g \quad \text{in } Q, \tag{2.1}$$

$$u(x,0) = a(x), \quad x \in \overline{\Omega}, \tag{2.2}$$

$$\frac{\partial u}{\partial n} + \sigma(\xi,t)u = h(\xi,t)f(\xi) + b(\xi,t), \quad (\xi,t) \in S, \tag{2.3}$$

$$l(u) = \chi(\xi), \quad \xi \in \Gamma, \tag{2.4}$$

where the functions $g(x,t), a(x), \sigma(\xi,t), h(\xi,t), b(\xi,t)$ and $\chi(\xi)$ are given, and n is the outward unit normal to the boundary Γ. Strictly speaking h should be equal to σ in order for f to represent the actual ambient temperature, but the boundary condition in (2.3) models a more general situation, which also include an additional heat flux contribution $b(\xi,t)$. In (2.4), the observation operator l has one of the following forms:

$$l(u) = u(\xi, T_1), \quad \xi \in \Gamma, \tag{2.5}$$

where T_1 is a fixed known time in $(0, T]$, or

$$l(u) = \int_0^T \omega(t)u(\xi,t)dt, \quad \xi \in \Gamma, \tag{2.6}$$

with ω being a given function in $L^1(0,T)$. The additional conditions (2.5) and (2.6) are called terminal and integral boundary observations, respectively.

Inverse Problem II. Find a pair of functions $\{u(x,t), f(t)\}$ such that

$$u_t - \Delta u = g \quad \text{in } Q, \tag{2.7}$$

$$u(x,0) = a(x), \quad x \in \overline{\Omega}, \tag{2.8}$$

$$\frac{\partial u}{\partial n} + \sigma(\xi,t)u = h(\xi,t)f(t) + b(\xi,t), \quad (\xi,t) \in S, \tag{2.9}$$

$$l_1(u) = \chi_1(t), \quad t \in [0,T], \tag{2.10}$$

where the functions $g(x,t), a(x), \sigma(\xi,t), h(\xi,t)$, and $\chi_1(t)$ are given. The observation operator $l_1(u)$ has one of the following forms:

$$l_1(u) = u(\xi_0,t), \quad t \in [0,T], \tag{2.11}$$

where ξ_0 is fixed known point in Γ, or

$$l_1(u) = \int_\Gamma \nu(\xi)u(\xi,t)d\xi, \quad t \in [0,T], \tag{2.12}$$

with $\nu(\xi)$ being a given function in $L^1(\Gamma)$. The additional conditions (2.11) and (2.12) are called point and boundary integral observations, respectively.

At this stage, it is worth mentioning that in practice conditions (2.6) and (2.12) are indeed measured by averaging a series of pointwise boundary temperature measurements. This is particularly advantageous to use in situations where the time pointwise or space pointwise boundary temperature measurements (2.5) or (2.6) posses different sensitivities with respect to the value of T_1 within the interval $(0,T]$ or, the boundary point ξ_0 along the boundary Γ, respectively. On the other hand, one can observe that equations (2.6) and (2.12) reduce to equations (2.5) (for $T_1 \in (0,T)$) and (2.11) if one takes the weights $\omega(t) = \delta(t-T_1)$ and $\nu(\xi) = \delta(\xi - \xi_0)$, respectively, where δ is the Dirac delta function. However, because ω and ν have to be L^1-integrable, these choices are not quite strictly possible. Approximations with Gaussian functions or employing cut-off weights, see later equations (5.1) and (6.1), can be alternatives to model pointwise measurements (thermocouples have non-zero width, or the time is never instant) as local averages.

The common feature in the above inverse problems is the Robin third kind boundary condition, see equations (2.3) and (2.9).

The notation for the spaces of functions involved in the following theorems follows [7]. With the assumptions that Ω is simply-connected and its boundary $\Gamma \in C^{1+\beta}$ with $\beta > 0$, $g \in C^{\beta,0}(\overline{Q})$, $a \in C^1(\overline{\Omega})$, $h, b \in C(\overline{S})$, Kostin and Prilepko [8, 9] proved the following results.

Theorem 2.1. *Suppose that σ is independent of t, $\sigma \in C(\Gamma)$, $0 \le \sigma(\xi)$ on Γ, $\omega(t) \ge 0$ on $[0,T]$, $l(h) > 0$ almost everywhere on Γ, and the function h is positive on S, monotone non-decreasing with respect to t. Then the solution $(u(x,t), f(\xi)) \in C^{2,1}(Q) \times C(\Gamma)$ to the inverse problem I is unique.*

Theorem 2.2. *Assume $\sigma \in C(\overline{S})$ and denote by $u^0 \in C^{2,1}(Q) \cap C^{\beta,\beta/2}(\overline{Q})$ the unique solution of the direct problem (2.7)–(2.9) with $f = 0$ (see [7]). Further, assume that the function $\chi_2(t) := \chi_1(t) - l_1(u^0) \in C^{1/2}[0,T]$ and $\chi_2(0) = 0$, $\frac{d}{dt}\int_0^t \frac{\chi_2(\tau)}{\sqrt{t-\tau}}d\tau \in C[0,T]$. Then, if $h \in C^{\beta,0}(\overline{S})$ and $|l_1(h)| > 0$ on $[0,T]$, there exists a unique solution $(u(x,t), f(t)) \in C^{2,1}(Q) \times C[0,T]$ to the inverse problem II.*

Although of theoretical interest, these uniqueness theorems cannot be used directly in the numerical analysis, since it is not straightforward how to use the space of continuous functions in a weak formulation. Therefore, in this paper we relax some assumptions on the smoothness of the data posed above so that we can work in the Hilbert space framework. Then we can solve the above inverse problems in the least-squares sense. We will report about this in the next section.

3 Direct problem

In this section, we suppose that Ω is a bounded Lipschitz domain and introduce the notion for standard Sobolev spaces as follows.

For a Banach space B, we define

$$L^2(0,T;B) = \{u : u(t) \in B \ \text{ a.e. } t \in (0,T) \text{ and } \|u\|_{L^2(0,T;B)} < \infty\},$$

with the norm

$$\|u\|^2_{L^2(0,T;B)} = \int_0^T \|u(t)\|^2_B dt.$$

In the sequel, we shall use the space $W(0,T)$ defined as

$$W(0,T) = \{u : u \in L^2(0,T;H^1(\Omega)), u_t \in L^2(0,T;(H^1(\Omega))')\},$$

equipped with the norm

$$\|u\|^2_{W(0,T)} = \|u\|^2_{L^2(0,T;H^1(\Omega))} + \|u_t\|^2_{L^2(0,T;(H^1(\Omega))')}.$$

Now consider the direct problem

$$u_t - \Delta u = g \quad \text{in } Q, \tag{3.1}$$

$$u(x,0) = a(x), \quad x \in \Omega, \tag{3.2}$$

$$\frac{\partial u}{\partial n} + \sigma(\xi,t)u = b(\xi,t), \quad (\xi,t) \in S, \tag{3.3}$$

with

$$g \in L^2(Q), a \in L^2(\Omega), \sigma \in L^\infty(S), \sigma \geq 0, b \in L^2(S). \tag{3.4}$$

Definition 3.1. *A function $u \in W(0,T)$ is called a weak solution to the direct problem* (3.1)–(3.3), *if*

$$\int_Q (u_t\eta + \nabla u \cdot \nabla \eta) dxdt + \int_S \sigma u \eta d\xi dt = \int_Q g\eta dxdt + \int_S b\eta d\xi dt \tag{3.5}$$

for all $\eta \in L^2(0,T;H^1(\Omega))$, and $u(\cdot,0) = a$.

The following theorem giving the existence and uniqueness of a weak solution to the direct problem (3.1)–(3.3) is given in [15].

Theorem 3.2. *Suppose that conditions* (3.4) *are satisfied. Then there exists a unique weak solution in $W(0,T)$ of the direct problem* (3.1)–(3.3)*. Moreover, there exists a constant $c_d > 0$ independent of g, b and a such that*

$$\|u\|_{W(0,T)} \leq c_d(\|g\|_{L^2(Q)} + \|b\|_{L^2(S)} + \|a\|_{L^2(\Omega)}). \tag{3.6}$$

Remark 3.3. *The constant c_d depends on σ. However, if we suppose that*

$$0 < \sigma_1 \le \sigma \le \sigma_2,$$

where σ_1 and σ_2 are given, then by examining the proof of this theorem in [10] and [15] we see that c_d depends on these two constants only.

