Limit shape of random convex polygonal lines: Even more universality

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To the memory of Yu. V. Prokhorov

Abstract

The paper is concerned with the limit shape (under some probability measure) of convex polygonal lines with vertices on $\mathbb{Z}_+^2$, starting at the origin and with the right endpoint $n = (n_1, n_2) \to \infty$. In the case of the uniform measure, an explicit limit shape $\gamma^* := \{(x_1, x_2) \in \mathbb{R}_+^2 : \sqrt{1-x_1} + \sqrt{x_2} = 1\}$ was found independently by Vershik [A.M. Vershik, The limit shape of convex lattice polygons and related topics, Funct. Anal. Appl. 28 (1994) 13–20], Bárány [I. Bárány, The limit shape of convex lattice polygons, Discrete Comput. Geom. 13 (1995) 279–295], and Sinai [Ya.G. Sinaĭ, Probabilistic approach to the analysis of statistics for convex polygonal lines, Funct. Anal. Appl. 28 (1994) 108–113]. Recently, Bogachev and Zarbaliev [L.V. Bogachev, S.M. Zarbaliev, Universality of the limit shape of convex lattice polygonal lines, Ann. Probab. 39 (2011) 2271–2317] proved that the limit shape γ^* is universal for a certain parametric family of multiplicative probability measures generalizing the uniform distribution. In the present work, the universality result is extended to a much wider class of multiplicative measures, including (but not limited to) analogs of the three meta-types of decomposable combinatorial structures — multisets, selections, and assemblies. This result is in sharp contrast with the one-dimensional case where the limit shape of Young diagrams associated with integer partitions heavily depends on the distributional type.

Keywords: Convex lattice polygonal line; Limit shape; Multiplicative measures; Local limit theorem; Möbius inversion formula; Generating function; Cumulants

2010 MSC: Primary 52A22; Secondary 05A17, 05D40, 60F05, 60G50

Contents

1. Introduction

In this paper, a *convex lattice polygonal line* $\varGamma$ is understood as a piecewise linear continuous path on the plane starting at the origin $0 = (0, 0)$, with vertices on the two-dimensional integer lattice $\mathbb{Z}^2$ and such that the inclination of its consecutive edges is strictly increasing, staying between 0 and $\pi/2$ (clearly, any such $\varGamma$ lies within the first coordinate quadrant). Let $\varPi$ be the set of all convex lattice polygonal lines *with finitely many edges*, and denote by $\varPi_n \subset \varPi$ the subset of polygonal lines $\varGamma \in \varPi$ with the right endpoint $\xi_\varGamma = (\xi_1, \xi_2)$ *fixed* at $n = (n_1, n_2) \in \mathbb{Z}^2_+ := \{(k_1, k_2) \in \mathbb{Z}^2 \colon k_j \ge 0\}$.

If each space $\varPi_n$ is endowed with a probability measure P_n, respectively (e.g., a uniform measure making all $\varGamma \in \varPi_n$ equiprobable), then one can speak of *random polygonal lines*, and it is of interest to study their asymptotic statistics as $n \to \infty$ (say, assuming that $n_2/n_1 \to c \in (0, \infty)$). In particular, the *limit shape* of random polygonal lines, whenever it exists, is defined as a planar curve γ^* such that, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P_n\{\varGamma \in \varPi_n \colon d(\tilde{\varGamma}_n, \gamma^*) \le \varepsilon\} = 1, \tag{1.1}$$

where $\tilde{\varGamma}_n := \mathfrak{s}_n(\varGamma)$, with a suitable scaling transform $\mathfrak{s}_n \colon \mathbb{R}^2 \to \mathbb{R}^2$, and $d(\cdot, \cdot)$ is some metric on the path space, for instance induced by the Hausdorff distance between compact sets,

$$d_{\mathcal{H}}(A, B) := \max\Bigl\{\max_{x\in A}\min_{y\in B}|x-y|,\ \max_{y\in B}\min_{x\in A}|x-y|\Bigr\}, \tag{1.2}$$

where $|\cdot|$ is the Euclidean vector norm in $\mathbb{R}^2$.

Remark 1.1. By definition, for a polygonal line $\varGamma \in \varPi_n$ the vector sum of its consecutive edges equals $n = (n_1, n_2)$; due to the convexity property, the order of parts in the sum is determined uniquely. Hence, any such $\varGamma$ represents a (two-dimensional) integer partition of $n \in \mathbb{Z}^2_+$ which is *strict* (i.e., without proportional parts; see [19, §3]). Let us remark that for ordinary (one-dimensional) integer partitions the limit shape problem is set out differently, in terms of the associated *Young diagrams* [20, 4, 22].

The limit shape and its very existence may depend on the family of probability laws P_n. With respect to the *uniform* distribution on $\varPi_n$, the problem was solved independently by Vershik [19], Bárány [3] and Sinai [16], who showed that under the natural scaling

$$\mathfrak{s}_n \colon (x_1, x_2) \mapsto (n_1^{-1}x_1, n_2^{-1}x_2), \qquad n = (n_1, n_2),\ \ n_1, n_2 > 0, \tag{1.3}$$

and with respect to the Hausdorff metric $d_{\mathcal{H}}$, the limit (1.1) holds with the limit shape γ^* given by a parabola arc

$$\sqrt{1-x_1}+\sqrt{x_2}=1, \qquad 0\le x_1,x_2\le 1. \tag{1.4}$$

Recently, Bogachev and Zarbaliev [6, 7] considered the limit shape problem for a more general class of "multiplicative" measures $\{P_n\}$ of the form

$$P_n(\varGamma):=\frac{b(\varGamma)}{B_n}, \qquad \varGamma\in\varPi_n, \tag{1.5}$$

with

$$b(\varGamma):=\prod_{e_i\in\varGamma} b_{k_i}, \qquad B_n:=\sum_{\varGamma\in\varPi_n} b(\varGamma), \tag{1.6}$$

