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Unified Derivation of the Limit Shape for Multiplicative Ensembles of Random Integer Partitions with Equiweighted Parts

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Dedicated to Professor Anatoly M. Vershik on the occasion of his 80th birthday

Abstract

We derive the limit shape of Young diagrams, associated with growing integer partitions, with respect to multiplicative probability measures underpinned by the generating functions of the form $\mathcal{F}(z)=\prod_{\ell=1}^{\infty}\mathcal{F}_0(z^\ell)$ (which entails equal weighting among possible parts $\ell\in\mathbb{N}$). Under mild technical assumptions on the function $H_0(u)=\ln(\mathcal{F}_0(u))$, we show that the limit shape $\omega^*(x)$ exists and is given by the equation $y=\gamma^{-1}H_0(\mathrm{e}^{-\gamma x})$, where $\gamma^2=\int_0^1 u^{-1}H_0(u)\,\mathrm{d}u$. The wide class of partition measures covered by this result includes (but is not limited to) representatives of the three meta-types of decomposable combinatorial structures — assemblies, multisets and selections. Our method is based on the usual randomization and conditioning; to this end, a suitable local limit theorem is proved. The proofs are greatly facilitated by working with the cumulants of sums of the part counts rather than with their moments.

Keywords: Integer partitions; Young diagrams; limit shape; local limit theorem; generating functions; cumulants

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Contents

1. Introduction

1.1. Integer partitions and the limit shape problem

An *integer partition* is a decomposition of a given natural number into an *unordered sum* of integers; for example, $12 = 4+2+2+2+1+1$. More formally, a collection of integers $\lambda = \{\lambda_1 \ge \lambda_2 \ge \cdots > 0,\ \lambda_i \in \mathbb{N}\}$ is a partition of $n \in \mathbb{N}$ if $n = \lambda_1 + \lambda_2 + \cdots$, which is sometimes written as $\lambda \vdash n$. We denote by $\varLambda_n$ the (finite) set of partitions $\lambda \vdash n \in \mathbb{N}$, and by $\varLambda := \cup_n \varLambda_n$ the collection of *all* integer partitions. The terms $\lambda_i \in \lambda$ are called *parts* of the partition λ. The alternative notation $\lambda = (1^{\nu_1} 2^{\nu_2} \dots)$ specifies the *multiplicities* (or *counts*) of the parts involved, $\nu_\ell := \#\{\lambda_i \in \lambda \colon \lambda_i = \ell\}$ ($\ell \in \mathbb{N}$), with zero counts usually omitted from the notation. (Here and below, $\#\{\cdot\}$ denotes the number of elements in a set.) It is evident that the part counts satisfy the condition $\sum_{\ell=1}^{\infty} \ell \nu_\ell = n$ for any partition $\lambda = (1^{\nu_1} 2^{\nu_2} \dots) \in \varLambda_n$.

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ is succinctly visualized by its *Young diagram* Υ_λ formed by (left- and bottom-aligned) row blocks with $\lambda_1, \lambda_2, \dots$ unit square cells (see Fig. 1a). If $\lambda \in \varLambda_n$ (i.e., $\lambda \vdash n$) then the area of the Young diagram Υ_λ equals n. The upper boundary of Υ_λ is a piecewise-constant function $Y_\lambda \colon [0,\infty) \to \mathbb{Z}_+ := \{0,1,2,\dots\}$ given by (see Fig. 1b)

$$Y_\lambda(x) := \sum_{\ell \ge x} \nu_\ell, \qquad \lambda = (1^{\nu_1} 2^{\nu_2} \dots) \in \varLambda. \tag{1.1}$$

In particular, $Y_\lambda(0) = \sum_{\ell \ge 0} \nu_\ell = \#\{\lambda_i \in \lambda\}$ is the total number of parts in partition $\lambda \in \varLambda$.

If the space $\varLambda_n$ is endowed with a probability measure P_n (e.g., the uniform measure whereby all $\lambda \in \varLambda_n$ are equiprobable) then one can speak of *random partitions* $\lambda \vdash n$. The *limit shape*, with respect to a family of probability measures P_n on $\varLambda_n$ as $n \to \infty$, is understood as (the graph of) a function $y = \omega^*(x)$ such that, for every $\delta > 0$ and any $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n\bigl\{\lambda \in \varLambda_n \colon \sup\nolimits_{x \ge \delta} \bigl|\widetilde{Y}_\lambda^n(x) - \omega^*(x)\bigr| > \varepsilon\bigr\} = 0, \tag{1.2}$$

where $\widetilde{Y}_\lambda^n(x) = A_n^{-1} Y_\lambda(x B_n)$ for suitable scaling constants A_n, B_n. It is natural to require that $A_n B_n = n$, which would render the area of the scaled Young diagram $\widetilde{\Upsilon}_\lambda^n$ to be normalized to unity; the most frequent choice is specified as $A_n = B_n = \sqrt{n}$.

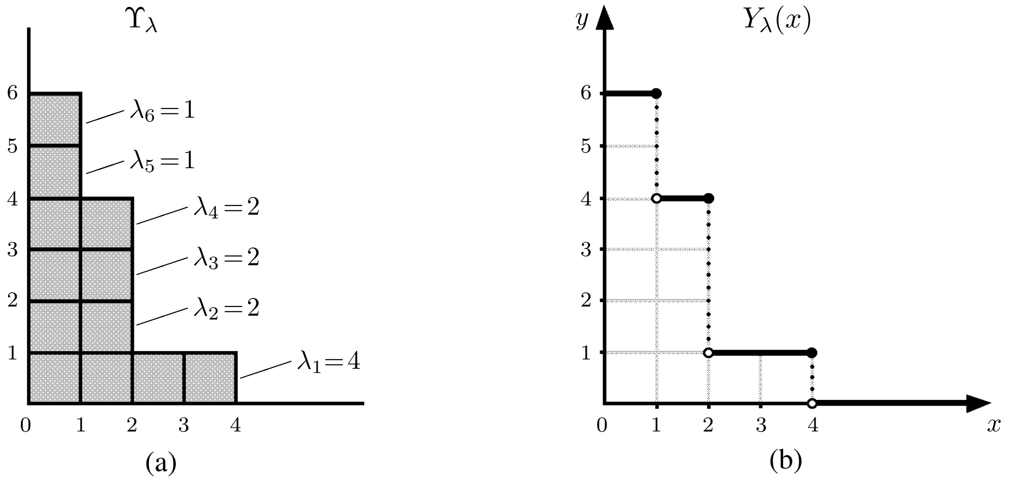

Figure 1: The Young diagram Υ_λ (a) and the graph of its upper boundary $Y_\lambda(x) = \sum_{\ell\ge x}\nu_\ell$ (b) for a partition $\lambda = (4, 2, 2, 2, 1, 1) \equiv (1^2\, 2^3 4^1) \vdash n = 12$, with the part counts $\nu_1 = 2$, $\nu_2 = 3$ and $\nu_4 = 1$.

Of course, the limit shape and its very existence depend on the chosen family of probability laws P_n on the partition spaces $\varLambda_n$ ($n \in \mathbb{N}$). With respect to the uniform (equiprobable) distribution on $\varLambda_n$, the limit shape $\omega^*(x)$ exists under the scaling $A_n = B_n = \sqrt{n}$ and is determined by the equation (see Fig. 2a)

$$\mathrm{e}^{-x\pi/\sqrt{6}} + \mathrm{e}^{-y\pi/\sqrt{6}} = 1, \qquad x, y \ge 0. \tag{1.3}$$

The limit shape (1.3) was first identified by Temperley [23] in relation to the equilibrium shape of a growing crystal, and derived more rigorously much later by Vershik (as noted in [29, p. 30]) using some asymptotic estimates from Szalay and Turán [22]. The proof in its modern form was outlined by Vershik in [26]; an alternative proof was given by Pittel [18].

Unlike [18] where only the uniform case was studied, Vershik's method was used in [26] to settle the limit shape problem for more general partition ensembles of the so-called *multiplicative type* (see Section 1.2 below), including the uniform distribution on the subset $\check{\varLambda}_n \subset \varLambda_n$ of *strict* partitions (i.e., with distinct parts, $\nu_\ell \le 1$ for all $\ell \in \mathbb{N}$), whereby the limit shape, under the same scaling, appears to be of the form (see Fig. 2b)

$$\mathrm{e}^{y\pi/\sqrt{12}} = 1 + \mathrm{e}^{-x\pi/\sqrt{12}}, \qquad x, y \ge 0. \tag{1.4}$$

1.2. Multiplicative measures on partition spaces

For a general discussion and plentiful examples of multiplicative probability measures on partitions, the reader may consult the classic work by Vershik [26, 27] and more recent papers by Erlihson and Granovsky [7], Su [21] and Yakubovich [32], with an abundance of further references therein. In the monograph by Arratia, Barbour and Tavaré [1], such measures are considered in the general context of decomposable combinatorial structures.

In short, multiplicative measures are underpinned by the generating functions of the form

$$\mathcal{F}(z) = \prod_{\ell=1}^{\infty} \mathcal{F}_\ell(z^\ell) = \prod_{\ell=1}^{\infty}\sum_{k=0}^{\infty} c_k^{(\ell)} z^{k\ell}, \quad \text{with } c_0^{(\ell)} \equiv 1,\ c_k^{(\ell)} \ge 0\ \ (k, \ell \in \mathbb{N}). \tag{1.5}$$

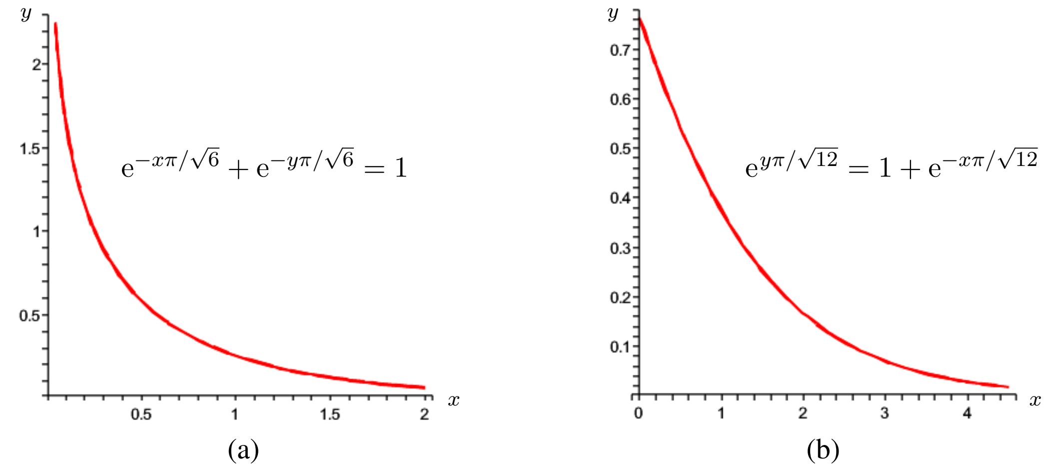

Figure 2: The limit shape $y = \omega^*(x)$ for two classical ensembles of uniform (equiprobable) random partitions: (a) unrestricted partitions (Λ_n); (b) partitions with distinct parts ($\check{\Lambda}_n$). In both cases, the normalization in (1.2) is specified by $A_n = B_n = \sqrt{n}$.

More precisely, the corresponding family of the measures P_n on the respective partition spaces Λ_n ($n \in \mathbb{N}$) is defined by setting

$$P_n(\lambda) := \mathfrak{C}_n^{-1} \prod_{\ell=1}^{\infty} c_{\nu_\ell}^{(\ell)}, \qquad \lambda = (1^{\nu_1} 2^{\nu_2} \dots) \in \Lambda_n, \tag{1.6}$$

where $\mathfrak{C}_n$ is the suitable normalization constant. For example, the generating function $\mathcal{F}_\ell(u) \equiv (1-u)^{-1} = \sum_{k=0}^{\infty} u^k$ defines the uniform measure on each Λ_n, whereas the choice $\mathcal{F}_\ell(u) \equiv 1 + u$ leads to the uniform measure on the space $\check{\Lambda}_n$ of strict partitions.

According to (1.6), each generating function $\mathcal{F}_\ell(\cdot)$ assigns some weights, relative to the uniform case with $c_k^{(\ell)} \equiv 1$, to specific values of the part count $\nu_\ell = \#\{\lambda_i = \ell\}$ in a random partition $\lambda = (1^{\nu_1} 2^{\nu_2} \dots) \in \Lambda$. Furthermore, possible variation of the functions $\mathcal{F}_\ell(\cdot)$ across $\ell \in \mathbb{N}$ determines a certain weighting among different parts that may contribute to a partition.

Definition 1.1. If the functions $\mathcal{F}_\ell(\cdot)$ do not depend on ℓ (hence, $c_k^{(\ell)} \equiv c_k$ for all $\ell \in \mathbb{N}$) then we say that the parts are *equiweighted* (which is alluded to in the title of the paper).

Remark 1.1. Note from the definition (1.6) that the marginal distribution of a random count ν_ℓ is ℓ-*biased*, being given by $P_n\{\nu_\ell = k\} = c_k^{(\ell)} \mathfrak{C}_{n-k\ell}^{(\ell)} / \mathfrak{C}_n$ ($0 \le k \le n/\ell$), where $\mathfrak{C}_m^{(\ell)} := \sum_{\lambda \in \Lambda_m} \prod_{j \ne \ell} c_{\nu_j}^{(j)}$ (with the convention $\mathfrak{C}_0^{(\ell)} := 1$). Thus, the assumption that the parts are equiweighted does not imply that their counts have the same distribution.

Building on Vershik's pioneering ideas, the limit shape problem was advanced in various directions (see [4, 7, 11, 12, 19, 21, 27, 30, 32] and further references therein). In a separate but related development, Logan and Shepp [17] and Vershik and Kerov [28, 29] found the limit shape for a different (non-multiplicative) ensemble of partitions endowed with the *Plancherel measure* emerging in relation with representation theory of the symmetric group. A recent review of both areas can be found in [21].

Returning to the multiplicative class of probability measures on partitions, note that most of the aforementioned papers on the limit shape problem have focused on the particular case

$$\mathcal{F}_\ell(u) = (\mathcal{F}_0(u))^{r_\ell}, \qquad \ell \in \mathbb{N}, \tag{1.7}$$

for some classes of sequences $r_\ell > 0$ (usually assumed to behave like $r_\ell \sim \mathrm{const} \cdot \ell^{p-1}$ as $\ell \to \infty$, with $p > 0$) but subject to a more limited choice of the basic generating function $\mathcal{F}_0(u)$, often borrowed from the standard equiprobable cases mentioned above (see, e.g., [26, 27, 7, 12, 21]).

A recent paper by Yakubovich [32] offers a more general treatment by considering a wider class of functions $\mathcal{F}_0(u)$; a typical condition imposed there (see, e.g., [32, Lemma 10]) is that $\mathcal{F}_0(u)$ be complex analytic in a disk centered at zero up to an isolated (real) singularity point $u_1 \ge 1$, which *must be a pole* if $u_1 = 1$. Some simple examples such as $\mathcal{F}_0(u) = (1-u)^{-r}$ with a real (non-integer) $r > 0$ do not formally conform to this requirement but none the less have a limit shape (see [26], where the assumption that r_ℓ's are integer is in fact not essential in the light of the Meinardus theorem, see [12]). On the other hand, one can write $\mathcal{F}_0(u) = (f_0(u))^r$, where the function $f_0(u) = (1-u)^{-1}$ has a required pole at $u_1 = 1$ and thus fits in the framework of [32].[1] However, there are examples with a genuine non-pole singularity of $\mathcal{F}_0(u)$ which do possess a limit shape (see such examples in Section 6 below).

1.3. An outline of the main result

In the present paper, we confine ourselves to the class of multiplicative ensembles of partitions *with equiweighted parts* (see Definition 1.1), specified by the simplest case $\mathcal{F}_\ell(u) \equiv \mathcal{F}_0(u)$ in (1.5) (which also corresponds to setting $r_\ell \equiv 1$ in the model (1.7)) but with a fairly general variety of permissible generating functions $\mathcal{F}_0(u)$. In particular, measures P_n covered by our method include (but are not limited to) representatives of the three classical meta-types of decomposable combinatorial structures — assemblies, multisets and selections (see [1, Ch. 2] for a general background and also concrete examples in Section 6 below).

A loose formulation of our main result about the limit shape is as follows.

Theorem 1.1. *Denote* $H_0(u) := \ln(\mathcal{F}_0(u))$, $\gamma := \sqrt{\int_0^1 u^{-1} H_0(u)\, \mathrm{d}u}$ *and*

$$\omega^*(x) := \gamma^{-1} H_0(\mathrm{e}^{-\gamma x}), \qquad x \ge 0. \tag{1.8}$$

Then, under mild technical conditions on the function $H_0(u)$, *for every* $\delta > 0$ *and any* $\varepsilon > 0$

$$\lim_{n\to\infty} P_n\bigl\{\lambda \in \mathit{\Lambda}_n\colon\ \sup_{x\ge\delta}\bigl|\widetilde{Y}_\lambda^n(x) - \omega^*(x)\bigr| > \varepsilon\bigr\} = 0, \tag{1.9}$$

where $\widetilde{Y}_\lambda^n(x) := n^{-1/2} Y_\lambda(x n^{1/2})$.