We introduce now the adjoint problem to (3.1)–(3.3) as follows:

$$-\psi_t - \Delta\psi = a_Q \quad \text{in } Q, \tag{3.7}$$

$$\psi(x,T) = a_\Omega(x), \quad x \in \Omega, \tag{3.8}$$

$$\frac{\partial \psi}{\partial n} + \sigma(\xi,t)\psi = a_S(\xi,t), \quad (\xi,t) \in S, \tag{3.9}$$

with

$$a_Q \in L^2(Q), a_\Omega \in L^2(\Omega), \sigma \in L^2(S), \sigma \ge 0, a_S \in L^2(S). \tag{3.10}$$

From Lemma 3.17 and Theorem 3.18 in [15] we have the following theorem giving the existence and uniqueness of a weak solution to the adjoint problem (3.7)–(3.9).

Theorem 3.4. *Suppose that conditions* (3.10) *are satisfied. Then there exists a unique weak solution in $W(0,T)$ of the adjoint problem* (3.7)–(3.9) *in the sense that*

$$\int_Q (-\psi_t\eta + \nabla\psi \cdot \nabla\eta) dxdt = \int_Q a_Q \eta dxdt + \int_S a_S \eta d\xi dt - \int_S \sigma\psi\eta d\xi dt$$

for all $\eta \in L^2(0,T;H^1(\Omega))$, and $\psi(\cdot,T) = a_\Omega(\cdot)$. Moreover, there exists a constant $c_a > 0$ independent of a_Q, a_Ω and a_S such that

$$\|\psi\|_{W(0,T)} \le c_a(\|a_Q\|_{L^2(Q)} + \|a_S\|_{L^2(S)} + \|a_\Omega\|_{L^2(\Omega)}).$$

Furthermore, if $u \in W(0,T)$ is the weak solution to the problem (3.1)–(3.3), *then*

$$\int_\Omega a_\Omega(x)u(x,T)dx + \int_Q a_Q u dxdt + \int_S a_S u d\xi dt = \int_\Omega a(x)\psi(x,0)dx + \int_Q g\psi dxdt + \int_S b\psi d\xi dt. \tag{3.11}$$

4 Boundary element method for the direct problem

The unknown Cauchy data $[w := \dfrac{\partial u}{\partial n}, u]$ on S of the direct problem (3.1)–(3.3) with the input data satisfying (3.4) can be found by the boundary integral equation approach of [2]. Indeed, the solution of the heat equation (3.1) is given by a representation formula, for $(\tilde{x},t) \in Q$,

$$u(\tilde{x},t) = \int_0^t \int_\Gamma \mathcal{E}(\tilde{x}-y,t-\tau)w(y,\tau)\, ds_y\, d\tau - \int_0^t \int_\Gamma \frac{\partial \mathcal{E}}{\partial n_y}(\tilde{x}-y,t-\tau)u(y,\tau)\, ds_y\, d\tau + \int_\Omega \mathcal{E}(\tilde{x}-y,t)a(y)\, dy + \int_0^t \int_\Omega \mathcal{E}(\tilde{x}-y,t-\tau)g(y,\tau)\, dy\, d\tau, \tag{4.1}$$

where $\mathcal{E}(x,t)$ is the fundamental solution of the heat equation as given in [2]:

$$\mathcal{E}(x,t) = \begin{cases} (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

We define the single and double layer heat potentials as

$$(Vw)(x,t) = \int_0^t \int_\Gamma \mathcal{E}(x-\xi, t-\tau) w(\xi,\tau)\, d\xi\, d\tau, \quad (Ku)(x,t) = \int_0^t \int_\Gamma \frac{\partial}{\partial n_\xi} \mathcal{E}(x-\xi, t-\tau) u(\xi,\tau)\, d\xi\, d\tau,$$

for $(x,t) \in S$, and the boundary integral operators N and W,

$$(Nw)(x,t) = \int_0^t \int_\Gamma \frac{\partial}{\partial n_x} \mathcal{E}(x-\xi, t-\tau) w(\xi,\tau)\, d\xi\, d\tau,$$

and

$$(Wu)(x,t) = -\frac{\partial}{\partial n_x} \int_0^t \int_\Gamma \frac{\partial}{\partial n_\xi} \mathcal{E}(x-\xi, t-\tau) u(\xi,\tau)\, d\xi\, d\tau$$

as in [2]. Moreover, we introduce the volume potentials, for $(x,t) \in S$,

$$(\mathcal{M}_0 a)(x,t) = \int_\Omega \mathcal{E}(x-y, t) a(y)\, dy, \quad (\mathcal{N}_0 g)(x,t) = \int_0^t \int_\Omega \mathcal{E}(x-y, t-\tau) g(y,\tau)\, dy\, d\tau$$

and

$$(\mathcal{M}_1 a)(x,t) = \frac{\partial}{\partial n_x} \int_\Omega \mathcal{E}(x-y, t) a(y)\, dy, \quad (\mathcal{N}_1 g)(x,t) = \frac{\partial}{\partial n_x} \int_0^t \int_\Omega \mathcal{E}(x-y, t-\tau) g(y,\tau)\, dy\, d\tau.$$

For the properties of the above operators, see [2, 14]. In particular, we have that N is the adjoint of the double layer potential K with respect to the "time-twisted" duality, see [2, p.541], i.e.,

$$\langle \kappa_T w, N\varphi \rangle = \langle \kappa_T \varphi, Kw \rangle,$$

where the time reversal map κ_T is defined by $\kappa_T v(x,t) := v(x, T-t)$.

As in [2], we obtain the boundary integral equations

$$(Vw)(x,t) = \left(\frac{1}{2} I + K\right) u(x,t) - (\mathcal{M}_0 a)(x,t) - (\mathcal{N}_0 g)(x,t) \quad \text{for } (x,t) \in S, \tag{4.2}$$

and

$$(Wu)(x,t) = \left(\frac{1}{2} I - N\right) w(x,t) - (\mathcal{M}_1 a)(x,t) - (\mathcal{N}_1 g)(x,t) \quad \text{for } (x,t) \in S. \tag{4.3}$$

From the boundary condition (3.3), we are now in a position to rewrite the boundary integral equations (4.2) and (4.3) as follows:

$$\mathcal{A} \begin{pmatrix} w \\ u \end{pmatrix} := \begin{pmatrix} V & -\left(\frac{1}{2} I + K\right) \\ \left(\frac{1}{2} I + N\right) & W + \sigma I \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} -\mathcal{M}_0 a - \mathcal{N}_0 g \\ b - \mathcal{M}_1 a - \mathcal{N}_1 g \end{pmatrix}. \tag{4.4}$$

Lemma 4.1. *The operator $\mathcal{A}$ is elliptic, i.e.,*

$$\langle \mathcal{A}\begin{pmatrix} w \\ u \end{pmatrix}, \begin{pmatrix} w \\ u \end{pmatrix} \rangle = \langle \begin{pmatrix} V & -K \\ N & W \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix}, \begin{pmatrix} w \\ u \end{pmatrix} \rangle + \langle \sigma u, u \rangle \geq C \left(\|w\|^2_{H^{-\frac{1}{2},-\frac{1}{4}}(S)} + \|u\|^2_{H^{\frac{1}{2},\frac{1}{4}}(S)} \right)$$

for all $w \in H^{-\frac{1}{2},-\frac{1}{4}}(S)$, $u \in H^{\frac{1}{2},\frac{1}{4}}(S)$ and some positive constant C.

Proof. See [2, Theorem 3.11] and use the condition $\sigma \geq 0$. □

By assumptions (3.4), the boundary integral equations (4.4) admit a unique solution $(w, u) \in H^{-\frac{1}{2},-\frac{1}{4}}(S) \times H^{\frac{1}{2},\frac{1}{4}}(S)$. Let us consider now the numerical discretization of (4.4).

Let $\mathcal{V}_h$ be the trial space of functions which are piecewise linear with respect to the space variables on a triangulation of Γ and piecewise constant with respect to the time variable. We also introduce a set of ansatz functions $\mathcal{U}_h$ consisting of piecewise constant basis functions both in space and in time, see [2, 14].