where the product is over all edges e_i of $\varGamma\in\varPi_n$, index k_i equals the number of lattice points on the edge e_i except its left endpoint, and $\{b_k\}$ is a given nonnegative real sequence. Specifically, it has been proved in [6, 7] that, under the scaling (1.3), the same limit shape (1.4) is valid for a parametric class of measures $P_n=P_n^{(r)}$ $(0<r<\infty)$ with the coefficients[1]

$$b_k=b_k^{(r)}:=\binom{r+k-1}{k}=\frac{r(r+1)\cdots(r+k-1)}{k!}. \tag{1.7}$$

This result has provided the first evidence in support of a conjecture on the *limit shape universality*, put forward independently by Vershik [19, Remark 2, p. 20][2] and Prokhorov [15]. The goal of the present paper is to show that the limit shape γ^* given by (1.4) is universal in a much wider class of probability measures of the multiplicative form (1.5), (1.6). For instance, along with the uniform measure on $\varPi_n$ this class contains the uniform measure on the subset $\breve{\varPi}_n\subset\varPi_n$ of polygonal lines that do not have any lattice points other than vertices. More generally, measures covered by our method include (but are not limited to) analogs of the three classical meta-types of decomposable combinatorial structures — multisets, selections, and assemblies [1, 2] (see examples in Section 6 below).

Remark 1.2. It should be stressed that our universality result is in sharp contrast with the one-dimensional case, where the limit shape of random Young diagrams heavily depends on the distributional type [4, 10, 20, 22]. Thus, the limit shape of (strict) vector partitions is a relatively "soft" property; such a distinction is essentially due to the different ways of geometrization used in the two models (i.e., convex polygonal lines vs. Young diagrams), resulting in similar but not identical functionals responsible for the limit shape (cf. [4, Sec. 1.1]).

Let us state our result more precisely. Using the *tangential parameterization* of convex paths introduced in [7, §A.1], consider the scaled polygonal line $\tilde{\varGamma}_n=\mathfrak{s}_n(\varGamma)$ (see (1.3)) and let $\tilde{\xi}_n(t)$ denote the right endpoint of the part of $\tilde{\varGamma}$ with the tangent slope (where it exists) not exceeding $t\in[0,\infty]$. Similarly, the tangential parameterization of the parabola arc γ^* (see (1.4)) is given by[3]

$$g^*(t)=\left(\frac{t^2+2t}{(1+t)^2},\frac{t^2}{(1+t)^2}\right), \qquad 0\le t\le\infty, \tag{1.8}$$

[1] Note that for $r=1$ the formula (1.7) gives $b_k\equiv 1$, which implies that $b(\varGamma)=1$ for any $\varGamma\in\varPi_n$ and hence the measure (1.5) is reduced to the uniform distribution on the space $\varPi_n$.

[2] Page reference is given to the English translation of [19].

[3] It is easy to check that the coordinate functions $(g_1^*(t),g_2^*(t))$ in (1.8) satisfy the equation (1.4) (and therefore parametrically define the curve γ^*) and, furthermore, $g_2^{*\prime}(t)/g_1^{*\prime}(t)\equiv t$, so that the parameter t has the meaning of the tangent slope at the corresponding point on the curve, as required.

with $g^*(\infty) := \lim_{t\to\infty} g^*(t) = (1,1)$. Then the *tangential distance* between $\tilde{\Gamma}_n$ and γ^* is defined as

$$d_{\mathcal{T}}(\tilde{\Gamma}_n, \gamma^*) := \sup_{0\le t\le\infty} |\tilde{\xi}_n(t) - g^*(t)|. \tag{1.9}$$

It is known [7, §A.1] that *the Hausdorff distance* $d_{\mathcal{H}}$ (see (1.2)) *is dominated by the tangential distance* $d_{\mathcal{T}}$.

A loose formulation of our result about the universality of the limit shape is as follows.[4]

Theorem 1.1. *Suppose that the family of measures P_n on the respective spaces $\varPi_n$ is defined via the multiplicative formulas* (1.5), (1.6) *with the coefficients $\{b_k\}$ satisfying some mild technical conditions expressed in terms of the power series expansion of the function $u \mapsto \ln\bigl(\sum_{k=0}^{\infty} b_k u^k\bigr)$. Then, under the scaling* (1.3)*, for any $\varepsilon > 0$*

$$\lim_{n\to\infty} P_n\{\varGamma \in \varPi_n \colon d_{\mathcal{T}}(\tilde{\varGamma}_n, \gamma^*) \le \varepsilon\} = 1.$$

Remark 1.3. Universality of the limit shape γ^* has its boundaries: as has been demonstrated by Bogachev and Zarbaliev [5, 8], any C^3-smooth, strictly convex curve γ starting at the origin may serve as the limit shape with respect to a suitable family of multiplicative probability measures $P_n = P_n^{\gamma}$ on $\varPi_n$.

Following [6, 7] our proof employs an elegant probabilistic approach based on randomization and conditioning (see [1, 2]) first used in the polygonal context by Sinai [16]. The idea is to randomize the right endpoint $\xi_\varGamma$ of the polygonal line $\varGamma$, originally fixed at $n = (n_1, n_2)$, by introducing a probability measure Q_z on the space $\varPi = \bigcup_n \varPi_n$ (conveniently depending on an auxiliary "free" parameter $z = (z_1, z_2)$, $0 < z_j < 1$), such that for each $n \in \mathbb{Z}_+^2$ the measure P_n on $\varPi_n$ is recovered as the conditional distribution $P_n(\cdot) = Q_z(\cdot\,|\,\varPi_n)$. By virtue of the multiplicativity of P_n (see (1.5), (1.6)), Q_z may be constructed as a *product measure*, under which the coefficients $\{k_i\}$ in (1.6) become independent (although not identically distributed) random variables, so that $\xi_\varGamma$ is represented as a sum of independent vectors. Thus the asymptotics of the probability $Q_z(\varPi_n) = Q_z\{\xi_\varGamma = n\}$, needed in order to return from Q_z to P_n, can be obtained by proving the corresponding (two-dimensional) local limit theorem. Let us point out that we find it more convenient to calibrate the parameter z from the asymptotic equation $E_z(\xi_\varGamma) = n\,(1 + o(1))$ as $n \to \infty$, rather than from the exact relation $E_z(\xi_\varGamma) = n$; however, this necessitates obtaining a refined asymptotic bound on the error term $E_z(\xi_\varGamma) - n$. Last but not least, the main technical novelty that has allowed us to extend and enhance the argumentation of [7] to a much more general setting considered here is that we work with *cumulants* rather than moments (see Section 2.2), which proves extremely efficient throughout.