Remark 1.2. The restriction $x \ge \delta > 0$ in (1.9) takes into account the possibility $\omega^*(0) = \infty$ (cf. (1.3), (1.4)). If $\omega^*(0) < \infty$ then the supremum in (1.9) can be extended to all $x \ge 0$.

[1] This substitution replaces the normalization $r_1 = 1$ adopted for convenience in [32, §1.1, p. 1254] by $r_1 = r > 0$, which is not essential for the validity of results in [32]. Incidentally, this remark shows that it is more natural to impose conditions on the function $H_0(u) := \ln(\mathcal{F}_0(u))$ rather than on $\mathcal{F}_0(u)$ itself.

Like in [10, 26, 27, 7, 32], our proof employs the elegant probabilistic approach in the theory of decomposable combinatorial structures based on randomization and conditioning, first applied in the context of random partitions by Fristedt [10] (see the monograph [1] and an earlier review [2] for a general discussion of the method and many examples). The idea is to introduce a suitable measure Q_z on the union space $\mathit{\Lambda} = \cup_n \mathit{\Lambda}_n$ (depending on an auxiliary "free" parameter $z \in (0,1)$), such that a given measure P_n on $\mathit{\Lambda}_n$ is recovered as the conditional distribution $P_n(\cdot) = Q_z(\cdot\,|\,\mathit{\Lambda}_n)$.

The great advantage of the multiplicativity property (1.5) is that Q_z can be constructed as a product measure, resulting in *mutually independent* random counts ν_ℓ. Clearly, such a device calls for the asymptotics of the probability $Q_z(\mathit{\Lambda}_n)$, which should be obtained by proving a suitable *local limit theorem*; the latter suggests that it is natural to calibrate the parameter z from the asymptotic equation $E_z(N_\lambda) = n\,(1 + o(1))$, where E_z is the expectation with respect to the measure Q_z and $N_\lambda := \lambda_1 + \lambda_2 + \cdots = \sum_{\ell=1}^{\infty} \ell\nu_\ell$ (so that $\mathit{\Lambda}_n = \{\lambda \in \mathit{\Lambda}\colon N_\lambda = n\}$). This is sufficient to ensure the (uniform) convergence of the *expectation* $E_z\bigl[\widetilde{Y}_\lambda^n(x)\bigr]$ to the limit $\omega^*(x)$ specified in (1.8), together with the corresponding convergence of the random paths $\widetilde{Y}_\lambda^n(\cdot)$ in Q_z-probability. However, in order to extend this to the original measure P_n using the local limit theorem, our methods require an improved estimate of the approximation error $E_z(N_\lambda) - n$ to at least $O(n^{3/4})$. Let us also point out that the proofs are greatly facilitated by working with the *cumulants* of sums of the part counts ν_ℓ rather than with their moments.

Layout. The rest of the paper is organized as follows. In Section 2.1, we define the multiplicative families of measures Q_z and P_n on the corresponding spaces of partitions with equiweighted parts. Important cumulant expansions and certain technical conditions on the generating function $\mathcal{F}_0(u)$ are discussed in Section 2.3. In Section 3.1, a suitable value of the parameter $z \in (0,1)$ is chosen (Theorem 3.1), which implies the convergence of "expected" (scaled) Young diagrams to the limit curve $y = \omega^*(x)$ (Theorem 3.2). Refined first-order moment asymptotics are obtained in Section 3.3 (Theorem 3.3), while higher-order cumulant sums are analyzed in Section 4. The local limit theorem (Theorem 5.1) is established in Section 5, which paves the way to the proof of the limit shape results in Section 5.4 with respect to both Q_z and P_n (Theorems 5.5 and 5.6, respectively). Finally, our results are illustrated by a number of examples in Section 6.

Some notation. We denote $\mathbb{Z}_+ := \{k \in \mathbb{Z}\colon k \ge 0\}$ and $\mathbb{R}_+ := [0,\infty)$. The real part of $s \in \mathbb{C}$ is denoted $\Re(s)$. The notation $x_n \asymp y_n$ signifies that $0 < \liminf_{n\to\infty} x_n/y_n \le \limsup_{n\to\infty} x_n/y_n < \infty$, whereas $x_n \sim y_n$ is a shorthand for $\lim_{n\to\infty} x_n/y_n = 1$. The standard symbols $\lfloor x \rfloor := \max\{k \in \mathbb{Z}\colon k \le x\}$ and $\lceil x \rceil := \min\{k \in \mathbb{Z}\colon k \ge x\}$ denote, respectively, the floor and ceiling integer parts of $x \in \mathbb{R}$.

2. Generating functions and cumulants

2.1. Global measure Q_z and conditional measure P_n

Let $\Phi := \mathbb{Z}_+^{\mathbb{N}}$ be the space of functions $\nu\colon \mathbb{N} \to \mathbb{Z}_+$ (i.e., sequences $\nu = \{\nu_\ell\}$ with nonnegative integer values), and consider the subspace $\Phi_0 := \{\nu \in \Phi\colon \#(\operatorname{supp}\nu) < \infty\}$ of functions with *finite support*, where $\operatorname{supp}\nu := \{\ell \in \mathbb{N}\colon \nu_\ell > 0\}$. The space Φ_0 is in one-to-one correspondence with the union set $\mathit{\Lambda} = \bigcup_{n\in\mathbb{Z}_+} \mathit{\Lambda}_n$ under the identification of the values ν_ℓ's (including zeroes) with the multiplicities of the virtual parts ℓ's, respectively, leading to a partition $\lambda = (1^{\nu_1} 2^{\nu_2} \dots)$ of the integer $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell \in \mathbb{Z}_+$.

Let $c_0 = 1, c_1, c_2, \dots$ be a sequence of nonnegative numbers such that *not all c_k's vanish for $k \ge 1$*, and assume that the corresponding power series (generating function)

$$\mathcal{F}_0(u) := \sum_{k=0}^{\infty} c_k u^k, \qquad u \in \mathbb{C}, \tag{2.1}$$

is convergent for all $|u| < 1$. For every $z \in (0,1)$, let us define a probability measure Q_z on the space $\Phi = \mathbb{Z}_+^{\mathbb{N}}$ as the distribution of a random sequence $\{\nu_\ell,\, \ell \in \mathbb{N}\}$ with *mutually independent values* and *marginal distributions*

$$Q_z\{\nu_\ell = k\} = \frac{c_k z^{k\ell}}{\mathcal{F}_0(z^\ell)}, \qquad k \in \mathbb{Z}_+. \tag{2.2}$$

Lemma 2.1. *For $z \in (0,1)$, the condition*

$$\mathcal{F}(z) := \prod_{\ell=1}^{\infty} \mathcal{F}_0(z^\ell) < \infty \tag{2.3}$$

is necessary and sufficient in order that $Q_z(\Phi_0) = 1$. Furthermore, if $\mathcal{F}_0(u)$ is finite for all $u \in (0,1)$ then the condition (2.3) *is satisfied for all $z \in (0,1)$.*

Proof. According to (2.2) we have $Q_z\{\nu_\ell > 0\} = 1 - 1/\mathcal{F}_0(z^\ell)$ ($\ell \in \mathbb{N}$). Hence, Borel–Cantelli's lemma (see, e.g., [8, Ch. VIII, §3]) implies that $Q_z\{\nu \in \Phi_0\} = 1$ if and only if $\sum_{\ell=1}^{\infty} \bigl(1 - 1/\mathcal{F}_0(z^\ell)\bigr) < \infty$. In turn, the latter bound is equivalent to (2.3).

To prove the second statement, observe using (2.1) that

$$\begin{aligned}\ln\bigl(\mathcal{F}(z)\bigr) = \sum_{\ell=1}^{\infty} \ln\bigl(\mathcal{F}_0(z^\ell)\bigr) \le \sum_{\ell=1}^{\infty} \bigl(\mathcal{F}_0(z^\ell) - 1\bigr) &= \sum_{k=1}^{\infty} c_k \sum_{\ell=1}^{\infty} z^{k\ell} \\ &= \sum_{k=1}^{\infty} \frac{c_k z^k}{1 - z^k} \le \frac{1}{1-z} \sum_{k=1}^{\infty} c_k z^k \le \frac{\mathcal{F}_0(z)}{1-z} < \infty,\end{aligned}$$

which implies the condition (2.3). □

Lemma 2.1 ensures that the random sequence $\{\nu_\ell\}$ defined above (see (2.2)) belongs to the space Φ_0 (Q_z-a.s.)[2] and therefore determines a *finite* (random) partition $\lambda \in \Lambda$. By the mutual independence of the values ν_ℓ, the corresponding Q_z-probability is given by

$$Q_z(\lambda) = \prod_{\ell=1}^{\infty} \frac{c_{\nu_\ell} z^{\ell \nu_\ell}}{\mathcal{F}_0(z^\ell)} = \frac{c(\lambda)\, z^{N_\lambda}}{\mathcal{F}(z)}, \qquad \lambda = (1^{\nu_1} 2^{\nu_2} \dots) \in \Lambda, \tag{2.4}$$

where $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell < \infty$ (Q_z-a.s.) and (see (2.1))

$$c(\lambda) = \prod_{\ell=1}^{\infty} c_{\nu_\ell} < \infty, \qquad \lambda = (1^{\nu_1} 2^{\nu_2} \dots) \in \Lambda. \tag{2.5}$$

[2] The abbreviation "a.s." stands for *almost surely*, that is, with probability 1.

Remark 2.1. The infinite product (2.5) defining $c(\lambda)$ contains only finitely many factors different from 1, because any $\ell \notin \operatorname{supp} \nu$ renders $\nu_\ell = 0$, so that $c_{\nu_\ell} = c_0 = 1$.

Remark 2.2. For the "empty" partition $\lambda_\emptyset \vdash 0$ formally associated with the configuration $\nu \equiv 0$, formula (2.4) yields $Q_z(\lambda_\emptyset) = 1/\mathcal{F}(z) > 0$. On the other hand, $Q_z(\lambda_\emptyset) < 1$, since $\mathcal{F}_0(u) > \mathcal{F}_0(0) = 1$ for $u > 0$ and hence, according to the definition (2.3), $\mathcal{F}(z) > 1$.

On the subspace $\Lambda_n \subset \Lambda$, the measure Q_z induces the conditional distribution

$$P_n(\lambda) := Q_z(\lambda|\Lambda_n) = \frac{Q_z(\lambda)}{Q_z(\Lambda_n)}, \qquad \lambda \in \Lambda_n. \tag{2.6}$$

The formula (2.6) is well defined as long as $Q_z(\Lambda_n) > 0$, that is, if there is at least one partition $\lambda \in \Lambda_n$ with $c(\lambda) > 0$ (see (2.4)). An obvious sufficient condition is as follows.

Lemma 2.2. *Suppose that* $c_1 > 0$. *Then* $Q_z(\Lambda_n) > 0$ *for all* $n \in \mathbb{Z}_+$.

The following key fact is a direct consequence of the definition (2.4).

Lemma 2.3. *The formula* (2.6) *for the measure* P_n *is reduced to the expression* (*cf.* (1.6))

$$P_n(\lambda) = \frac{c(\lambda)}{\mathfrak{C}_n} \quad (\lambda \in \Lambda_n), \qquad \mathfrak{C}_n = \sum_{\lambda' \in \Lambda_n} c(\lambda'), \tag{2.7}$$

where $c(\lambda)$ *is defined in* (2.5). *In particular,* P_n *does not depend on* z.

Proof. If $\Lambda_n \ni \lambda \leftrightarrow \nu \in \Phi_0$ then $N_\lambda = n$ and the formula (2.4) is reduced to $Q_z(\lambda) = c(\lambda) z^n/\mathcal{F}(z)$. In turn, the ratio in (2.6) amounts to the expression in (2.7), which is z-free. □

Specific examples of multiplicative measures P_n with equiweighted parts will be given below in Section 6, together with the corresponding limit shapes determined by Theorem 1.1.

2.2. Expansion of the logarithm of the generating function $\mathcal{F}_0(u)$

Recalling the power series expansion (2.1) for $\mathcal{F}_0(u)$, consider the corresponding expansion of its logarithm,

$$H_0(u) := \ln(\mathcal{F}_0(u)) = \sum_{k=1}^{\infty} a_k u^k, \qquad u \in \mathbb{C}, \tag{2.8}$$

assuming that the series (2.8) is (absolutely) convergent for all $|u| < 1$. Here $\ln(\cdot)$ means the principal branch of the logarithm specified by the value $\ln(\mathcal{F}_0(0)) = \ln 1 = 0$.

Remark 2.3. Substituting (2.1) into (2.8), it is evident that $a_1 = c_1$; more generally, if $j_* := \min\{j \ge 1\colon a_j \ne 0\}$ and $k_* := \min\{k \ge 1\colon c_k > 0\}$ then $j_* = k_*$ and $a_{j_*} = c_{k_*} > 0$. In particular, it follows that the first non-vanishing coefficient in the power series (2.8) is *positive*.

Differentiating (2.8), we get the standard formulas for the power sums

$$\sum_{k=1}^{\infty} k a_k u^k = u H_0'(u), \tag{2.9}$$

$$\sum_{k=1}^{\infty} k^2 a_k u^k = u\big(u H_0'(u)\big)' = u^2 H_0''(u) + u H_0'(u), \tag{2.10}$$

with similar expressions available for the higher-order sums $\sum_{k=1}^{\infty} k^q a_k u^k$ ($q \in \mathbb{N}$).

For $s \in \mathbb{C}$ such that $\sigma := \Re(s) > 0$, consider the Dirichlet series

$$A(s) := \sum_{k=1}^{\infty} \frac{a_k}{k^s}, \qquad A^+(\sigma) := \sum_{k=1}^{\infty} \frac{|a_k|}{k^\sigma}, \tag{2.11}$$

where a_k's are the coefficients in the power series expansion of $H_0(u)$ (see (2.8)). Although some of the coefficients a_k may be negative, it turns out that the quantity $A(1) = \sum_{k=1}^{\infty} a_k\, k^{-1}$, whenever it is finite, cannot vanish or take a negative value.

Lemma 2.4. *If* $A^+(1) < \infty$ *then* $0 < A(1) < \infty$ *and the following equality holds,*

$$A(1) = \int_0^1 u^{-1} H_0(u)\, \mathrm{d}u. \tag{2.12}$$

In particular, the integral in (2.12) *is convergent.*

Proof. From the assumptions on the coefficients c_k's in the expansion (2.1), it is evident that for all $u \in (0,1)$ we have $\mathcal{F}_0(u) = 1 + \sum_{k=1}^{\infty} c_k u^k > 1$, and hence $H_0(u) = \ln(\mathcal{F}_0(u)) > 0$. Furthermore, substituting the expansion (2.8) for $H_0(u)$ and integrating term by term (which is permissible for power series inside the interval of convergence), we get for any $s \in (0,1)$

$$\int_0^s u^{-1} H_0(u)\, \mathrm{d}u = \sum_{k=1}^{\infty} a_k \int_0^s u^{k-1}\, \mathrm{d}u = \sum_{k=1}^{\infty} \frac{a_k s^k}{k}.$$

Passing here to the limit as $s \uparrow 1$ and applying to the right-hand side Abel's theorem on the boundary value of a power series (see [24, §1.22, pp. 9–10]), we obtain the identity (2.12). $\square$

The quantity $A(1)$ will play a major role in our argumentation; in particular, it is involved in a suitable calibration of the "free" parameter z in the definition (2.2) of the measure Q_z (see Section 3.1 below).

2.3. Cumulants of the part counts

Let us now turn to the random variables ν_ℓ (i.e., the counts of parts $\ell \in \mathbb{N}$ in a partition $\lambda \in \Lambda$). Under the probability measure Q_z (see (2.2)), the characteristic function of ν_ℓ is given by[3]

$$\varphi_{\nu_\ell}(t) := E_z(\mathrm{e}^{\mathrm{i}t\nu_\ell}) = \frac{\mathcal{F}_0(z^\ell \mathrm{e}^{\mathrm{i}t})}{\mathcal{F}_0(z^\ell)}, \qquad t \in \mathbb{R}. \tag{2.13}$$

Hence, the (principal branch of the) logarithm of $\varphi_{\nu_\ell}(t)$ is expanded using (2.8) as

$$\ln(\varphi_{\nu_\ell}(t)) = H_0(z^\ell \mathrm{e}^{\mathrm{i}t}) - H_0(z^\ell) = \sum_{k=1}^{\infty} a_k (\mathrm{e}^{\mathrm{i}kt} - 1) z^{k\ell}, \qquad t \in \mathbb{R}. \tag{2.14}$$

[3] For notational simplicity, we suppress the dependence on z, which should cause no confusion.