The Galerkin variational formulation of (4.4) is to find $(w_h, u_h) \in \mathcal{V}_h \times \mathcal{U}_h$ such that

$$\langle \mathcal{A}\begin{pmatrix} w_h \\ u_h \end{pmatrix}, \begin{pmatrix} \tau_h \\ v_h \end{pmatrix} \rangle = \langle \begin{pmatrix} -\mathcal{M}_0 a - \mathcal{N}_0 g \\ b - \mathcal{M}_1 a - \mathcal{N}_1 g \end{pmatrix}, \begin{pmatrix} \tau_h \\ v_h \end{pmatrix} \rangle \quad \text{for all } (\tau_h, v_h) \in \mathcal{V}_h \times \mathcal{U}_h.$$

This is equivalent to

$$\begin{cases} \langle V w_h, \tau_h \rangle_S - \langle \left(\frac{1}{2} I + K\right) u_h, \tau_h \rangle_S & = -\langle \mathcal{M}_0 a + \mathcal{N}_0 g, \tau_h \rangle_S, \\ \langle \left(\frac{1}{2} I + N\right) w_h, v_h \rangle_S + \langle W u_h, v_h \rangle_S + \langle \sigma u_h, v_h \rangle_S & = \langle b - \mathcal{M}_1 a - \mathcal{N}_1 g, v_h \rangle_S, \end{cases} \tag{4.5}$$

for all $(\tau_h, v_h) \in \mathcal{V}_h \times \mathcal{U}_h$.

Let

$$u_h(x,t) = \sum_{\ell=0}^{n-1} \sum_{i=0}^{m_1-1} u_{i\ell} \varphi_i^1(x) \psi_\ell^0(t), \quad w_h(x,t) = \sum_{\ell=0}^{n-1} \sum_{j=0}^{m_0-1} w_{j\ell} \varphi_j^0(x) \psi_\ell^0(t).$$

Here, $m_0 = m_1$ in two dimensional case, n is the number of time steps, $\varphi_j^0(x)$ and $\varphi_i^1(x)$ are piecewise constant and piecewise linear basis functions in space, respectively, and $\psi_\ell^0(t)$ are piecewise constant basis functions in time.

With these approximations we obtain the following linear system of equations:

$$\begin{cases} V_h \underline{w} - (\frac{1}{2} M_h + K_h) \underline{u} = \underline{f}^1 \\ (\frac{1}{2} M_h^\top + N_h) \underline{w} + W_h \underline{u} + M_h^\sigma \underline{u} = \underline{f}^2 \end{cases}$$

where V_h, K_h, N_h and W_h are the Galerkin matrices corresponding to the boundary integral operators V, K, N and W, and M_h is the mass matrix, see [5, 14]. The vectors $\underline{f}^1$ and $\underline{f}^2$ are the related vectors to the right hand sides.

Moreover, we introduce the mass matrix entries

$$M_{k\ell}^\sigma[j][i] = \langle \sigma(x,t) \varphi_i^1(x) \psi_\ell^0(t), \varphi_j^1(x) \psi_k^0(t) \rangle = \int_0^T \int_\Gamma \sigma(x,t) \varphi_i^1(x) \psi_\ell^0(t) \varphi_j^1(x) \psi_k^0(t) \, ds_x \, dt,$$

which are zero whenever $k \neq \ell$. For $k = \ell$, we denote the matrix $M_{\ell\ell}^\sigma$ by M_ℓ^σ.

Note that the matrix $\frac{1}{2}M_h^\top + N_h$ can be obtained as follows. We first have the following (block) lower triangular matrices:

$$V_h = \begin{pmatrix} V_0 & & & & \\ V_1 & V_0 & & & \\ ... & ... & ... & & \\ V_{n-2} & V_{n-3} & & V_0 & \\ V_{n-1} & V_{n-2} & & & V_0 \end{pmatrix}, \quad W_h = \begin{pmatrix} W_0 & & & & \\ W_1 & W_0 & & & \\ ... & ... & ... & & \\ W_{n-2} & W_{n-3} & & W_0 & \\ W_{n-1} & W_{n-2} & & & W_0 \end{pmatrix}$$

$$\frac{1}{2}M_h + K_h = \begin{pmatrix} K_0 & & & & \\ K_1 & K_0 & & & \\ ... & ... & ... & & \\ K_{n-2} & K_{n-3} & & K_0 & \\ K_{n-1} & K_{n-2} & & & K_0 \end{pmatrix}$$

and then

$$\frac{1}{2}M_h^\top + N_h = \begin{pmatrix} K_0^\top & & & & \\ K_1^\top & K_0^\top & & & \\ ... & ... & ... & & \\ K_{n-2}^\top & K_{n-3}^\top & & K_0^\top & \\ K_{n-1}^\top & K_{n-2}^\top & & & K_0^\top \end{pmatrix}.$$

The linear system

$$\begin{pmatrix} V_h & -(\frac{1}{2}M_h + K_h) \\ (\frac{1}{2}M_h^\top + N_h) & W_h + M_h^\sigma \end{pmatrix} \begin{pmatrix} \underline{w} \\ \underline{u} \end{pmatrix} = \begin{pmatrix} \underline{f}^1 \\ \underline{f}^2 \end{pmatrix}$$

can be rewritten as follows:

$$\begin{pmatrix} V_0 & & & & \\ V_1 & V_0 & & & \\ ... & ... & ... & & \\ V_{n-2} & V_{n-3} & & V_0 & \\ V_{n-1} & V_{n-2} & & & V_0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ ... \\ w_{n-2} \\ w_{n-1} \end{pmatrix} - \begin{pmatrix} K_0 & & & & \\ K_1 & K_0 & & & \\ ... & ... & ... & & \\ K_{n-2} & K_{n-3} & & K_0 & \\ K_{n-1} & K_{n-2} & & & K_0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ ... \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_0^1 \\ f_1^1 \\ ... \\ f_{n-2}^1 \\ f_{n-1}^1 \end{pmatrix}$$

and

$$\begin{pmatrix} K_0^\top & & & & \\ K_1^\top & K_0^\top & & & \\ ... & ... & ... & & \\ K_{n-2}^\top & K_{n-3}^\top & & K_0^\top & \\ K_{n-1}^\top & K_{n-2}^\top & & & K_0^\top \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ ... \\ w_{n-2} \\ w_{n-1} \end{pmatrix} +$$

$$+ \begin{pmatrix} W_0 + M_0^\sigma & & & & \\ W_1 & W_0 + M_1^\sigma & & & \\ ... & ... & ... & & \\ W_{n-2} & W_{n-3} & & W_0 + M_{n-2}^\sigma & \\ W_{n-1} & W_{n-2} & & & W_0 + M_{n-1}^\sigma \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ ... \\ u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} f_0^2 \\ f_1^2 \\ ... \\ f_{n-2}^2 \\ f_{n-1}^2 \end{pmatrix}.$$

From the first two equations of the above systems, we obtain

$$\begin{pmatrix} V_0 & -K_0 \\ K_0^\top & W_0 + M_0^\sigma \end{pmatrix} \begin{pmatrix} w_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} f_0^1 \\ f_0^2 \end{pmatrix} \Longrightarrow \left[K_0^\top V_0^{-1} K_0 + W_0 + M_0^\sigma \right] u_0 = f_0^2 - K_0^\top V_0^{-1} f_0^1,$$

since the matrix V_0 is invertible, and then we can solve for u_0 and w_0. Therefore, w_k and u_k can be found from the system

$$\begin{cases} V_k w_0 + V_{k-1} w_1 + ... + V_0 w_k - K_k u_0 - K_{k-1} u_1 - ... - K_0 u_k = f_k^1 \\ K_k^\top w_0 + K_{k-1}^\top w_1 + ... + K_0^\top w_k + W_k u_0 + ... + W_1 u_{k-1} + (W_0 + M_k^\sigma) u_k = f_k^2 \end{cases}$$

for $k = 1, ..., n-1$. This system can be re-arranged as follows:

$$\begin{cases} V_0 w_k - K_0 u_k = f_k^1 + K_k u_0 + K_{k-1} u_1 + ... + K_1 u_{k-1} - V_k w_0 - V_{k-1} w_1 - ... - V_1 w_{k-1} \\ K_0^\top w_k + (W_0 + M_k^\sigma) u_k = f_k^2 - K_k^\top w_0 - ... - K_1^\top w_{k-1} - W_k u_0 - ... - W_1 u_{k-1}. \end{cases}$$

Observe that the matrices $\mathbb{R}^{m_1 \times m_1} \ni A_h^k := K_0^\top V_0^{-1} K_0 + W_0 + M_k^\sigma$, $k = 0, ..., n-1$, are symmetric and positive definite and the corresponding system of linear equations can be solved efficiently using standard methods of inversion.

