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Bilinear Systems in the 'Frequency Domain' and an Application to Optimal Control

by

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Abstract

A 'frequency domain' approach to the optimal control of bilinear systems is presented . This is based on a new representation of the input-output map of the system in terms of a basis of $L^2[0,T]$.

1. Introduction

The theory of linear systems has been developed at different times using one of two basic approaches - (i) the frequency domain , transfer function method and (ii) the state space (time domain) technique . These two methods can be regarded as being equivalent because of the isomorphism between the time domain and the frequency domain provided by the Fourier transform . This isomorphism maps $L^2[0,\infty]$ onto $L^2[-\infty,\infty]$ or more generally from (some) space of distributions onto another . The Fourier transformed system then exhibits the frequency response behaviour of the original system .

In the theory of nonlinear systems it is no longer of much significance what the response of the system to a sine wave may be , since this response will not be a sine wave and the original system has no superposition principle , in general . However , if we consider such a nonlinear system to be defined on a finite time interval $[0,T]$ (merely for convenience , the infinite time interval can be considered similarly) and the inputs belong to $L^2[0,T]$ then each input may be regarded as an infinite sum of 'special' functions . For if $\{e_i\}_{i\geq 1}$ is a basis of $L^2[0,T]$, then any input u may be written

$$u = \sum_{i=1}^{\infty} u_i e_i \quad .$$

Note , however , that as we have stated above , there is now no 'canonical' basis since the sine and cosine functions have no special significance for general nonlinear systems .

If the nonlinear system S is input-output stable in the usual sense , then $S:L^2[0,T] \to L^2[0,T]$, where S is the input-output map of the system . Thus ,

$$y = S(u)$$

and so

$$y_i = s_i(u_1, u_2, \ldots) \tag{1.1}$$

where

$$s_i(u_1, u_2, \ldots) = \langle S(\Sigma u_j e_j), e_i \rangle .$$

We shall call the sequence $\{s_i\}_{i\geq 1}$ the **generalized frequency response** of S . In [1] and [2] we have presented a realization theory based on the generalized frequency response . This paper will be concerned mainly with the application of this approach to the optimal control of bilinear systems . The bilinear-quadratic regulator problem was solved recently [3],[4] with the solution being given as a power series with tensor-operator coefficients . This solution has two drawbacks - first it is only valid for sufficiently small initial states and secondly the tensor-operator valued 'Riccati-like' equations have expanding dimension making computation very difficult . In this paper we shall show that , if we take a finite number of basis vectors , say $e_1, \ldots, e_k$, then the solution of the optimal control problem is given in terms of a set of polynomial equations .

Note finally that there have been other approaches to the 'frequency domain' behaviour of nonlinear systems ([5-7]) but these are based mainly on the Volterra series and the generalized Laplace transform . The method proposed here is a direct generalization of the linear frequency domain approach , since it is based on an isomorphism between the 'time' and 'frequency' domains .

2. Notation and Terminology

In this paper we shall denote by $L^{2,n}[0,T]$ the space of square integrable functions $x:[0,T] \to \mathbb{R}^n$; i.e. the integral

$$\int_0^T \| x(t) \|^2 dt$$

is finite , where $\|.\|$ is the usual Euclidean norm on $\mathbb{R}^n$. If n=1 we shall write simply $L^2[0,T]$. If $\{e_i\}$ is a basis of $L^2[0,T]$, then there is an obvious isomorphism $\mathcal{J}:L^2[0,T] \to \ell^2$ induced by this basis , where ℓ^2 is the usual space of square summable sequences . Let $\ell^{2,n}$ denote the space of square summable sequences with values in $\mathbb{R}^n$, so that if $\{\xi(i)\} \in \ell^{2,n}$, where $\xi(i) \in \mathbb{R}^n$, for each i , then

$$\sum_{i=1}^{\infty} \| \xi(i) \|^2 < \infty .$$

Then $\mathcal{J}$ induces an isomorphism $\mathcal{J}^n:L^{2,n}[0,T] \to \ell^{2,n}$ given by

