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DEPARTMENT OF AUTOMATIC CONTROL AND SYSTEMS ENGINEERING

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The validity of polynomial models for the tracking of a physical field

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Abstract

This report is the fourth in a series and expands the investigation of the feasibility of using the linear relationship for the (n-1)th spatial derivative of an n'th order field model for locating a sensor within the sampled field. Taking the electrical (or magnetic) field between two meandering parallel conductors as an example, the effect of a non polynomial field equation for generating the field measurements is explored. Criteria for the best polynomial fit for position-location purposes are examined and promising results achieved.

1. Introduction

Previous research reports (1), (2), (3) in this series have concentrated on the determination or estimation of position y_0 in a physical field $H(y)$ given n values of $H(y)\{= H(y_0 + i\Delta y)$ where $0 \le i \le n-1\}$ of known spacing Δy. The field has been assumed to be modelled exactly by a polynomial function, i. e.

$$H(y) = a_0 + a_1 y + a_2 y^2 \cdots\cdots a_n y^n \tag{1}$$

where constant coefficients $a_0 \cdots a_n$ are known and the positional determination made using the derived equation:

$$\frac{\Delta^{n-1} H}{\Delta y^{n-1}} = D^{n-1} H + \frac{(n-1)}{2} D^n H \Delta y \tag{2}$$

where $D^i H$ denotes $d^i H / dy^i$ (3)

$$\Delta^i H\big|_y = \Delta^{i-1} H\big|_{y+\Delta y} - \Delta^{i-1} H\big|_y \tag{4}$$

and

$$\Delta H\big|_y = H(y+\Delta y) - H(y) \tag{5}$$

Since

$$D^{n-1} H = (n-1)!\,a_{n-1} + n!\,a_n y \tag{6}$$

and

$$D^n H = n!\,a_n \tag{7}$$

equation (2) can be written as a simple linear formula for the calculation of y thus:

$$y = \frac{1}{n!\,a_n}\left[\frac{\Delta^{n-1} H}{\Delta y^{n-1}} - (n-1)!\,a_{n-1} - \left(\frac{n-1}{2}\right) n!\,a_n \Delta y\right] \tag{8}$$

For finding the first value y_0 of the equispaced sequence y_0, $y_0 + \Delta y$, $\cdots$ $y_0 + (n+1)\Delta y$, $\Delta^{n-1} H$ is evaluated at the first sampling point (i.e. $\Delta^{n-1} H$ in (8) is interpreted as $\Delta^{n-1} H\big|_{y_0}$).

The previous reports have concentrated on the effect of white noise on the measurements of H(y) on the accuracy of the y_0 - estimate obtained from formula (8) and on the improvements possible by combining estimates from using the formula on measurements of H exceeding the deterministic minimum number n. The white noise could be interpreted as measurement noise or as deviations between an idealised, deterministic field model (described by eq^n. (1)) and its real life, part-random, part-deterministic realisation. Correlated (i.e. coloured) random deviations were not considered. White noise effects are more readily analysed and are likely to be more serious because the finite differencing process (to form $\Delta^{n-1}H$) should be more immune to the more slowly changing disturbances that coloured noise describes. The argument could also be made that, slow variation (from the polynomial model H(y)) would be present in successive scans as the machine passes through the field. This is because field isotropy considerations would make it unlikely for random deviations to occur rapidly in the (x-) direction of machine travel yet slowly in the (y-) direction of scanning. Thus, slow changes in the y- direction are probably better modelled by an improved, à priori choice of polynomial rather than by adding coloured noise to the original model.

The question therefore arises as to the best choice of polynomial model for a physical field. Previous reports were motivated by the problem of tracking layered strata fields comprising bands of material of different hardness repeatedly scanned and sensed by a strain-gauged cutting tool as the cutting machine progresses along the strata field. Here, a field of n-1 turning points (i.e. maxima, minima and inflexions) within the y - range of interest would be modelled by an n'th order polynomial calculated to pass through n+1 points selected close to the turning points. The question of whether increase of order n might generate a better positional estimate is interesting. In this report this question is addressed by consideration of a much simpler physical field in order to assess the potential benefit of higher-order field modelling before proceeding (if necessary) to the complexity of geological fields in subsequent reports.

2. The Electric Field Example

The simple field considered is the electrostatic field $\xi(x,y)$ created between two long, cylindrical, charged conductors 1 and 2 lying parallel to one another in the flat plane z=0 and separated by distance d. The small 'vehicle' carrying a field strength detector is constrained to travel only in the plane of the two conductors and the problem posed is the determination of position y, $0<y<d$ from conductor 1 from measurements of the field strength. This hypothetical situation is illustrated in Fig 1. Since the conductors are of constant cross-section (having radii r_1 and r_2 respectively), long (compared to d), parallel and subject to only gentle lateral undulations (i.e. of curvature radius>>d) then the field will be independent of x i.e.

$$\xi(x,y) = \xi(y) \tag{9}$$

Provided vehicle position y is always such that

$$r_1 << y << d - r_2 \tag{10}$$

and the vehicle itself is of small dimensions (compared to conductor spacing, d) so that the field remains sensibly unaffected by it its presence, then the field strength at any chosen point (e.g. point P in Fig 1) will be given by

$$\xi(y) = \frac{1}{2\pi\varepsilon_0}\left[\frac{\sigma_1}{y} + \frac{\sigma_2}{y-d}\right] \tag{11}$$