Layout. The rest of the paper is organized as follows. In Section 2, we define the families of measures Q_z and P_n. In Section 3, suitable values of the parameter $z = (z_1, z_2)$ are chosen (Theorem 3.2), which implies convergence of "expected" polygonal lines to the limit curve γ^* (Theorems 3.3 and 3.4). Refined first-order moment asymptotics are obtained in Section 3.3 (Theorem 3.6), while higher-order moment sums are analyzed in Section 4. Most of Section 5 is devoted to the proof of the local limit theorem (Theorem 5.1). Finally, the limit shape results, with respect to both Q_z and P_n, are proved in Section 5.4 (Theorems 5.5 and 5.6).

[4] For an exact statement and its proof, see Theorem 5.6 in Section 5.4 below.

Some general notation. We denote $\mathbb{Z}_+ := \{k \in \mathbb{Z}\colon k \ge 0\}$, $\mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$, and similarly $\mathbb{R}_+ := \{x \in \mathbb{R}\colon x \ge 0\}$, $\mathbb{R}_+^2 := \mathbb{R}_+ \times \mathbb{R}_+$. The notation $\#(\cdot)$ stands for the number of elements in a set. The symbol $\lfloor x \rfloor := \max\{k \in \mathbb{Z}\colon k \le x\}$ denotes the (floor) integer part of $x \in \mathbb{R}$. The real part of a complex number $s = \sigma + \mathrm{i}t \in \mathbb{C}$ is denoted $\Re(s) = \sigma$. For a (row-)vector $x = (x_1, x_2) \in \mathbb{R}^2$, its Euclidean norm is defined as $|x| := \sqrt{x_1^2 + x_2^2}$, and $\langle x, y\rangle := x\,y^\top = x_1 y_1 + x_2 y_2$ is the corresponding inner product of vectors $x, y \in \mathbb{R}^2$, where $y^\top = \binom{y_1}{y_2}$ is the transpose of $y = (y_1, y_2)$. More generally, $A^\top = (a_{ji})$ is the transpose of matrix $A = (a_{ij})$. The matrix norm induced by the vector norm $|\cdot|$ is defined by $\|A\| := \sup_{|x|=1}|xA|$. For $x = (x_1, x_2) \in \mathbb{Z}_+^2$ and $z = (z_1, z_2) \in \mathbb{R}_+^2$ with $z_1, z_2 > 0$, we use the multi-index notation $z^x := z_1^{x_1} z_2^{x_2}$. The gamma function is denoted $\Gamma(s) = \int_0^\infty u^{s-1}\,\mathrm{e}^{-u}\,\mathrm{d}u$, and $\zeta(s) = \sum_{k=1}^\infty k^{-s}$ is the Riemann zeta function.

Throughout the paper, the notation $n \to \infty$ (with $n = (n_1, n_2) \in \mathbb{Z}_+^2$) is understood as $n_1, n_2 \to \infty$ in such a way that the ratio n_2/n_1 stays bounded, that is, $c_* \le n_2/n_1 \le c^*$ with some constants $0 < c_* \le c^* < \infty$. The asymptotic relation $x_n \asymp y_n$ between real-valued sequences $\{x_n\}$ and $\{y_n\}$ ($n \in \mathbb{Z}_+^2$) signifies that $0 < \liminf_{n\to\infty} x_n/y_n \le \limsup_{n\to\infty} x_n/y_n < \infty$, whereas $x_n \sim y_n$ is a standard shorthand for $\lim_{n\to\infty} x_n/y_n = 1$. Thus, the limit $n \to \infty$ defined above can itself be characterized via the asymptotic condition $n_1 \asymp n_2$; in particular, this implies that $n_1 \asymp |n|$, $n_2 \asymp |n|$, where $|n| = \sqrt{n_1^2 + n_2^2} \to \infty$.

2. Probability measures on spaces of convex polygonal lines

2.1. Global measure Q_z and conditional measure P_n

2.1.1. Encoding of polygonal lines. Let $\mathcal{X} \subset \mathbb{Z}_+^2$ be the subset of integer vectors with co-prime coordinates,

$$\mathcal{X} := \{x = (x_1, x_2) \in \mathbb{Z}_+^2\colon\ \gcd(x_1, x_2) = 1\}, \tag{2.1}$$

where "gcd" stands for *greatest common divisor*. Note that the set $\mathbb{Z}_+^2$ can be viewed as an integer cone (i.e., with nonnegative integer multipliers) generated by $\mathcal{X}$ as a base; more precisely, $\mathbb{Z}_+^2$ is a disjoint union of the multiples of $\mathcal{X}$,

$$\mathbb{Z}_+^2 = \bigsqcup_{k=0}^{\infty} k\mathcal{X}. \tag{2.2}$$

That is, for each nonzero $y \in \mathbb{Z}_+^2$ there are unique $x \in \mathcal{X}$ and $k \in \mathbb{N}$ such that $y = kx$.