$$\mathcal{J}^n(x) = \mathcal{J}^n\left(\sum_{i=1}^{\infty} x(i)e_i\right) = (\mathcal{J}x_1,\ldots,\mathcal{J}x_n)^T = \{x(i)\}$$

where

$$x(i) = (x_1(i),\ldots,x_n(i))$$

and

$$x(t) = (x_1(t),\ldots,x_n(t)) .$$

We shall then denote the sequence $\{x(i)\}$ by $\mathbf{x}$. Finally , I will denote the identity operator without further comment . It will be clear from the context on which space this operates .

3. 'Frequency Domain' Representation of Bilinear Systems

Consider a bilinear system

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x \quad , \quad x(0)=x_0 \in \mathbb{R}^n \tag{3.1}$$

defined on the interval $[0,T]$, where T is arbitrary , but finite . This system gives rise to an input-output relation given by the well-known Volterra series . In this section we shall present an alternative representation of the input-output relation which has a connection with the frequency domain transfer function for linear systems . Let

$$x = h(u) \tag{3.2}$$

denote the input-output relation of (3.1) . Then we require

Lemma 3.1 As a nonlinear operator , h maps $L^{2,m}[0,T] \cap L^{\infty,m}[0,T]$ into $L^{2,m}[0,T]$.

Proof Define the sequence of functions $\xi_i(t)$ by

$$\xi_0(t) = e^{At}x_0$$

$$\xi_i(t) = \int_0^t e^{A(t-s)} \sum_{j=1}^{m} u_j(s) B_j \xi_{i-1}(s)ds \quad , \quad i \geq 1 .$$

We can write

$$\| e^{At} \| \leq Ke^{\omega t}$$

for some real constants K, ω and so

$$\| \xi_i(t) \| \leq \int_0^t Ke^{\omega(t-s)} \sum_{j=1}^{m} |u_j(s)| \cdot \| B_j \| \cdot \| \xi_{i-1}(s) \| ds .$$

Thus , if $\beta = \max_{1 \leq j \leq m} \| B_j \|$ and $\nu = \sum_{j=1}^{m} \sup_{0 \leq t \leq T} |u_j(t)|$, then

$$\| \xi_i(t) \| \leq \int_0^t Ke^{\omega(t-s)} \nu\beta \, \| \xi_{i-1}(s) \| ds$$

$$\leq (K\nu\beta)^i \frac{t^i}{i!} e^{\omega t} \| x_0 \|$$

by iteration . Hence , if $x(t) = \sum_{i=0}^{\infty} \xi_i(t)$, then

$$\|x(t)\| \leq e^{(\omega+K\nu\beta)t} \|x_0\|$$

and the series is uniformly convergent . It is easy to check that

$$h(u) = \sum_{i=0}^{\infty} \xi_i(t)$$

and the result follows . □

Let $\{e_i\}_{i\geq 1}$ be an orthonormal basis of $L^2[0,T]$. From (3.1) we have

$$x(t) = e^{At}x + \sum_{i=1}^{m} \int_0^t e^{A(t-s)} u_i(s) B_i x(s) ds \quad ,$$

and by lemma 3.1 , if $u \in L^{2,m}[0,T] \cap L^{\infty,m}[0,T]$ then $x \in L^{2,n}[0,T]$ and so we can write

$$u = \sum_{i=1}^{\infty} u(i) e_i \quad , \qquad x = \sum_{i=1}^{\infty} x(i) e_i$$

where

$$\mathbf{u} \triangleq \{u(i)\} \in \ell^{2,n} \quad , \qquad \mathbf{x} \triangleq \{x(i)\} \in \ell^{2,n} \quad .$$

Thus ,

$$\sum_{i=1}^{\infty} x(i) e_i = e^{At}x_0 + \sum_{\ell=1}^{m} \int_0^t e^{A(t-s)} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} u_\ell(i) B_\ell x(j) e_i(s) e_j(s) ds \quad .$$

Now assume that the basis $\{e_i\}$ satisfies the assumption

(A) the function $t \to \int_0^t e^{A(t-s)} e_i(s) e_j(s) ds \in L^2[0,T]$ for each i,j .