[As y approaches r_1 or d-r_2, then equation (10) will become progressively invalid since it is based on the assumption of ideal linear conductors of zero cross-section: Hence the necessity for constraint (10)]. ε_0 is the permittivity of the insulating medium between the conductors[1] . For simplicity of notation in subsequent analysis and for consistency with previous notation in this sequence of reports, we here define field function H(y) to be:

$$H(y) = 2\pi\varepsilon_0\xi(y) \tag{12}$$

so that

$$H(y) = \frac{\sigma_1}{y} + \frac{\sigma_2}{y-d} \tag{11}$$

and

$$DH(y) = -\left\{\frac{\sigma_1}{y^2} + \frac{\sigma_2}{(d-y)^2}\right\} \tag{12}$$

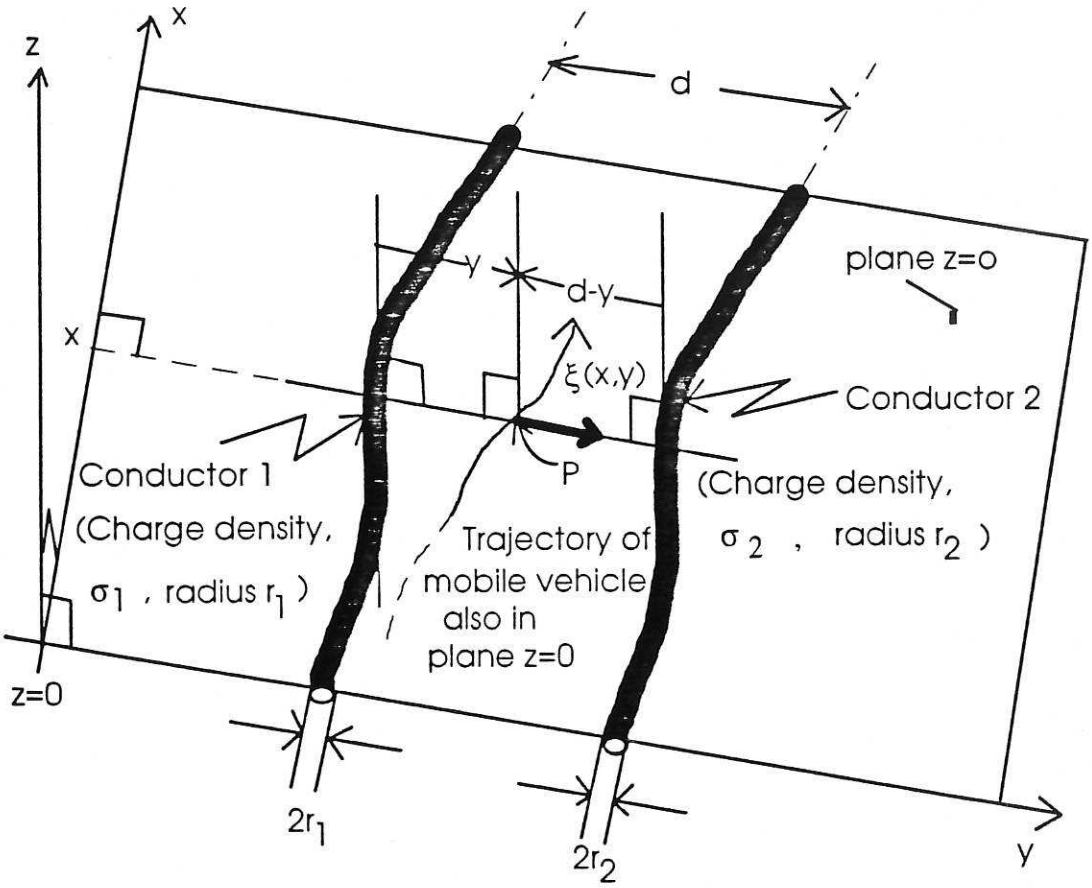

Fig. 1. Electrostatic Field between two charged parallel conductors lying in plane z=0

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[1] An equation similar in form to (11) above could be used to describe the magnetic field $\beta(y)$ resulting from currents I_1 and I_2 flowing along the conductors 1 & 2 but with I_1 and I_2 replacing σ_1 and σ_2 and with medium permeability μ_o replacing $(1/\varepsilon_o)$.