Let $\Phi := (\mathbb{Z}_+)^{\mathcal{X}}$ be the space of functions on $\mathcal{X}$ with nonnegative integer values, and consider the subspace of functions with *finite support*, $\Phi_0 := \{\nu \in \Phi\colon \#(\operatorname{supp}\nu) < \infty\}$, where $\operatorname{supp}\nu := \{x \in \mathcal{X}\colon \nu(x) > 0\}$. It is easy to see that the space Φ_0 is in one-to-one correspondence with the space $\varPi = \bigcup_{n\in\mathbb{Z}_+^2} \varPi_n$ of all (finite) convex lattice polygonal lines. Indeed, given a configuration $\nu = \{\nu(x)\} \in \Phi_0$, each $x \in \mathcal{X}$ specifies the *direction* of a potential edge, only utilized if $x \in \operatorname{supp}\nu$, in which case the value $\nu(x) = k > 0$ specifies the *scaling factor*, altogether yielding a vector edge kx; finally, assembling (a finite set of) all such edges into a polygonal line is uniquely determined by fixation of the starting point at the origin and the convexity property. Conversely, via the same interpretation of vector edges it is evident, in view of the decomposition (2.2), that any finite polygonal line $\varGamma \in \varPi$ determines uniquely a finitely supported configuration $\nu \in \Phi_0$. Let us point out that the case $\nu(x) \equiv 0$ corresponds to the "trivial" polygonal line $\varGamma_\emptyset$ with no edges (and with coinciding endpoints).

Under the association $\Pi \ni \Gamma \leftrightarrow \nu \in \Phi_0$ described above, the vector

$$\xi \equiv \xi_\Gamma := \sum_{x\in\mathcal{X}} x\nu(x) \tag{2.3}$$

has the meaning of the *right endpoint* of the corresponding polygonal line Γ. In particular, the space Π_n ($n \in \mathbb{Z}_+^2$) is identified as $\Pi_n = \{\Gamma \in \Pi\colon \xi_\Gamma = n\}$.

2.1.2. Family of multiplicative measures Q_z. Let $b_0, b_1, b_2, \dots$ be a sequence of nonnegative numbers such that $b_0 > 0$ (without loss of generality, we put $b_0 = 1$) and *not all b_k vanish for* $k \ge 1$, and assume that the corresponding (ordinary) generating function

$$\beta(u) := 1 + \sum_{k=1}^{\infty} b_k u^k, \qquad u \in \mathbb{C}, \tag{2.4}$$

is finite for $|u| < 1$ (i.e., the radius of convergence of the power series (2.4) is not smaller than 1). Let us now define a family of probability measures Q_z on the space $\Phi = \mathbb{Z}_+^{\mathcal{X}}$, indexed by the parameter $z = (z_1, z_2) \in (0,1) \times (0,1)$, as the distribution of a random field $\nu = \{\nu(x)\}_{x\in\mathcal{X}}$ with mutually independent values and marginal distributions

$$Q_z\{\nu(x) = k\} = \frac{b_k z^{kx}}{\beta(z^x)}, \qquad k = 0,1,2,\dots \qquad (x \in \mathcal{X}). \tag{2.5}$$

Lemma 2.1. *For each $z \in (0,1)^2$, the condition*

$$\tilde\beta(z) := \prod_{x\in\mathcal{X}} \beta(z^x) < \infty \tag{2.6}$$

is necessary and sufficient in order that $Q_z(\Phi_0) = 1$. Furthermore, if $\beta(u)$ is finite for all $|u| < 1$ then the condition (2.6) *is satisfied.*

Proof. According to (2.5), $Q_z\{\nu(x) > 0\} = 1 - \beta(z^x)^{-1}$ ($x \in \mathcal{X}$). Hence, Borel–Cantelli's lemma implies that $Q_z\{\nu \in \Phi_0\} = 1$ if and only if $\sum_{x\in\mathcal{X}} \bigl(1 - \beta(z^x)^{-1}\bigr) < \infty$. In turn, the latter inequality is equivalent to (2.6).

To prove the second statement, observe using (2.4) that

$$\ln \tilde\beta(z) = \sum_{x\in\mathcal{X}} \ln \beta(z^x) \le \sum_{x\in\mathcal{X}} \bigl(\beta(z^x) - 1\bigr) = \sum_{k=1}^{\infty} b_k \sum_{x\in\mathcal{X}} z^{kx}. \tag{2.7}$$

Furthermore, for any $k \ge 1$

$$\begin{aligned}\sum_{x\in\mathcal{X}} z^{kx} &\le \sum_{x_1=1}^{\infty} z_1^{kx_1} + \sum_{x_1=0}^{\infty} z_1^{kx_1} \sum_{x_2=1}^{\infty} z_2^{kx_2}\\ &= \frac{z_1^k}{1 - z_1^k} + \frac{z_2^k}{(1 - z_1^k)(1 - z_2^k)} \le \frac{z_1^k}{1 - z_1} + \frac{z_2^k}{(1 - z_1)(1 - z_2)}.\end{aligned}$$

Substituting this into (2.7) and recalling (2.4), we obtain

$$\ln \tilde\beta(z) \le \frac{\beta(z_1)}{1 - z_1} + \frac{\beta(z_2)}{(1 - z_1)(1 - z_2)} < \infty,$$

which implies (2.6). $\square$

Lemma 2.1 ensures that a sample configuration of the random field $\nu(\cdot)$ belongs (Q_z-almost surely) to the space Φ_0, and therefore determines a *finite* polygonal line $\varGamma \in \varPi$. By the mutual independence of the random values $\nu(x)$ ($x \in \mathcal{X}$), the corresponding Q_z-probability is given by

$$Q_z(\varGamma) = \prod_{x\in\mathcal{X}} \frac{b_{\nu(x)}\, z^{x\nu(x)}}{\beta(z^x)} = \frac{b(\varGamma)\, z^{\xi}}{\tilde\beta(z)}, \qquad \varGamma \in \varPi, \tag{2.8}$$

where $\xi := \sum_{x\in\mathcal{X}} x\nu(x)$ (see the definition (2.3)) and

$$b(\varGamma) := \prod_{x\in\mathcal{X}} b_{\nu(x)} < \infty, \qquad \varGamma \in \varPi. \tag{2.9}$$

Remark 2.1. The infinite product in (2.9) contains only finitely many terms different from 1 (since $b_{\nu(x)} = b_0 = 1$ for $x \notin \operatorname{supp} \nu$).

In particular, for the trivial polygonal line $\varGamma_\emptyset \leftrightarrow \nu \equiv 0$ (see Section 2.1.1) the formula (2.8) yields

$$Q_z(\varGamma_\emptyset) = \tilde\beta(z)^{-1} > 0.$$

On the other hand, we have $Q_z(\varGamma_\emptyset) < 1$, since $\beta(u) > \beta(0) = 1$ for all $u > 0$ and hence, according to the definition (2.6), $\tilde\beta(z) > 1$ for any $z \in (0,1)^2$.