Then , under assumption (A) we can write

$$\sum_{i=1}^{\infty} x(i) e_i = e^{At}x_0 + \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t e^{A(t-s)} e_i(s) e_j(s) ds\, u_\ell(i) B_\ell x(j)$$

Again , by assumption (A) , we have

$$\int_0^t e^{A(t-s)} e_i(s) e_j(s) ds = \sum_{k=0}^{\infty} \alpha_{ijk} e_k(t) \quad .$$

Hence

$$x(k) = \xi(k) + \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ijk} u_\ell(i) B_\ell x(j)$$

where

$$\xi(k) = \langle e^{A\cdot} x_0, e_k \rangle \ . \tag{3.3}$$

If we define the operator $K(\mathbf{u})\colon \ell^{2,n} \to \ell^{2,n}$ by

$$(K(\mathbf{u})\mathbf{x})(k) = \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ijk} u_\ell(i) B_\ell x(j) \quad , \tag{3.4}$$

where $\mathbf{u} = \{u(i)\}$, $\mathbf{x} = \{x(i)\}$, then we have

$$(I-K(\mathbf{u}))\mathbf{x} = \boldsymbol{\xi} \quad ,$$

where $\boldsymbol{\xi} = \{\xi(k)\}$. Hence we have

Lemma 3.2 If $\mathbf{u} \in \ell^{2,m}$ then the solution of the bilinear system (3.1) with input $\mathbf{u}$ is given by

$$\mathbf{x} = (I-K(\mathbf{u}))^{-1} \boldsymbol{\xi} \tag{3.5}$$

where $\boldsymbol{\xi}$ and $K(\mathbf{u})$ are given by (3.3) and (3.4) , respectively . □

The 'inverse system' (3.5) can be written in a different way which is convenient for applications . We have

$$(K(\mathbf{u})\mathbf{x})(k) = \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ijk} u_\ell(i) B_\ell x(j)$$

Define the operators $L_i\colon \ell^{2,n} \to \bigoplus_{\ell=1}^{m} \ell^{2,n}$ by

$$(L_i\mathbf{x})(k) = \left(\sum_{j=1}^{\infty} \alpha_{ijk} B_1 x(j) \ , \ \dots \ , \ \sum_{j=1}^{\infty} \alpha_{ijk} B_m x(j) \right)$$

Then

$$K(\underline{u}) = \sum_{i=1}^{\infty} u(i)L_i \; .$$

Hence , by lemma 3.2 we have

$$\mathbf{x} = (I - \sum_{i=1}^{\infty} u(i)L_i)^{-1} \underline{\xi} \; . \tag{3.6}$$

In particular , if $u \epsilon P_k L^{2,m}[0,T]$, where P_k is the projection of $L^{2,m}[0,T]$ onto the subspace generated by the basis $\{e_1, \ldots, e_k\}$, then

$$\mathbf{x} = (I - \sum_{i=1}^{k} u(i)L_i)^{-1} \underline{\xi} \; . \tag{3.7}$$

Theorem 3.3 If the input u satisfies the inequality

$$\sum_{i=1}^{k} \| u(i) \| \, \| L_i \| < 1$$

then we have the expansion

$$\mathbf{x} = \{I + \sum_{i=1}^{k} u(i)L_i + \sum_{i=1}^{k} \sum_{j=1}^{k} u(i)u(j)L_iL_j + \ldots\} \; . \tag{3.8}$$

Proof This follows directly from the Neumann series . □

Remark In contrast to the Volterra series , which follows from lemma 3.1 and is valid for $u \epsilon L^{\infty,m}[0,T]$, the expansion (3.8) is only valid for sufficiently small $\|u\|$. However , lemma 3.1 shows that (3.5.) and (3.7) are valid for $u \epsilon L^{2,m}[0,T] \cap L^{\infty,m}[0,T]$ and so (3.7) is an analytic continuation of (3.8) .