2.1. Some Field Characteristics

It is readily shown that a minimum value exits at

$$y = \frac{d\{\sigma_1 - \sqrt{(-\sigma_1\sigma_2)}\}}{\sigma_1 + \sigma_2} \tag{13}$$

which yield a real solution only if

$$sign\sigma_1 = -sign\sigma_2 \tag{14}$$

If (14) is not satisfied (i.e. if the conductors are charged in the same sense, rather than opposite senses) then the value of H(y) will move between near-infinite values of opposite sense as y increases from near zero to near d and passing through zero (i.e. through a null point) at

$$y = \sigma_1 d / (\sigma_1 + \sigma_2) \tag{15}$$

without passing through a turning point. The curve of H(y) is therefore monotonic in the case of charges of similar sign. We shall therefore concentrate on the more challenging case of opposite sign charges which involves duplicate solutions for y for any given H(y) is the range (10). In particular we will restrict attention to $\sigma_1>0$ and $\sigma_2 < 0$ so that:

$$\sigma_1 = |\sigma_1| \tag{16}$$

$$\sigma_2 = -|\sigma_2| \tag{17}$$

Under there circumstances, equation (13) predicts a minimum field strength of :

$$H(y) = H_{\min} = (\sqrt{\sigma_1} + \sqrt{|\sigma_2|})^2 / d \tag{18}$$

occurring at:

$$y = \frac{\sqrt{\sigma_1} d}{(\sqrt{\sigma_1} + \sqrt{|\sigma_2|})} \tag{19}$$

2.2. A Numerical Example

As a test example we choose , rather arbitrarily, d=20. Noting constraint(10) we shall assume that the vehicle track remains within the bounds

$$4 \le y \le 17 \tag{20}$$

The explored width of field will be a fairly substantial fraction (0.65) of the conductor spacing, d. To introduce a good measure of asymmetry we choose σ_1=4.0 and $\sigma_2 = -1.0$ so that

$$H(y) = \frac{4.0}{y} + \frac{1.0}{20 - y} \tag{21}$$

yielding a minimum value, H_{min}=0.4500 @ y=13.333. Curves of H(y) and DH(y) are shown in Fig 2.

It is clear from Fig 2 that a polynomial of minimal order=2.0 i.e. a parabola would be needed as an approximation to generate the single turning point in H(y) but it is also clear that associated straight line approximation to DH(y) might involve significant errors that would reflect in estimates of y derived using formula (8) and measurements of H(y) and H(y+Δy) (needed to form ΔH(y)) and the parameters a_1 and a_2 used to describe the best straight - line fit to the curve of DH(y) shown in Fig 2.

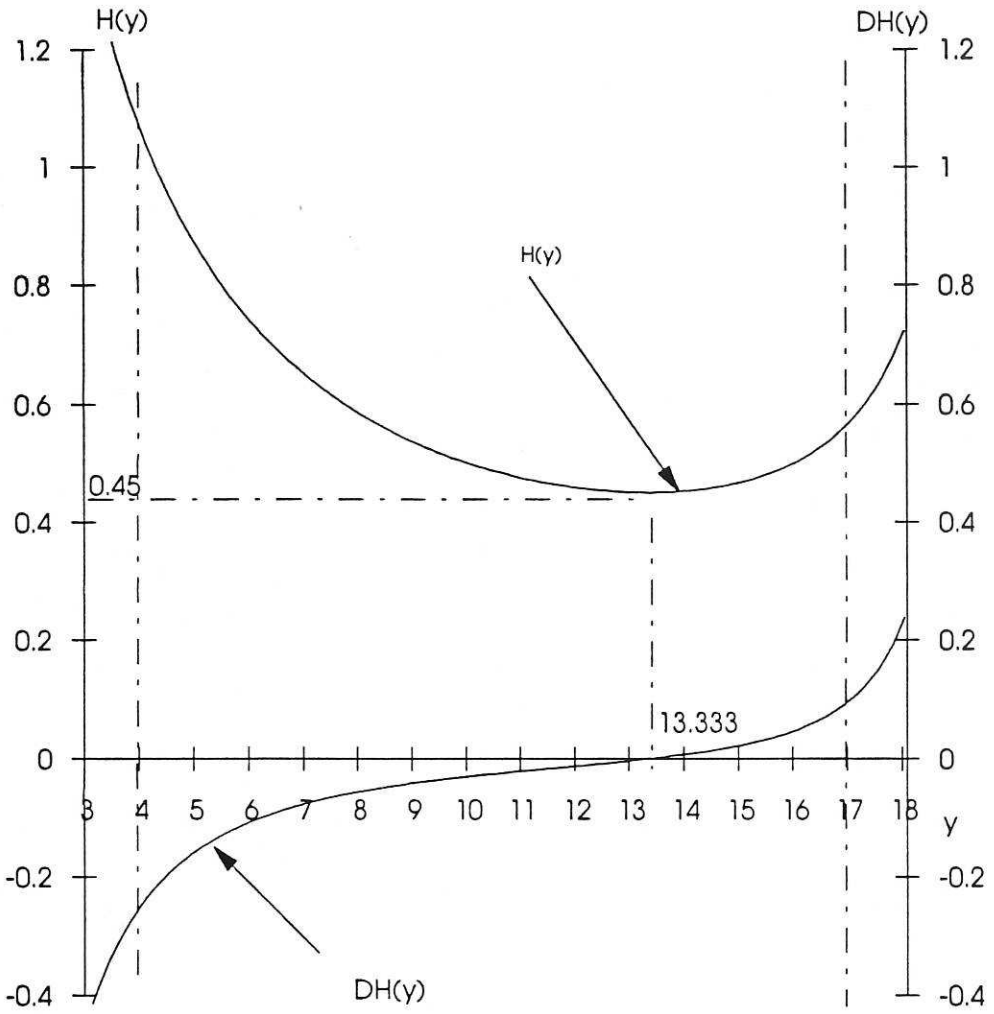

Fig 2. Curves of H(y) and DH(y) for σ_1=4, σ_2=-1 and d=20

3. Choosing a polynomial to approximate the real field function

For reasons of analytical and computational convenience (and possibly also for greater noise immunity) a polynomial of minimal order is desirable as a field function approximation provided the main features of the field are reproduced within the desired range