2.1.3. Conditional measure P_n. On the subspace $\varPi_n \subset \varPi$ of polygonal lines with the right endpoint fixed at $n \in \mathbb{Z}_+^2$, the measure Q_z ($z \in (0,1)^2$) induces the conditional distribution

$$P_n(\varGamma) := Q_z(\varGamma \,|\, \varPi_n) = \frac{Q_z(\varGamma)}{Q_z(\varPi_n)}, \qquad \varGamma \in \varPi_n. \tag{2.10}$$

The formula (2.10) is well defined as long as $Q_z(\varPi_n) > 0$, that is, there is at least one polygonal line $\varGamma \in \varPi_n$ with $b(\varGamma) > 0$ (see (2.8), (2.9)). A simple sufficient condition is as follows.

Lemma 2.2. *Suppose that $b_1 > 0$. Then $Q_z(\varPi_n) > 0$ for all $n \in \mathbb{Z}_+^2$ such that $n_1, n_2 > 0$.*

Proof. Observe that $n = (n_1, n_2) \in \mathbb{Z}_+^2$ (with $n_1, n_2 \ge 1$) can be represented as

$$(n_1, n_2) = (n_1 - 1, 1) + (1, n_2 - 1), \tag{2.11}$$

where both points $x^{(1)} = (n_1 - 1, 1)$ and $x^{(2)} = (1, n_2 - 1)$ belong to the set $\mathcal{X}$. Moreover, $x^{(1)} \ne x^{(2)}$ unless $n_1 = n_2 = 2$, in which case instead of (2.11) we can write $(2,2) = (1,0) + (1,2)$, where again $x^{(1)} = (1,0) \in \mathcal{X}$, $x^{(2)} = (1,2) \in \mathcal{X}$. If $\varGamma^* \in \varPi_n$ is a polygonal line with two edges determined by the values $\nu(x^{(1)}) = 1$, $\nu(x^{(2)}) = 1$ (and $\nu(x) = 0$ otherwise), then, according to the definition (2.8), $Q_z(\varPi_n) \ge Q_z(\varGamma^*) = b_1^2 z^n \tilde\beta(z)^{-1} > 0$. ☐

The parameter z may be dropped in the notation (2.10) due to the following key fact.

Lemma 2.3. *The measure P_n in* (2.10) *does not depend on z.*

Proof. If $\varPi_n \ni \varGamma \leftrightarrow \nu_\varGamma \in \Phi_0$ then $\xi_\varGamma = n$ (see (2.3)) and the formula (2.8) is reduced to

$$Q_z(\varGamma) = \frac{b(\varGamma)\, z^n}{\tilde\beta(z)}, \qquad \varGamma \in \varPi_n.$$

Accordingly, using (2.10) we get the expression (cf. (1.5))

$$P_n(\varGamma) = \frac{b(\varGamma)}{\sum_{\varGamma' \in \varPi_n} b(\varGamma')}, \qquad \varGamma \in \varPi_n, \tag{2.12}$$

which is z-free. ☐

2.2. Generating functions and cumulants

2.2.1. Cumulant expansions. Recalling the expansion (2.4) for the generating function $\beta(u)$ (with $\beta(0) = b_0 = 1$), consider the corresponding power series expansion of its logarithm,

$$\ln \beta(u) = \sum_{k=1}^{\infty} a_k u^k, \qquad u \in \mathbb{C}, \tag{2.13}$$

assuming that the series (2.13) is (absolutely) convergent for all $|u| < 1$. Here and below, $\ln s$ with $s \in \mathbb{C}$ means the principal branch of the logarithm specified by the value $\ln 1 = 0$.

Remark 2.2. On substituting the expansion (2.4) into (2.13), it is evident that $a_1 = b_1$; more generally, if $j^* := \min\{j \geq 1\colon a_j \neq 0\}$ and $k^* := \min\{k \geq 1\colon b_k > 0\}$ then $j^* = k^*$ and $a_{j^*} = b_{k^*} > 0$.

Under the measure Q_z defined in (2.5), the characteristic function $\varphi_{\nu(x)}(t) := E_z(\mathrm{e}^{\mathrm{i}t\nu(x)})$ of the random variable $\nu(x)$ ($x \in \mathcal{X}$) is given by

$$\varphi_{\nu(x)}(t) = \frac{\beta(z^x \mathrm{e}^{\mathrm{i}t})}{\beta(z^x)}, \qquad t \in \mathbb{R}. \tag{2.14}$$

[For notational simplicity, we suppress the dependence on z in the notation, which should cause no confusion.] Hence, with the help of (2.13) the (principal branch of the) logarithm of $\varphi_{\nu(x)}(t)$ is expanded as

$$\ln \varphi_{\nu(x)}(t) = \ln \beta(z^x \mathrm{e}^{\mathrm{i}t}) - \ln \beta(z^x) = \sum_{k=1}^{\infty} a_k (\mathrm{e}^{\mathrm{i}kt} - 1)\, z^{kx}, \qquad t \in \mathbb{R}. \tag{2.15}$$

For a generic random variable X, let $\varkappa_q = \varkappa_q[X]$ denote its *cumulants* of order $q \in \mathbb{N}$ (see [14, §3.12, p. 69]), defined by the formal identity in indeterminant t

$$\ln \varphi(t) = \sum_{q=1}^{\infty} \frac{(\mathrm{i}t)^q}{q!}\, \varkappa_q, \tag{2.16}$$

where $\varphi(t) = E(\mathrm{e}^{\mathrm{i}tX})$ is the characteristic function of X. By differentiating (2.16) at $t = 0$, it is easy to see (cf. [14, §3.14, Eq. (3.37), p. 71]) that

$$E(X) = \varkappa_1, \qquad \mathrm{Var}(X) = \varkappa_2. \tag{2.17}$$

Let us also point out the standard expressions for the next few central moments of X through the cumulants (see [14, §3.14, Eq. (3.38), p. 72]): if $X_0 := X - E(X)$ then

$$\begin{aligned} E(X_0^3) &= \varkappa_3, \\ E(X_0^4) &= \varkappa_4 + 3\varkappa_2^2, \\ E(X_0^5) &= \varkappa_5 + 10\varkappa_3\varkappa_2, \\ E(X_0^6) &= \varkappa_6 + 15\varkappa_4\varkappa_2 + 10\varkappa_3^2 + 15\varkappa_2^3. \end{aligned} \tag{2.18}$$

Let us now turn to the cumulants $\varkappa_q[\nu(x)]$ of the random variables $\nu(x)$ (under the probability distribution Q_z). The following simple lemma will be instrumental in our analysis.