5 Variational method for the inverse problem I

Now we return to the inverse problem I consisting of determining $\{u(x,t), f(\xi)\}$ from the system of equations (2.1)–(2.4). If we suppose that $g \in L^2(Q), a \in L^2(\Omega), h \in L^\infty(S), f \in L^2(\Gamma), b \in L^2(S)$ and $\sigma \in L^\infty(S), \sigma \geq 0$, then from Theorem 3.2, there exists a unique solution in $W(0,T)$ of the direct problem (2.1)–(2.3). Since $u \in W(0,T)$, we cannot determine the trace $u(\xi, T_1)$, $\xi \in \Gamma, 0 < T_1 \leq T$ in (2.5). Therefore, in this setting, we take the observation operator l as in (2.6). Afterwards we use

$$\frac{1}{\gamma} \int_{T_1 - \gamma}^{T_1} u(\xi, t) dt, \tag{5.1}$$

where $\gamma > 0$ small, as an approximation to $u(\xi, T_1)$, if it exists. Here and thereafter, for simplicity, we suppose that the weight $\omega \in L^2(0,T)$.

To emphasize the dependence of the solution u of (2.1)–(2.3) on the boundary data f, sometimes we write it by $u(x,t;f)$ or $u(f)$. Now, the variational approach to the first inverse problem can be considered as the problem of minimizing the functional

$$\begin{aligned} J_\alpha(f) &= \frac{1}{2} \int_\Gamma \Big(\int_0^T \omega(t) u(\xi, t; f) dt - \chi(\xi) \Big)^2 d\xi + \frac{\alpha}{2} \|f\|^2_{L^2(\Gamma)} \\ &= \frac{1}{2} \int_\Gamma |l(u(f)) - \chi|^2 d\xi + \frac{\alpha}{2} \|f\|^2_{L^2(\Gamma)} \end{aligned} \tag{5.2}$$

over $L^2(\Gamma)$, where $u(x,t;f)$ solves (2.1)–(2.3) and α is the regularization parameter.

Since the mapping from $f \in L^2(\Gamma)$ to $l(u)$ is affine, by the standard reasoning, we see that the above minimization problem admits a unique solution, if $\alpha > 0$.

We note that since the imbedding of the trace of $W(0,T)$ on S into $L^2(0,T; L^2(\Gamma))$ is compact, the mapping from $f \in L^2(\Gamma)$ to $l(u(f)) \in L^2(\Gamma)$ is compact. Hence the inverse problem in this setting is ill-posed and so is the minimization problem for J_0.

Now we find the gradient of J_α.

Take a variation $\delta f \in L^2(\Gamma)$ and consider problem (2.1)–(2.3) with the data $f + \delta f$ instead of f. We have the unique solution $u(x,t; f + \delta f) \in W(0,T)$.

Set $v = u(x,t; f+\delta f) - u(x,t;f)$. Then, $v \in W(0,T)$ is the weak solution of

$$v_t - \Delta v = 0 \quad \text{in } Q, \tag{5.3}$$

$$v(x,0) = 0, \quad x \in \Omega, \tag{5.4}$$

$$\frac{\partial v}{\partial n} + \sigma(\xi,t) v = h(\xi,t) \delta f(\xi), \quad (\xi,t) \in S. \tag{5.5}$$

We have

$$\begin{aligned}
J_0(f+\delta f)-J_0(f) &= \frac{1}{2}\int_\Gamma\Big(\int_0^T \omega(\tau)\big(u(\xi,\tau;f)+v(\xi,\tau;\delta f)\big)d\tau-\chi(\xi)\Big)^2 d\xi \\
&\quad -\frac{1}{2}\int_\Gamma\Big(\int_0^T \omega(\tau)u(\xi,\tau;f)d\tau-\chi(\xi)\Big)^2 d\xi \\
&= \int_\Gamma\Big(\int_0^T \omega(\tau)v(\xi,\tau;\delta f)d\tau\Big)\Big(\int_0^T \omega(\tau)u(\xi,\tau;f)d\tau-\chi(\xi)\Big)d\xi \\
&\quad +\frac{1}{2}\int_\Gamma\Big(\int_0^T \omega(\tau)v(\xi,\tau;\delta f)d\tau\Big)^2 d\xi.
\end{aligned}$$

To evaluate the first item in the last equation, we introduce the adjoint problem

$$-\psi_t-\Delta\psi=0 \quad \text{in } Q, \tag{5.6}$$

$$\psi(x,T)=0, \quad x\in\Omega, \tag{5.7}$$

$$\frac{\partial\psi}{\partial n}+\sigma(\xi,t)\psi=\omega(t)\Big(\int_0^T \omega(\tau)u(\xi,\tau)d\tau-\chi(\xi)\Big) \quad \text{on } S. \tag{5.8}$$

There exists a unique weak solution in $W(0,T)$ of this problem and applying Theorem 3.4 to (5.3)–(5.5) and (5.6)–(5.7), we have that the identity (3.11) yields

$$\begin{aligned}
\int_S\Big(\int_0^T \omega(\tau)u(\xi,\tau;f)d\tau-\chi(\xi)\Big)\omega(t)v(\xi,t)d\xi dt &= \int_S h(\xi,t)\delta f(\xi)\psi(\xi,t)d\xi dt \\
&= \int_\Gamma\Big(\int_0^T \omega(t)u(\xi,t;f)dt-\chi(\xi)\Big)\Big(\int_0^T \omega(t)v(\xi,t)\Big)d\xi.
\end{aligned} \tag{5.9}$$

On the other hand, in virtue of Theorem 3.2,

$$\|v\|_{W(0,T)}\le c_d\|h\|_{L^\infty(Q)}\|\delta f\|_{L^2(\Gamma)}.$$

Hence

$$J_0(f+\delta f)-J_0(f)=\int_S h(\xi,t)\psi(\xi,t)\delta f(\xi)d\xi dt+o(\|\delta f\|_{L^2(\Gamma)}).$$

Thus, we conclude that the functional J_0 is Fréchet differentiable and its gradient has the form

$$J_0'(f)=\int_0^T h(\xi,t)\psi(\xi,t)dt. \tag{5.10}$$

We immediately have

$$J_\alpha'(f)=\int_0^T h(\xi,t)\psi(\xi,t)dt+\alpha f. \tag{5.11}$$

Thus, the optimality condition for the problem (5.2), (2.1)–(2.3) is

$$\int_0^T h(\xi,t)\psi(\xi,t)dt+\alpha f=0. \tag{5.12}$$

If $\alpha>0$, then there is a unique solution f^α to this problem.

5.1 Boundary element method for the variational problem

Denoting the solution of the direct problem (2.1)–(2.3) with $f = 0$ by u_0 and that with $g = 0, a = 0, b = 0$ by $\bar{u}$, then the solution of (2.1)–(2.3) is $u = u_0 + \bar{u}$. The operator $A_0 f = l(\bar{u}(f))$ is linear and bounded, and the operator $Af = l(u(f)) = A_0 f + l(u_0)$ is affine. Thus, the functional (5.2) can be written in the form

$$\begin{aligned} J_\alpha(f) &= \frac{1}{2}\|Af - \chi\|^2_{L^2(\Gamma)} + \frac{1}{2}\alpha\|f\|^2_{L^2(\Gamma)} \\ &= \frac{1}{2}\|A_0 f - (\chi - l(u_0))\|^2_{L^2(\Gamma)} + \frac{1}{2}\alpha\|f\|^2_{L^2(\Gamma)} \\ &:= \frac{1}{2}\|A_0 f - \overline{\chi}\|^2_{L^2(\Gamma)} + \frac{1}{2}\alpha\|f\|^2_{L^2(\Gamma)}. \end{aligned}$$

It follows that the gradient of J_α can be represented as

$$J'_\alpha(f) = A_0^*(Af - \chi) + \alpha f. \tag{5.13}$$

Here $A_0^* : L^2(\Gamma) \to L^2(\Gamma)$ is the adjoint operator of A_0 defined by $A_0^* q = \int_0^T h(\xi,t)\psi(\xi,t)dt$, where ψ is the solution of the adjoint problem

$$-\psi_t - \Delta\psi = 0 \quad \text{in } Q, \tag{5.14}$$

$$\psi(x,T) = 0, \quad x \in \Omega, \tag{5.15}$$

$$\frac{\partial\psi}{\partial n} + \sigma(\xi,t)\psi = \omega(t)q(\xi), \quad (\xi,t) \in S. \tag{5.16}$$

Now, the optimality condition (5.12) can be rewritten in the form

$$A_0^*(Af - \chi) + \alpha f = 0, \tag{5.17}$$

from which we see immediately that there exists a unique solution f^α of it, if $\alpha > 0$.