For $q \in \mathbb{N}$, denote by $m_q[\nu_\ell] := E_z(\nu_\ell^q)$ the *moments* of the random variable ν_ℓ about zero, and let $\varkappa_q[\nu_\ell]$ be the *cumulants*, or *semi-invariants* of ν_ℓ (see, e.g., [15, §3.12, p. 69]), defined by the following formal identity in indeterminant t,

$$\ln E_z(\mathrm{e}^{\mathrm{i}t\nu_\ell}) = \sum_{q=1}^{\infty} \frac{(\mathrm{i}t)^q}{q!}\,\varkappa_q[\nu_\ell]. \tag{2.15}$$

From (2.15) it is easy to see (e.g., by taking the derivative at $t = 0$) that the expected value of ν_ℓ coincides with its first-order cumulant (see [15, §3.14, Eq. (3.37), p. 71]),

$$E_z(\nu_\ell) = m_1[\nu_\ell] = \varkappa_1[\nu_\ell]. \tag{2.16}$$

Let us also point out the standard expressions for the first few *central* moments (including the variance) through the cumulants (see [15, §3.14, Eq. (3.38), p. 72]),

$$\mathrm{Var}_z(\nu_\ell) = E_z\bigl[(\nu_\ell - m_1[\nu_\ell])^2\bigr] = \varkappa_2[\nu_\ell], \tag{2.17}$$

$$E_z\bigl[(\nu_\ell - m_1[\nu_\ell])^3\bigr] = \varkappa_3[\nu_\ell], \tag{2.18}$$

$$E_z\bigl[(\nu_\ell - m_1[\nu_\ell])^4\bigr] = \varkappa_4[\nu_\ell] + 3(\varkappa_2[\nu_\ell])^2. \tag{2.19}$$

Remark 2.4. The cumulants $\varkappa_q[X]$ of any random variable X are defined similarly to (2.15); needless to say, the formulas analogous to (2.16) – (2.19) also hold true in the general case (as long as the corresponding moments exist).

The next lemma will be instrumental in our analysis.

Lemma 2.5. *The cumulants $\varkappa_q[\nu_\ell]$ are given by*

$$\varkappa_q[\nu_\ell] = \sum_{k=1}^{\infty} k^q a_k z^{k\ell}, \qquad q \in \mathbb{N}. \tag{2.20}$$

In particular,

$$m_1[\nu_\ell] = \sum_{k=1}^{\infty} k a_k z^{k\ell}. \tag{2.21}$$

Proof. Taylor expanding the exponential function in (2.14), we get

$$\ln\bigl(\varphi_{\nu_\ell}(t)\bigr) = \sum_{k=1}^{\infty} a_k z^{k\ell} \sum_{q=1}^{\infty} \frac{(\mathrm{i}kt)^q}{q!} = \sum_{q=1}^{\infty} \frac{(\mathrm{i}t)^q}{q!} \sum_{k=1}^{\infty} k^q a_k z^{k\ell}, \tag{2.22}$$

where the interchange of the order of summation in the double series (2.22) is justified by its absolute convergence. Now, by a comparison of the expansion (2.22) with the identity (2.15), the formulas (2.20) for the coefficients $\varkappa_q[\nu_\ell]$ readily follow. □

By virtue of the expression (2.21) for the expected value of ν_ℓ, it is easy to obtain a formula for the expectation of $N_\lambda = \sum_{\ell=1}^{\infty} \ell \nu_\ell$,

$$E_z(N_\lambda) = \sum_{\ell=1}^{\infty} \ell\, m_1[\nu_\ell] = \sum_{\ell=1}^{\infty} \ell \sum_{k=1}^{\infty} k a_k z^{k\ell}. \tag{2.23}$$

More generally, the expressions (2.20) for the cumulants $\varkappa_q[\nu_\ell]$ furnish a representation of the cumulants of N_λ of any order; namely, using the rescaling relation $\varkappa_q[\ell\nu_\ell] = \ell^q\varkappa_q[\nu_\ell]$ (see [15, §3.13, p. 70]) and the additivity property of the cumulants for independent summands (see [15, §7.18, pp. 201–202]), we obtain

$$\varkappa_q[N_\lambda] = \sum_{\ell=1}^{\infty} \ell^q \varkappa_q[\nu_\ell] = \sum_{\ell=1}^{\infty} \ell^q \sum_{k=1}^{\infty} k^q a_k z^{k\ell}, \qquad q \in \mathbb{N}. \tag{2.24}$$

Similarly, recalling that the upper boundary $Y_\lambda(x)$ of the Young diagram Υ_λ is given by the formula (1.1), we obtain for any $x \geq 0$

$$\varkappa_q[Y_\lambda(x)] = \sum_{\ell\geq x} \varkappa_q[\nu_\ell] = \sum_{\ell\geq x} \sum_{k=1}^{\infty} k^q a_k z^{k\ell}, \qquad q \in \mathbb{N}, \tag{2.25}$$

and in particular (with $q = 1$)

$$E_z[Y_\lambda(x)] = \sum_{\ell\geq x} m_1[\nu_\ell] = \sum_{\ell\geq x} \sum_{k=1}^{\infty} k a_k z^{k\ell}. \tag{2.26}$$

2.4. Estimates for power-exponential sums

In what follows, we frequently encounter power-exponential sums of the form

$$S_q(t) := \sum_{\ell=1}^{\infty} \ell^{q-1} \mathrm{e}^{-t\ell}, \qquad t > 0. \tag{2.27}$$

Lemma 2.6. *For $q \in \mathbb{N}$, the function $S_q(t)$ admits the representation*

$$S_q(t) = \sum_{j=1}^{q} c_{j,q} \frac{\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j}, \qquad t > 0, \tag{2.28}$$

with some constants $c_{j,q} > 0$ $(j = 1, \dots, q)$; in particular, $c_{q,q} = (q-1)!$.

Proof. If $q = 1$ then the expression (2.27) is reduced to a geometric series

$$S_1(t) = \sum_{\ell=1}^{\infty} \mathrm{e}^{-t\ell} = \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}},$$

which is a particular case of (2.28) with $c_{1,1} = 1$. Assume now that (2.28) is valid for some $q \geq 1$. Then, differentiating the identities (2.27) and (2.28) with respect to t, we obtain

$$\begin{aligned} S_{q+1}(t) = -S_q'(t) &= \sum_{j=1}^{q} c_{j,q} \left(\frac{j\,\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j} + \frac{j\,\mathrm{e}^{-t(j+1)}}{(1-\mathrm{e}^{-t})^{j+1}} \right) \\ &= \sum_{j=1}^{q+1} c_{j,q+1} \frac{\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j}, \end{aligned}$$

where we set

$$c_{j,q+1} := \begin{cases} c_{1,q}, & j = 1, \\ j\, c_{j,q} + (j-1)\, c_{j-1,q}, & 2 \le j \le q, \\ q\, c_{q,q}, & j = q+1. \end{cases}$$

In particular, $c_{q+1,q+1} = q\, c_{q,q} = q(q-1)! = q!$. Thus, the formula (2.28) holds for $q+1$ and hence, by induction, for all $q \ge 1$. $\square$

Lemma 2.7. *For any $q > 0$, there is a constant $C_q > 0$ such that*

$$\frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^q} \le C_q\, t^{-q}, \qquad t > 0. \tag{2.29}$$

Proof. Set $f(t) := t^q \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{-q}$ and note that

$$\lim_{t\to 0^+} f(t) = 1, \qquad \lim_{t\to +\infty} f(t) = 0.$$

By continuity, the function $f(t)$ is bounded on $(0,\infty)$, and the inequality (2.29) follows. $\square$

Lemma 2.8. (a) *For any $q \in \mathbb{N}$, there is a constant $\tilde{C}_q > 0$ such that*

$$S_q(t) \le \tilde{C}_q\, t^{-q}, \qquad t > 0. \tag{2.30}$$

(b) *Moreover,*

$$S_q(t) \sim \frac{(q-1)!}{t^q}, \qquad t \to 0^+. \tag{2.31}$$

Proof. (a) Observe, using Lemma 2.7, that for $j = 1, \dots, q$

$$\frac{\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j} \le \frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^q} \le C_q\, t^{-q}, \qquad t > 0.$$

Substituting this inequality into (2.28) and recalling that the coefficients $c_{j,q}$ are positive, we obtain the bound (2.30) with $\tilde{C}_q := C_q \sum_{j=1}^q c_{j,q} > 0$.

(b) For each term in the expansion (2.28) we have $\mathrm{e}^{-tj}(1-\mathrm{e}^{-t})^{-j} \sim t^{-j}$ as $t \to 0^+$. Hence, the overall asymptotic behavior of $S_q(t)$ is determined by the term with $j = q$ and the corresponding coefficient $c_{q,q} = (q-1)!$ (see Lemma 2.6), and the formula (2.31) follows. $\square$

3. Asymptotics of the expectation

3.1. Calibration of the parameter z

Our aim is to find a suitable parameter $z = z_n \in (0,1)$ in the definition (2.4) of the probability measure Q_z, subject to the asymptotic condition

$$E_z(N_\lambda) \sim n, \qquad n \to \infty, \tag{3.1}$$

where $N_\lambda = \sum_{\ell=1}^\infty \ell \nu_\ell$. To this end, let us seek z in the form

$$z = \mathrm{e}^{-\alpha}, \qquad \alpha = \alpha_n := \gamma n^{-1/2}, \tag{3.2}$$

where the constant $\gamma > 0$ is to be fitted. Hence, the formula (2.23) takes the form

$$E_z(N_\lambda) = \sum_{\ell=1}^{\infty} \ell \sum_{k=1}^{\infty} k a_k \mathrm{e}^{-k\alpha\ell}. \tag{3.3}$$

Let us state our main result in this section.

Theorem 3.1. *Suppose that $A^+(1) < \infty$. Then, under the parameterization* (3.2)*, the asymptotic condition* (3.1) *is satisfied with the choice*

$$\gamma = \sqrt{A(1)} > 0. \tag{3.4}$$

Proof. By Lemma 2.4, we know that $A(1) > 0$ and hence the inequality (3.4) holds true.

Let us now investigate the asymptotics of the expectation $E_z(N_\lambda)$ under the parameterization $z = \mathrm{e}^{-\alpha}$ with $\alpha \to 0^+$ (cf. (3.2)). Interchanging the order of summation in (3.3) and using the notation (2.27), we obtain

$$E_z(N_\lambda) = \sum_{k=1}^{\infty} k a_k \sum_{\ell=1}^{\infty} \ell\, \mathrm{e}^{-k\alpha\ell} = \sum_{k=1}^{\infty} k a_k S_2(k\alpha). \tag{3.5}$$

According to Lemma 2.8(b) (with $q = 2$),[4] for each $k \in \mathbb{N}$ we have $S_2(k\alpha) \sim (k\alpha)^{-2}$ as $\alpha \to 0^+$. Moreover, by Lemma 2.8(a) the general summand in the series (3.5) is bounded, uniformly in k, by $O(\alpha^{-2})|a_k|k^{-1}$, which is a term of a convergent series since $A^+(1) < \infty$ by the theorem's hypothesis; in particular, this justifies the above interchange of the order of summation. Hence, by Lebesgue's dominated convergence theorem we obtain from (3.5)

$$\lim_{\alpha\to 0^+} \alpha^2 E_z(N_\lambda) = \sum_{k=1}^{\infty} k a_k \lim_{\alpha\to 0^+} \alpha^2 S_2(k\alpha) = \sum_{k=1}^{\infty} \frac{a_k}{k} = A(1). \tag{3.6}$$

Thus, putting $\alpha = \gamma n^{-1/2}$ with $\gamma = \sqrt{A(1)}$ (see (3.4)), the limit (3.6) is reduced to (3.1). □

The expression (2.12) for $A(1)$ directly in terms of the generating function $H_0(u)$ is sometimes useful (e.g., for computer calculations of the coefficient $\gamma = \sqrt{A(1)}$, see Example 6.6 in Section 6; cf. also the formulation of Theorem 1.1 in the Introduction).

Assumption 3.1. Throughout the rest of the paper, we assume that $A^+(1) < \infty$ and the parameter z is chosen according to the formulas (3.2) with $\gamma > 0$ defined by (3.4).

Remark 3.1. Under Assumption 3.1 the measure Q_z becomes dependent on n, as well as the Q_z-probabilities and the corresponding expected values.

3.2. The "expected" limit shape

Theorem 3.2. *For any $\delta > 0$, we have uniformly in $x \in [\delta, \infty)$*

$$E_z\big[Y_\lambda(xn^{1/2})\big] = n^{1/2}\omega^*(x) + O(1), \qquad n \to \infty, \tag{3.7}$$

where the limit shape function $\omega^*(x)$ is defined in (1.8).

[4] This can also be seen directly, without Lemma 2.8, from the explicit expression $S_2(t) = \mathrm{e}^{-t}(1 - \mathrm{e}^{-t})^{-2}$.

Proof. Setting $\ell^* = \ell_n^* := \lceil x n^{1/2} \rceil$, in view of (3.2) we have

$$0 \le \alpha \ell^* - \gamma x < \alpha, \qquad n \in \mathbb{N}, \tag{3.8}$$

and hence, uniformly in x,

$$\alpha \ell^* \to \gamma x, \qquad n \to \infty. \tag{3.9}$$

With this notation, from (2.26) we have for $x > 0$

$$\gamma n^{-1/2} E_z\big[Y_\lambda(x n^{1/2})\big] = \alpha \sum_{\ell=\ell^*}^{\infty} \sum_{k=1}^{\infty} k a_k \mathrm{e}^{-k\alpha\ell} = \alpha \sum_{\ell=\ell^*}^{\infty} g_0(\alpha\ell), \tag{3.10}$$

where (see (2.9))

$$g_0(t) := \sum_{k=1}^{\infty} k a_k \mathrm{e}^{-kt} = \mathrm{e}^{-t} H_0'(\mathrm{e}^{-t}), \qquad t > 0. \tag{3.11}$$

Note that (cf. (2.10))

$$g_0'(t) = -\sum_{k=1}^{\infty} k^2 a_k \mathrm{e}^{-kt} = -\left\{ \mathrm{e}^{-2t} H_0''(\mathrm{e}^{-t}) + \mathrm{e}^{-t} H_0'(\mathrm{e}^{-t}) \right\}, \qquad t > 0. \tag{3.12}$$

The right-hand side of (3.10) can be viewed as a Riemann integral sum for $g_0(t)$ (over $[\gamma x, \infty)$ with mesh size $\alpha \to 0^+$), suggesting its convergence to the corresponding integral as $n \to \infty$. More precisely, noting that $g_0(t)$ is continuously differentiable on $(0, \infty)$ and $g_0(\infty) = 0$, by Euler–Maclaurin's summation formula (see, e.g., [5, §12.2]) we get

$$\sum_{\ell=\ell^*}^{\infty} g_0(\alpha\ell) = \int_{\ell^*}^{\infty} g_0(\alpha t)\,\mathrm{d}t + \tfrac{1}{2} g_0(\alpha\ell^*) + \alpha \int_{\ell^*}^{\infty} B_1(t)\, g_0'(\alpha t)\,\mathrm{d}t, \tag{3.13}$$

where $B_1(t) := t - \lfloor t \rfloor - \frac{1}{2}$ $(t \in \mathbb{R})$. Furthermore, observing that the term $\frac{1}{2} g_0(\alpha\ell^*)$ can be included in the last integral yields a shorter form of (3.13),

$$\sum_{\ell=\ell^*}^{\infty} g_0(\alpha\ell) = \int_{\ell^*}^{\infty} g_0(\alpha t)\,\mathrm{d}t + \alpha \int_{\ell^*}^{\infty} \tilde{B}_1(t)\, g_0'(\alpha t)\,\mathrm{d}t, \tag{3.14}$$

with $\tilde{B}_1(t) := B_1(t) - \frac{1}{2} \equiv t - \lfloor t \rfloor - 1$ $(t \in \mathbb{R})$.

For the first integral in (3.14), on substituting (3.11) and using (3.9) we obtain

$$\int_{\ell^*}^{\infty} g_0(\alpha t)\,\mathrm{d}t = \int_{\ell^*}^{\infty} \mathrm{e}^{-\alpha t} H_0'(\mathrm{e}^{-\alpha t})\,\mathrm{d}t = \alpha^{-1} H_0(\mathrm{e}^{-\alpha\ell^*}) \sim \alpha^{-1} H_0(\mathrm{e}^{-\gamma x}) \tag{3.15}$$

uniformly in $x \ge \delta$, since by Lagrange's mean value theorem, on account of (3.8) and (3.11),

$$|H_0(\mathrm{e}^{-\alpha\ell^*}) - H_0(\mathrm{e}^{-\gamma x})| \le (\alpha\ell^* - \gamma x) \max_{t \ge \gamma\delta} |g_0(t)| = O(\alpha).$$

Next, noting that $\sup_{t\in\mathbb{R}} |\tilde{B}_1(t)| \le 1$, recalling that $\alpha\ell^* \ge \gamma x$ (see (3.8)) and substituting (3.11), the last term in (3.14) is bounded in absolute value, again uniformly in $x \ge \delta$, by

$$\alpha \int_{\ell^*}^{\infty} |g_0'(\alpha t)|\,\mathrm{d}t \le \int_0^{\mathrm{e}^{-\gamma\delta}} |u H_0''(u) + H_0'(u)|\,\mathrm{d}u = O(1), \tag{3.16}$$

where we used the expression (3.12) and the change of variables $u = \mathrm{e}^{-\alpha t}$. Thus, substituting the estimates (3.15), (3.16) into (3.14) and returning to (3.10), we obtain

$$\lim_{n\to\infty} n^{-1/2} E_z\bigl[Y_\lambda(xn^{1/2})\bigr] = \gamma^{-1} H_0(\mathrm{e}^{-\gamma x}) \equiv \omega^*(x),$$

where the convergence is uniform in $x \ge \delta$, as claimed. □

Remark 3.2. As was mentioned in Remark 1.2, the asymptotic formula (3.7) may be extended, with obvious adjustments of the proof, to the case $x = 0$ including the uniform convergence in $x \ge 0$ — *provided that* $\omega^*(0) < \infty$ (more precisely, if the function $H_0(u)$ and its first two derivatives are finite at $u = 1$).

