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Frequency-Domain Theory and Optimisation for Nonlinear Systems

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ABSTRACT

The frequency-domain theory of linear systems, including the root locus, is generalised to nonlinear analytic systems. The spectrum turns out to be a subbundle of a fibre bundle attached to the state-space of the system. We shall approach the problem via the Lie series and on the way show how to apply local Lie series solutions to generate a Lyapunov function for a nonlinear stable system. Moreover, a numerical method for computing the spectrum of a nonlinear system will be given and a number of examples will illustrate the method.

Research Report No 696

1. Introduction

The frequency-domain theory of linear systems is well known and widely applied in control systems engineering. A spectral theory for nonlinear input-output systems also exists by identifying kernels of the Volterra series and using a multi-dimensional Laplace transform. However, the theory is not particularly easy to apply because of the multi-dimensional nature of the frequency responses. (See [1,2,3].) In this paper we propose a single variable frequency-domain theory which can be applied to real systems and which directly generalises the linear theory. In the case of nonlinear oscillations (such as the Van der Pol oscillator) the theory gives the expected spectrum (i.e. an infinite number of poles on the imaginary axis) and we shall prove a stability theorem for certain systems with poles in the left-half plane.

We begin with a discussion about Lie series and show, as an example of the application of the theory, how to determine Lyapunov functions for a stable system. In the following section we shall define the frequency-domain theory for nonlinear differential equations and in the final section we show how to extend the theory to nonlinear systems with inputs and generalise the linear root locus to nonlinear systems. The choice of the parameters in the root locus will be made by an optimisation technique.

2. Lie Series

Consider the nonlinear differential equation

$$\dot{x} = f(x) \ , \ x(0) = x_0 \tag{2.1}$$

We can find a local representation (in infinite-dimensional space) of the solution in the following way. Define

$$\begin{aligned} g_1(x) &= x \\ g_i(x) &= f_i \frac{\partial g_{i-1}}{\partial x} = L_f g_{i-1} \ , \ i \geq 2. \end{aligned}$$

Then

$$\dot{g}_i = \operatorname{grad} g_i \cdot f = L_f g_i = g_{i+1}.$$

Hence, if we introduce the infinite vector $G = (g_1, g_2, \cdots)^T$, we obtain the system

$$\dot{G} = AG \ , \ G(0) = G_0 \tag{2.2}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & & \\ & 0 & 1 & 0 & \cdots & \\ & & & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

i.e. A is the left-shift operator [4], and the initial value G_0 of G is given by

$$G_0 = \left(x_0, L_f x|_{x=x_0}, (L_f)^2 x|_{x=x_0}, \cdots \right). \tag{2.3}$$

Since

$$e^{At} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \cdots & \cdots \\ & 1 & t & \frac{t^2}{2!} & \cdots & \cdots \\ & & 1 & t & \frac{t^2}{2!} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{2.4}$$

we see that

$$x(t; x_0) = (G)_1 = (e^{At} G_0)_1 = \sum_{i=0}^{\infty} \frac{t^i}{i!} (L_f)^i x \Big|_{x=x_0} \tag{2.5}$$

which is simply the Lie series solution of (2.1). (Here we denote the solution of (2.1) through x_0 by $x(t; x_0)$.) The Lie series is simply the Taylor series of the solution $x(t; x_0)$ in t (and x_0). For each x_0, the solution $e^{At} G_0$ will therefore be valid only up to the radius of convergence of the Taylor series of $x(t; x_0)$ with respect to t. Hence the general solution will be given by

$$x(t) = \left\{ \left(e^{At_1} e^{At_2} \cdots e^{At_k} \right) G_0 \right\}_1$$

where $t_1 + \cdots t_k = t$ and t_i <radius of convergence of the solution $x(t; \bar{x})$ where $\bar{x} = \left\{ \left(e^{At_{i+1}} \cdots e^{At_k} \right) G_0 \right\}_1$.

Remark Note that although

$$e^{At_1} e^{At_2} \cdots e^{At_k} = e^{A(t_1 + \cdots t_k)}$$

as an operator on ℓ^2, it is not true on the space of Taylor monomials. Of course, if $x(t; x_0)$ has infinite radius of convergence in t, then

$$x(t; x_0) = (e^{At} G_0)_1 \ , \ t \geq 0. \qquad \square$$

3. Application to Lyapunov Functions

We can use the Lie series to find a Lyapunov function for a stable system. The idea is based on the following result:

Lemma 2.1 If the solutions of the system

$$\dot{x} = f(x)\ ,\ f(0) = 0$$

are asymptotically stable in a region R and satisfy

$$\|x(t;x_0)\| = O\left(\frac{1}{t^{1/p}}\right)\ ,\ \text{for all } x \in R \text{ as } t \to \infty \tag{3.1}$$

for some integer $p > 0$, then

$$V(\xi) = \int_0^\infty \|x(t;\xi)\|^{2q}\,dt\ ,\ \xi \in R$$

is a Lyapunov function for the system in R for any $q \geq p$.