Using the ansatz functions $\mathcal{V}_h \times \mathcal{U}_h$ as described in Section 4 with mesh size h in space variable and $\sim \sqrt{h}$ in time variable, we can derive the error estimate

$$\|\overline{u} - \overline{u}_h\|_{L^2(\Gamma)} \le ch\|f\|_{L^2(\Gamma)} \tag{5.18}$$

with c being a positive constant. Defining

$$A_{0,h} f = \int_0^T \omega(t)\overline{u}_h(\xi,t;f)dt,$$

we conclude that

$$\|A_0 f - A_{0,h} f\|_{L^2(\Gamma)} \le ch\|f\|_{L^2(\Gamma)}.$$

Hence the discrete version of the optimal control problem (5.2), (2.1)–(2.3) reads

$$\min_{f\in L^2(\Gamma)} \Big(\frac{1}{2}\|A_{0,h} f - \overline{\chi}\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|f\|^2_{L^2(\Gamma)}\Big) \tag{5.19}$$

which is characterized by the first-order optimality condition

$$A_{0,h}^*(A_{0,h} f_h^\alpha - \overline{\chi}) + \alpha f_h^\alpha = 0. \tag{5.20}$$

Here $A_{0,h}^*: L^2(\Gamma) \to L^2(\Gamma)$ is the adjoint operator of $A_{0,h}$ defined by
$A_{0,h}^*(A_{0,h}f_h - \overline{\chi}) = \int_0^T h(\xi,t)\psi(\xi,t)dt$, where ψ is the solution of the adjoint problem

$$-\psi_t - \Delta\psi = 0 \quad \text{in } Q, \tag{5.21}$$

$$\psi(x,T) = 0, \quad x \in \Omega, \tag{5.22}$$

$$\frac{\partial\psi}{\partial n} + \sigma(\xi,t)\psi = \omega(t)(A_{0,h}f_h - \overline{\chi})(\xi), \quad (\xi,t) \in S. \tag{5.23}$$

If we solve the last problem by the BEM, then we get an approximation $\hat{A}_{0,h}^*$ of $A_{0,h}^*$ for which

$$\|A_{0,h}^* - \hat{A}_{0,h}^*\| \le ch. \tag{5.24}$$

Thus, we arrive at the variational problem

$$\hat{A}_{0,h}^*(A_{0,h}\hat{f}_h^\alpha - \overline{\chi}_\epsilon) + \alpha\hat{f}_h^\alpha = 0 \tag{5.25}$$

with a perturbation $\overline{\chi}_\epsilon$ of $\overline{\chi}$ satisfying

$$\|\overline{\chi} - \overline{\chi}_\epsilon\|_{L^2(\Gamma)} \le \epsilon. \tag{5.26}$$

By the same technique as in the proof of Theorem 1 in [5] we can prove that if f^α is the solution of the problem (5.12) and $\alpha > 0$, then

$$\|f^\alpha - \hat{f}_h^\alpha\|_{L^2(\Gamma)} \le c(h + \epsilon) \tag{5.27}$$

with c being a constant depending on f^α, χ and α.

5.2 Conjugate gradient method for problem (2.1)–(2.4)

1. Initialization

1.1. Choose an initial guess $f_0 \in L^2(\Gamma)$.

1.2. Calculate the residual $\tilde{r}_0 = A_h f_0 - \chi_\epsilon$ by solving the direct problem (2.1)–(2.3) with $f = f_0$ by BEM.

1.3. Calculate $J_\alpha(f_0) = \frac{1}{2}\|\tilde{r}_0\|^2 + \frac{\alpha}{2}\|f_0\|^2$.

1.4. Calculate the gradient r_0 by solving the adjoint problem (5.14)–(5.16) with $q = \tilde{r}_0$ and set

$$r_0 = \int_0^T h(\xi,t)\psi_0(\xi,t)dt + \alpha f_0.$$

1.5. Define $d_0 = -r_0$.

2. For $n = 1, 2, \ldots$

2.1. Solve (2.1)–(2.3) with $g = 0, a = 0, b = 0$ and $f = d_n$ for calculating $A_{0,h}d_n$. Calculate

$$\alpha_n = \frac{\|r_n\|^2}{\|A_{0,h}d_n\|^2 + \alpha\|d_n\|^2}.$$

2.2. Update $f_{n+1} = f_n + \alpha_n d_n$.

2.3. Calculate the residual $\tilde{r}_{n+1} = \tilde{r}_n + \alpha_n A_{0,h}d_n$.

2.4. Calculate the gradient r_{n+1} by solving the adjoint problem (5.14)–(5.16) with $q = \tilde{r}_{n+1}$ and set

$$r_{n+1} = \int_0^T h(\xi,t)\psi_{n+1}(\xi,t)dt + \alpha f_{n+1}.$$

2.5. $J_\alpha(f_{n+1}) = \frac{1}{2}\|\tilde{r}_{n+1}\|^2 + \frac{1}{2}\alpha\|f_{n+1}\|^2$.

2.6. $\beta_n = \frac{\|r_{n+1}\|^2}{\|r_n\|^2}$.

2.7. Update $d_{n+1} = -r_{n+1} + \beta_n d_n$.

When $\alpha = 0$, stop at the first n such that $\|\tilde{r}_n\| \leq \gamma_1\epsilon$, where γ_1 is some number greater than 1, or when $\|r_n\| < \epsilon$. This discrepancy principle stopping criterion is required in order to achieve a stable solution, [11]. We can also choose $\alpha > 0$ as the regularization parameter in Tikhonov's method and stop the algorithm with a tolerance error. Of course, as the CGM is in itself a regularizing method, there is, in principle, no need to include a regularization term in the functional (5.2) that is minimized. As recently investigated in [5], both methods with or without α included produce similar results. However, the choice of $\gamma_1 > 1$ in the CGM with $\alpha = 0$ is not obvious and moreover, the inclusion of $\alpha > 0$ in the CGM tends to achieve a more robust stability than when $\alpha = 0$. Finally, we mention that the Tikhonov functional (5.2) with $\alpha > 0$ is recommended when used in conjunction with other iterative algorithms for minimization which do not necessarily have a regularizing effect. This is because otherwise, when $\alpha = 0$, stopping the iterations at a threshold given by the discrepancy principle, for example, does not guarantee that a stable solution is obtained.

5.3 Numerical examples

The one-dimensional spacewise ambient temperature case has been numerically investigated at length in [13] and therefore, in this subsection the emphasis is put on the multi-dimensional (two-dimensional) framework. We consider three examples in decreasing order of smoothness, namely: smooth, piecewise smooth and discontinuous functions.

In all examples in this subsection, $\Omega = (0,1) \times (0,1), T = 1, g = 0, a = 0, \sigma(\xi,t) = \xi_1^2 + \xi_2^2 + 1$, $h(\xi,t) = \xi_1 + \xi_2 + \sin(\frac{t}{2} + 1)$, where $\xi = (\xi_1, \xi_2)$. For the temperature we take the exact solution be given by, see [2],

$$u(x,t) = \frac{100}{4\pi t} e^{-\frac{|x-x_0|^2}{4t}}, \tag{5.28}$$

where $x_0 = (-1,-1)$. Then prescribing f we can take b given by

$$b(\xi,t) := \frac{\partial u}{\partial n} + \sigma(\xi,t)u - h(\xi,t)f(\xi), \quad (\xi,t) \in S. \tag{5.29}$$

The measurement (2.4) is obtained directly from (5.28), via (2.6) or (5.1). In the case of the integral measurement (2.6) we take $\omega(t) = t^2 + 1$. In the case of the terminal-integral measurement (5.1), $\gamma = 10^{-5}$ is fixed throughout, and the terminal time T_1 is varied within the interval $(0, T]$.

In order to investigate the stability of the numerical solution we add noise to the measurement (2.4), as

$$\chi_{noisy} = \chi + \epsilon \times rand(1), \tag{5.30}$$

where $rand(1)$ gives random variables in the interval $[-1, 1]$.

The number of boundary elements is taken as $M = 256$ and the number of time steps is taken as $N = 128$. These numbers are found sufficiently large to ensure that any further increase in them did not significantly affect the accuracy of the numerical results.

For simplicity, we illustrate the results obtained with $\alpha = 0$ and the CGM stopped according to the discrepancy principle with $\gamma_1 = 1.05$ starting with the initial guess $f_0 = 0$. We have also tested the unstopped CGM, but regularized with a positive α such as $\alpha = 10^{-5}$, and we have found similar results. Therefore, these latter results are not illustrated.