3.3. Refined asymptotics of the expectation of N_λ

We need to sharpen the asymptotics $E_z(N_\lambda) - n = o(n)$ provided by Theorem 3.1 (see (3.1)). The aim of this section is to prove the following refinement.

Theorem 3.3. *Under the condition $A^+(\sigma) < \infty$ with some $\sigma \in (0,1)$, we have*

$$E_z(N_\lambda) - n = O\bigl(n^{(\sigma+1)/2}\bigr), \qquad n \to \infty.$$

3.3.1. Preliminaries. For the proof of Theorem 3.3, some preparations are required. Let $\psi(x)$ be a continuous function on $\mathbb{R}_+$ such that $\limsup_{x\to\infty} x^\beta \psi(x) < \infty$ with some $\beta > 1$, which ensures that $\psi(x)$ is integrable on $\mathbb{R}_+$. It is easy to see that the series

$$\varPsi(h) := \sum_{\ell=1}^{\infty} \psi(\ell h), \qquad h > 0, \tag{3.17}$$

is absolutely convergent, and moreover

$$\varPsi(h) = O(1) \sum_{\ell=1}^{\infty} (\ell h)^{-\beta} = O(h^{-\beta}), \qquad h \to \infty. \tag{3.18}$$

Let us also assume that

$$\varPsi(h) = O(h^{-1}), \qquad h \to 0^+. \tag{3.19}$$

Remark 3.3. Note that $h\varPsi(h) = h\sum_{\ell=1}^{\infty} \psi(\ell h)$ is a Riemann integral sum for the function $\psi(x)$ over $\mathbb{R}_+$ with mesh size h, so typically (including a specific example emerging in the proof of Theorem 3.3) it will converge, as $h \to 0^+$, to the (finite) integral $\int_0^\infty \psi(x)\,\mathrm{d}x$,[5] thus automatically ensuring the bound (3.19). A sufficient condition for such a convergence, which can be verified by using Euler–Maclaurin's summation formula similar to (3.14), is that $\psi(x)$ be continuously differentiable and the derivative $\psi'(x)$ absolutely integrable on $\mathbb{R}_+$.

Let us now consider the *Mellin transform* of $\varPsi(h)$ (see, e.g., [31, Ch. VI, §9])

$$\widehat{\varPsi}(s) := \int_0^\infty h^{s-1} \varPsi(h)\,\mathrm{d}h, \qquad 1 < \Re(s) < \beta. \tag{3.20}$$

[5] Functions $\psi(x)$ satisfying this property are called *directly Riemann integrable* (see [9, Ch. XI, §1, p. 362]).

On substituting (3.17) into (3.20) we find

$$\widehat{\Psi}(s)=\int_0^\infty h^{s-1}\sum_{\ell=1}^\infty \psi(\ell h)\,\mathrm{d}h=\sum_{\ell=1}^\infty\int_0^\infty h^{s-1}\psi(\ell h)\,\mathrm{d}h$$
$$=\sum_{\ell=1}^\infty \ell^{-s}\int_0^\infty x^{s-1}\psi(x)\,\mathrm{d}x=\zeta(s)\int_0^\infty x^{s-1}\psi(x)\,\mathrm{d}x, \tag{3.21}$$

where $\zeta(s)=\sum_{\ell=1}^\infty \ell^{-s}$ is the Riemann zeta function. The interchange of summation and integration in this computation is justified by the absolute convergence of the integral on the right-hand side of (3.21). By the well-known properties of $\zeta(s)$ (see, e.g., [25, §2.1, p. 13]), from (3.18) and (3.20) it follows that the function $\widehat{\Psi}(s)$ is meromorphic in the strip $0<\Re(s)<\beta$, with a single pole at $s=1$. Set

$$\Delta_\psi(h):=\Psi(h)-\frac{1}{h}\int_0^\infty \psi(x)\,\mathrm{d}x, \qquad h>0. \tag{3.22}$$

Then the Müntz lemma (see [25, §2.11, pp. 28–29]) gives

$$\widehat{\Psi}(s)=\int_0^\infty h^{s-1}\Delta_\psi(h)\,\mathrm{d}h, \qquad 0<\Re(s)<1,$$

and the inversion formula for the Mellin transform (see, e.g., [31, Ch. VI, §9, Theorem 9a, pp. 246–247]) implies

$$\Delta_\psi(h)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} h^{-s}\,\widehat{\Psi}(s)\,\mathrm{d}s, \qquad 0<c<1. \tag{3.23}$$

3.3.2. Proof of Theorem 3.3. With the representation (3.23) at hand, set $\psi(x):=x\mathrm{e}^{-\alpha x}$, then the series (3.17) is explicitly given by

$$\Psi(h)=h\sum_{\ell=1}^\infty \ell\,\mathrm{e}^{-\alpha h\ell}=\frac{h\,\mathrm{e}^{-\alpha h}}{(1-\mathrm{e}^{-\alpha h})^2}, \qquad h>0. \tag{3.24}$$

Clearly, $\psi(x)=O(x^{-\beta})$ with any $\beta>0$, and from (3.24) it is evident that $\Psi(h)$ satisfies the condition (3.18) (with α fixed). Furthermore, the formula (3.21) for $\widehat{\Psi}(s)$ is specialized to

$$\widehat{\Psi}(s)=\zeta(s)\int_0^\infty x^s\,\mathrm{e}^{-\alpha x}\,\mathrm{d}x=\alpha^{-s-1}\zeta(s)\,\Gamma(s+1), \qquad 1<\Re(s)<\infty, \tag{3.25}$$

where $\Gamma(s)=\int_0^\infty x^{s-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is the gamma function. Since $\Gamma(s+1)$ is analytic for $\Re(s)>-1$ (cf. [24, §4.41, p. 148]) and, as already mentioned, $\zeta(s)$ has a single (simple) pole at point $s=1$, it follows that the expression (3.25) is meromorphic in the half-plane $\Re(s)>-1$, thus providing an analytic continuation of the function $\widehat{\Psi}(s)$ into the strip $-1<\Re(s)<1$.

Combining (3.5) and (3.24) we get

$$E_z(N_\lambda)=\sum_{k=1}^\infty a_k\Psi(k). \tag{3.26}$$

On the other hand, according to the notation (2.11) and Assumption 3.1 we have the identity

$$\sum_{k=1}^{\infty} \frac{a_k}{k\alpha^2} = \frac{n A(1)}{\gamma^2} \equiv n. \tag{3.27}$$

Consequently, subtracting (3.27) from (3.26) we obtain the representation

$$E_z(N_\lambda) - n = \sum_{k=1}^{\infty} a_k \left(\varPsi(k) - \frac{1}{k\alpha^2} \right) = \sum_{k=1}^{\infty} a_k \, \Delta_\psi(k), \tag{3.28}$$

recalling the notation (3.22) and observing that

$$\int_0^\infty \psi(x) \, \mathrm{d}x = \int_0^\infty x \, \mathrm{e}^{-\alpha x} \, \mathrm{d}x = \frac{1}{\alpha^2}.$$

Furthermore, using the representation (3.23) with $c = \sigma \in (0, 1)$ (see the hypothesis of the theorem) and substituting the expression (3.25), we can rewrite (3.28) in the form

$$\begin{aligned} E_z(N_\lambda) - n &= \frac{1}{2\pi \mathrm{i}} \sum_{k=1}^{\infty} a_k \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \frac{\zeta(s) \, \Gamma(s+1)}{\alpha^{s+1} k^s} \, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A(\sigma + \mathrm{i}t) \, \zeta(\sigma + \mathrm{i}t) \, \Gamma(\sigma + 1 + \mathrm{i}t)}{\alpha^{\sigma + 1 + \mathrm{i}t}} \, \mathrm{d}t, \end{aligned} \tag{3.29}$$

using the change of variables $s = \sigma + \mathrm{i}t$. To justify the interchange of summation and integration deployed in (3.29), note that

$$|A(\sigma + \mathrm{i}t)| \le A^+(\sigma) < \infty, \qquad |\alpha^{-\sigma - 1 - \mathrm{i}t}| \le \alpha^{-\sigma - 1}.$$

We can also use the following classical estimates as $t \to \infty$ (see [13, Theorem 1.9, p. 25] and [24, §4.42, p. 151], respectively),

$$\zeta(\sigma + \mathrm{i}t) = O\bigl(|t|^{(1-\sigma)/2} \ln(|t| + 2)\bigr), \qquad \Gamma(\sigma + 1 + \mathrm{i}t) = O\bigl(|t|^{\sigma + 1/2} \, \mathrm{e}^{-\pi |t|/2}\bigr).$$

Hence, the last integral in (3.29) is bounded, uniformly in $n \in \mathbb{N}$, by

$$O(\alpha^{-\sigma-1}) \int_{-\infty}^{\infty} |t|^{1 + \sigma/2} \, \mathrm{e}^{-\pi |t|/2} \ln(|t| + 2)) \, \mathrm{d}t = O(\alpha^{-\sigma - 1}) < \infty, \tag{3.30}$$

which validates the formula (3.29).

Moreover, combining (3.29) and (3.30) we get, on account of (3.2),

$$E_z(N_\lambda) - n = O(\alpha^{-\sigma-1}) = O\bigl(n^{(\sigma+1)/2}\bigr), \qquad n \to \infty,$$

and the proof of Theorem 3.3 is complete.

4. Asymptotic estimates for higher-order moments

4.1. The cumulants of N_λ

Substituting $z = \mathrm{e}^{-\alpha}$ (see (3.2)) into the formulas (2.24) for the cumulants of N_λ, we get

$$\varkappa_q[N_\lambda] = \sum_{\ell=1}^{\infty} \ell^q \sum_{k=1}^{\infty} k^q a_k \,\mathrm{e}^{-k\alpha\ell}, \qquad q \in \mathbb{N}. \tag{4.1}$$

Recall that Assumption 3.1 is presumed to be satisfied throughout.

Theorem 4.1. *For each $q \in \mathbb{N}$,*

$$\varkappa_q[N_\lambda] \sim \frac{q!}{\gamma^{q-1}}\, n^{(q+1)/2}, \qquad n \to \infty. \tag{4.2}$$

In particular, the variance of N_λ satisfies

$$\mathrm{Var}_z(N_\lambda) \sim \frac{2}{\gamma}\, n^{3/2}, \qquad n \to \infty. \tag{4.3}$$

Proof. The proof follows the same lines as that of Theorem 3.1 (i.e., with $q = 1$). Namely, again using Lemma 2.8 and Lebesgue's dominated convergence theorem, from (4.1) we get

$$\alpha^{q+1}\varkappa_q[N_\lambda] = \sum_{k=1}^{\infty} k^q a_k \bigl(\alpha^{q+1} S_{q+1}(k\alpha)\bigr) \to q! \sum_{k=1}^{\infty} \frac{a_k}{k} = q!\, A(1) \equiv q!\, \gamma^2. \tag{4.4}$$

But $\alpha^{q+1} \sim \gamma^{q+1} n^{-(q+1)/2}$ (see (3.2)), and the limit (4.4) is reduced to (4.2).

The second claim of the theorem (i.e., the asymptotic formula (4.3)) immediately follows from (4.2) with $q = 2$ by noting that $\mathrm{Var}_z(N_\lambda) = \varkappa_2[N_\lambda]$ (cf. (2.17)). □

4.2. The cumulants of $Y_\lambda(x)$

With the substitution $z = \mathrm{e}^{-\alpha}$, the representations (2.25) are rewritten in the form

$$\varkappa_q[Y_\lambda(x)] = \sum_{\ell \ge x} \sum_{k=1}^{\infty} k^q a_k \,\mathrm{e}^{-k\alpha\ell}, \qquad q \in \mathbb{N}. \tag{4.5}$$

Let us first consider the case $q = 2$, where $\varkappa_2[Y_\lambda(x)] = \mathrm{Var}_z[Y_\lambda(x)]$ (see (2.17)).

Theorem 4.2. *For every $x > 0$,*

$$\lim_{n\to\infty} n^{-1/2}\,\mathrm{Var}_z\bigl[Y_\lambda(xn^{1/2})\bigr] = \gamma^{-1}\mathrm{e}^{-\gamma x} H_0'(\mathrm{e}^{-\gamma x}), \tag{4.6}$$

where the convergence is uniform in $x \in [\delta, \infty)$ for any $\delta > 0$.

Proof. With the help of the notation $g_0(t)$ defined in (3.11) (see also (3.12)), formula (4.5) (with $q = 2$) takes the form

$$\mathrm{Var}_z\bigl[Y_\lambda(xn^{1/2})\bigr] = \sum_{\ell \ge xn^{1/2}} \sum_{k=1}^{\infty} k^2 a_k \,\mathrm{e}^{-k\alpha\ell} = -\sum_{\ell \ge xn^{1/2}} g_0'(\alpha\ell). \tag{4.7}$$

Interpreting the right-hand side of (4.7) as a Riemann integral sum and arguing as in the proof of Theorem 3.2, we deduce that the equation (4.7) converges, uniformly in $x \ge \delta$, to

$$\begin{aligned}\lim_{n\to\infty} n^{-1/2}\,\mathrm{Var}_z\big[Y_\lambda(xn^{1/2})\big] &= -\gamma^{-1}\int_{\gamma x}^{\infty} g_0'(t)\,\mathrm{d}t\\ &= \gamma^{-1} g_0(\gamma x) = \gamma^{-1}\mathrm{e}^{-\gamma x} H_0'(\mathrm{e}^{-\gamma x}),\end{aligned}$$

according to (3.11). Thus, the theorem is proved. □

It is straightforward to adapt the proof of Theorem 4.2 to the case $q \ge 3$, which only requires a standard generalization of the differential formulas (2.9), (2.10) to higher orders. This way, one can obtain the asymptotics of the form

$$\lim_{n\to\infty} n^{-1/2}\varkappa_q\big[Y_\lambda(xn^{1/2})\big] = \chi_\gamma(x), \qquad x > 0,$$

where the function $\chi_\gamma(x)$ is expressed in terms of the derivatives $H_0^{(j)}(\mathrm{e}^{-\gamma x})$ ($j = 1, \dots, q$).

For the purposes of the present paper (more precisely, for the proof of Lemma 4.4 below), we only need an *upper estimate* as follows.

Lemma 4.3. *For every $q \in \mathbb{N}$ and any $\delta > 0$ we have, uniformly in $x \in [\delta, \infty)$,*

$$\varkappa_q\big[Y_\lambda(xn^{1/2})\big] = O(n^{1/2}), \qquad n \to \infty. \tag{4.8}$$

In Section 5.4 we will require the asymptotics (in fact, an asymptotic bound) for the fourth *central moment* of $Y_\lambda(xn^{1/2})$, which is established next.

Lemma 4.4. *Set $Y_\lambda^0(t) := Y_\lambda(t) - E_z[Y_\lambda(t)]$. Then for any $\delta > 0$, uniformly in $x \in [\delta, \infty)$,*

$$\lim_{n\to\infty} n^{-1} E_z\big[\big(Y_\lambda^0(xn^{1/2})\big)^4\big] = 3\big\{\gamma^{-1}\mathrm{e}^{-\gamma x} H_0'(\mathrm{e}^{-\gamma x})\big\}^2. \tag{4.9}$$

Proof. Using the formula (2.19) (which is valid for any random variable) we have

$$\begin{aligned}E_z\big[\big(Y_\lambda^0(xn^{1/2})\big)^4\big] &= \varkappa_4\big[Y_\lambda(xn^{1/2})\big] + 3\big\{\varkappa_2\big[Y_\lambda(xn^{1/2})\big]\big\}^2\\ &= O(n^{1/2}) + 3n\big\{\gamma^{-1}\mathrm{e}^{-\gamma x} H_0'(\mathrm{e}^{-\gamma x})\big\}^2 (1 + o(1)), \qquad n \to \infty,\end{aligned}$$

on account of the (uniform) estimates (4.6) and (4.8). Hence, the limit (4.9) follows. □

Remark 4.1. Similarly to Remark 3.2, all the results above are valid also for $x = 0$ provided that the function $H_0(u)$ and the corresponding derivatives are finite at $u = 1$.

4.3. The Lyapunov ratio

Let us introduce the *Lyapunov ratio* (of the third order)

$$L_z := \frac{1}{\sigma_z^3}\sum_{\ell=1}^{\infty} \ell^3 \mu_3[\nu_\ell], \tag{4.10}$$

where we denote for short $\sigma_z := \sqrt{\mathrm{Var}_z(N_\lambda)}$ and

$$\mu_3[\nu_\ell] := E_z\big[|\nu_\ell^0|^3\big], \qquad \nu_\ell^0 := \nu_\ell - m_1[\nu_\ell]$$

(i.e., $\mu_3[\nu_\ell]$ is the third-order absolute central moment of ν_ℓ). The next asymptotic estimate will play an important role in the proof of the local limit theorem in Section 5.3 below.