Proof By condition (3.1), the integral exists and $V(\xi) > 0$ for $\xi \neq 0$ and $V(0) = 0$. Also, if $x_1 = x(t_1;\xi), t_1 > 0$, then, by the group property of solutions,

$$\begin{aligned}
V(\xi) &= \int_0^\infty \|x(t;\xi)\|^{2q}\,dt \\
&= \int_0^{t_1} \|x(t;\xi)\|^{2q}\,dt + \int_{t_1}^\infty \|x(t;\xi)\|^{2q}\,dt \\
&> \int_{t_1}^\infty \|x(t;\xi)\|^{2q}\,dt \\
&= \int_0^\infty \|x(t+t_1;\xi)\|^{2q}\,dt \\
&= \int_0^\infty \|x(t;x(t_1;\xi))\|^{2q}\,dt \\
&= V(x_1)
\end{aligned}$$

and so V decreases along trajectories. □

Of course, to apply the above result we need the solution of the system. However, the Lie series gives an expression for the solution inside the radius of convergence. Hence we propose the approximation

$$V(x) = \int_0^T \left\| \sum_{i=0}^m \frac{t^i}{i!}\left(L_f^i x\right) \right\|^{2q} dt$$

for some q, where T is less than the radius of convergence of the Lie series. In some cases, when T is small, we may require a number of Lie series expansions about several points along the trajectory. Thus, put

$$P_m(t;x) = \sum_{i=0}^m \frac{t^i}{i!}\left(L_f^i x\right)$$

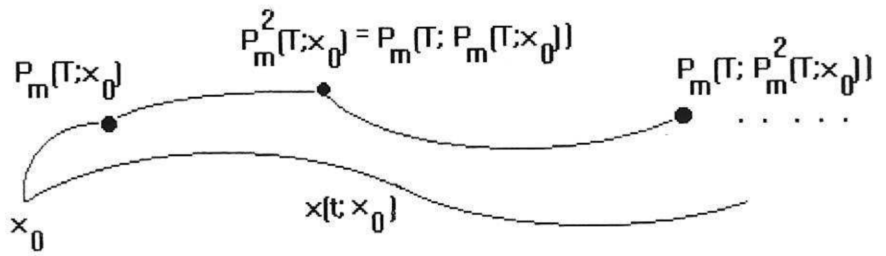
and define

$$\begin{aligned}
P_m^0(t;x) &= x \\
P_m^k(t;x) &= P_m(t;P_m(t;\cdots;P_m(t;x)\cdots))\ ,\ k \geq 1
\end{aligned}$$

The approximate Lyapunov function now becomes

$$\begin{aligned} V(x) &= \int_0^T \|P_m(t;x)\|^{2q}\, dt + \int_0^T \|P_m(t;P_m(t;x))\|^{2q}\, dt + \cdots \\ &= \sum_{k=0}^{N} \int_0^T \left\|P_m(t;P_m^k(t;x))\right\|^{2q} dt \end{aligned}$$

where $N+1$ is the number of Lie series expansions used (see the diagram below).