$$y_1 < y < y_2 \tag{22}$$

(In our numerical example the chosen limits y_1 and y_2 are 4 and 17 respectively). The most prominent features of a field function are its turning points so that, if there are, say, n_t of these within the range of interest, then the order n of the polynomial should be at least=n_t+1. We will use this minimum number initially (i.e. set n=2) for the chosen electric field example and consider the merits of increasing the order of the polynomial later in Section (5).

3.1. From a Taylor Expansion

Having selected the value for n there remains an infinitely free choice of curves all described by different combinations of parameter values a_0 to a_n. Where excursions of y are expected to remain very close to some nominal working point y_n then an nth - order Taylor expansion about $y=y_n$ would be a worthy candidate for the best polynomial to use. In our example for instance, choosing y_n=13.333 (=the turning point location in this case - see Fig. 2), then the 2nd-order Taylor expansion polynomial would be

$$\hat{H}(y) = 1.35 - 0.135y + 5.0625y^2 \times 10^{-3} \tag{23}$$

$$\text{and } D\hat{H}(y) = -0.135 + 10.125y \times 10^{-3} \tag{24}$$

The symbol $\hat{H}(y)$ is now used to denote the polynomial approximation for a given y, whilst retaining H(y) to denote the true field function. As a check note that equation (24) yields $D\hat{H}(y) = 0.0$ at y=13.333 and the associated minimum value of $\hat{H}(y) = 0.45$ which matches the minimum value of H(y), all as expected. The value of parameters a_1 and a_2 defining $D\hat{H}(y)$ in this case are therefore

$$a_1 = -0.1350 \ , \ a_2 = 5.0625 \times 10^{-3} \tag{25}$$

For small deviations (say $\delta_y = \pm 2.0$) either side y_n (here=13.333), using these values in position predictor formula (8) (in conjunction with a $\Delta H(y)$ value measured or derived from the true field function H(y)) does yield reasonable predictions of location y_0 as demonstrated in Section (4.1). In general however we shall be interested in estimating excursions in a wider range (22) i.e. within limits $4 \le y \le 17$ in our chosen example, for which the Taylor approximation is less adequate.

3.2. Orthogonal Least Squares Approximation the D^{n-1}H Characteristic

At first sight, the use of computational least squares (with respect to $a_0 \cdots a_n$) to minimise the error function

$$\overline{e_1^2} = \int_{y_1}^{y_2} \frac{\{H(y) - \hat{H}(y)\}^2}{(y_2 - y_1)} dy \tag{25}$$

where estimate $\hat{H}(y)$ is given by

$$\hat{H}(y) = a_0 + a_1 y + a_2 y^2 \cdots a_n y^n \tag{26}$$

would appear to offer a valid method for determining the best parameter values for the exercise. However it should be borne in mind that only a_{n-1} and a_n are needed in formula (8) indicating that minimisation of

$$\overline{e_{n-1}^2} = \int_{y_1}^{y_2} \frac{\{D^{n-1}H(y) - D^{n-1}\hat{H}(y)\}^2}{(y_2 - y_1)} dy \tag{27}$$

with respect to a_{n-1} and a_n, would be not only simple, but also would generate just the two parameters needed for formula (8). The least squares operation would be a simple straight line regression fitting process since $D^{n-1}\hat{H}(y)$ is linear in y. Such an operation yields the result

$$\hat{D}H(y) = -0.2436 + 19.306 \times 10^{-3} y \tag{28}$$

the fit of which to DH(y) is compared to that from the Taylor expansion result (24) in Fig 3. In this case the approximation is better towards the limits of the y - range and less good around in the neighbourhood of the H(y) turning point. Indeed, the intersection no longer occurs at DH(y)=0 (i.e. at y=13.333) as in the case of the Taylor expansion, but remains close @ $y = -a_1 / 2a_2 = 243.6 / 19.306 = 12.618$. The parameter values in this case are

$$a_1 = -0.2436, \ \ a_2 = 9.653 \times 10^{-3} \tag{29}$$

compared to those given in eq^n. (25) for the Taylor expansion.