Lemma 4.5. *Suppose that $A^{+}(\frac{1}{2}) < \infty$. Then*

$$L_z \asymp n^{-1/4}, \qquad n \to \infty. \tag{4.11}$$

Proof. In view of the definition (4.10) and the asymptotics $\sigma_z \asymp n^{3/4}$ provided by Theorem 4.1 (see (4.3)), for the proof of (4.11) it suffices to show that

$$M_3 := \sum_{\ell=1}^{\infty} \ell^3 \mu_3[\nu_\ell] \asymp n^2, \qquad n \to \infty. \tag{4.12}$$

Starting with a *lower bound* for M_3, observe using the relation (2.18) that

$$\mu_3[\nu_\ell] \ge m_3[\nu_\ell^0] = \varkappa_3[\nu_\ell]. \tag{4.13}$$

Hence, on account of the formula (2.24) and Theorem 4.1 (with $q = 3$), from (4.12) we get

$$M_3 \ge \sum_{\ell=1}^{\infty} \ell^3 \varkappa_3[\nu_\ell] = \varkappa_3[N_\lambda] \asymp n^2, \qquad n \to \infty, \tag{4.14}$$

which is in agreement with the claim (4.12).

To obtain a suitable *upper bound* on M_3, note that for any $u, v \ge 0$,

$$|u - v|^3 = (u - v)^2 |u - v| \le (u - v)^2 (u + v) = (u - v)^3 + 2v(u - v)^2. \tag{4.15}$$

Setting $u = \nu_\ell$, $v = m_1[\nu_\ell]$ in (4.15) and taking the expectation, we get the inequality

$$\mu_3[\nu_\ell] \le m_3[\nu_\ell^0] + 2m_1[\nu_\ell] \cdot m_2[\nu_\ell^0] = \varkappa_3[\nu_\ell] + 2\varkappa_1[\nu_\ell] \cdot \varkappa_2[\nu_\ell], \tag{4.16}$$

according to the identities (2.16) – (2.18). Note that the term $\varkappa_3[\nu_\ell]$ here is the same as in (4.13), and so gives the contribution of the order of n^2 into the corresponding upper bound for M_3, which is consistent with the lower bound (4.14).

The remaining product term on the right-hand side of (4.16), when elaborated using (2.20) (with $q = 1$ and $q = 2$, respectively) and substituted into (4.12), yields

$$\begin{aligned} \sum_{\ell=1}^{\infty} \ell^3 \varkappa_1[\nu_\ell]\, \varkappa_2[\nu_\ell] &= \sum_{\ell=1}^{\infty} \ell^3 \sum_{k=1}^{\infty} k a_k \mathrm{e}^{-k\alpha\ell} \sum_{m=1}^{\infty} m^2 a_m \mathrm{e}^{-m\alpha\ell} \\ &= \sum_{k,m\ge 1} k|a_k|\, m^2 |a_m|\, S_4\bigl((k+m)\alpha\bigr) \\ &= O(\alpha^{-4}) \sum_{k,m\ge 1} \frac{k|a_k|\, m^2|a_m|}{(k+m)^4}, \end{aligned} \tag{4.17}$$

according to Lemma 2.8. Observing that for $k, m \ge 1$

$$(k+m)^4 = (k+m)^{3/2} (k+m)^{5/2} \ge k^{3/2} m^{5/2},$$

the right-hand side of (4.17) is further estimated by

$$O(\alpha^{-4}) \sum_{k=1}^{\infty} \frac{|a_k|}{k^{1/2}} \sum_{m=1}^{\infty} \frac{|a_m|}{m^{1/2}} = O(\alpha^{-4}) \bigl(A^{+}(\tfrac{1}{2})\bigr)^2 = O(n^2),$$

according to the lemma's hypothesis and the asymptotics $\alpha \asymp n^{-1/2}$ (see (3.2)).

Thus, we have shown that $M_3 = O(n^2)$, and together with the lower bound (4.14) this completes the proof of (4.12). □

5. A local limit theorem and the limit shape

5.1. Statement of the local limit theorem

The role of a local limit theorem in our approach is to yield the asymptotics of the probability $Q_z\{N_\lambda = n\} \equiv Q_z(\Lambda_n)$ appearing in the representation of the measure P_n as a conditional distribution, $P_n(\cdot) = Q_z(\cdot\,|\Lambda_n) = Q_z(\cdot)/Q_z(\Lambda_n)$.

To prove such a theorem (see Theorem 5.1 below), we will require a technical condition on the generating function $\mathcal{F}_0(u)$ as follows.

Assumption 5.1. There exists a constant $\delta_* > 0$ such that for any $\theta \in (0,1)$ the function $H_0(u) = \ln(\mathcal{F}_0(u))$ ($u \in \mathbb{C}$) satisfies the inequality

$$H_0(\theta) - \Re(H_0(\theta\,\mathrm{e}^{\mathrm{i}t})) \ge \delta_*\theta(1-\cos t), \qquad t \in \mathbb{R}. \tag{5.1}$$

Remark 5.1. In terms of the coefficients $\{a_k\}$ in the expansion (2.1), the left-hand side of (5.1) is expressed as $\sum_{k=1}^{\infty} a_k\theta^k(1-\cos kt)$. Consequently, if $a_1 > 0$ and $a_k \ge 0$ for all $k \ge 2$ then the inequality (5.1) is satisfied with $\delta_* = a_1 > 0$.

As before, we denote $\mu_z = E_z(N_\lambda)$, $\sigma_z = \sqrt{\mathrm{Var}_z(N_\lambda)}$. Consider the probability density of a normal distribution $\mathcal{N}(\mu_z, \sigma_z^2)$ (i.e., with mean μ_z and variance σ_z^2),

$$f_{\mu_z,\sigma_z}(x) = \frac{1}{\sqrt{2\pi}\,\sigma_z}\,\exp\left\{-\tfrac{1}{2}(x-\mu_z)^2/\sigma_z^2\right\}, \qquad x \in \mathbb{R}. \tag{5.2}$$

Theorem 5.1. *Let $A^+(\frac{1}{2}) < \infty$ and Assumption* 5.1 *hold. Then, uniformly in $m \in \mathbb{Z}_+$,*

$$Q_z\{N_\lambda = m\} = f_{\mu_z,\sigma_z}(m) + O(n^{-1}), \qquad n \to \infty. \tag{5.3}$$

In fact we will only need a particular case with $m = n$.

Corollary 5.2. *Under the conditions of Theorem* 5.1*,*

$$Q_z\{N_\lambda = n\} \asymp n^{-3/4}, \qquad n \to \infty. \tag{5.4}$$

With the asymptotic results of Sections 3.3 and 4.2 at hand, it is not difficult to deduce the corollary from the theorem.

Proof of Corollary 5.2. By Theorem 3.3 with $\sigma = \frac{1}{2}$, we have $\mu_z = n + O(n^{3/4})$. Together with Theorem 4.1 (see (4.3)) this implies that $(n-\mu_z)/\sigma_z = O(1)$. Hence,

$$f_{\mu_z,\sigma_z}(n) = \frac{1}{\sqrt{2\pi}\,\sigma_z}\,\exp\left\{-\tfrac{1}{2}(n-\mu_z)^2/\sigma_z^2\right\} \asymp \sigma_z^{-1} \sim n^{-3/4}, \qquad n \to \infty,$$

and (5.4) now readily follows from (5.3). □

5.2. Estimates of the characteristic functions

For the proof of Theorem 5.1, we need some technical preparations. Recall from Section 2.1 that the random variables $\{\nu_\ell,\,\ell\in\mathbb{N}\}$ are mutually independent under the measure Q_z. Hence, the characteristic function $\varphi_{N_\lambda}(t)=E_z(\mathrm{e}^{\mathrm{i}tN_\lambda})$ of the sum $N_\lambda=\sum_{\ell=1}^{\infty}\ell\nu_\ell$ is given by

$$\varphi_{N_\lambda}(t)=\prod_{\ell=1}^{\infty}\varphi_{\nu_\ell}(t\ell)=\prod_{\ell=1}^{\infty}\frac{\mathcal{F}_0(z^\ell\mathrm{e}^{\mathrm{i}t\ell})}{\mathcal{F}_0(z^\ell)},\qquad t\in\mathbb{R},\tag{5.5}$$

where $\varphi_{\nu_\ell}(\cdot)$ is the characteristic function of ν_ℓ (see (2.13)). The next lemma provides a useful estimate for $\varphi_{N_\lambda}(t)$ essentially proved in [3, Lemma 7.12].[6] Recall that the Lyapunov ratio L_z is defined in (4.10).

Lemma 5.3. *For all $t\in\mathbb{R}$ such that $|t|\le(L_z\sigma_z)^{-1}$ we have*

$$\bigl|\varphi_{N_\lambda}(t)-\exp\bigl\{\mathrm{i}t\mu_z-\tfrac{1}{2}t^2\sigma_z^2\bigr\}\bigr|\le 16|t|^3L_z\sigma_z^3\exp\bigl\{-\tfrac{1}{6}t^2\sigma_z^2\bigr\}.$$

Let us also prove the following global bound.

Lemma 5.4. *Suppose that Assumption* 5.1 *is satisfied* (*with* $\delta_*>0$). *Then*

$$|\varphi_{N_\lambda}(t)|\le\exp\bigl\{-\delta_*J_\alpha(t)\bigr\},\qquad t\in\mathbb{R},\tag{5.6}$$

where

$$J_\alpha(t):=\sum_{\ell=1}^{\infty}\mathrm{e}^{-\alpha\ell}(1-\cos t\ell).\tag{5.7}$$

Proof. From (5.5) it follows that

$$\ln|\varphi_{N_\lambda}(t)|=\sum_{\ell=1}^{\infty}\ln|\varphi_{\nu_\ell}(t\ell)|,\qquad t\in\mathbb{R}.\tag{5.8}$$

Furthermore, using (2.14) and Assumption 5.1 with $\theta=z^\ell$ (see (5.1)), for each $\ell\in\mathbb{N}$ we have

$$\begin{aligned}\ln|\varphi_{\nu_\ell}(t\ell)|=\Re\bigl(\ln(\varphi_{\nu_\ell}(t\ell))\bigr)&=\Re(H_0(z^\ell\mathrm{e}^{\mathrm{i}t\ell}))-H_0(z^\ell)\\&\le-\delta_*z^\ell(1-\cos t\ell),\qquad t\in\mathbb{R}.\end{aligned}\tag{5.9}$$

Setting here $z=\mathrm{e}^{-\alpha}$ (see (3.2)) and returning from (5.9) to (5.8), we obtain the inequality $\ln|\varphi_{N_\lambda}(t)|\le-\delta_*J_\alpha(t)$, which is equivalent to (5.6). □

5.3. Proof of Theorem 5.1

By definition, the characteristic function $\varphi_{N_\lambda}(t)=E_z\bigl[\mathrm{e}^{\mathrm{i}tN_\lambda}\bigr]$ is given by

$$\varphi_{N_\lambda}(t)=\sum_{m=0}^{\infty}Q_z\{N_\lambda=m\}\,\mathrm{e}^{\mathrm{i}tm},\qquad t\in\mathbb{R}.\tag{5.10}$$

[6] A "two-dimensional" proof in [3, Lemma 7.12] can be easily adapted to the one-dimensional case.

Hence, the coefficients of the Fourier series (5.10) are expressed as

$$Q_z\{N_\lambda = m\} = \frac{1}{2\pi}\int_T \mathrm{e}^{-\mathrm{i}tm}\,\varphi_{N_\lambda}(t)\,\mathrm{d}t, \qquad m\in\mathbb{Z}_+, \tag{5.11}$$

where $T := [-\pi,\pi]$. On the other hand, the characteristic function of the normal distribution $\mathcal{N}(\mu_z,\sigma_z^2)$ (see (5.2)) is given by

$$\int_{-\infty}^{\infty} f_{\mu_z,\sigma_z}(x)\,\mathrm{e}^{\mathrm{i}tx}\,\mathrm{d}x = \mathrm{e}^{\mathrm{i}t\mu_z - t^2\sigma_z^2/2}, \qquad t\in\mathbb{R},$$

so by the inversion formula we have

$$f_{\mu_z,\sigma_z}(m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tm}\,\mathrm{e}^{\mathrm{i}t\mu_z - t^2\sigma_z^2/2}\,\mathrm{d}t, \qquad m\in\mathbb{Z}_+. \tag{5.12}$$

Denote $D_z := \{t\in\mathbb{R}\colon |t| > (L_z\sigma_z)^{-1}\}$. By the asymptotic formula (4.3) and Lemma 4.5, we have $(L_z\sigma_z)^{-1}\asymp n^{1/4}n^{-3/4} = n^{-1/2} = o(1)$, which implies that $D_z^c := \mathbb{R}\setminus D_z \subset T$ for all n large enough. Furthermore, since $\alpha\asymp n^{-1/2}$ (see (3.2)), it follows that $(L_z\sigma_z)^{-1} > \eta\alpha$ with a suitable (small) constant $\eta>0$, hence $D_z\subset\{t\in\mathbb{R}\colon |t|>\eta\alpha\}$. Thus, subtracting (5.12) from (5.11) we get, uniformly in $m\in\mathbb{Z}_+$,

$$\bigl|Q_z\{N_\lambda = m\} - f_{\mu_z,\sigma_z}(m)\bigr| \le \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{5.13}$$

where

$$\mathcal{I}_1 := \frac{1}{2\pi}\int_{D_z^c}\bigl|\varphi_{N_\lambda}(t) - \mathrm{e}^{\mathrm{i}t\mu_z - t^2\sigma_z^2/2}\bigr|\,\mathrm{d}t, \qquad \mathcal{I}_2 := \frac{1}{2\pi}\int_{D_z}\mathrm{e}^{-t^2\sigma_z^2/2}\,\mathrm{d}t, \tag{5.14}$$

$$\mathcal{I}_3 := \frac{1}{2\pi}\int_{T\cap D_z}|\varphi_{N_\lambda}(t)|\,\mathrm{d}t. \tag{5.15}$$

By Lemma 5.3 and on the substitution $t = y\sigma_z^{-1}$, the integral $\mathcal{I}_1$ in (5.14) is estimated by

$$\mathcal{I}_1 = O(L_z\sigma_z^{-1})\int_0^{\infty} y^3\,\mathrm{e}^{-y^2/6}\,\mathrm{d}y = O(n^{-1}), \tag{5.16}$$

according to the asymptotics of σ_z and L_z (see (4.3) and (4.11), respectively). Similarly, for the integral $\mathcal{I}_2$ (see (5.14)) we obtain, again using (4.3) and (4.11),

$$\begin{aligned}\mathcal{I}_2 = O(\sigma_z^{-1})\int_{L_z^{-1}}^{\infty}\mathrm{e}^{-y^2/2}\,\mathrm{d}y &= O(L_z\sigma_z^{-1})\int_{L_z^{-1}}^{\infty} y\,\mathrm{e}^{-y^2/2}\,\mathrm{d}y\\ &= O(n^{-1/2})\,\mathrm{e}^{-L_z^{-2}/2} = O\bigl(n^{-1/2}\mathrm{e}^{-\mathrm{const}\sqrt{n}}\bigr) = o(n^{-1}).\end{aligned} \tag{5.17}$$

Finally, let us turn to the integral $\mathcal{I}_3$ in (5.15). By Lemma 5.4 and a remark about the domain D_z made before display (5.13), we have

$$\mathcal{I}_3 \le \frac{1}{2\pi}\int_{T\cap D_z}\mathrm{e}^{-\delta_* J_\alpha(t)}\,\mathrm{d}t \le \frac{1}{\pi}\int_{\eta\alpha}^{\pi}\mathrm{e}^{-\delta_* J_\alpha(t)}\,\mathrm{d}t. \tag{5.18}$$

Furthermore, evaluating the sum in (5.7) (where for convenience we include the vanishing term with $\ell = 0$) we obtain

$$
\begin{aligned}
J_\alpha(t) = \sum_{\ell=0}^{\infty} \mathrm{e}^{-\alpha\ell}\bigl(1 - \Re(\mathrm{e}^{\mathrm{i}t\ell})\bigr) &= \frac{1}{1-\mathrm{e}^{-\alpha}} - \Re\biggl(\frac{1}{1-\mathrm{e}^{-\alpha+\mathrm{i}t}}\biggr) \\
&\ge \frac{1}{1-\mathrm{e}^{-\alpha}} - \frac{1}{|1-\mathrm{e}^{-\alpha+\mathrm{i}t}|}, \qquad (5.19)
\end{aligned}
$$

because $\Re(s) \le |s|$ for any $s \in \mathbb{C}$. Observe that for $t \in [\eta\alpha, \pi]$

$$
|1-\mathrm{e}^{-\alpha+\mathrm{i}t}| \ge |1-\mathrm{e}^{-\alpha+\mathrm{i}\eta\alpha}| \sim \alpha\,|1+\mathrm{i}\eta| = \alpha\sqrt{1+\eta^2} \qquad (\alpha \to 0^+).
$$

Substituting this estimate into (5.19), we conclude that $J_\alpha(t)$ is asymptotically bounded below by $C(\eta)\alpha^{-1} \asymp n^{1/2}$ (with $C(\eta) = 1 - (1+\eta^2)^{-1/2} > 0$), uniformly in $t \in [\eta\alpha, \pi]$. Thus, the integral in (5.18) is bounded by $O\bigl(\mathrm{e}^{-\mathrm{const}\cdot\sqrt{n}}\bigr) = o(n^{-1})$.

Hence, recalling also the estimates (5.16) and (5.17), we see that the right-hand side of (5.13) admits an asymptotic bound $O(n^{-1})$, which completes the proof of Theorem 5.1.