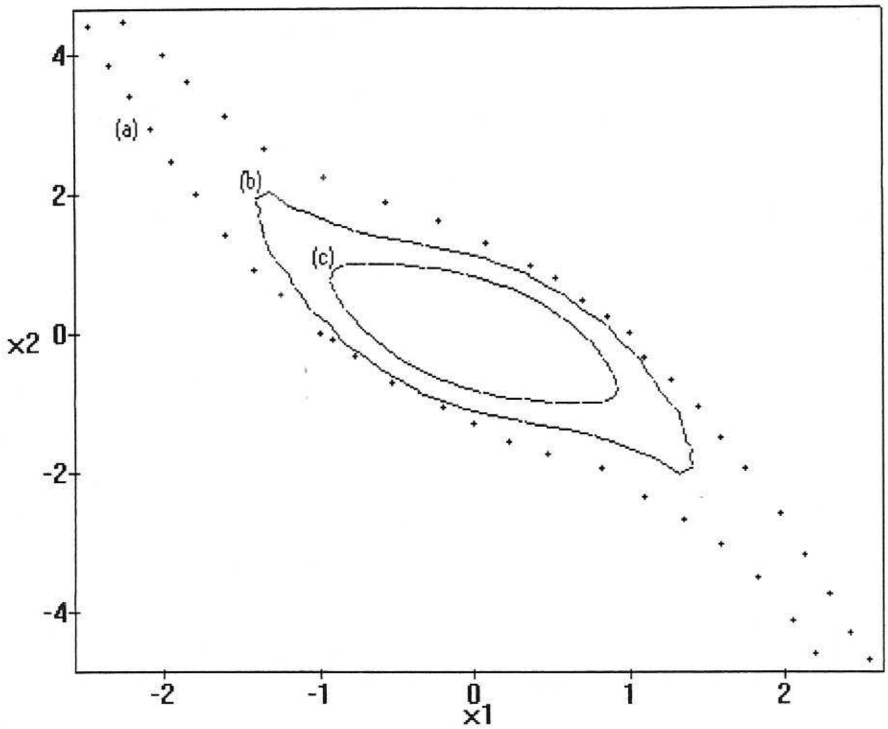


True solution and Lie series approximations

Example 2.1 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + x_1^3 \end{aligned}$$

Then we obtain the approximate Lyapunov functions shown below:

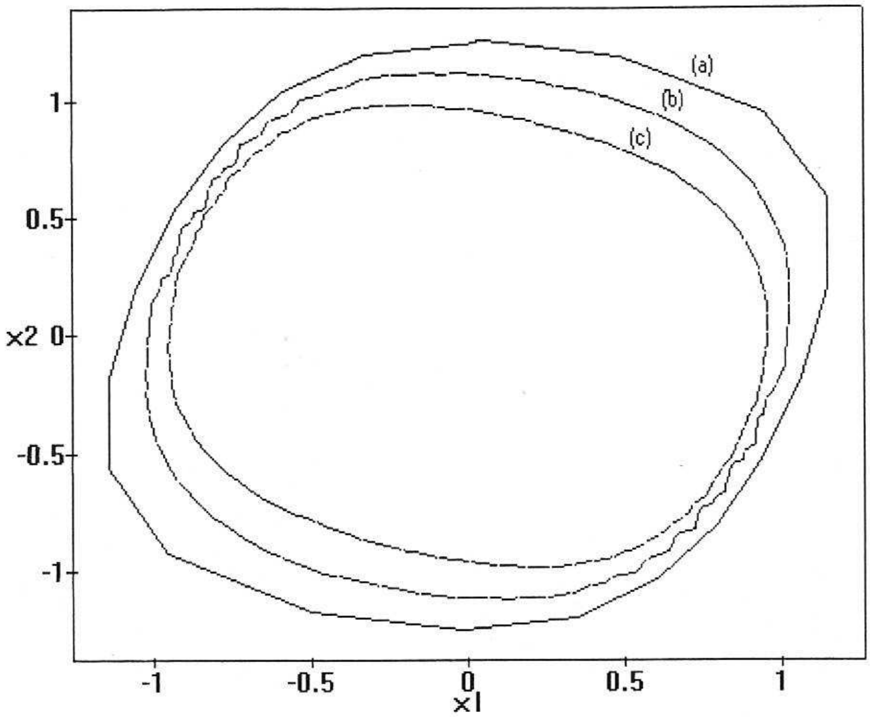


Approximations to a Lyapunov function for the above system

Example 2.2 The time-reversed Van der Pol oscillator

$$\begin{aligned}\dot{x}_1 &= x_1^3 - x_1 - x_2 \\ \dot{x}_2 &= x_1\end{aligned}$$

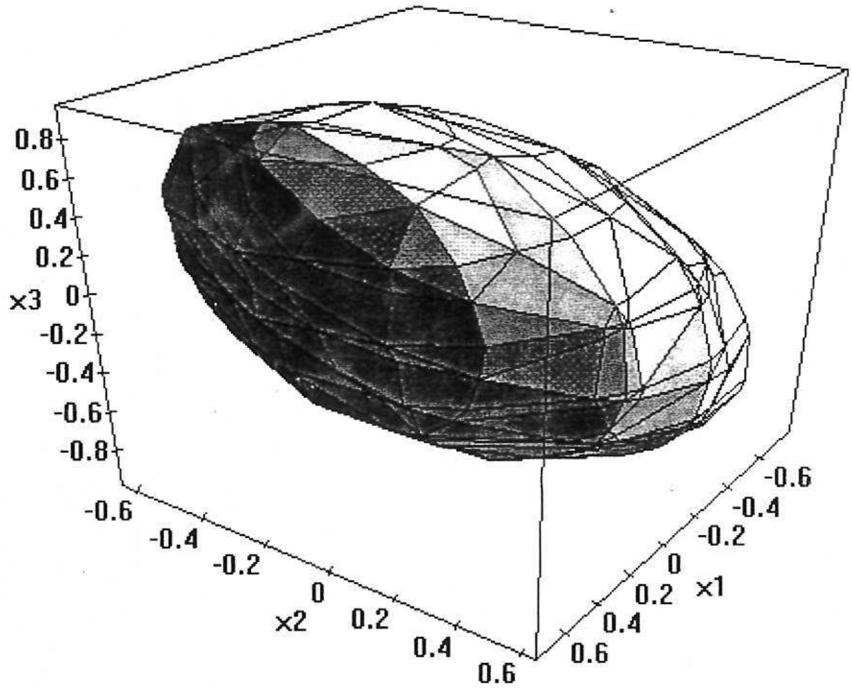
has the following approximations:



Approximate Lyapunov functions for inverse Van del Pol

Example 2.3 A third order system is also shown below:

$$\begin{aligned}\dot{x}_1 &= x_1^3 - x_1 - x_2 - x_3\\ \dot{x}_2 &= x_1\\ \dot{x}_3 &= -x_3\end{aligned}$$



Approximate domain of attraction for a third order system

4. Exponential Representation of Solutions and the Frequency Domain

In this section we shall consider the representation of the equation

$$\dot{x} = f(x) \ , \ x(0) = x_0 \tag{4.1}$$

in the frequency domain. In order to develop a frequency domain theory we shall assume that all solutions are exponentially bounded:

$$|x(t; x_0)| \leq M e^{\omega t}$$

for some M and ω, so that we can define their Laplace transforms. Let $X(s; x_0) = \mathcal{L}(x(t; x_0))$. Then we define the **spectrum** of the system (3.1) to be the zeros of

$$\det \left[\left(\frac{\partial X(s; x_0)}{\partial x_0} \right)^{-1} \right].$$

To justify this definition we first consider systems for which the Lie series has infinite radius of convergence. Then,

$$x(t; x_0) = \sum_{i=0}^{\infty} \frac{t^i}{i!} (L_f)^i x \bigg|_{x=x_0}$$

so

$$X(s;x_0) = \sum_{i=0}^{\infty} \frac{1}{s^{i+1}} (L_f)^i x_0. \tag{4.2}$$

We may write this in symbolic form

$$X(s;x_0) = \left(\frac{1}{s - L_f} \right) x_0.$$

Note that, in the linear case $\dot{x} = Ax$, we get

$$\begin{aligned} X(s;x_0) &= \sum_{i=0}^{\infty} \frac{1}{s^{i+1}} (L_{Ax_0})^i x_0 \\ &= (sI - A)^{-1} x_0, \end{aligned}$$

and so

$$\frac{\partial X(s;x_0)}{\partial x_0} = (sI - A)^{-1}.$$

Of course, in general the Lie series does not always converge for all $t > 0$ and so we cannot use (3.2) to obtain the frequency spectrum. In certain cases one can use the method of Fourier series, however. Suppose that the system has a limit cycle through x_0. Then, if the period of oscillation is T we can write, for example,

$$x_1(t;x_0) = \sum_{k=0}^{\infty} a_k \cos \frac{2k\pi t}{T} + \sum_{k=1}^{\infty} b_k \sin \frac{2k\pi t}{T} \tag{4.3}$$

where $\sum_{k=0}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2 < \infty$, with similar representations for $x_2, \cdots, x_n$.

Lemma (3.1) The spectrum at x_0 of a system with a periodic orbit of period T through x_0 consists of an infinite sequence of poles on the imaginary axis at the points $\pm i\frac{2k\pi}{T}$, $1 \le k < \infty$.

Proof From (3.3) it follows that the poles of $x_1(t;x_0)$ can occur only at the points $\pm i\frac{2k\pi}{T}$ and clearly all these values *are* poles of $x_1(t;x_0)$. Since $\sum_{k=0}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2 < \infty$, the function

$$X_1(s;x_0) = \sum_{k=0}^{\infty} a_k \frac{s}{s^2 + \left(\frac{2k\pi}{T}\right)^2} + \sum_{k=1}^{\infty} b_k \frac{T/2k\pi}{s^2 + \left(\frac{2k\pi}{T}\right)^2}$$

exists in the sense that the right hand side converges on all compact subsets of $\mathbb{C} \setminus \cup_k \left\{ i\frac{2k\pi}{T} \right\}$.□

Note, however, that it is not necessary for a system to have its singularities in the left half plane for stability. For example, the equation

$$\ddot{x} = -\frac{\dot{x}}{t} - x \ , \ x(0) = 1, \dot{x}(0) = 0$$

has solution $J_0(t)$ (the zeroth order Bessel function) and

$$\mathcal{L}(J_0)(s) = \frac{1}{\sqrt{s^2+1}}.$$

This is a function which is analytic on $\mathbb{C}$ apart from on a cut joining $-i$ to i. The main result for applications of this theory is, of course, a sufficient condition for stability. Clearly arbitrary distributions of poles strictly in the left half plane do not imply stability. For example,

$$\begin{aligned} e^{2t} &= e^{-2t}e^{4t} \\ &= e^{-2t}\sum_{k=0}^{\infty}\frac{(4t)^k}{k!} \end{aligned}$$