Whereas the use of the values given in eq^n(29) produces generally better estimates of position y_0 across the range $4<y_0<17$ using formula (8), as demonstrated in Section (4.2), further improvement remains possible. This is because our true purpose is the estimation of y from $D^{n-1}H$, <u>not</u> the other way around! Hence the regression line really needed is that which minimises

$$\overline{e_q^2} = \int_{q_1}^{q_2} \frac{\{y - \hat{y}\}^2 dq}{q_2 - q_1} \tag{30}$$

where y is the value derived from the true curve $D^{n-1}H(y)$, whereas $\hat{y}$ is the value derived from the linear approximation

$$\hat{y} = b_1 + b_2 D^{n-1}H(y) \tag{31}$$

or

$$\hat{y} = b_1 + b_2 q \tag{32}$$

where

$$q(y) = D^{n-1}H(y) \tag{33}$$

and limits

$$q_1 = q(y_1), \ \ q_2 = q(y_2) \tag{34}$$

In our second order example, model (31) could be written alternatively in the from

$$DH(y) = a_1 + 2a_2\hat{y} \tag{35}$$

where the desired pair of polynomial parameters are obtained from those resulting from of least squares process simply thus:

$$a_2 = 1/2b_2 \tag{36}$$

and

$$a_1 = -b_1 / b_2 \tag{37}$$

In our example we get

$$DH(\hat{y}) = -0.26697 + 22.0416 \times 10^{-3} \hat{y} \tag{38}$$

i.e. parameters a_1 and a_2 are given by

$$a_1 = -0.26697, \quad a_2 = 11.0208 \times 10^{-3} \tag{39}$$

the values obtained from this "orthogonal q on y regression process" clearly differing slightly but significantly from those obtained by the conventional "y on q" process. The orthogonal regression line is also shown in Fig 3. The turning point of the polynomial (i.e. the zero of $DH(\hat{y})$ now occurs at $\hat{y} = 12.112$ compared to y=12.618 for the conventional regression line (and y=13.33 for the Taylor expansion) and the orthogonal regression line deviates slights more from the Taylor expansion line than does the conventional.

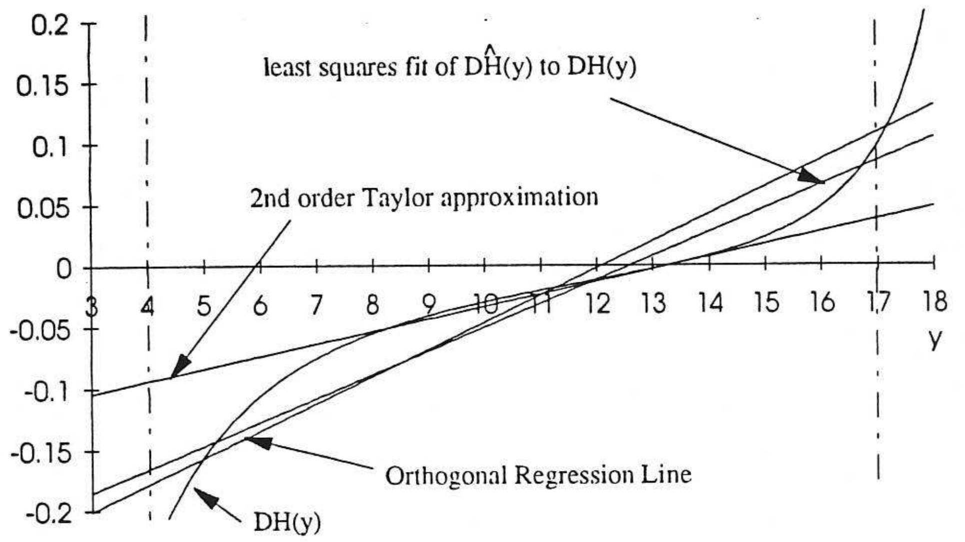

Fig. 3 Comparing alternative linear approximations to DH(y) for the Electrostatic Field Example

4. Specimen Results

To find an estimate $\hat{y}_0$ of the position y_0 from a sequence of n measurements $H(y_0)$, $H(y_0 + \Delta y) \cdots H\{y_0 + (n-1)\Delta y\}$ equispaced at Δy from real field function H(y) implicitly approximated by the n'th order polynomial

$$\hat{H}(y) = a_0 + a_1 y + \cdots a_{n-1} y^{n-1} + a_n y^n$$

we merely need to utilise formula (8) written in the form

$$\hat{y}_0 = \frac{1}{n!a_n}\left[\frac{\Delta^{n-1}H}{\Delta y^{n-1}}\bigg|_{y_{(0)}} - (n-1)!a_{n-1} - \frac{(n-1)}{2}n!a_n\Delta y\right] \tag{40}$$

where $\Delta^{n-1}H\big|_{y_{(0)}}$ is the (n-1)th the order finite difference of H(y) calculated at the unknown location y_0 from the above mentioned measured samples of H(y). In our electrostatic field example the measurements are obtained from H(y) defined by eq^n.(20)

i.e. $$H(y)=\frac{4.0}{y}+\frac{1.0}{20-y} \qquad (41)$$

at preselected values of y_0 and intersample spacing Δy and the alternative values for parameter pairs a_1 and a_2 used in (40) for the purpose of comparing the different estimates $\hat{y}_0$ produced.