Lemma 2.4. *The cumulants of* $\nu(x)$ $(x \in \mathcal{X})$ *are given by*

$$\varkappa_q[\nu(x)] = \sum_{k=1}^{\infty} k^q a_k z^{kx}, \qquad q \in \mathbb{N}. \tag{2.19}$$

Proof. Taylor expanding the exponential function in (2.15), we get

$$\ln \varphi_{\nu(x)}(t) = \sum_{k=1}^{\infty} a_k z^{kx} \sum_{q=1}^{\infty} \frac{(\mathrm{i}kt)^q}{q!} = \sum_{q=1}^{\infty} \frac{(\mathrm{i}t)^q}{q!} \sum_{k=1}^{\infty} k^q a_k z^{kx}, \tag{2.20}$$

where the interchange of the order of summation in the double series (2.20) is justified by its absolute convergence. Comparing the expansion (2.20) with the identity (2.16), we obtain the formulas (2.19) for the coefficients $\varkappa_q[\nu(x)]$. □

Lemma 2.4 allows us to obtain series representations for the cumulants of the components $\xi_j = \sum_{x\in\mathcal{X}} x_j\, \nu(x)$ of the random vector $\xi = (\xi_1, \xi_2)$ (see (2.3)). Namely, using the rescaling relation $\varkappa_q[cX] = c^q \varkappa_q[X]$ (see [14, §3.13, p. 70]) and the additivity property of the cumulants for independent summands (see [14, §7.18, pp. 201–202]), from (2.19) we get for $q \in \mathbb{N}$

$$\varkappa_q[\xi_j] = \sum_{x\in\mathcal{X}} x_j^q \varkappa_q[\nu(x)] = \sum_{x\in\mathcal{X}} x_j^q \sum_{k=1}^{\infty} k^q a_k z^{kx} \qquad (j = 1, 2). \tag{2.21}$$

In particular, the expected value and the variance of ξ_j are given by (see (2.17))

$$E_z(\xi_j) = \sum_{x\in\mathcal{X}} x_j \sum_{k=1}^{\infty} k a_k z^{kx},$$
$$\mathrm{Var}_z(\xi_j) = \sum_{x\in\mathcal{X}} x_j^2 \sum_{k=1}^{\infty} k^2 a_k z^{kx}.$$

2.2.2. Dirichlet series associated with $\ln \beta(u)$. For $s \in \mathbb{C}$ such that $\Re(s) =: \sigma > 0$, consider the Dirichlet series

$$A(s) := \sum_{k=1}^{\infty} \frac{a_k}{k^s}, \qquad A^+(\sigma) := \sum_{k=1}^{\infty} \frac{|a_k|}{k^\sigma}, \tag{2.22}$$

where $\{a_k\}$ are the coefficients in the power series expansion of $\ln \beta(u)$ (see (2.13)).

Although some of the coefficients $\{a_k\}$ may be negative, it turns out that the quantity $A(2) = \sum_{k=1}^{\infty} a_k k^{-2}$, whenever it is finite, cannot vanish or take a negative value.

Lemma 2.5. *If* $A^+(2) < \infty$ *then* $0 < A(2) < \infty$ *and the following integral formula holds*

$$A(2) = \int_0^1 u^{-1} \left(\int_0^u v^{-1} \ln \beta(v)\, \mathrm{d}v \right) \mathrm{d}u. \tag{2.23}$$

Proof. From (2.4) it is evident that $\ln \beta(u) > \ln 1 = 0$ for all $u \in (0, 1)$, and it readily follows that the double integral on right-hand side of (2.23) is positive (and possibly infinite). Furthermore, substituting the expansion (2.13) and integrating term by term (which is permissible for power series inside the interval of convergence), we obtain for $s \in (0,1)$

$$\int_0^s u^{-1}\left(\int_0^u v^{-1}\ln\beta(v)\,\mathrm{d}v\right)\mathrm{d}u = \int_0^s u^{-1}\sum_{k=1}^{\infty} a_k\left(\int_0^u v^{k-1}\,\mathrm{d}v\right)\mathrm{d}u$$
$$= \sum_{k=1}^{\infty}\frac{a_k}{k}\int_0^s u^{k-1}\,\mathrm{d}u = \sum_{k=1}^{\infty}\frac{a_k}{k^2}\,s^k. \qquad (2.24)$$

Passing here to the limit as $s \uparrow 1$ and applying to the right-hand side of (2.24) Abel's theorem on power series (see [17, §1.22, pp. 9–10]), we obtain the identity (2.23). □

Remark 2.3. The condition $A^+(2) < \infty$ and the quantity $A(2)$ will play a major role in our argumentation; in particular, $A(2)$ is involved in a suitable calibration of the "free" parameter $z = (z_1, z_2)$ in the definition (2.5) of the measure Q_z (see Section 3.1 below). However, some results (such as Theorem 3.6, Lemma 4.7 and Theorem 5.1) will require a stronger condition $A^+(1) < \infty$. Our main result on the limit shape under the measure P_n (see Theorem 5.6) is dependent on these statements, and therefore is stated and proved under the latter condition.