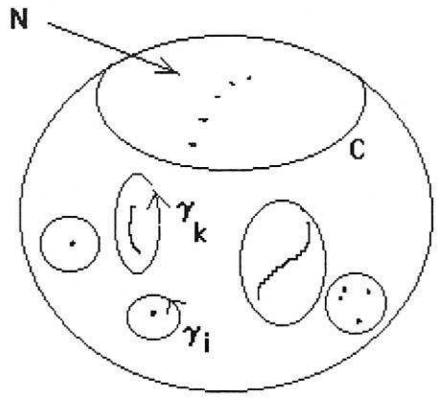
and so

$$\mathcal{L}(e^{2t}) = \sum_{k=0}^{\infty}\frac{4^k}{(s+2)^{k+1}}$$

and the right hand side can be approximated by rational functions with $k+1$ poles at $s=-2$. In fact, we have the following result:

Theorem 4.1 Suppose that the analytic function $F(s)$ has a finite number of isolated singularities each of finite multiplicity (which may include cuts) in any compact subset of $\mathbb{C}$ and assume that they are all contained in some strict left half plane, i.e. $\{z : Re(z) < -\varepsilon\}$ for some $\varepsilon > 0$. Moreover, suppose that for all sufficiently small neighbourhoods N of ∞ there exists a function $G(s)$ with the same principal as $F(s)$ in N (and only those poles) and $g(t) = (\mathcal{L}^{-1}G)(t) \to 0$ as $t \to \infty$. Then $f(t) = (\mathcal{L}^{-1}F)(t) \to 0$ as $t \to \infty$.

Proof Consider the function $F(s)$ on the Riemann sphere, as shown below.



Choose a neighbourhood N of ∞ and a function g as in the statement of the theorem. Then,

$$\begin{aligned} (t) &= (\mathcal{L}^{-1}F)(t) \\ &= \int_C F(s)e^{st}ds + \sum_{k=1}^{M}\int_{\gamma_k}F(s)e^{st}ds \end{aligned}$$

where $C \subset N$ and each γ_k is strictly in the left half plane and surrounds a pole (or cut) of $F(s)$. By Runge's theorem [5], we can approximate F in $\mathbb{C}\backslash\{N \cup_k \Gamma_k\}$ by a rational function, where Γ_k is the interior of γ_k. Hence,

$$f(t) = g(t) + \sum_{k=1}^{M}\int_{\gamma_k}R_\delta(s)e^{st}ds + \sum_{k=1}^{M}\int_{\gamma_k}\left(F(s)-R_\delta(s)\right)e^{st}ds$$

where $|F(s) - R_\delta(s)| < \delta$ for any $\delta > 0$, and so

$$|f(t)| \leq |g(t)| + p(t)e^{-\alpha t} + Ke^{-\beta t}$$

as $t \to \infty$. Here, $p(t)$ is some polynomial function, $\alpha, \beta > 0$ and $K = \delta M \max_k\{\text{length}(\gamma_k)\}$.□

5. Evaluation of the Spectrum of a Nonlinear System

We have defined the spectrum of a nonlinear differential equation

$$\dot{x} = f(x) \ , \ x(0) = x_0$$

by the roots of the determinant of the matrix

$$\frac{\partial X(s; x_0)}{\partial x_0}$$

which requires a knowledge of the Laplace transform of the solution of the equation. (Note that we can also obtain this matrix as the Laplace transform of the solution of the variational system

$$\dot{\Phi} = \frac{\partial f(x(t; x_0))}{\partial x} \Phi.)$$

We can only use the Lie series directly if it converges for all time and even then a closed form for the Laplace transform of the solution is likely to be difficult to obtain. Here we shall introduce an algorithm to find rational approximations to $X(s; x_0)$ for systems of the form

$$\dot{x} = Ax + f(x, t) \ , \ x(0) = x_0$$

where f is a polynomial function in the x variables with coefficients bounded in t and $f(0, t) = 0$ for all $t \geq 0$. The algorithm is simply

$$\begin{aligned} X^{[0]}(s) &= (sI - A)^{-1} x_0 \\ X^{[i]}(s) &= (sI - A)^{-1} \left(x_0 + \mathcal{L}(f(x^{[i-1]}(t), t)) \right) \ , \ i \geq 1 \end{aligned} \tag{5.1}$$