We aim to retrieve the following functions representing the spacewise dependent ambient temperature:

$$f(\xi) = \xi_1 + \xi_2 \quad \text{for } \textbf{Example 1}, \tag{5.31}$$

$$f(\xi) = \begin{cases} -\left|\xi_1 - \frac{1}{2}\right| + \frac{1}{2} & \text{if } \xi_2 \in \{0,1\}, \xi_1 \in (0,1), \\ -\left|\xi_2 - \frac{1}{2}\right| + \frac{1}{2} & \text{if } \xi_1 \in \{0,1\}, \xi_2 \in (0,1) \end{cases} \quad \text{for } \textbf{Example 2}, \tag{5.32}$$

$$f(\xi) = \begin{cases} 1 & \text{if } \xi_2 \in \{0,1\}, \xi_1 \in (0,1), \\ 0 & \text{elsewhere} \end{cases} \quad \text{for } \textbf{Example 3}. \tag{5.33}$$

Example	ϵ	n^*	$\|f - f_{n^*}\|_{L^2(\Gamma)}$
1	10^{-3}	6	0.014053
1	10^{-2}	3	0.043466
1	10^{-1}	2	0.136111
2	10^{-3}	7	0.013738
2	10^{-2}	4	0.053866
2	10^{-1}	3	0.094848
3	10^{-3}	13	0.223736
3	10^{-2}	7	0.376212
3	10^{-1}	2	0.665324

Table 1: The stopping CGM iteration numbers n^* and the $L^2(\Gamma)$-errors $\|f - f_{n^*}\|_{L^2(\Gamma)}$ for Examples 1–3 of the inverse problem I with the integral observation (2.6) perturbed by various levels of noise $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$.

Figure 1 shows the the comparison between the exact and numerical solutions of the inverse problem I with the integral observation (2.6) perturbed by various levels of noise $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$ for Examples 1–3. These levels of noise yield the stopping CGM iteration number n^* and the $L^2(\Gamma)$-errors $\|f - f_{n^*}\|_{L^2(\Gamma)}$ given in Table 1. From Figure 1 and Table 1 it can be seen that the numerical solutions for all three Examples 1–3 are stable and they become more accurate as the level of noise ϵ decreases. Obviously, Examples 2 and 3 are more difficult to retrieve accurately because the functions (5.32) and (5.33) are less regular than the smooth function (5.31). Finally, the low values of the stopping iteration numbers n^* reported in Table 1 show that the CGM rapidly achieves the required level of stability and accuracy.

Next we discuss the numerical results obtained for the inverse problem I with the terminal observation (2.5). As previously mentioned at the beginning of Section 5, since the trace (2.5) is not defined for the weak solution, we use instead the measurement (5.1), which is of the integral type (2.6) with

$$\omega(t) = \begin{cases} \frac{1}{\gamma}, & \text{if } t \in [T_1 - \gamma, T_1], \\ 0, & \text{otherwise.} \end{cases}$$

Example	T_1	n^*	$\|f - f_{n^*}\|_{L^2(\Gamma)}$
1	$T/3$	2	0.128147
1	$2T/3$	2	0.141655
1	T	2	0.132975
2	$T/3$	3	0.104567
2	$2T/3$	3	0.096957
2	T	3	0.086336
3	$T/3$	9	0.276654
3	$2T/3$	9	0.274843
3	T	10	0.274150

Table 2: The stopping CGM iteration numbers n^* and the $L^2(\Gamma)$-errors $\|f - f_{n^*}\|_{L^2(\Gamma)}$ for Examples 1–3 of the inverse problem I with terminal-integral observation (5.1) perturbed by $\epsilon = 10^{-1}$ noise for Example 1, $\epsilon = 10^{-2}$ noise for Example 2, and $\epsilon = 10^{-3}$ noise for Example 3, for various terminal times $T_1 \in \{T/3, 2T/3, T\}$.

Letting $\gamma > 0$ small, such as $\gamma = 10^{-5}$, we expect (5.1) to become a good approximation to (2.5). Figure 2 shows the comparison between the exact and numerical solutions for the spacewise dependent ambient temperature of the inverse problem I with the terminal-integral observation (5.1), with $\gamma = 10^{-5}$, perturbed by $\epsilon = 10^{-2}$ noise for various terminal times $T_1 \in \{T/3, 2T/3, T\}$ for Examples 1-3. The stopping CGM iteration numbers n^* and the $L^2(\Gamma)$-errors $\|f - f_{n^*}\|_{L^2(\Gamma)}$ are given in Table 2. From Figure 2 and Table 2 it can be seen that the numerical solutions for all three Examples 1–3 are stable and they are quite insensitive to the choice of the terminal time T_1.

6 Variational method for the inverse problem II

As in Section 5, since $u \in W(0,T)$ we cannot determine the trace $u(\xi_0, t), t \in [0,T], \xi_0 \in \Gamma$ in (2.11). Therefore, in this setting, we take the observation operator l_1 as in (2.12). Afterwards, we use

$$\frac{1}{2\gamma} \int_{\Gamma(\xi_0,\gamma)=\{\xi\in\Gamma \mid |\xi-\xi_0|\le\gamma\}} u(\xi,t)d\xi, \tag{6.1}$$

where $\gamma > 0$ is small, as an approximation to $u(\xi_0, t)$, if it exists. Here and thereafter, for simplicity we suppose that the weight $\nu \in L^2(\Gamma)$.

The variational setting of the inverse problem II given by equations (2.7)–(2.10) and (2.12) is as follows.

Minimize the functional

$$\begin{aligned} J_\alpha(f) &= \frac{1}{2}\|l_1(u(f) - \chi_1\|^2_{L^2(0,T)} + \frac{\alpha}{2}\|f\|^2_{L^2(0,T)} \\ &= \frac{1}{2}\int_0^T \Big| \int_\Gamma \nu(\xi)u(\xi,t;f)d\xi - \chi_1(t)\Big|^2 dt + \frac{\alpha}{2}\|f\|^2_{L^2(0,T)}, \end{aligned} \tag{6.2}$$

where $u = u(x,t;f)$ is the solution in $W(0,T)$ of the problem (2.7)–(2.9) with $g \in L^2(Q), a \in L^2(\Omega), \sigma \in L^\infty(S), \sigma \ge 0, h \in L^2(S), \nu \in L^2(\Gamma)$, and $\chi_1 \in L^2(0,T)$ being given.

There exists a unique solution in $W(0,T)$ of problem (2.7)–(2.9) for $f \in L^2(0,T)$, therefore, the problem setting has a meaning. Furthermore, since the trace of the space $W(0,T)$ on S is compactly imbedded into $L^2(0,T)$, the problem (6.2), (2.7)–(2.9) is ill-posed when $\alpha = 0$.

By the same arguments in the variational method for the inverse problem I, as described in Section 5, we can prove that there exists a solution of this minimization problem, the functional (6.2) is Fréchet differentiable and if ψ is the solution of the adjoint problem

$$-\psi_t - \Delta\psi = 0 \quad \text{in } Q, \tag{6.3}$$

$$\psi(x,T) = 0, \quad x \in \Omega, \tag{6.4}$$

$$\frac{\partial\psi}{\partial n} + \sigma(\xi,t)\psi = \nu(\xi)\Big(\int_\Gamma \nu(\xi)u(\xi,t;f)d\xi - \chi_1(t)\Big), \quad (\xi,t) \in S, \tag{6.5}$$

then

$$J_0'(f) = \int_\Gamma h(\xi,t)\psi(\xi,t)d\xi.$$

and

$$J_\alpha'(f) = J_0'(f) + \alpha f.$$

Denote the solution the direct problem (2.7)–(2.9) with $f = 0$ by u_0 and that with $g = 0, a = 0, b = 0$ by $\bar{u}$, then the solution of (2.7)–(2.9) is $u = u_0 + \bar{u}$. The operator $A_0 f = l_1(\bar{u}(f))$ is linear and bounded, and the operator $Af = l(u(f)) = A_0 f + l_1(u_0)$ is affine.

6.1 Conjugate gradient method for problem (6.2), (2.7)–(2.9)

1. Initialization

1.1. Choose an initial guess $f_0 \in L^2(0,T)$.

1.2. Calculate the residual $\tilde{r}_0 = Af_0 - \chi_\epsilon$ by solving the direct problem (2.7)–(2.9) with $f = f_0$.