4. Application to Optimal Control

The solution of the linear-quadratic regulator problem has recently been extended to bilinear systems ([3],[4]) . The major drawback with the solution of the bilinear-quadratic regulator problem is that the feedback control is a tensorial power series in the state with tensor operator coefficients , the ranks of which tend to infinity . Hence , even for low-order approximations , we must perform considerable amounts of computation and to increase the accuracy of the control we must compute tensor operators of higher rank .

We can obviate this difficulty by expressing the problem in the 'frequency domain' . Thus , if we consider the optimal control problem :

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x \quad ,$$

$$\text{minimize} \quad J \triangleq \int_0^T (\|x\|^2 + \|u\|^2) dt \quad ,$$

then we can write

$$\begin{aligned} J &= \|x(.)\|_{L^{2,n}[0,T]} + \|u(.)\|_{L^{2,m}[0,T]} \\ &= \sum_{i=1}^{\infty} \|x(i)\|^2 + \sum_{i=1}^{\infty} \|u(i)\|^2 \\ &= \|\mathbf{x}\|^2 + \|\mathbf{u}\|^2 \quad . \end{aligned}$$

By lemma 3.2 we have

$$\mathbf{x} = (I-K(\mathbf{u}))^{-1} \underline{\xi}$$

and so

$$J = ((I-K(\mathbf{u}))^{-1}\underline{\xi})^T ((I-K(\mathbf{u}))^{-1}\underline{\xi}) + \|\mathbf{u}\|^2 \; .$$

Hence

$$\frac{\partial J}{\partial u_j(k)} = 2\left(\frac{\partial}{\partial u_j(k)} (I-K(\mathbf{u}))^{-1} \underline{\xi} \right)^T ((I-K(\mathbf{u}))^{-1} \underline{\xi}) + 2u_j(k)$$

However ,

$$\frac{\partial}{\partial u_j(k)}(I-K(\underline{u}))^{-1}\underline{\xi} = -(I-K(\underline{u}))^{-1}\frac{\partial}{\partial u_j(k)}(I-K(\underline{u}))(I-K(\underline{u}))^{-1}\underline{\xi}$$

and so

$$\frac{\partial J}{\partial u_j(k)} = -2\left((I-K(\underline{u}))^{-1}\frac{\partial}{\partial u_j(k)}(I-K(\underline{u}))(I-K(\underline{u}))^{-1}\underline{\xi}\right)^T((I-K(\underline{u}))^{-1}\underline{\xi}) + 2u_j(k) .$$

Also , we have

$$\frac{\partial}{\partial u_j(k)}(I-K(\underline{u})) = -\frac{\partial}{\partial u_j(k)} K(\underline{u}) = -L_{j,k}$$

where

$$L_{j,k} = P(k)L_j$$

and

$$P(k) : \bigoplus_{\ell=1}^{m} \ell^{2,n} \to \ell^{2,n}$$

is the projection on the k^{th} factor . We have therefore proved

Theorem 4.1 A necessary condition for u to be an optimal solution of the bilinear-quadratic regulator control problem

$$\dot{x} = Ax + \sum_{i=1}^{m} u_i B_i x$$

$$\min J = \int_0^T (\|x\|^2 + \|u\|^2)dt$$

is that u satisfies the equations

$$u_j(k) = -((I-K(\underline{u}))^{-1}L_{j,k}(I-K(\underline{u}))^{-1}\underline{\xi})^T((I-K(\underline{u}))^{-1}\underline{\xi}) \qquad (4.1)$$

for $1 \leq j \leq m$, $1 \leq k < \infty$, where

$$u_j(k) = \langle u_j, e_k \rangle \quad , \quad \xi(k) = \langle e^{A\cdot}x_0, e_k \rangle \quad .$$