where

$$x^{[i-1]}(t) = \mathcal{L}^{-1}(X^{[i-1]}(s)).$$

It is clear that each $X^{[i]}(s)$ is a rational function of s, since f is a polynomial. If we show that the sequence of functions $X^{[i]}(s)$ (or $x^{[i]}(t)$) converges in some sense, then we will have obtained a rational approximation to $X(s; x_0)$. We shall consider the case where A is stable (similar arguments applied to systems with solutions multiplied by $e^{-\alpha t}$ for some $\alpha > 0$ will cover the unstable case; or alternatively use weighted L^2 norms). Thus,

$$\left\| e^{At} \right\| \leq M e^{-\omega t}$$

for some positive constants M, ω. We shall prove that $x^{[i]}(t)$ converges in $L^2[0, \infty]$ under certain conditions on f. First we show that this will imply that $X^{[i]}(s)$ converges in H^2 i.e. the Hardy space of functions F which are analytic in the open right half plane and satisfy

$$\left[\sup_{\xi > 0} (2\pi)^{-1} \int_{-\infty}^{\infty} |F(\xi + i\omega)|^2 d\omega \right]^{1/2} < \infty.$$

Lemma 4.1 Suppose that the sequence of functions $x^{[i]}(t)$ given by the iteration procedure (4.1) converges in $L^2[0,\infty]$ where f is a polynomial function. Then the sequence $X^{[i]}(s)$ converges in H^2.

Proof As stated above, each function $X^{[i]}(s)$ is proper and rational, since f is a polynomial. For such functions it is well known that

$$\left\|X^{[j]}(s)\right\| = \left[(2\pi)^{-1}\int_{-\infty}^{\infty}|X^{[j]}(i\omega)|^2 d\omega\right]^{1/2}$$

and so by Plancherel's theorem we have

$$\left\|x^{[m]}(t) - x^{[n]}(t)\right\|_{L^2[0,\infty]} = \left\|X^{[m]}(s) - X^{[n]}(s)\right\|_{H^2}$$

which shows that $X^{[i]}(s)$ is a Cauchy sequence in H^2.□

To show that the sequence $x^{[i]}(t)$ converges in $L^2[0,\infty]$ it is enough to prove that the map

$$x \to \mathcal{F}x = \mathcal{L}^{-1}\left((sI-A)^{-1}(x_0 + \mathcal{L}(f(x,t))\right)$$

maps a bounded closed set of $L^2[0,\infty]$ to itself where it is a contraction, by the contraction mapping theorem. Note first that since f is a polynomial in x with $f(0,t) = 0$ we have

$$\begin{aligned}\|f(x,t)\| &\leq p(x)\|x\| \text{ , for all } t\\ \|f(x,t) - f(y,t)\| &\leq q(x,y)\|x-y\| \text{ , for all } t\end{aligned}$$

for some functions $p(x), q(x,y)$ and we assume that $p(x) \leq \gamma, q(x,y) \leq \eta$ for $\|x\| \leq \delta$. Hence

$$\|f(x(\cdot),t) - f(y(\cdot),t)\|_{L^2[0,\infty;\mathbb{R}^n]} \leq \eta\|x(\cdot) - y(\cdot)\|_{L^2[0,\infty;\mathbb{R}^n]}$$

for $x(\cdot), y(\cdot) \in B(0,\delta;L^\infty[0,\infty;\mathbb{R}^n]) \cap L^2[0,\infty;\mathbb{R}^n]$. Now,

$$\begin{aligned}\mathcal{F}x &= \mathcal{L}^{-1}\left((sI-A)^{-1}(x_0 + \mathcal{L}(f(x,t))\right)\\ &= e^{At}x_0 + \int_0^t e^{A(t-\tau)}f(x(\tau),\tau)d\tau\end{aligned}$$

so that, if $x \in B(0,\delta;L^\infty[0,\infty;\mathbb{R}^n])$, we have

$$\|\mathcal{F}x\|_{L^\infty[0,\infty;\mathbb{R}^n]} \leq M\|x_0\| + \frac{\gamma}{\omega}.$$

Hence if

$$M\|x_0\| + \frac{\gamma}{\omega} \leq \delta$$

then $\mathcal{F}$ maps the closed subset $S \triangleq B(0,\delta;L^\infty[0,\infty;\mathbb{R}^n]) \cap L^2[0,\infty;\mathbb{R}^n] \cap C_{x_0}(0,\infty;\mathbb{R}^n)$ into itself where $C_{x_0}(0,\infty;\mathbb{R}^n)$ is the set of continuous functions z with $z(0) = x_0$. To prove that $\mathcal{F}$ is a contraction on this set note that