5.4. The limit shape results

Recall the definition $\omega^*(x) := \gamma^{-1} H_0(\mathrm{e}^{-\gamma x})$ (see (1.8)), where $H_0(u) = \ln(\mathcal{F}_0(u))$ and $\gamma = \bigl(\int_0^1 u^{-1} H_0(u)\,\mathrm{d}u\bigr)^{1/2}$ (see (3.4) and (2.12)).

Theorem 5.5. *Under Assumption* 3.1 *we have, for every* $\delta > 0$ *and any* $\varepsilon > 0$,

$$
\lim_{n\to\infty} Q_z\bigl\{\lambda \in \Lambda\colon\ \sup\nolimits_{x\ge\delta} |n^{-1/2} Y_\lambda(xn^{1/2}) - \omega^*(x)| > \varepsilon\bigr\} = 1.
$$

Proof. By virtue of Theorem 3.2, letting $Y_\lambda^0(t) := Y_\lambda(t) - E_z[Y_\lambda(t)]$ it suffices to check that

$$
\lim_{n\to\infty} Q_z\bigl\{\sup\nolimits_{x\ge\delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2}\bigr\} \to 0. \qquad (5.20)
$$

Put $Z_\lambda(t) := Y_\lambda(t^{-1})$ $(t > 0)$. From the definition (1.1) of $Y_\lambda(\cdot)$, for any $0 < s < t$ we have

$$
Z_\lambda(t) - Z_\lambda(s) = Y_\lambda(t^{-1}) - Y_\lambda(s^{-1}) = \sum_{t^{-1}\le\ell<s^{-1}} \nu_\ell,
$$

and it follows that the random process $Z_\lambda(t)$ $(t > 0)$ has independent increments. Hence, $Z_\lambda^0(t) := Z_\lambda(t) - E_z[Z_\lambda(t)]$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\nu_\ell,\, \ell \ge t^{-1}\}$. From (1.1) it is also evident that $Z_\lambda^0(t)$ is *càdlàg* (i.e., its paths are everywhere right-continuous and have left limits, cf. Fig. 1a). Therefore, by the Doob–Kolmogorov submartingale inequality (see, e.g., [33, Theorem 6.14, p. 99] we obtain

$$
\begin{aligned}
Q_z\bigl\{\sup\nolimits_{x\ge\delta} |Y_\lambda^0(xn^{1/2})| > \varepsilon n^{1/2}\bigr\} &\equiv Q_z\bigl\{\sup\nolimits_{y\le\delta^{-1}} |Z_\lambda^0(yn^{-1/2})| > \varepsilon n^{1/2}\bigr\} \\
&\le \frac{\sup_{y\le\delta^{-1}} \mathrm{Var}_z\bigl[Z_\lambda(yn^{-1/2})\bigr]}{\varepsilon^2 n} \\
&\le \frac{\mathrm{Var}_z\bigl[Z_\lambda(\delta^{-1}n^{-1/2})\bigr]}{\varepsilon^2 n} \\
&\equiv \frac{\mathrm{Var}_z\bigl[Y_\lambda(\delta n^{1/2})\bigr]}{\varepsilon^2 n} = O(n^{-1/2}), \qquad (5.21)
\end{aligned}
$$

in view of Theorem 4.2. Thus, the claim (5.20) follows and the theorem is proved. □

We are finally ready to prove our main result about the limit shape under the measure P_n (cf. Theorem 1.1 stated in the Introduction).

Theorem 5.6. *Suppose that* $A^+(\frac{1}{2})<\infty$ *and that Assumption* 5.1 *is satisfied. Then, for every* $\delta>0$ *and any* $\varepsilon>0$,

$$\lim_{n\to\infty} P_n\bigl\{\lambda\in\varLambda_n\colon \sup\nolimits_{x\ge\delta}\bigl|n^{-1/2}Y_\lambda(xn^{1/2})-\omega^*(x)\bigr|>\varepsilon\bigr\}=0.$$

Proof. Like in the proof of Theorem 5.5, the claim is reduced to the limit

$$\lim_{n\to\infty} P_n\bigl\{\sup\nolimits_{x\ge\delta}|Y_\lambda^0(xn^{1/2})|>\varepsilon n^{1/2}\bigr\}=0, \tag{5.22}$$

with $Y^0_\lambda(t)=Y_\lambda(t)-E_z[Y_\lambda(t)]$. Recalling the definition (2.6) of $P_n(\cdot)$, it is easy to see that

$$P_n\bigl\{\sup\nolimits_{x\ge\delta}|Y_\lambda^0(xn^{1/2})|>\varepsilon n^{1/2}\bigr\}\le\frac{Q_z\bigl\{\sup\nolimits_{x\ge\delta}|Y_\lambda^0(xn^{1/2})|>\varepsilon n^{1/2}\bigr\}}{Q_z\{N_\lambda=n\}}. \tag{5.23}$$

Again using the time reversal $t\mapsto t^{-1}$ as in the proof of Theorem 5.5 and applying the Doob–Kolmogorov submartingale inequality (now with the fourth moment), we obtain (cf. (5.21))

$$Q_z\bigl\{\sup\nolimits_{x\ge\delta}|Y_\lambda^0(xn^{1/2})|>\varepsilon n^{1/2}\bigr\}\le\frac{E_z\bigl[\bigl(Y_\lambda^0(\delta n^{1/2})\bigr)^4\bigr]}{\varepsilon^4 n^2}=O(n^{-1}),$$

by Lemma 4.4. On the other hand, for the denominator in (5.23) we have $Q_z\{N_\lambda=n\}\asymp n^{-3/4}$ by Corollary 5.2. As a result, the right-hand side of (5.23) is dominated by $O(n^{-1/4})=o(1)$, and the limit (5.22) readily follows. □

6. Examples

We now proceed to a few illustrative examples of multiplicative ensembles of random partitions with equiweighted parts. As we will see, some of the examples entail simple representatives of the three meta-classes of decomposable combinatorial structures known as *assemblies*, *multisets* and *selections* (see [1, §2.2]). More specifically, Example 6.1 below belongs to the class of weighted partitions, including the case of unrestricted partitions under the uniform (equiprobable) distribution; Example 6.2 leads to (weighted) partitions with bounds on the part counts, including uniformly distributed strict partitions (i.e., with distinct parts); Example 6.3 includes set partitions with labeled elements and ordered contents. Examples 6.4 and 6.5, as well as Example 6.3, are instances of the so-called *exponential structures* (see, e.g., [20, §5.5]). To the best of our knowledge, Example 6.6 appears to be new in the context of random partitions; interestingly, it furnishes a *branch point singularity* of the generating function $\mathcal{F}_0(u)$ at $u=1$ (see a discussion at the end of Section 1.2).

6.1. Assemblies, multisets and selections: a synopsis

A brief account below essentially follows the classic book [1] (see also the earlier paper [2]).

A decomposable combinatorial structure defined on $n\in\mathbb{N}$ elements is characterized by the (non-ordered) collection of its components of sizes $\ell=1,2,\dots$ with the corresponding counts (multiplicities) $\nu_1, \nu_2, \dots$, so that $\sum_{\ell=1}^{n} \ell \nu_\ell = n$. Consequently, the counts $\{\nu_\ell\}$ determine a partition $\lambda = (1^{\nu_1} 2^{\nu_2} \dots)$ of the integer n. The specific composition of each component may or may not be relevant, depending on whether the elements are distinguishable ("labeled") or not. Furthermore, suppose that components of the same size may vary by their type; more specifically, given a sequence of natural numbers $\{m_\ell\}$, suppose that a component of size $\ell \in \mathbb{N}$ may be colored in m_ℓ different colors, irrespectively of any other components.

Let $\mathcal{S}_n = \{s\}$ be the set of all admissible instances s of such a structure of size $n \in \mathbb{Z}_+$, and denote their number by $p(n) := \#\mathcal{S}_n$ (by convention, $\mathcal{S}_0 = \emptyset$ and $p(0) := 1$). Suppose that the space $\mathcal{S}_n$ is endowed with a uniform probability measure, whereby all $p(n)$ instances $s \in \mathcal{S}_n$ are equally likely; in turn, this induces a certain probability distribution P_n on the corresponding random counts $(\nu_1, \dots, \nu_n)$ and, consequently, on the partition space $\varLambda_n$.

This general scheme is exemplified by the three aforementioned meta-types of decomposable combinatorial structures. In brief, *assemblies* are formed of labeled exchangeable elements, whereas in *multisets* the elements are unlabeled and therefore indistinguishable; furthermore, *selections* are like multisets but with distinct components. In what follows, we elaborate on that by giving formulas for the respective generating functions (which in all cases enjoy a product decomposition of the form (2.1)), as well as for the corresponding joint distributions of the random counts ν_ℓ's under the uniform parent measure on the space $\mathcal{S}_n$ (which should be compared with the general multiplicative formula (1.6)).

6.1.1. Assemblies. This class is characterized by the formula [1, §2.2, p. 46] (cf. [20, §5.1])

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} \frac{p(n)\, z^n}{n!} = \exp\left(\sum_{\ell=1}^{\infty} \frac{m_\ell\, z^\ell}{\ell!} \right), \tag{6.1}$$

which fits in the definition (2.1) of multiplicative measures with the constituent generating functions $\mathcal{F}_\ell(z) := \exp(m_\ell z/\ell!)$ and the corresponding power expansion coefficients

$$c_k^{(\ell)} = \left(\frac{m_\ell}{\ell!}\right)^k \frac{1}{k!}, \qquad k \in \mathbb{Z}_+.$$

It is easy to show (see [1, Eq. (2.2), p. 46]) that the number of assemblies of size n which have prescribed counts $\nu_\ell = k_\ell$, $\ell = 1, \dots, n$ (satisfying the condition $\sum_{\ell=1}^{n} \ell k_\ell = n$) is equal to

$$n! \prod_{\ell=1}^{n} \left(\frac{m_\ell}{\ell!}\right)^{k_\ell} \frac{1}{k_\ell!}, \tag{6.2}$$

and it follows that the joint distribution of ν_ℓ's in this model is given by [1, Eq. (2.6), p. 48]

$$P_n\{\nu_\ell = k_\ell,\ \ell = 1, \dots, n\} = \frac{n!}{p(n)} \prod_{\ell=1}^{n} \left(\frac{m_\ell}{\ell!}\right)^{k_\ell} \frac{1}{k_\ell!}, \qquad \sum_{\ell=1}^{n} \ell k_\ell = n.$$

A simple subclass of assemblies is obtained by setting $m_\ell \equiv m \in \mathbb{N}$, which may be interpreted as equiprobable colored set partitions with labeled elements $\{1, \dots, n\}$; the case $m = 1$ thus corresponds to plain set partitions with uniform distribution.

6.1.2. Multisets. This class is determined by the generating function [1, §2.2, p. 47]

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} p(n)\, z^n = \prod_{\ell=1}^{\infty} (1 - z^\ell)^{-m_\ell},$$

which satisfies the multiplicative definition (2.1) with

$$\mathcal{F}_\ell(z) := (1 - z)^{-m_\ell} = \exp\left(m_\ell \sum_{j=1}^{\infty} \frac{z^j}{j} \right), \qquad \ell \in \mathbb{N},$$

and the corresponding coefficients

$$c_k^{(\ell)} = \binom{m_\ell + k - 1}{k}, \qquad k \in \mathbb{Z}_+.$$

Here, the joint distribution of ν_ℓ's is given by (see [1, Eqs. (2.3), (2.9)])

$$P_n\{\nu_\ell = k_\ell,\ \ell = 1, \dots, n\} = \frac{1}{p(n)} \prod_{\ell=1}^{n} \binom{m_\ell + k_\ell - 1}{k_\ell}, \qquad \sum_{\ell=1}^{n} \ell k_\ell = n.$$

The particular case $m_\ell \equiv m \in \mathbb{N}$ corresponds to weighted integer partitions, which for $m = 1$ is reduced to the plain (unrestricted) partitions with uniform distribution on the space Λ_n.

6.1.3. Selections. This class is defined by the generating function [1, §2.2, p. 47]

$$\mathcal{F}(z) := \sum_{n=0}^{\infty} p(n)\, z^n = \prod_{\ell=1}^{\infty} (1 + z^\ell)^{m_\ell}.$$

Hence, $\mathcal{F}(z)$ satisfies the definition (2.1) with

$$\mathcal{F}_\ell(z) := (1 + z)^{m_\ell} = \exp\left(m_\ell \sum_{j=1}^{\infty} \frac{(-z)^j}{j} \right), \qquad \ell \in \mathbb{N},$$

and the coefficients

$$c_k^{(\ell)} = \binom{m_\ell}{k}, \qquad k = 0, 1, \dots, m_\ell.$$

The joint distribution of ν_ℓ's is given by (see [1, Eqs. (2.4), (2.12)])

$$P_n\{\nu_\ell = k_\ell,\ \ell = 1, \dots, n\} = \frac{1}{p(n)} \prod_{\ell=1}^{n} \binom{m_\ell}{k_\ell}, \qquad \sum_{\ell=1}^{n} \ell k_\ell = n.$$

The case $m_\ell \equiv m \in \mathbb{N}$ entails integer partitions with part counts capped by m; for $m = 1$ this is further reduced to strict partitions (i.e., with distinct parts) under the uniform distribution on the corresponding space $\check{\Lambda}_n$.

6.2. The generating functions

In this section, we introduce six examples by specifying the generating function $\mathcal{F}_0(u) = \sum_{k=0}^{\infty} c_k u^k$ and the corresponding function $H_0(u) = \ln(\mathcal{F}_0(u)) = \sum_{k=1}^{\infty} a_k u^k$. Although the associated multiplicative measures Q_z and P_n are defined primarily in terms of the coefficients $\{c_k\}$ (see (2.4) and (2.7), respectively), the explicit expressions for c_k's may be complicated, so we will not always attempt to give such expressions.

For our purposes, it is more important to focus on the function $H_0(u)$ and its power expansion coefficients $\{a_k\}$, since these are the ingredients that determine the existence and exact form of the limit shape $\omega^*(x) = \gamma^{-1} H_0(\mathrm{e}^{-\gamma x})$ (see (1.8)), including the parameter γ (see (3.4) and (2.12)). In particular, we have to check the basic condition $A^+(1) < \infty$ (see Assumption 3.1), as well as the refined condition $A^+(\frac{1}{2}) < \infty$ and Assumption 5.1, both needed for the limit shape result under the measure P_n (see Theorem 5.6).

Example 6.1. For $r \in (0, \infty)$, $\rho \in (0, 1]$, set

$$\mathcal{F}_0(u) := (1 - \rho u)^{-r}, \qquad |u| < \rho^{-1}. \tag{6.3}$$

By the binomial formula, the coefficients in the power series expansion (2.1) are given by

$$c_k := \binom{r+k-1}{k} \rho^k = \frac{r(r+1)\cdots(r+k-1)}{k!}\, \rho^k, \qquad k \in \mathbb{Z}_+. \tag{6.4}$$

In particular, $c_0 = 1$ and, moreover, $c_k > 0$ for all $k \in \mathbb{N}$.

Remark 6.1. The parameter $\rho < 1$ introduces exponential weights of the part counts, which discourages multiple occurrences of the same part as compared to the neutral case $\rho = 1$. The parameter r also contributes to the weighting; e.g., if $\rho = 1$ then $c_{k+1}/c_k > 1$ whenever $r > 1$. The combined effect of the parameters $\rho < 1$ and $r > \rho^{-1} > 1$ is more interesting: it is easy to see that the maximum of the sequence c_k is attained for (integer) $k = k^*$ near $(r\rho - 1)/(1-\rho)$.

For $\rho = 1$ and $r = m \in \mathbb{N}$, the formula (6.3) pinpoints a multiset structure (see Section 6.1.2) arising via partitioning an integer $n \in \mathbb{N}$ into parts, each of which is then colored in one of m different colors, irrespectively of its size. The simplest case $\rho = 1$, $r = 1$ thus corresponds to the classical ensemble of uniform integer partitions mentioned in the Introduction (Sections 1.1, 1.2).

Note that formula (2.2) for the Q_z-distribution of the part counts ν_ℓ ($\ell \in \mathbb{N}$) specializes to

$$Q_z\{\nu_\ell = k\} = \binom{r+k-1}{k} \rho^k z^{k\ell} (1 - \rho z^\ell)^r, \qquad k \in \mathbb{Z}_+, \tag{6.5}$$

which is a negative binomial distribution with parameters r and $p = 1 - \rho z^\ell$ [8, Ch. VI, §8, p. 165]. If $r = 1$ then $\mathcal{F}_0(u) = (1 - \rho u)^{-1}$, $c_k = \rho^k$, and (6.5) is reduced to a geometric distribution

$$Q_z\{\nu_\ell = k\} = \rho^k z^{k\ell} (1 - \rho z^\ell), \qquad k \in \mathbb{Z}_+.$$

In the latter case, from (1.6) we get

$$P_n(\lambda) = \mathfrak{C}_n^{-1} \rho^{Y_\lambda(0)}, \qquad \lambda \in \varLambda_n, \tag{6.6}$$

where $Y_\lambda(0) = \sum_{\ell=1}^{\infty} \nu_\ell$ is the total number of parts in partition $\lambda = (1^{\nu_1} 2^{\nu_2} \dots)$ (cf. (1.1)). If also $\rho = 1$ then (6.6) is further reduced to the uniform distribution on $\varLambda_n$.