4.1 Taylor Expansion Model: Narrow field

Table 1: Specimen Result using the Taylor Expansion Model

y_0	Δy	$H(y_0)$	$H(y_0+\Delta y)$	$\Delta H\|y_0$	$\hat{y}_0$	Error $\hat{y}_0 - y_0$
11.333	2.0	0.468	0.450	-0.018	11.444	0.111
11.333	4.0	0.468	0.475	+0.007	11.506	0.173
13.333	2.0	0.450	0.475	+0.025	13.568	0.235
15.333	-2.0	0.475	0.450	-0.025	15.568	0.235

The 2nd-order Taylor approximation (24) is first validated for values of y_0 chosen close to the turning point (y=13.333) of the true field function H(y) and using a small sample spacing Δy. The results are shown in Table 1.

Errors range from 2.775% to 5.875% of the narrow field-width (=4.0) under exploration. Errors will clearly increase significantly if the same a_1 and a_2 parameter values are used in the wider field $4 \le y_0 \le 17$ for which regression models are more appropriate and the performance of which are now examined.

4.2 Regression Model for DH(y)

Applying the previous four tests to the conventional regression model of eq^n.(28) yields the results of Table 2.

The absolute errors are clearly some 4.5 times larger than with the Taylor expansion model in the region of the turning point but as a percentage of the field width $(17-4=13)$ the errors range only between -8.22 to +7.96%.

Table 2: Specimen Result using conventional Regression Model

y_0	Δy	$H(y_0)$	$H(y_0+\Delta y)$	$\Delta H\|y_0$	$\hat{y}_0$	Error $\hat{y}_0 - y_0$
11.333	2.0	0.468	0.450	-0.018	11.152	-0.182
11.333	4.0	0.468	0.475	+0.007	12.368	1.035
13.333	2.0	0.450	0.475	+0.025	12.265	-1.068
15.333	-2.0	0.475	0.450	-0.025	14.265	-1.068

Table 3 shows the variation of prediction $\hat{y}_0$ with y_0 using Δy=2.0 over the range $4 \le y_0 \le 15$. Errors range from -3.4% to 16.72% of the field range at its extremes but are clearly much smaller near to centre of range. Of course the window of observation for Table 3 is very small (=Δy=2.0). Increasing Δy to say 7.0 improves the estimation somewhat. For instance, with y_0=6.0, using Δy=7.0 yields an estimate $\hat{y}_0 = 7.0036$ which is a clear improvement on the value (7.748) obtained using Δy=2.0. See Table 3.

Table 3. Variation of Regression Model Prediction with y_0 (Δy=2.0)

y_0	$H(y_0)$	$\Delta H = H(y_0+2) - H(y_0)$	$\hat{y}_0 = 25\Delta H + 11.618$
4.0	1.0625	-0.3224	3.558
5.0	0.8667	-0.2184	6.158
6.0	0.7381	-0.1548	7.748
7.0	0.6483	-0.1134	8.783
8.0	0.5833	-0.0833	9.535
9.0	0.5349	-0.0602	10.113
10.0	0.5000	-0.0417	10.575
11.0	0.4747	-0.0223	11.060
12.0	0.4583	+0.0084	11.828
13.0	0.4524	+0.0476	12.808
14.0	0.4667	+0.1196	14.608
15.0	0.5000	+0.2222	17.173
16.0	0.5863		
17.0	0.7222		

4.3 Orthogonal regression Model

As anticipated, the orthogonal regression model yields improved estimation as illustrated, again for Δy=2.0, in Table 4:

The error band is somewhat reduced in extent compared to the conventional regression model. It ranges from -6.22% to +12.31% and is more symmetrically disposed about zero than before.

Increasing Δy from 2 to 7 improves the estimate $\hat{y}_0$ at y_0=6.0 from 7.600 (Table 4) to 6.760 and this is a significant improvement also on the prediction of 7.0036 using the conventional regression model with Δy=7.0.

4.4 Using Extra Measurements

In a previous report (3) it has been shown that using formula (8) on several sets of n measurements all within a given observation window yields an averaged estimate that is a significant statistical improvement on a single estimate for the same window size but utilising the deterministic minimum number n of measurements. That investigation was confined to white noise deviations however.

We here assess the benefits of using additional measurements to improve errors resulting from the systematic discrepancies that result from the inevitable mismatch between a non-polynomial H(y) and its polynomial approximation. The electrostatic field example is again used as the simple case study.

Table 4. Variation of Orthogonal Regression Model with y_0 (Δy=2.0)

y_0	$H(y_0)$	$\Delta H=H(y_0+2)-H(y_0)$	$\hat{y}_0 = 22.684\Delta H + 11.112$
4.0	1.0625	-0.3224	3.799
5.0	0.8667	-0.2184	6.158
6.0	0.7381	-0.1548	7.600
7.0	0.6483	-0.1134	8.540
8.0	0.5833	-0.0833	9.222
9.0	0.5349	-0.0602	9.746
10.0	0.5000	-0.0417	10.166
11.0	0.4747	-0.0223	10.606
12.0	0.4583	+0.0084	11.303
13.0	0.4524	+0.0476	12.192
14.0	0.4667	+0.1196	13.825
15.0	0.5000	+0.2222	16.152
16.0	0.5863	-	-
17.0	0.7222	-	-