$$\begin{aligned}\|\mathcal{F}x - \mathcal{F}y\|_S &= \left\|\mathcal{L}^{-1}\left((sI-A)^{-1}\mathcal{L}(f(x,t)\right) - \mathcal{L}^{-1}\left((sI-A)^{-1}\mathcal{L}(f(y,t)\right)\right\|_S\\ &= \left\|\mathcal{L}^{-1}\left[(sI-A)^{-1}\left(\mathcal{L}\left(f(x,t) - f(y,t)\right)\right)\right]\right\|_S\\ &= \left\|\int_0^t e^{-\omega(t-\tau)}\left(f(x(\tau),\tau) - f(y(\tau),\tau)\right)\right\|_S\\ &\leq \left\|e^{-\omega t}\right\|_{L^1(0,\infty)} \cdot \|f(x,t) - f(y,t)\|_{L^2[0,\infty;\mathbb{R}^n]} \text{ (by Young's inequality)}\\ &\leq \frac{1}{\omega}\eta\|x-y\|_{L^2[0,\infty;\mathbb{R}^n]}\end{aligned}$$

and so, if $\eta < \omega$, then $\mathcal{F}$ is a contraction. Hence we have proved

Theorem 4.1 If A is a stable matrix with

$$\left\| e^{At} \right\| \leq Me^{-\omega t}$$

and $f(x,t)$ is a polynomial function satisfying

$$\begin{aligned} \|f(x,t)\| &\leq \gamma \|x\| \ , \text{ for } \|x\| \leq \delta \\ \|f(x,t) - f(y,t)\| &\leq \eta \|x - y\| \ , \text{ for } \|x\| \leq \delta \end{aligned}$$

and

$$M \|x_0\| + \frac{\gamma}{\omega} \leq \delta \ , \ \eta < \omega$$

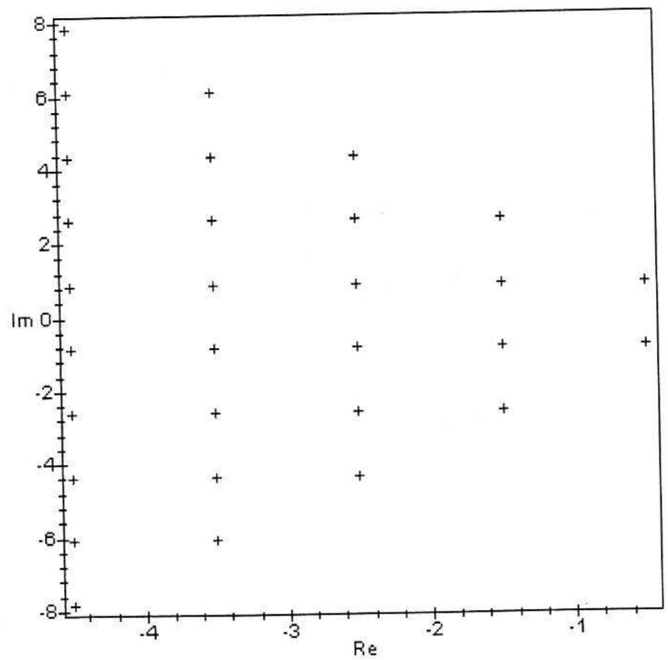
then the set of rational approximations given by (4.1) converges in H^2 and the limit is the Laplace transform of the solution of

$$\dot{x} = Ax + f(x,t) \ , \ x(0) = x_0.\square$$

Examples Consider the time-reversed Van der Pol oscillator:

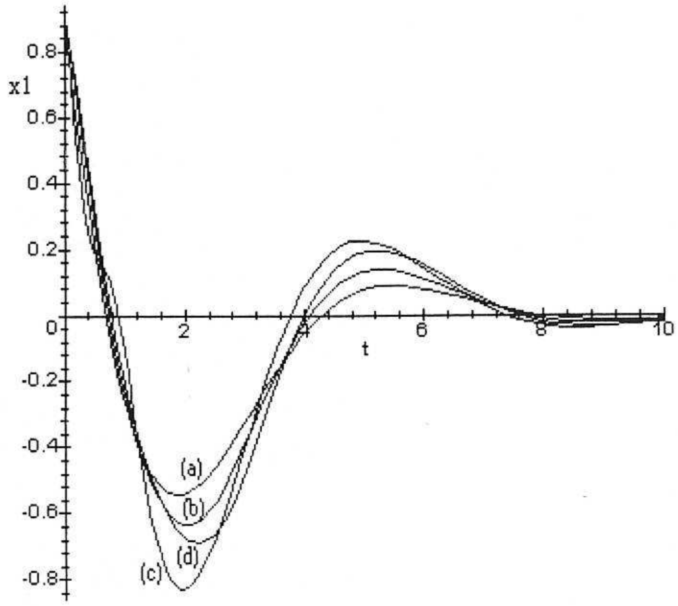
$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1^3 - x_1 \\ \dot{x}_2 &= x_1 \end{aligned}$$