1.3. Calculate $J_\alpha(f_0) = \frac{1}{2}\|\tilde{r}_0\|^2 + \frac{\alpha}{2}\|f_0\|^2$.

1.4. Calculate the gradient r_0 by solving the adjoint problem (6.3)–(6.5) with the right hand side of (6.5) equal to $\nu(\xi)\tilde{r}_0$ and set

$$r_0 = \int_\Gamma h(\xi,t)\psi_0(\xi,t)d\xi + \alpha f_0.$$

1.5. Define $d_0 = -r_0$.

2. For $n = 1, 2, \ldots$

2.1. Solve (2.7)–(2.9) with $g = 0, a = 0, b = 0$ and $f = d_n$ for calculating $A_0 d_n$. Calculate

$$\alpha_n = \frac{\|r_n\|^2}{\|A_0 d_n\|^2 + \alpha\|d_n\|^2}.$$

2.2. Update $f_{n+1} = f_n + \alpha_n d_n$.

2.3. Calculate residual $\tilde{r}_{n+1} = \tilde{r}_n + \alpha_n A_0 d_n$.

2.4. Calculate the gradient r_{n+1} by solving the adjoint problem (6.3)–(6.5) with the right hand side of (6.5) equal to $\nu(\xi)\tilde{r}_{n+1}$ and set

$$r_{n+1} = \int_\Gamma h(\xi,t)\psi_{n+1}(\xi,t)d\xi + \alpha f_{n+1}.$$

2.5. $J_\alpha(f_{n+1}) = \frac{1}{2}\|\tilde{r}_{n+1}\|^2 + \frac{1}{2}\alpha\|f_{n+1}\|^2$.

2.6. $\beta_n = \frac{\|r_{n+1}\|^2}{\|r_n\|^2}$.

2.7. Update $d_{n+1} = -r_{n+1} + \beta_n d_n$.

When $\alpha = 0$ stop at the first n such that $\|\tilde{r}_n\| \le \gamma_1 \epsilon$, or when $\|r_n\| < \epsilon$. Otherwise, choose $\alpha > 0$ as the regularization parameter in Tikhonov' method and stop the algorithm with a tolerance error.

6.2 Numerical example

The one-dimensional timewise ambient temperature case has been numerically investigated at length in [12] and therefore, in this subsection the emphasis is put on the two-dimensional framework.

We take $\Omega = (0,1) \times (0,1), T = 1, g = 0, a = 0, \sigma(\xi, t) = \xi_1^2 + \xi_2^2 + 1, h(\xi, t) = \sin(\xi_1 + \xi_2) + t^2 + 1$. For the temperature we take the exact solution (5.28). Then prescribing f we can take b given by

$$b(\xi, t) := \frac{\partial u}{\partial n} + \sigma(\xi, t)u - h(\xi, t)f(t), \quad (\xi, t) \in S. \tag{6.6}$$

The measurement (2.10) is obtained directly from (5.28), via (2.12) or (6.1). In the case of the integral measurement (2.12) we take $\nu(\xi) = \xi_1 + \xi_2 + 1$. In the case of the point-integral measurement (6.1), $\gamma = 10^{-5}$ is fixed throughout, and $\xi_0 \in \Gamma$ is taken arbitrary, for example $\xi_0 = (0.5, 0)$ or $\xi_0 = (0.9375, 0)$.

In order to investigate the stability of the numerical solution we add noise to the measurement (2.10), similarly as in (5.30).

As in subsection 5.3, we take $M = 256, N = 128, \alpha = 0, \gamma_1 = 1.05$ and $f_0 = 0$. In order to avoid repetition with the previous spacewise dependent case discussed at length in subsection 5.3 we only present numerical results for retrieving a severe discontinuous time-dependent ambient temperature given by

$$f(t) = \begin{cases} 1, & \text{if } t \in (1/3, 2/3), \\ 0, & \text{otherwise} \end{cases} \quad \text{for } \textbf{Example 4}. \tag{6.7}$$

Although not illustrated, it is reported that for smoother examples, e.g. $f(t) = \sin(2\pi t)$, we obtained excellent numerical results which were found in good agreement and stability with the available exact solutions.

Figures 3(a)–3(c) show the comparison between the exact and numerical solutions for the timewise varying ambient temperature (6.7) of the inverse problem II with the integral observations (2.12), (6.1) with $\gamma = 10^{-5}$, $\xi_0 = (0.5, 0)$ and $\xi_0 = (0.9375, 0)$, respectively, perturbed by various levels of noise $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$ for Example 4. These levels of noise yield the stopping CGM iteration numbers n^* and the $L^2(0,T)$-errors $\|f - f_{n^*}\|_{L^2(0,T)}$ given in Table 3. From this table and by comparing Figure 3(a) with Figures 3(b) and 3(c) it can be seen that the integral observation (2.12) yields more accurate results than the point-integral observation (6.1). Also, changing the boundary point $\xi_0 \in \Gamma$ at which a thermocouple/sensor takes the measurement (2.11) shows some slight sensitivity in the numerically retrieved results, see Table 3 and compare Figures 3(b) and 3(c). Overall, from Figure 3 and Table 3 it can be seen that the numerical solution for Example 4 is stable and becomes more accurate as the level of noise ϵ decreases.

Observation	ϵ	n^*	$\|f - f_{n^*}\|_{L^2(\Gamma)}$
(2.12)	10^{-3}	23	0.059861
(2.12)	10^{-2}	13	0.060511
(2.12)	10^{-1}	7	0.081467
(6.1) at $\xi_0 = (0.5, 0)$	10^{-3}	10	0.066628
(6.1) at $\xi_0 = (0.5, 0)$	10^{-2}	4	0.120731
(6.1) at $\xi_0 = (0.5, 0)$	10^{-1}	2	0.191686
(6.1) at $\xi_0 = (0.9375, 0)$	10^{-3}	9	0.062297
(6.1) at $\xi_0 = (0.9375, 0)$	10^{-2}	4	0.080687
(6.1) at $\xi_0 = (0.9375, 0)$	10^{-1}	2	0.153459

Table 3: The stopping CGM iteration numbers n^* and the $L^2(0,T)$-errors $\|f - f_{n^*}\|_{L^2(0,T)}$ for Example 4 of the inverse problem II with integral observation (2.12) or (6.1) perturbed by various levels of noise $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$.

7 Conclusions

Multi-dimensional inverse heat conduction problems which require determining the space- or time-dependent ambient temperature appearing in the convective Robin boundary conditions of the third-kind from additional terminal, point or integral measurements have been investigated. The problems have been formulated as least-squares problems and formulae for the gradients have been delivered. A numerical method based on the CGM+BEM has been developed for obtaining a stable numerical solution when the input data is subject to noise. Numerical results for several benchmark test examples were presented in order to illustrate the feasibility of the approach. Intuitively, in the dimension > 2, the spacewise retrieval of the ambient temperature considered in the inverse problem I is more difficult than the timewise retrieval considered in the inverse problem II since we have more unknowns. But clearly, for a reliable comparison one would need to estimate the rate of decay of the singular values of the linear/affine operators involved in expressions (5.2) and (6.2) for the inverse problems I and II. This difficult task is deferred to a future work. Analogous multi-dimensional, but nonlinear inverse problems which require determining the space- and time-dependent heat transfer coefficient will be investigated in a separate future work, [6].