Of course , we have an infinite number of equations in an infinite number of variables and so the computational problem is no easier than before . However , since we are considering Fourier series representations of functions , we have an obvious approximation by applying the projections P_k defined above . Thus , we define the operator $K_k(\underline{u}):L^{2,n}[0,T] \to L^{2,n}[0,T]$ by

$$K_k(\underline{u}) = P_k K(\underline{u}) P_k$$

for each $\underline{u} \in P_k L^{2,m}[0,T]$. Then we have

Lemma 4.2 The solution of the bilinear system (3.1) (specified in (3.5)) is given by

$$\lim_{k\to\infty} \underline{x}^k$$

where

$$\underline{x}^k = (I-K_k(\underline{u}^k))^{-1}\, \underline{\xi}^k ,$$

and

$$\underline{u}^k = P_k\underline{u} \quad , \quad \underline{\xi}^k = P_k\underline{\xi} \quad , \quad \text{and} \quad I:P_k L^{2,n}[0,T] \to P_k L^{2,n}[0,T]$$

is the identity map .

Proof We have seen that (3.5) is equivalent to the equation

$$x(\nu) = \xi(\nu) + \sum_{\ell=1}^{m} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij\nu} u_\ell(i) B_\ell x(j) \quad , \quad \nu \geq 1 .$$

Hence ,

$$x(\nu) = \xi(\nu) + \sum_{\ell=1}^{m} \sum_{i=1}^{k} \sum_{j=1}^{k} \alpha_{ij\nu} u_\ell(i) B_\ell x(j) + \varepsilon(\nu) \quad , \quad \nu \geq 1$$

where

$$\varepsilon(\nu) = \sum_{\ell=1}^{m} \sum_{i=k+1}^{\infty} \sum_{j=k+1}^{\infty} \alpha_{ij\nu} u_\ell(i) B_\ell x(j) .$$

Hence

$$P_k\underline{x} = (I-K_k(\underline{u}^k))^{-1}\, \underline{\xi}^k + (I-K_k(\underline{u}^k))^{-1}\, \underline{\varepsilon}^k$$

where $\underline{\varepsilon}^k = P_k\underline{\varepsilon}$, $\underline{\varepsilon} = (\varepsilon(1), \varepsilon(2), \ldots.)$. The result now follows from the fact that $I-K(\underline{u})$ is invertible , $I-K_k(\underline{u}^k)$ is arbitrarily close to $I-K(\underline{u})$ for large k and $\underline{\varepsilon}^k$ is arbitrarily small for large k . □

It follows from lemma 4.2 that we can approximate the cost functional J arbitrarily closely by writing

$$J_k = \|P_k \mathbf{x}\|^2 + \|P_k \mathbf{u}\|^2 .$$

From theorem 4.1 we have

Corollary 4.3 A necessary condition for u to be an optimal solution of the approximate bilinear-quadratic control problem

$$\begin{aligned} \dot{\overline{P_k x}} &= P_k A P_k x + \sum_{i=1}^{m} (P_k u_i) P_k B_i P_k x \\ \min J &= \int_0^T (\|P_k x\|^2 + \|P_k u\|^2) dt \end{aligned} \tag{4.2}$$

is that $P_k u$ satisfies the equations

$$u_j(\ell) = -((I-K_k(\mathbf{u}^k))^{-1} L_{j,\ell}(I-K_k(\mathbf{u}^k))^{-1}\underline{\xi}_k)^T((I-K_k(\mathbf{u}^k))^{-1}\underline{\xi}_k) \tag{4.3}$$

for $1 \leqslant \ell \leqslant k$, $1 \leqslant j \leqslant m$. □

Consider the operator-valued function $(I-K_k(\mathbf{u}^k))$ in the km variables $u_j(\ell)$, $1 \leqslant \ell \leqslant k, 1 \leqslant j \leqslant m$. This function has a matrix representation on $\mathbb{R}^{nk}$, the diagonal elements of which are affine functions of $\mathbf{u}^k$. Hence the determinant of this matrix representation of $(I-K_k(\mathbf{u}^k))$ is a polynomial in $\mathbf{u}^k$ of order $\leqslant nk$. It follows that the equations (4.3) can be written in the form