Returning to the general case, from (6.3) we have

$$H_0(u) = -r \ln(1-\rho u) = r\sum_{k=1}^{\infty} \frac{\rho^k}{k}\, u^k. \tag{6.7}$$

Since the coefficients in the expansion (6.7) are positive, Assumption 5.1 is satisfied by Remark 5.1; also, it readily follows that $A^+(\sigma)<\infty$ for any $\sigma>0$ (and each $\rho\in(0,1]$).

Example 6.2. For $m\in\mathbb{N}$, $\rho\in(0,1]$, consider the generating function

$$\mathcal{F}_0(u) := (1+\rho u)^m, \qquad u\in\mathbb{C}, \tag{6.8}$$

with the coefficients

$$c_k = \binom{m}{k}\rho^k = \frac{m(m-1)\cdots(m-k+1)}{k!}\,\rho^k, \qquad k=0,1,\dots,m.$$

Consequently, formula (2.2) gives a binomial distribution

$$Q_z\{\nu_\ell = k\} = \binom{m}{k}\frac{\rho^k z^{k\ell}}{(1+\rho z^\ell)^m}, \qquad k=0,1,\dots,m, \tag{6.9}$$

with parameters m and $p=\rho z^\ell(1+\rho z^\ell)^{-1}$.

Setting $\rho=1$ in (6.8) yields selections (see Section 6.1.3) corresponding to integer partitions with multiplicities $\nu_\ell\le m$ ($\ell\in\mathbb{N}$); in particular, $m=1$ corresponds to strict partitions (see Sections 1.1, 1.2). More generally, for $m=1$ and $0<\rho\le 1$, the measure Q_z is concentrated on the subspace $\check{\Lambda}\subset\Lambda$ with the distribution (6.9) reduced to

$$Q_z\{\nu_\ell=k\} = \frac{\rho^k z^{k\ell}}{1+\rho z^\ell}, \qquad k=0,1.$$

Accordingly, formula (1.6) specifies on $\check{\Lambda}_n$ the weighted distribution (cf. (6.6))

$$P_n(\lambda) = \check{\mathfrak{C}}_n^{-1}\rho^{Y_\lambda(0)}, \qquad \lambda\in\check{\Lambda}_n,$$

which is reduced to the uniform distribution if $\rho=1$, as already mentioned.

In the general case, from (6.8) it follows

$$H_0(u) = m\ln(1+\rho u) = m\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\rho^k}{k}\,u^k, \tag{6.10}$$

and it is evident that $A^+(\sigma)<\infty$ for each $\sigma>0$ (and any $\rho\in(0,1]$). Finally, let us verify Assumption 5.1. Using (6.10) we obtain, for any $\theta\in(0,1)$ and all $t\in\mathbb{R}$,

$$\begin{aligned}
H_0(\theta)-\Re(H_0(\theta\,\mathrm{e}^{\mathrm{i}t})) &= m\ln(1+\rho\theta) - m\Re\bigl(\ln(1+\rho\theta\,\mathrm{e}^{\mathrm{i}t})\bigr)\\
&= m\ln(1+\rho\theta) - m\ln|1+\rho\theta\,\mathrm{e}^{\mathrm{i}t}|\\
&= -\frac{m}{2}\ln\left(\frac{1+2\rho\theta\cos t+\rho^2\theta^2}{(1+\rho\theta)^2}\right)\\
&\ge -\frac{m}{2}\left(\frac{1+2\rho\theta\cos t+\rho^2\theta^2}{(1+\rho\theta)^2}-1\right)\\
&= \frac{m\rho\,\theta\,(1-\cos t)}{(1+\rho\theta)^2} \ge \frac{m\rho}{(1+\rho)^2}\,\theta\,(1-\cos t).
\end{aligned}$$

Thus, the inequality (5.1) holds with $\delta_*=m\rho/(1+\rho)^2>0$.

Example 6.3. For $b \in (0,\infty)$, $\rho \in [0,1]$, consider the generating function

$$\mathcal{F}_0(u) := \exp\left(\frac{bu}{1-\rho u}\right), \qquad |u| < \rho^{-1}. \tag{6.11}$$

Noting that $(1-t)^{-1} = \sum_{k=0}^{\infty} t^k$ (with $t = \rho s$), it is evident that the coefficients c_k's in the power series expansion of the function (6.11) are positive, with $c_0 = 1$, $c_1 = b$, $c_2 = b\rho + \frac{1}{2}b^2$, etc. More systematically, by Faà di Bruno's formula generalizing the chain rule of differentiation to higher derivatives (see [14, Ch. I, §12, p. 34]) we obtain

$$c_k = \sum_{m=1}^{k} b^m \rho^{k-m} \sum_{(j_1,\dots,j_k)\in\mathcal{J}_m} \frac{1}{j_1!\cdots j_k!}, \qquad k \in \mathbb{N}, \tag{6.12}$$

where $\mathcal{J}_m$ is the set of all nonnegative integer k-tuples $(j_1,\dots,j_k)$ such that $j_1+\dots+j_k = m$ and $j_1 + 2j_2 + \dots + kj_k = k$.

Remark 6.2. Note that the k-tuples $(j_1,\dots,j_k) \in \mathcal{J}_m$ are in one-to-one correspondence with partitions of k involving precisely m different integers as parts, where each element j_ℓ has the meaning of the multiplicity of part $\ell \in \{1,\dots,k\}$.

Remark 6.3. For $b = \rho = 1$, the formula (6.12) is reduced, on account of Remark 6.2, to

$$c_k = \sum_{m=1}^{k} \sum_{(j_1,\dots,j_k)\in\mathcal{J}_m} \frac{1}{j_1!\cdots j_k!} = \sum_{\lambda\vdash k} \frac{1}{\nu_1!\cdots\nu_k!}, \qquad \lambda = (1^{\nu_1}2^{\nu_2}\dots) \in \varLambda_k.$$

From the formula (6.2) with $m_\ell = \ell!$, it follows that the quantity $p(k) = k!c_k$ equals the number of partitions of the set $\{1,\dots,k\}$ into components with *ordered* contents. Such a structure may be visualized as a *forest of linear rooted trees* (i.e., a disjoint union of connected directed acyclic graphs, where each vertex has at most two neighbors), with labeled vertices.

The observation made in Remark 6.3 can be explained without calculations using the general theory of assemblies (see Section 6.1.1). Namely, the function (6.11) with $b = \rho = 1$ may be represented in the exponential form (6.1) by setting $m_\ell := \ell!$ ($\ell \in \mathbb{N}$),

$$\mathcal{F}_0(u) = \exp\left(\frac{u}{1-u}\right) = \exp\left(\sum_{\ell=1}^{\infty} u^\ell\right) = \exp\left(\sum_{\ell=1}^{\infty} \frac{m_\ell\, u^\ell}{\ell!}\right). \tag{6.13}$$

In the terminology of Section 6.1, that is to say that in the corresponding assembly each part of size ℓ is colored in one of $m_\ell = \ell!$ different colors, which is equivalent to ordering the content of this part in one of $\ell!$ ways. Hence, on comparing the power series expansions (2.1) and (6.1) for $\mathcal{F}_0(u)$, we conclude that $c_k = p(k)/k!$, where $p(k)$ equals the total number of instances of such an assembly of size n, in accord with Remark 6.3.

If $\rho = 0$ then the generating function (6.11) is reduced to $\mathcal{F}_0(u) = \mathrm{e}^{bu}$, with the expression (6.12) simplified to $c_k = b^k/k!$ ($k \in \mathbb{Z}_+$). Hence, according to (2.2) the counts ν_ℓ in a random partition $\lambda = (1^{\nu_1}2^{\nu_2}\dots) \in \varLambda$ have a Poisson distribution with parameter bz^ℓ,

$$Q_z\{\nu_\ell = k\} = \frac{b^k z^{k\ell}}{k!}\,\mathrm{e}^{-bz^\ell}, \qquad k \in \mathbb{Z}_+,$$

which leads to the distribution on $\varLambda_n$ of the form (see (1.6))

$$P_n(\lambda) = \mathfrak{C}_n^{-1} \prod_{\ell=1}^{\infty} \frac{b^{\nu_\ell}}{\nu_\ell!}, \qquad \lambda \in \varLambda_n.$$

The parameter $b > 0$ here determines an exponential weighting: having more parts of each size is either encouraged or discouraged according as $b > 1$ or $b < 1$.

In the special case $\rho = 0$, $b \in \mathbb{N}$, the multiplicative ensemble defined via the product formula (1.5) admits a simple combinatorial interpretation. Indeed, similarly to (6.13) the exponential identity (6.1) with $m_\ell := b\ell!$ ($\ell \in \mathbb{N}$) determines an assembly of size $n \in \mathbb{N}$ obtained by partitioning the set $\{1, \dots, n\}$ into non-empty blocks with ordered contents, each block colored in one of b different colors irrespectively of its size.

Returning to the general formula (6.11), we get

$$H_0(u) = \frac{bu}{1 - \rho u} = b \sum_{k=1}^{\infty} \rho^{k-1} u^k, \tag{6.14}$$

and hence Assumption 5.1 is automatic (see Remark 5.1); moreover, $A^+(\sigma) < \infty$ for any $\sigma > 0$, except for $\rho = 1$ whereby $A^+(\sigma) < \infty$ only with $\sigma > 1$.

Example 6.4. Extending Example 6.3 (for simplicity, with $b = 1$), let us set for $r > 0$, $r \neq 1$ and $\rho \in (0, 1]$

$$\mathcal{F}_0(u) := \exp\left(\frac{u}{(1 - \rho u)^r}\right), \qquad |u| < \rho^{-1}. \tag{6.15}$$

Taking the logarithm of (6.15) we get the power series expansion (cf. (6.3))

$$H_0(u) = \frac{u}{(1 - \rho u)^r} = \sum_{k=1}^{\infty} \binom{r + k - 2}{k - 1} \rho^{k-1} u^k, \tag{6.16}$$

which has positive coefficients a_k (cf. (6.4)). Hence, Assumption 5.1 is satisfied by virtue of Remark 5.1. To check the condition $A^+(\sigma) < \infty$, observe using Stirling's asymptotic formula for the gamma function (see [5, §12.5, p. 130]) that

$$a_k = \binom{r + k - 2}{k - 1} \rho^{k-1} = \frac{\Gamma(k + r - 1)}{\Gamma(r)\,\Gamma(k)}\, \rho^{k-1} \sim \frac{k^{r-1}}{\Gamma(r)}\, \rho^{k-1} \qquad (k \to \infty),$$

hence $A^+(\sigma) < \infty$ for any $\sigma > 0$ if $\rho < 1$, whereas if $\rho = 1$ then $A^+(\sigma) < \infty$ only for $\sigma > r$.

On substituting (6.16) into Taylor's expansion of the exponential function in (6.15), it is evident that the corresponding coefficients c_k in the power series (2.1) of $\mathcal{F}_0(u)$ are also positive, with $c_0 = c_1 = 1$, $c_2 = r\rho + \frac{1}{2}$, etc.; more generally, c_k's can be evaluated using Faà di Bruno's formula like in Example 6.3, but we omit the details.

However, the special case $\rho = 1$, $r = m \in \mathbb{N}$ may be given a combinatorial interpretation as follows. Substituting the expansion (6.16) into (6.15) and setting $m_\ell := \ell! \binom{m+\ell-2}{\ell-1}$ for $\ell \in \mathbb{N}$ (cf. (6.13)), the identity (6.1) applied to $\mathcal{F}_0(u)$ yields that the coefficients c_k in the power series expansion of (6.15) are expressed as $c_k = p(k)/k!$, where $p(k)$ is the total number of instances of the corresponding assembly of size k. Construction of such an assembly comprises three steps: (i) the set $\{1, \dots, k\}$ is partitioned into non-empty blocks; (ii) a block with ℓ elements is represented as a linear rooted tree (see Remark 6.3) distinguished by $\ell!$ permutations of its vertices; (iii) $\ell-1$ edges of such a tree are colored using m colors, subject to the convention that if j-th color is used $i_j \ge 0$ times (with $i_1+\cdots+i_m=\ell-1$) then the color schemes are distinguishable only if the corresponding m-tuples $(i_1,\dots,i_m)$ are not identical, making the total number of such schemes equal to $\binom{m+\ell-2}{\ell-1}$ (see [8, Ch. II, §5, p. 38]).

Example 6.5. Combining the exponential form of Example 6.4 with the generating function from Example 6.2, for $\rho\in[0,1],\ m\in\mathbb{N}$ consider

$$\mathcal{F}_0(u):=\exp\bigl(u(1+\rho u)^{m-1}\bigr),\qquad u\in\mathbb{C}. \tag{6.17}$$

Since $u\mapsto u(1+\rho u)^{m-1}$ is a polynomial of degree m with positive coefficients, it follows that the coefficients c_k in the power series expansion of the function (6.17) are positive for all $k\in\mathbb{Z}_+$.

Remark 6.4. Caution is needed with a *non-integer* $r>1$ replacing $m\in\mathbb{N}$ in (6.17): e.g., for $\rho=1,\ r=1.5$ we obtain (with the help of Maple™) $c_9=-921479/92897280<0$.

From (6.17) by the binomial formula we obtain

$$H_0(u)=u(1+\rho u)^{m-1}=\sum_{k=1}^{m}\binom{m-1}{k-1}\rho^{k-1}u^k, \tag{6.18}$$

so that the corresponding coefficients a_k's are positive for $k=1,\dots,m$ and vanish for $k\ge m+1$. Hence, Assumption 5.1 is satisfied and $A^+(\sigma)<\infty$ for any $\sigma>0$.

In the special case $\rho=1$, it is not hard to give a combinatorial interpretation of the coefficients c_k by adapting considerations in Examples 6.3 and 6.4. Indeed, substituting the expansion (6.18) back into (6.17) and defining $m_\ell:=\ell!\binom{m-1}{\ell-1}$ for $\ell=1,\dots,m$ and $m_\ell\equiv 0$ for $\ell\ge m+1$, similarly as above we can use the exponential identity (6.1) to conclude that $c_k=p(k)/k!$, where $p(k)$ is the total number of assemblies of size k constructed as follows: (i) the set $\{1,\dots,k\}$ is partitioned into blocks of size *not bigger than* m each; (ii) a block of size ℓ is arranged as a rooted linear tree with labeled vertices (resulting in $\ell!$ possible permutations); (iii) the total of m unlabeled tokens is allocated to ℓ consecutive vertices on such a tree according to an integer ℓ-tuple $(m_1,\dots,m_\ell)$ subject to the conditions $m_1+\cdots+m_\ell=m$ and $m_i\ge 1$ for all $i=1,\dots,\ell$ (so that all m tokens are allocated and each vertex gets at least one token); the total number of such (strict) allocations is known to be given by $\binom{m-1}{\ell-1}$ (see [8, Ch. II, §5, p. 38]).

Example 6.6. For $\rho\in(0,1]$, $r\in(0,\infty)$, consider the generating function

$$\mathcal{F}_0(u):=\left(\frac{-\ln(1-\rho u)}{\rho u}\right)^{r}\equiv(f_0(u))^r,\qquad |u|<\rho^{-1}, \tag{6.19}$$

where

$$f_0(u):=\frac{-\ln(1-\rho u)}{\rho u}=1+\sum_{k=1}^{\infty}\frac{\rho^k u^k}{k+1}. \tag{6.20}$$

If $r=m\in\mathbb{N}$ then from (6.20) it is evident that the coefficients c_k in the power series expansion of $\mathcal{F}_0(u)$ in (6.19) are positive for all $k\in\mathbb{Z}_+$; however, for non-integer $r>0$ this is not so clear, since the binomial expansion of $t\mapsto(1+t)^r$ involves negative terms (cf.

Remark 6.4). Yet, as a matter of fact, the positivity of c_k's holds for *any real* $r > 0$ — this will be established in Corollary 6.2.

By a term-by-term comparison, it is also clear that, for any $r > 0$,

$$\mathcal{F}_0(u) = \left(1 + \sum_{k=1}^{\infty} \frac{\rho^k u^k}{k+1}\right)^r \le \left(1 + \sum_{k=1}^{\infty} \rho^k u^k\right)^r = (1-\rho u)^{-r}, \qquad 0 \le u < \rho^{-1}, \tag{6.21}$$

with the inequality being strict for $u > 0$. That is to say, the function $\mathcal{F}_0(u)$ is bounded by a multiset-type generating function (6.3) considered in Example 6.1. Moreover, *for integer* $r = m \in \mathbb{N}$, by expanding both parts in (6.21) it is evident that the coefficients c_k in the power series expansion of $\mathcal{F}_0(u)$ are dominated by the coefficients of the multiset generating function $(1-\rho u)^{-m}$ (cf. (6.4)),

$$c_k < \binom{m+k-1}{k} \rho^k = \frac{m(m+1)\cdots(m+k-1)}{k!}\,\rho^k, \qquad k \in \mathbb{N}. \tag{6.22}$$

Thus, the multiplicative ensemble determined by (6.19) may be viewed (at least for integer $r = m$) as a *discounted multiset ensemble*, whereby larger values of each count $\nu_\ell = k$ are progressively discouraged. Again, this statement turns out to be true for *any real* $r > 0$, which will be explained below (see Corollary 6.2). On the other hand, a direct combinatorial interpretation of the generating function (6.19) (say, in the spirit of the previous examples) is not clear, even in the simplest case $r = \rho = 1$.