For an n'th order polynomial field , the minimum number of (accurate) measurements required to determine location y_0 is n. However, if the number of H-samples available is n+r, then, using the same value of Δy in each case, formula (40) may be used to estimate y_0, $y_0 + \delta y$, $y_0 + 2\delta y \cdots y_0 + r\delta y$, where δy is the spacing between successive samples and Δy is retained to mean the distance over which finite differences are taken. Thus r+1 calculations of y_0 could be made from the following (modified) version of formula (40).

$$\hat{y}_0(i, \Delta y) = \frac{1}{n!a_n}\left[\left. \frac{\Delta^{n-1}H}{\Delta y^{n-1}} \right|_{(y_0+i\delta_y)} + (n-1)!a_{n-1} - \frac{(n-1)}{2} n!a_n \Delta y \right] - i\delta_y \qquad (42)$$

where $$0 \le i \le r \qquad (43)$$

Now, as illustrated in Fig 4, the observation window width required will be

$$W = (n-1)|\Delta y| + r\delta_y$$

and if the additional H-sampling points are partly interleaved with the originals (that were spaced at intervals Δy) such that

$$\delta_y = |\Delta y| / r_1 \qquad (44)$$

then

$$W = \{(n-1)r_1 + r\}\delta_y = \left\{ n - 1 + \frac{r}{r_1} \right\} |\Delta y| \qquad (45)$$

The total number N of H-samples thus required for the r+1 estimates of y_0 is thus:

$$N = \frac{W}{\delta_y} + 1 = (n-1)r_1 + r + 1 \tag{46}$$

Where the field is truly described by the known n'th order polynomial, then the r additional calculations (i.e. the N-n additional measurements) would be redundant since $\hat{y}_0(i, \Delta y)$ will be independent of i and = y_0 itself. Where the polynomial is only an approximate model, then the averaging of $\hat{y}_0(i, \Delta y)$ across the range $1 \le i \le r$ should assist in reducing the average estimation error across the range of exploration.

As an example we here set r_1=2 and r=5 so that N=8 ordinates (compared to n=2) and W=7 compared to $|\Delta y| = 2.0$, and $\delta_y = 1.0$. Table 5 shows the results of averaging the $\hat{y}_0(i, \Delta y)$ ordinates for a range of y_0.

Table 5. Showing the effect of averaging the 6 $\hat{y}_0$-estimates calculated over the wider window (=7.0)

y_0	$\hat{y}_0$ (From Table 4)	Fwd. Averaged $\hat{y}_0$ (i.e. Δy=+2.0)	Rev. Averaged $\hat{y}_0$ (i.e. Δy=-2.0)
4.0	3.799	5.011	-
5.0	6.158	6.072	-
6.0	7.600	6.813	-
7.0	8.540	7.431	-
8.0	9.222	8.039	-
9.0	9.746	8.806	10.011
10.0	10.166	9.874	11.072
11.0	10.606	-	11.813
12.0	11.303	-	12.431
13.0	12.192	-	12.039
14.0	13.825	-	13.806
15.0	16.152	-	14.874
16.0	-	-	-
17.0	-	-	-

The averaging process has further reduced the error band which now ranges from -1.49% to +8.25% of the field width (=13.0) investigated. The blank entries in Table 5 merely indicate that forward estimates of y_0 are not possible beyond y_0=10.0 with a window of 7 in a field limited by y_2=17 and , similarly, reverse estimates are not possible below y_0=9 because of the lower limit y_1=4.

The opposite discrepancies in the forward and reverse averaged estimates in the overlap region of y_0=9.0 and 10.0 is interesting and suggests that further improvements in accuracy may be achievable in the combining of predictions from (deterministically) redundant data. There is possible scope for optimisation in this area.