$$\begin{aligned} &p_1(u_1(1),\ldots,u_m(1),\ldots,u_1(k),\ldots,u_m(k)) = 0 \\ &\ldots\ldots\ldots\ldots\ldots \\ &p_{km}(u_1(1),\ldots,u_m(1),\ldots,u_1(k),\ldots,u_m(k)) = 0 \end{aligned} \tag{4.4}$$

where p_i, $1 \leqslant i \leqslant km$ is a polnomial of order 3nk+1 . An explicit expression for p_i is given by

$$p_i = u_j(\ell) D_k + (A_k L_{j,\ell} A_k \underline{\xi}_k)^T (A_k \underline{\xi}_k) ,$$

where $i=(\ell-1)k+j$, and D_k and A_k are , respectively , the determinant and adjugate matrix of the matrix representation of $(I-K_k(\mathbf{u}^k))$.

5. Example

In this section we shall consider the simple example

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ .75 & 1 \end{pmatrix} x \quad + \quad u \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} x \quad , \quad x_0 = (1 \quad 1)^T$$

where $u(.)\in\mathbb{R}$, $x = (x_1, x_2)^T \in\mathbb{R}^2$, and evaluate the polynomial equations (4.4) in the cases k=1 , k=2 to illustrate the method . Let $T=2\pi$ and choose the standard basis $\{(1/2\pi), (1/\pi)\cos t, (1/\pi)\sin t, (1/\pi)\cos 2t, (1/\pi)\sin 2t, \ldots\}$ of $L^2[0,2\pi]$.

We have

$$e^{At} = \begin{pmatrix} (1/4)e^{(3/2)t}+(3/4)e^{-(1/2)t} & (1/2)e^{(3/2)t}-(1/2)e^{-(1/2)t} \\ (3/8)e^{(3/2)t}-(3/8)e^{-(1/2)t} & (3/4)e^{(3/2)t}+(1/4)e^{-(1/2)t} \end{pmatrix}$$

and

$$\alpha_{ijk} = \int_0^{2\pi} \int_0^t e^{A(t-s)} e_i(s) e_j(s) ds e_k(t) dt \ .$$

Thus ,

$$\alpha_{111} = \frac{1}{(2\pi)^3} \int_0^{2\pi} \int_0^t e^{A(t-s)} dsdt$$

$$\cong \begin{pmatrix} 5.6 & 11 \\ 8.3 & 17 \end{pmatrix} .$$

Also ,

$$\xi(1) = \frac{1}{2\ \pi} \int_0^{2\pi} e^{At} x_0 dt \cong \begin{pmatrix} 986 \\ 1479 \end{pmatrix}$$

and

$$K_1(u^1) = u(1) \begin{pmatrix} 5.6 & 11 \\ 8.3 & 17 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= u(1) \begin{pmatrix} 11 & 11 \\ 17 & 17 \end{pmatrix} .$$

Hence ,

$$x(1) = \begin{pmatrix} 1-11u(1) & -11u(1) \\ -17u(1) & 1-17u(1) \end{pmatrix}^{-1} \begin{pmatrix} 986 \\ 1479 \end{pmatrix} .$$

The cost functional in (4.2) becomes

$$J = \alpha^T (P^{-1}(u))^T (P^{-1}(u)\alpha) + u(1)^2$$

where

$$P(u) = \begin{pmatrix} 1-11u(1) & -11u(1) \\ -17u(1) & 1-17u(1) \end{pmatrix} , \quad \alpha = \begin{pmatrix} 986 \\ 1479 \end{pmatrix} .$$