2.3. Auxiliary estimates for power-exponential sums

In what follows, we frequently encounter power-exponential sums of the form

$$S_q(t) := \sum_{k=1}^{\infty} k^{q-1}\mathrm{e}^{-tk}, \qquad t > 0. \qquad (2.25)$$

For the first few integer values of q, explicit expressions of $S_q(t)$ are easily available,

$$S_1(t) = \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}, \qquad S_2(t) = \frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^2}, \qquad S_3(t) = \frac{\mathrm{e}^{-t}\,(1+\mathrm{e}^{-t})}{(1-\mathrm{e}^{-t})^3}. \qquad (2.26)$$

The purpose of this subsection is to obtain estimates on $S_q(t)$ with any integer q.

Lemma 2.6. *For $q \in \mathbb{N}$, the function $S_q(t)$ admits the representation*

$$S_q(t) = \sum_{j=1}^{q} c_{j,q}\,\frac{\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j}, \qquad t > 0, \qquad (2.27)$$

with some constants $c_{j,q} > 0$ $(j = 1, \dots, q)$; in particular, $c_{1,q} = 1$ and $c_{q,q} = (q-1)!$.

Proof. For $q = 1$, the expression for $S_1(t)$ from (2.26) is a particular case of (2.27) with $c_{1,1} = 1$. Assume now that the expansion (2.27) is valid for some $q \ge 1$ (including the "boundary" values $c_{1,q} = 1$, $c_{q,q} = (q-1)!$). Then, differentiating the identities (2.25) and (2.27) with respect to t, we obtain

$$S_{q+1}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}S_q(t) = \sum_{j=1}^{q} c_{j,q}\left(\frac{j\,\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j} + \frac{j\,\mathrm{e}^{-t(j+1)}}{(1-\mathrm{e}^{-t})^{j+1}}\right)$$
$$= \sum_{j=1}^{q+1} c_{j,q+1}\,\frac{\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j},$$

where we set

$$c_{j,q+1} := \begin{cases} c_{1,q}, & j = 1, \\ j\, c_{j,q} + (j-1)\, c_{j-1,q}, & 2 \le j \le q, \\ q c_{q,q}, & j = q+1. \end{cases}$$

In particular, $c_{1,q+1} = c_{1,q} = 1$ and $c_{q+1,q+1} = q c_{q,q} = q(q-1)! = q!$. Thus, the formula (2.27) holds for $q+1$ and hence, by induction, for all $q \ge 1$. □

Lemma 2.7. (a) *For each $q \in \mathbb{N}$, there exists an absolute constant $\bar{c}_q > 0$ such that*

$$S_q(t) \le \frac{\bar{c}_q\, \mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^q}, \qquad t > 0. \tag{2.28}$$

(b) *Moreover,*

$$S_q(t) \sim \frac{(q-1)!}{t^q}, \qquad t \to 0. \tag{2.29}$$

Proof. (a) Observe that for $j = 1, \dots, q$ and all $t > 0$

$$\frac{\mathrm{e}^{-tj}}{(1-\mathrm{e}^{-t})^j} \le \frac{\mathrm{e}^{-t}}{(1-\mathrm{e}^{-t})^q}.$$

Substituting these inequalities into (2.27) and recalling that all $c_{j,q} > 0$, we obtain (2.28) with $\bar{c}_q := \sum_{j=1}^q c_{j,q}$.

(b) For each term in the expansion (2.27) we have $\mathrm{e}^{-tj}(1-\mathrm{e}^{-t})^{-j} \sim t^{-j}$ as $t \to 0$. Hence, the overall asymptotic behavior of $S_q(t)$ is determined by the term with $j = q$ and the corresponding coefficient $c_{q,q} = (q-1)!$ (see Lemma 2.6), and the formula (2.29) follows. □

The next general lemma can be used to obtain a simplified polynomial estimate for the right-hand side of the bound (2.28), which is sometimes convenient.

Lemma 2.8. *For any $q > 0$ and $\theta > 0$, there is a constant $C_q(\theta) > 0$ such that*

$$\frac{\mathrm{e}^{-\theta t}}{(1-\mathrm{e}^{-t})^q} \le C_q(\theta)\, t^{-q}, \qquad t > 0. \tag{2.30}$$

Proof. Set $f(t) := t^q\, \mathrm{e}^{-\theta t}(1-\mathrm{e}^{-t})^{-q}$ and note that

$$\lim_{t \downarrow 0} f(t) = 1, \qquad \lim_{t \to +\infty} f(t) = 0.$$

By continuity, the function $f(t)$ is bounded on $(0, \infty)$, and the inequality (2.30) follows. □

3. Asymptotics of the expectation

3.1. Calibration of the parameter z

Our aim in this section is to adjust the parameter $z = (z_1, z_2)$, as a suitable function of $n = (n_1, n_2)$, in such a way that under the corresponding measure Q_z the following asymptotic conditions are satisfied,

$$\lim_{n \to \infty} n_1^{-1} E_z(\xi_1) = \lim_{n \to \infty} n_2^{-1} E_z(\xi_2) = 1, \tag{3.1}$$

where $\xi_j = \sum_{x\in\mathcal{X}} x_j\,\nu(x)$ (see (2.3)) and E_z denotes expectation with respect to Q_z. Let us use the ansatz

$$z_j = \mathrm{e}^{-\alpha_j}, \qquad \alpha_j = \delta_j\, n_j^{-1/3} \qquad (j = 1, 2), \tag{3.2}$$

where the quantities $\delta_1, \delta_2 > 0$, possibly depending on n, are presumed to be bounded and separated from zero (i.e., $\delta_1, \delta_2 \asymp 1$ as $n \to \infty$). Hence, using the formula (2.21) with $q = 1$, we get (in vector form)

$$E_z(\xi) = \sum_{k=1}^{\infty} k a_k \sum_{x\in\mathcal{X}} x\,\mathrm{e}^{-k\langle\alpha,x\rangle}. \tag{3.3}$$

3.1.1. Evaluating sums over $\mathcal{X}$ via the Möbius inversion formula. Recall that the *Möbius function* $\mu\colon \mathbb{N} \to \{-1, 0, 1\}$ is defined as follows (see [11, §16.3, p. 304]): $\mu(1) := 1$, $\mu(m) := (-1)^d$ if m is a product of d different prime numbers, and $\mu(m) := 0$ otherwise; in particular, $|\mu(m)| \le 1$ for all $m \in \mathbb{N}$.