The algorithm above gives the poles shown below:



Poles of the Time-Reversed Van der Pol Oscillator

The corresponding approximations to the solution are shown below:



Approximations to the Solution

6. Feedback Systems

Consider now systems of the form

$$\dot{x} = f(x, u) \ , \ x(0) = x_0 \tag{6.1}$$

We shall develop a nonlinear root locus theory, so we shall assume that u is a rational function of x. The reason is that, in the linear case, we use a linear feedback in the state x and so in the nonlinear case the feedback will, in many cases, be an analytic function of x which can be approximated arbitrarily closely by a rational function. Thus, we put

$$u = \frac{\sum_{|\mathbf{i}|=0}^{K_1} p_{\mathbf{i}} x^{\mathbf{i}}}{\sum_{|\mathbf{j}|=0}^{K_2} q_{\mathbf{j}} x^{\mathbf{j}}}$$

where $\mathbf{i} = (i_1, \cdots, i_n)$, $|\mathbf{i}| = i_1 + \cdots + i_n$ and $x^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n}$. Substituting this into (5.1) we have

$$\dot{x} = f\left(x, \frac{\sum_{|\mathbf{i}|=0}^{K_1} p_{\mathbf{i}} x^{\mathbf{i}}}{\sum_{|\mathbf{j}|=0}^{K_2} q_{\mathbf{j}} x^{\mathbf{j}}}\right) \ , \ x(0) = x_0 \tag{6.2}$$

Let $\sigma(p_{\mathbf{i}}, q_{\mathbf{j}}; 0 \le |\mathbf{i}| \le K_1, 0 \le |\mathbf{j}| \le K_2) = \sigma(p_{\mathbf{i}}, q_{\mathbf{j}}) \subseteq \mathbb{C}$ denote the spectrum of the system (5.2) as defined above. Then we define the **root locus** of (5.2) to be the set

$$\sum = \cup_{p_{\mathbf{i}}, q_{\mathbf{j}}} \sigma(p_{\mathbf{i}}, q_{\mathbf{j}}).$$

If u is constrained so that $|u| \leq u_{\max}$, then p_i and q_j must belong to some parameter set Ξ, i.e.

$$|u| \leq u_{\max} \text{ iff } \{p_i, q_j\} \in \Xi$$

and we denote the **constrained root locus** by

$$\sum_{\Xi} = \cup_{p_i, q_j \in \Xi} \sigma(p_i, q_j).$$

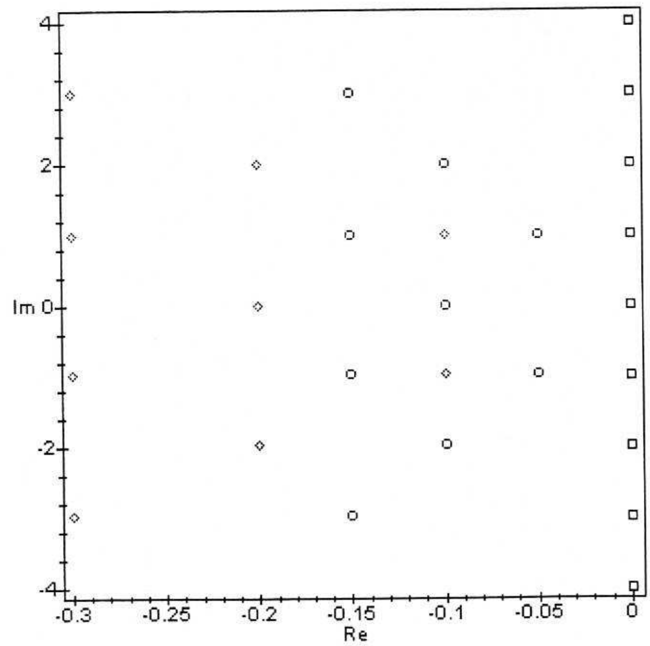
If a system satisfies the conditions of theorem 5.1 for all $\{p_i, q_j\} \in \Xi$ then it follows that the following optimisation problem will stabilise the system:

$$\min_{z \in \sum_{\Xi}} \text{Re } z.$$

Example 6.1. Consider the control system

$$\begin{aligned} \dot{x}_1 &= x_2 + u(1 - x_1^2) \\ \dot{x}_2 &= -x_1 \end{aligned}$$

If we choose $u = (\alpha x_1 + \beta)$ then we have the following pole configurations for $\alpha = -1$ and $\beta = 0, -0.1, -0.2$. In the figure, the boxes correspond to roots with $\beta = 0$, the circles to $\beta = -0.1$ and the diamonds to $\beta = -0.2$. The latter gives the most stable configuration for $|u| \leq 1.2$.



Root Locus with beta =0,-0.1,-0.2

7. References

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