Hence ,

$$\frac{dJ}{du(1)} = 2\alpha^T (P^{-1}(u))^T \left(\frac{dP^{-1}(u)}{du(1)} \alpha \right) + u(1)$$

$$= -2\alpha^T (P^{-1}(u))^T \left(P^{-1}(u) \begin{pmatrix} -11 & -11 \\ -17 & -17 \end{pmatrix} P^{-1}(u)\alpha \right) + u(1)$$

$$= 0$$

when u(1) satisfies the polynomial equation

$$-(986 \quad 1479) \begin{pmatrix} 1-17u & 17u \\ 11u & 1-11u \end{pmatrix} \begin{pmatrix} 1-17u & 11u \\ 17u & 1-11u \end{pmatrix} \begin{pmatrix} -11 & -11 \\ -17 & -17 \end{pmatrix} \begin{pmatrix} 1-17u & 11u \\ 17u & 1-11u \end{pmatrix} \begin{pmatrix} 968 \\ 1479 \end{pmatrix}$$

$$+ u(1-28u)^3 = 0 .$$

This has a real root when $u=u(1) \cong -6.54$.

Now consider the case $k=2$, so that the basis of $P_k L[0,2\pi]$ is $\{(1/2\pi), (1/\pi)\cos t\}$. Then

$$K_2(u^2) = \begin{pmatrix} u(1)\begin{pmatrix} 11 & 11 \\ 17 & 17 \end{pmatrix} + u(2)\begin{pmatrix} 15 & 15 \\ 23 & 23 \end{pmatrix} & u(1)\begin{pmatrix} 15 & 15 \\ 23 & 23 \end{pmatrix} + u(2)\begin{pmatrix} 30 & 30 \\ 45 & 45 \end{pmatrix} \\ u(1)\begin{pmatrix} 15 & 15 \\ 23 & 23 \end{pmatrix} + u(2)\begin{pmatrix} 21 & 21 \\ 32 & 32 \end{pmatrix} & u(1)\begin{pmatrix} 21 & 21 \\ 32 & 32 \end{pmatrix} + u(2)\begin{pmatrix} 42 & 42 \\ 63 & 63 \end{pmatrix} \end{pmatrix}$$

and

$$\alpha \triangleq \begin{pmatrix} \underline{\xi}(1) \\ \\ \underline{\xi}(2) \end{pmatrix} = \begin{pmatrix} 986 \\ 1479 \\ 1365 \\ 2048 \end{pmatrix} .$$

Hence ,

$$\begin{pmatrix} \underline{x}(1) \\ \underline{x}(2) \end{pmatrix} = P^{-1}(u) \begin{pmatrix} \underline{\xi}(1) \\ \underline{\xi}(2) \end{pmatrix} ,$$

where

$$P(u) = 1 - K_2(\underline{u}^2) .$$

The cost function becomes

$$J = \alpha^T (P^{-1}(u))^T (P^{-1}(u)\alpha) + u(1)^2 + u(2)^2$$

and so

$$\frac{\partial J}{\partial u(1)} = -2\alpha^T (P^{-1}(u))^T \left(P^{-1}(u) \frac{\partial P}{\partial u(1)} P^{-1}(u)\alpha \right) + 2u(1) = 0$$

$$\frac{\partial J}{\partial u(2)} = -2\alpha^T (P^{-1}(u))^T \left(P^{-1}(u) \frac{\partial P}{\partial u(2)} P^{-1}(u)\alpha \right) + 2u(2) = 0 .$$

In this case we obtain $u(1) \cong -1.7$ and $u(2) \cong -1.95$ and it is easily checked that the cost is reduced .

6. Conclusions In this paper we have given a 'frequency domain' approach to the study of bilinear systems and applied it to the optimal control problem . The method is based on the isomorphism of $L^2[0,T]$ with ℓ^2 , defined by a basis of $L^2[0,T]$, and the fact that a quadratic cost may be written in the form

$$\begin{aligned} J &= \|x\|_{L^2[0,T;\mathbb{R}^n]} + \|u\|_{L^2[0,T;\mathbb{R}^m]} \\ &= \|\underline{x}\|_{\ell^{2,n}} + \|\underline{u}\|_{\ell^{2,m}} . \end{aligned}$$

If we approximate the system in terms of a finite number of basis elements , then it has been seen that the optimal control problem reduces to the solution of a system of polynomial equations .

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