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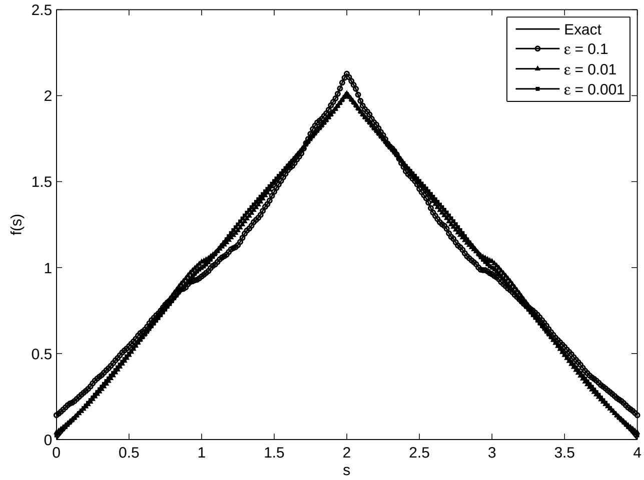
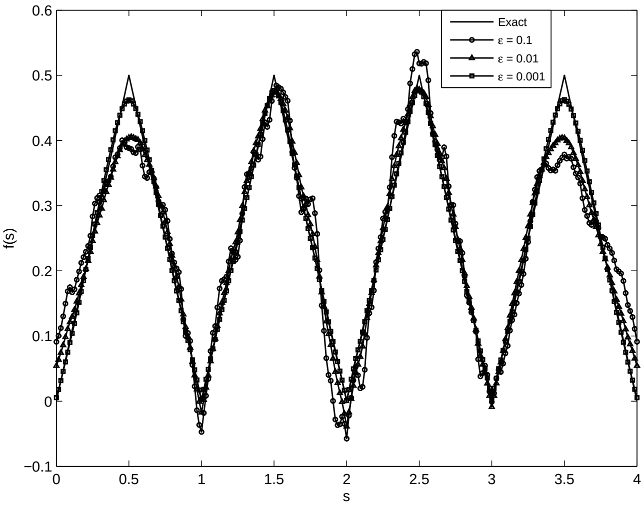
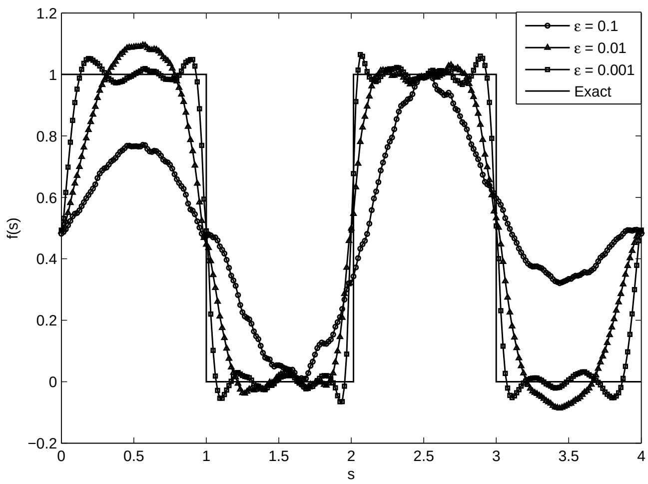


(a) Example 1

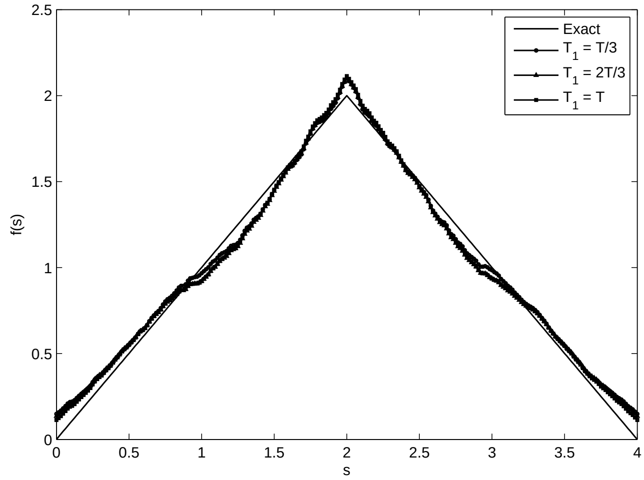


(b) Example 2

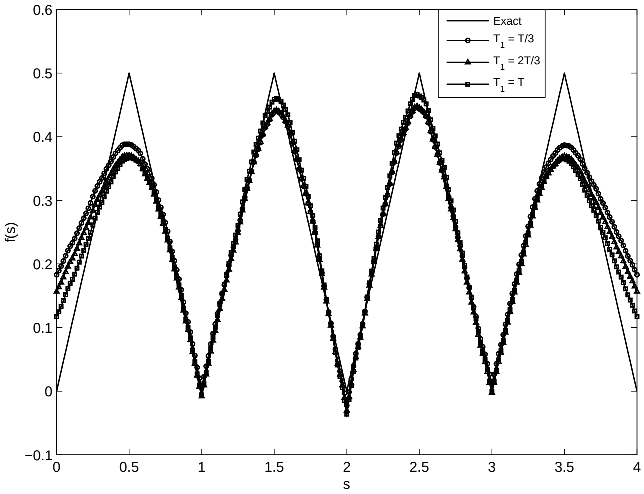


(c) Example 3

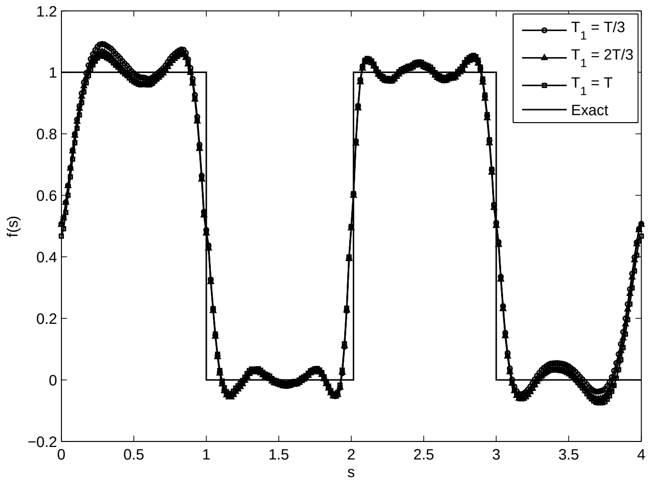
Figure 1: The exact and numerical spacewise dependent ambient temperature, as a function of the arclength s along the boundary Γ (starting from the origin), for various levels of noise $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$, for Examples 1–3 (inverse problem I with the integral observation (2.6)).



(a) Example 1

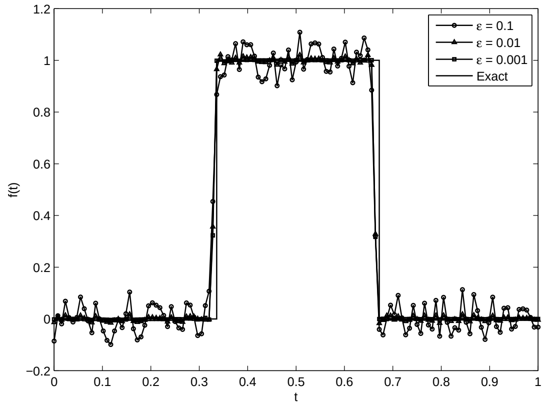


(b) Example 2

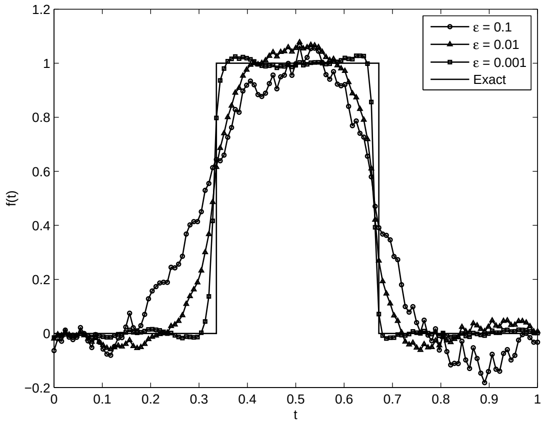


(c) Example 3

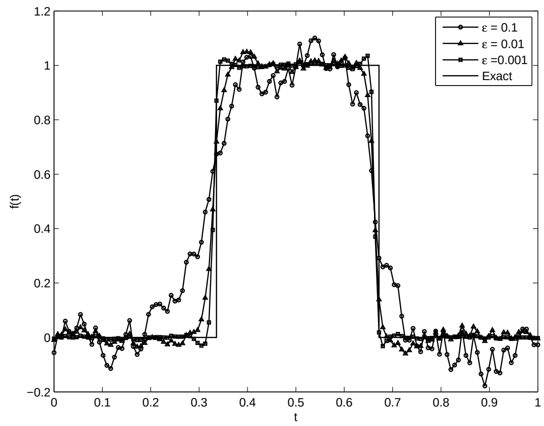
Figure 2: The exact and numerical spacewise dependent ambient temperature, as a function of the arclength s along the boundary Γ (starting from the origin), for $\epsilon = 10^{-2}$ and various instants $T_1 \in \{T/3, 2T/3, T\}$, for Examples 1–3 (inverse problem I with the terminal-integral observation (5.1) and $\gamma = 10^{-5}$).



(a) Example 4 with the integral observation (2.12)



(b) Example 4 with the point-integral observation (6.1) at $\xi_0 = (0.5, 0)$ and $\gamma = 10^{-5}$



(c) Example 4 with the point-integral observation (6.1) at $\xi_0 = (0.9375, 0)$ and $\gamma = 10^{-5}$

Figure 3: The exact and numerical timewise varying ambient temperature, for various levels of noise $\epsilon \in \{10^{-3}, 10^{-2}, 10^{-1}\}$ for Example 4, inverse problem II.