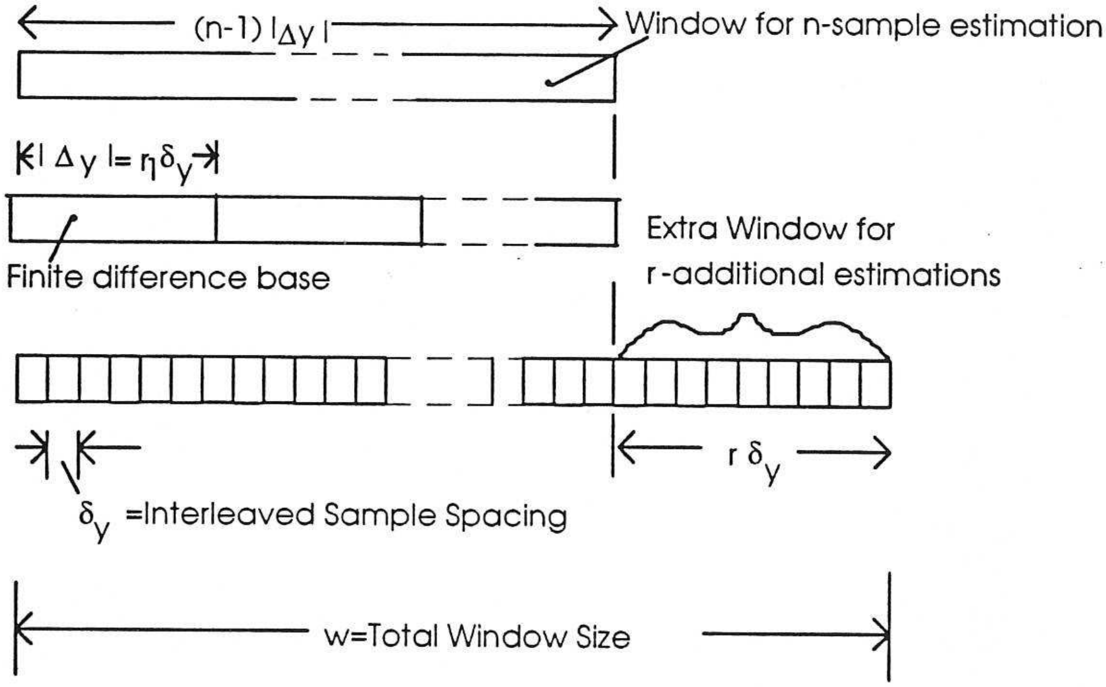

Fig 4 . Showing composition of enlarged observation window W necessary to accommodate additional estimations

5. Higher Order Polynomial Approximations

The results of Section 4 indicate that a reasonable predication, $\hat{y}_0$, of sensor-location, y_0, is possible from n measurements of H(y) using the three parameters a_{n-1}, a_n and n of a minimal-order polynomial approximation $\hat{H}(y)$. However, it is reasonable to enquire whether or not the parameters of a polynomial of higher order might yield a more accurate positional estimate.

In the chosen example, it is clear that any increase in the chosen value of n must be made in increments of 2. i.e. polynomials of order only 2,4,6,8 ......should be utilised in this case. This is because the (n-1)th derivative function is given by:

$$D^{n-1}H(y) = (n-1)!\left[\frac{(-1)^{n-1}4}{y^n} + \frac{1}{(20-y)^n}\right] \qquad (47)$$

so that choosing n=any odd integer yields a curve for $D^{n-1}H(y)$ that is nonmonotonic in the region of interest, $0<y<20$ thus precluding a straight-line approximation. Setting n=2,4,6,8......, however, yields a monotonic curve for $D^{n-1}H(y)$ for which a straight-line might be a reasonable approximation. [Note that a n'th order polynomial $\hat{H}(y)$ produces a curve for $D^{n-1}\hat{H}(y)$ that is truly linear].

Inspection of (42), however, reveals that, for n even, increasing n in steps of 2 makes the curve of y versus $D^{n-1}H(y)$ progressively more sigmoid in shape i.e. progressively less linear over a given range of y. Thus the polynomial approximation becomes less-and-less appropriate for the purpose of position-finding (in this type of field at least) and numerical experiments readily confirm this observation. Thus, as in the case of experiments with more-undulating field profiles (1,2,3), utilising a polynomial of minimal order (i.e. just sufficient to produce the number of actual turning points in H(y)) would again seem to be the best policy to adopt.

6. Conclusions

Previous reports (1),(2),(3) in the series have examined the feasibility of determining location y_0 within a known field H(y) from a sequence of measurements $H(y_0)$, $H(y_0+\Delta y)$, $\cdots\cdots H\{y_0+(N-1)\Delta y\}$. Thoughout the field was assumed to be described by a polynomial in y, having known coefficients. These reports showed such an exercise to be entirely feasible even in the presence of white noise on the measurements of H(y). In the noisy situation the necessity of over-sampling, i.e. setting number of samples N>polynomial order n was demonstrated in order to retain a reasonable level of prediction accuracy. The basic technique relies on the use of the formula for $D^{n-1}H(y)$ that is, of course, linear in y and thus unambiguous in its solution (for the deterministic case).

In this fourth report, the question has been addressed of whether a polynomial approximation $\hat{H}(y)$, might be used for a field governed by other physical laws. The related questions of the best fit polynomial of given order n, and the necessary value of n have been investigated also. The physical field used as a case study has been the electrical field set up between two oppositely charged parallel conductors (or the magnetic field between two parallel counter-current-carrying conductors described by a similar mathematical equations).

There being only a single turning point in H(y), a parabolic polynomial approximation $\hat{H}(y)$ has been used initially to provide the parameters a_{n-1} and a_n needed by the position estimation formula. The "best-fit" model employed was the least squares regression, straight-line fit of y on DH(y). This is here termed the orthogonal regression fit, as opposed to the usual regression fit of DH(y) on y. It is found to provide better overall estimates $\hat{y}_0$ of position y_0 than either the usual regression line or that based on a Taylor expansion about the H(y) turning point. The latter improves considerably as the range of exploration is reduced.

Further, improvements in position prediction have been demonstrated using oversampling within a given observation-window size. The improvements are noticeable although not so significant as the benefits of oversampling in the white noise situation (3).

As a rough rule of thumb, this study indicates that, over a range of around 4<y<17, within a conductor spacing of 20, then position estimates accurate to better than ±10% of the range-width are feasible across the whole range. This is using a minimal order polynomial approximation : i.e. setting n-1=number of turning points in the range. It has been reasoned that any increase in polynomial order will spoil the position-predicting accuracy in fields of this general type.

As with the noisy immunity studies, increasing the window of observation results in improved positional prediction accuracy. Clearly there exists scope for optimisation of predictors of this type.

7. References

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