Let us now look at the function $H_0(u) = \ln(\mathcal{F}_0(u))$ (see (2.8)): according to (6.19),

$$H_0(u) = r \ln\left(\frac{-\ln(1-\rho u)}{\rho u}\right) = r \ln(f_0(u)), \qquad |u| < \rho^{-1}. \tag{6.23}$$

The next proposition implies that $A^+(\sigma) < \infty$ for any $\sigma > 0$ (including the case $\rho = 1$); furthermore, since all $a_k > 0$, by Remark 5.1 it follows that Assumption 5.1 is satisfied.

Proposition 6.1. *The coefficients* $\{a_k\}$ *in the power series expansion* (2.8) *of the function* (6.23) *satisfy the inequalities*

$$\frac{r\rho^k}{k^2(k+1)} \le a_k \le \frac{r\rho^k}{k+1}, \qquad k \in \mathbb{N}. \tag{6.24}$$

In particular, $a_k > 0$ *for all* $k \in \mathbb{N}$.

Proof. Differentiation of the identity $r \ln(f_0(u)) = \sum_{j=1}^{\infty} a_j u^j$ (see (6.23)) gives

$$r f_0'(u) = f_0(u) \sum_{j=1}^{\infty} j a_j u^{j-1}. \tag{6.25}$$

Differentiating (6.25) again $k-1$ times ($k \ge 1$), by the Leibniz rule we obtain

$$f_0^{(k)}(0) = \frac{1}{r} \sum_{i=0}^{k-1} \binom{k-1}{i} f_0^{(k-1-i)}(0)\,(i+1)!\,a_{i+1}, \qquad k \in \mathbb{N}. \tag{6.26}$$

Noting from (6.20) that $f_0^{(j)}(0) = \rho^j j!/(j+1)$ ($j \in \mathbb{Z}_+$) and using the shorthand notation $\tilde{a}_k := k a_k \rho^{-k}/r$ ($k \in \mathbb{N}$), the system of equations (6.26) is reduced to

$$\frac{k}{k+1} = \sum_{i=0}^{k-1} \frac{\tilde{a}_{i+1}}{k-i}, \qquad k \in \mathbb{N}, \tag{6.27}$$

while the inequalities (6.24) are rewritten as

$$\frac{1}{k(k+1)} \le \tilde{a}_k \le \frac{k}{k+1}, \qquad k \in \mathbb{N}. \tag{6.28}$$

Let us prove (6.28) by induction in $k \in \mathbb{N}$. For $k = 1$, from (6.27) we find $\tilde{a}_1 = \frac{1}{2}$ and the claim (6.28) is obviously satisfied. Suppose now that the inequalities (6.28) hold for $\tilde{a}_1, \dots, \tilde{a}_{k-1}$ ($k \ge 2$), which entails that $\tilde{a}_i > 0$ ($i = 1, \dots, k-1$). Then the recursion (6.27) (with k replaced by $k-1$) implies

$$\frac{k}{k+1} = \sum_{i=0}^{k-2} \frac{\tilde{a}_{i+1}}{k-i} + \tilde{a}_k \le \sum_{i=0}^{k-2} \frac{\tilde{a}_{i+1}}{k-1-i} + \tilde{a}_k = \frac{k-1}{k} + \tilde{a}_k,$$

and it follows that

$$\tilde{a}_k \ge \frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)},$$

which gives the lower bound in (6.28). On the other hand, again using that $\tilde{a}_1, \dots, \tilde{a}_{k-1} > 0$, from (6.27) we get

$$\frac{k}{k+1} = \tilde{a}_k + \sum_{i=0}^{k-2} \frac{\tilde{a}_{i+1}}{k-i} \ge \tilde{a}_k,$$

which proves the upper bound in (6.28). Thus, the claim (6.28) is verified for the $\tilde{a}_k$, and therefore it is valid with all $k \in \mathbb{N}$. ☐

Corollary 6.2. *For any real $r > 0$, the coefficients c_k in the power series expansion of the generating function* (6.19) *satisfy the two-sided bounds* (*cf.* (6.22))

$$0 < c_k < \binom{r+k-1}{k} \rho^k, \qquad k \in \mathbb{N}. \tag{6.29}$$

Proof. Using the expansion $H_0(u) = \sum_{k=1}^{\infty} a_k u^k$ we have

$$\mathcal{F}_0(u) = \exp\left(H_0(u)\right) = \exp\left(\sum_{k=1}^{\infty} a_k u^k\right) = 1 + \sum_{k=1}^{\infty} c_k u^k. \tag{6.30}$$

By Proposition 6.1 all $a_k > 0$, and since Taylor's coefficients of the exponential function are positive as well, it is evident from (6.30) that $c_k > 0$ for all $k \in \mathbb{N}$.

Furthermore, from the bounds (6.24) we get

$$0 < a_k \le \frac{r\rho^k}{k+1} < \frac{r\rho^k}{k}, \qquad k \in \mathbb{N}. \tag{6.31}$$

Considering the corresponding power series and their exponentials

$$\exp\left(\sum_{k=1}^{\infty} a_k u^k\right) = \exp\bigl(H_0(u)\bigr) \equiv \mathcal{F}_0(u)$$

and

$$\exp\left(\sum_{k=1}^{\infty} \frac{r\rho^k}{k}\, u^k\right) = \exp\bigl(-r\ln(1-\rho u)\bigr) = (1-\rho u)^{-r} =: \widetilde{\mathcal{F}}(u),$$

it follows from the term-by-term subordination (6.31) that the coefficients in the respective power series expansions $\mathcal{F}_0(u) = \sum_k c_k u^k$ and $\widetilde{\mathcal{F}}(u) = \sum_k \tilde{c}_k u^k$ inherit the same (strict) subordination, that is, $c_k < \tilde{c}_k$ for all $k \in \mathbb{N}$. It remains to notice that (cf. (6.3), (6.4)) $\tilde{c}_k = \binom{r+k-1}{k}\rho^k$, which yields the upper bound in (6.29), as claimed. □

For convenience, the results of Section 6.1.2 are summarized in Table 1.

Table 1: The generating functions in Examples 6.1–6.6 $(0 < \rho \le 1,\ r > 0,\ m \in \mathbb{N},\ b > 0)$. Third column shows the type of singularity of $\mathcal{F}_0(u)$ (with $\rho = 1,\ r = m \in \mathbb{N}$) at point $u = 1$.

No.	$\mathcal{F}_0(u)$	$u=1$	$H_0(u)$	$a_k\ (k \in \mathbb{N})$	$A^+(\sigma) < \infty$
6.1	$(1-\rho u)^{-r}$	pole of order m	$-r\ln(1-\rho u)$	$r k^{-1}\rho^k$	$\sigma > 0$
6.2	$(1+\rho u)^m$	regular point	$m\ln(1+\rho u)$	$(-1)^{k-1} r k^{-1}\rho^k$	$\sigma > 0$
6.3	$\exp\left(\dfrac{bu}{1-\rho u}\right)$	essential singularity	$\dfrac{bu}{1-\rho u}$	$b\rho^{k-1}$	$\sigma > 0\ (\rho < 1)$ $\sigma > 1\ (\rho = 1)$
6.4	$\exp\left(\dfrac{u}{(1-\rho u)^r}\right)$	essential singularity	$\dfrac{u}{(1-\rho u)^r}$	$\binom{r+k-2}{k-1}\rho^{k-1}$ $\sim k^{r-1}\rho^{k-1}/\Gamma(r)$	$\sigma > 0\ (\rho < 1)$ $\sigma > r\ (\rho = 1)$
6.5	$\exp\bigl(u(1+\rho u)^{m-1}\bigr)$	regular point	$u(1+\rho u)^{m-1}$	$\binom{m-1}{k-1}\rho^{k-1}$ $(k = 1,\dots,m)$	$\sigma > 0$
6.6	$\left(\dfrac{-\ln(1-\rho u)}{\rho u}\right)^r$	branch point	$r\ln\left(\dfrac{-\ln(1-\rho u)}{\rho u}\right)$	$a_k = O(k^{-1}\rho^k)$	$\sigma > 0$

6.3. The limit shapes

In this section, for each of Examples 6.1–6.6 we evaluate the parameter $\gamma = \sqrt{A(1)}$ (see (3.4)) using for $A(1)$ either the definition (2.11) or the equivalent integral expression (2.12), and then apply Theorems 5.5 and 5.6 to obtain the explicit limit shape $\omega^*(x)$ as identified by the general formula (1.8). For the reader's convenience, the limit shape function in Example 6.i is denoted by $\omega_i^*(x)$ $(i = 1,\dots,6)$, and the results are summarized in Table 2.

Starting with Example 6.1, from (3.4) and (6.7) we have

$$\gamma^2 = r\sum_{k=1}^{\infty} \frac{\rho^k}{k^2} = r\,\mathrm{Li}_2(\rho), \tag{6.32}$$

Table 2: The limit shapes in Examples 6.1–6.6 $(0<\rho\le 1,\ r>0,\ m\in\mathbb{N},\ b>0)$.

No.	$H_0(u)$	γ^2	$\omega_i^*(x)$	Special case
6.1	$-r\ln(1-\rho u)$	$r\,\mathrm{Li}_2(\rho)$	$-\dfrac{r\ln(1-\rho\mathrm{e}^{-\gamma x})}{\gamma}$	$r=\rho=1$: $\gamma=\pi/\sqrt{6}$
6.2	$m\ln(1+\rho u)$	$-m\,\mathrm{Li}_2(-\rho)$	$\dfrac{m\ln(1+\rho\mathrm{e}^{-\gamma x})}{\gamma}$	$\rho=1,\ m=1$: $\gamma=\pi/\sqrt{12}$
6.3	$\dfrac{bu}{1-\rho u}$	$-\dfrac{b\ln(1-\rho)}{\rho}$	$\dfrac{b\,\mathrm{e}^{-\gamma x}}{\gamma(1-\rho\mathrm{e}^{-\gamma x})}$	$\rho\to 0^+$: $\gamma=\sqrt{b}$
6.4	$\dfrac{u}{(1-\rho u)^r}$	$\dfrac{1-(1-\rho)^{1-r}}{\rho(1-r)}$	$\dfrac{\mathrm{e}^{-\gamma x}}{\gamma(1-\rho\mathrm{e}^{-\gamma x})^r}$	$\rho=1,\ r<1$: $\gamma=1/\sqrt{1-r}$
6.5	$u(1+\rho u)^{m-1}$	$\dfrac{(1+\rho)^m-1}{\rho m}$	$\dfrac{\mathrm{e}^{-\gamma x}(1+\rho\mathrm{e}^{-\gamma x})^{m-1}}{\gamma}$	$\rho=1,\ m=2$: $\gamma=\sqrt{3/2}$
6.6	$r\ln\!\left(\dfrac{-\ln(1-\rho u)}{\rho u}\right)$	$\int_0^1 u^{-1}H_0(u)\,\mathrm{d}u$	$\dfrac{r}{\gamma}\ln\!\left(\dfrac{-\ln(1-\rho\mathrm{e}^{-\gamma x})}{\rho\mathrm{e}^{-\gamma x}}\right)$	$r=\rho=1$: $\gamma\doteq 0.853636$

where $\mathrm{Li}_2(\cdot)$ is the *dilogarithm* (see [16]). Hence, the limit shape is given by (see (1.8))

$$\boxed{\omega_1^*(x)=-\frac{r}{\gamma}\,\ln(1-\rho\,\mathrm{e}^{-\gamma x})} \tag{6.33}$$

For $r=\rho=1$, (6.32) gives $\gamma^2=\mathrm{Li}_2(1)=\pi^2/6$ and from (6.33) we recover the classical formula (1.3) for the limit shape of partitions under the uniform distribution on $\varLambda_n$.

Next, in Example 6.2 we have, according to (3.4) and (6.10),

$$\gamma^2=m\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\rho^k}{k^2}=-m\,\mathrm{Li}_2(-\rho)\equiv m\bigl(\mathrm{Li}_2(\rho)-\tfrac{1}{2}\,\mathrm{Li}_2(\rho^2)\bigr), \tag{6.34}$$

and the limit shape (1.8) specializes to

$$\boxed{\omega_2^*(x)=\frac{m}{\gamma}\,\ln(1+\rho\,\mathrm{e}^{-\gamma x})} \tag{6.35}$$

If $m=\rho=1$ then from (6.34) we find $\gamma^2=\frac{1}{2}\,\mathrm{Li}_2(1)=\pi^2/12$ and the equation (6.35) is reduced to the limit shape (1.4) of uniformly distributed strict partitions.

In Example 6.3, according to (3.4) and (6.14) we have for $\rho\in(0,1)$

$$\gamma^2=b\sum_{k=1}^{\infty}\frac{\rho^{k-1}}{k}=-\frac{b}{\rho}\,\ln(1-\rho)<\infty, \tag{6.36}$$

while if $\rho=0$ then $\gamma^2=b$. Alternatively, we can obtain the same result as (6.36) by using the integral formula (2.12) with the expression (6.14) for $H_0(u)$,

$$\gamma^2=\int_0^1\frac{b}{1-\rho u}\,\mathrm{d}u=-\frac{b}{\rho}\,\ln(1-\rho). \tag{6.37}$$

In turn, equation (1.8) for the limit shape is reduced to

$$\boxed{\omega_3^*(x) = \frac{b\,\mathrm{e}^{-\gamma x}}{\gamma(1-\rho\,\mathrm{e}^{-\gamma x})}}$$

Likewise, in Example 6.4 we get, similarly to (6.37),

$$\gamma^2 = \int_0^1 \frac{1}{(1-\rho u)^r}\,\mathrm{d}u = \frac{1-(1-\rho)^{1-r}}{\rho(1-r)} < \infty, \tag{6.38}$$

which holds for $0 < \rho < 1$ (and any $r \neq 1$). In the special case $\rho = 1$ the computation in (6.38) is modified as follows,

$$\gamma^2 = \int_0^1 \frac{1}{(1-s)^r}\,\mathrm{d}s = \frac{1}{1-r} < \infty,$$

provided that $r < 1$. Next, substituting (6.16) into (1.8), we get the limit shape

$$\boxed{\omega_4^*(x) = \frac{\mathrm{e}^{-\gamma x}}{\gamma\,(1-\rho\,\mathrm{e}^{-\gamma x})^r}}$$

In Example 6.5 we have, according to (2.12) and (6.18),

$$\gamma^2 = \int_0^1 (1+\rho u)^{m-1}\,\mathrm{d}u = \frac{(1+\rho)^m - 1}{\rho m},$$

and the limit shape (1.8) specializes to

$$\boxed{\omega_5^*(x) = \frac{\mathrm{e}^{-\gamma x}\,(1+\rho\,\mathrm{e}^{-\gamma x})^{m-1}}{\gamma}}$$

In particular, if $\rho = 1$, $m = 2$, then $\gamma^2 = \frac{3}{2}$.

Finally, in Example 6.6 the parameter γ may only be computed numerically, to which end it is more convenient to use the formula (2.12). The limit shape can then be plotted using the explicit equation (1.8) with the function $H_0(\mathrm{e}^{-\gamma x})$ evaluated from the formula (6.23),

$$\boxed{\omega_6^*(x) = \frac{r}{\gamma}\Bigl\{\gamma x - \ln\rho + \ln\bigl(-\ln(1-\rho\,\mathrm{e}^{-\gamma x})\bigr)\Bigr\}}$$

For instance, taking $r = 1$, $\rho = 0.5$ we computed[7] $\gamma \doteq 0.532202$, with the corresponding limit shape shown in Fig. 3a. For a comparison, we also plotted the limit shape with parameters $r = 1$, $\rho = 1$, giving $\gamma \doteq 0.853636$ (see Fig. 3b).

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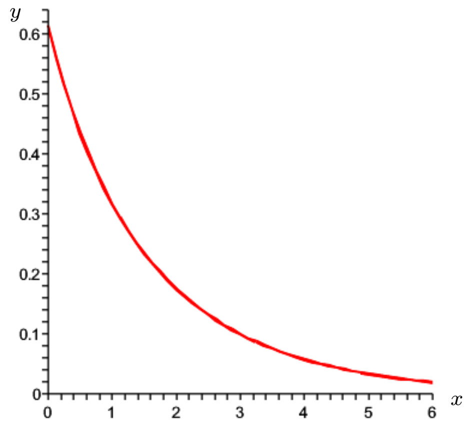
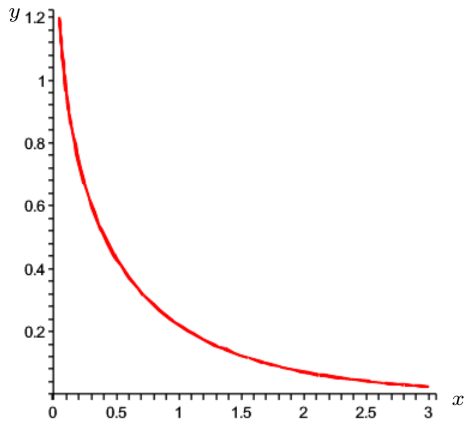
[7] Numerical computations and graphical outputs were obtained using `Maple™`.



(a)



(b)

Figure 3: The limit shape $y = \omega^*(x)$ in Example 6.6, with the function $H_0(u)$ given by equation (6.23): (a) $r = 1$, $\rho = 0.5$ ($\gamma \doteq 0.532202$); (b) $r = \rho = 1$ ($\gamma \doteq 0.853636$).

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