To deal with sums over the set $\mathcal{X}$ (see (2.1)), the following lemma will be instrumental.

Lemma 3.1. *Let $f\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ be a function such that $f(x)|_{x=(0,0)} = 0$ and for all $h > 0$*

$$F(h) := \sum_{x\in\mathbb{Z}_+^2} f(hx) < \infty. \tag{3.4}$$

Moreover, assume that

$$\sum_{k=1}^{\infty} F(hk) < \infty, \qquad h > 0. \tag{3.5}$$

Then the function

$$F^\sharp(h) := \sum_{x\in\mathcal{X}} f(hx), \qquad h > 0, \tag{3.6}$$

satisfies the identity

$$F^\sharp(h) = \sum_{m=1}^{\infty} \mu(m) F(mh), \qquad h > 0. \tag{3.7}$$

Proof. Recalling the decomposition (2.2) and using that $f(x)$ vanishes at the origin, observe from (3.4) and (3.6) that

$$F(h) = \sum_{m=1}^{\infty} F^\sharp(mh), \qquad h > 0. \tag{3.8}$$

Then the identity (3.7) follows by the Möbius inversion formula (see [11, §16.5, Theorem 270, p. 307]), provided that $\sum_{k,m} F^\sharp(kmh) < \infty$ ($h > 0$). Indeed, the latter condition is satisfied,

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} F^\sharp(kmh) = \sum_{k=1}^{\infty} F(kh) < \infty,$$

according to (3.8) and the hypothesis (3.5). This completes the proof. □

3.1.2. The basic parameterization.

Theorem 3.2. *Suppose that* $A^{+}(2) < \infty$ (*see* (2.22)), *and choose* δ_1, δ_2 *in* (3.2) *as follows*

$$\delta_1 = \kappa \tau^{1/3}, \qquad \delta_2 = \kappa \tau^{-1/3}, \tag{3.9}$$

where

$$\tau \equiv \tau_n := \frac{n_2}{n_1}, \qquad \kappa := \left(\frac{A(2)}{\zeta(2)}\right)^{1/3}. \tag{3.10}$$

Then the asymptotic conditions (3.1) *are satisfied.*

Remark 3.1. According to our convention about the limit $n \to \infty$ (see the end of the Introduction), we have $\tau \asymp 1$. Observe also that (3.2), (3.9) and (3.10) imply the scaling relations

$$\alpha_1^2 \alpha_2 n_1 = \alpha_1 \alpha_2^2 n_2 = \kappa^3, \qquad \alpha_2 = \alpha_1/\tau. \tag{3.11}$$

Proof of Theorem 3.2. Let us prove (3.1) for ξ_1 (the proof for ξ_2 is similar). Setting

$$f(x) := x_1 \mathrm{e}^{-\langle \alpha, x\rangle} = x_1 \mathrm{e}^{-\alpha_1 x_1 - \alpha_2 x_2}, \qquad x = (x_1, x_2) \in \mathbb{R}_+^2, \tag{3.12}$$

and following the notation (3.6) of Lemma 3.1, a projection of the equation (3.3) onto the first coordinate takes the form

$$E_z(\xi_1) = \sum_{k=1}^{\infty} k a_k \sum_{x \in \mathcal{X}} x_1 \mathrm{e}^{-\langle \alpha, kx\rangle} = \sum_{k=1}^{\infty} a_k \sum_{x \in \mathcal{X}} f(kx) = \sum_{k=1}^{\infty} a_k F^{\sharp}(k). \tag{3.13}$$

On the other hand, substituting (3.12) into (3.6) and using the expression (2.26) for $S_q(\cdot)$ with $q = 2$, we obtain

$$F(h) = h \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-h\alpha_1 x_1} \sum_{x_2=0}^{\infty} \mathrm{e}^{-h\alpha_2 x_2} = \frac{h\,\mathrm{e}^{-h\alpha_1}}{(1-\mathrm{e}^{-h\alpha_1})^2 (1-\mathrm{e}^{-h\alpha_2})}. \tag{3.14}$$

It is evident that $F(h)$ satisfies the condition (3.5), hence by Lemma 3.1 the function $F^{\sharp}(\cdot)$ (see (3.6)) can be expressed via the formula (3.7). Thus, substituting also (3.14), we can rewrite (3.13) as

$$E_z(\xi_1) = \sum_{k=1}^{\infty} a_k \sum_{m=1}^{\infty} \mu(m) F(km) = \sum_{k,m=1}^{\infty} \frac{k a_k m \mu(m)\,\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2 (1-\mathrm{e}^{-km\alpha_2})}. \tag{3.15}$$

Now, using the representation (3.15) we can obtain the asymptotics of $E_z(\xi_1)$ as $n \to \infty$. Recall that $\alpha_1 = \alpha_2 \tau$ (see (3.11)), where $\tau \equiv \tau_n \asymp 1$ (see (3.10) and Remark 3.1) and so $\tau \geq \tau_*$ for some $\tau_* > 0$ and all n large enough. Applying Lemma 2.8 twice (with $q = 2$, $\theta = 1/2$ and $q = 1$, $\theta = \tau_*/2$, respectively), we obtain, uniformly in $k, m \geq 1$,

$$\begin{aligned} \frac{\alpha_1^2 \alpha_2\,\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2 (1-\mathrm{e}^{-km\alpha_2})} &= \frac{\alpha_1^2\,\mathrm{e}^{-km\alpha_1/2}}{(1-\mathrm{e}^{-km\alpha_1})^2} \cdot \frac{\alpha_2\,\mathrm{e}^{-km\alpha_2\tau/2}}{1-\mathrm{e}^{-km\alpha_2}} \\ &\leq \frac{C_2(1/2)}{(km)^2} \cdot \frac{C_1(\tau_*/2)}{km} = \frac{O(1)}{k^3 m^3}, \qquad n \to \infty. \end{aligned} \tag{3.16}$$

Thus, remembering that $|\mu(m)| \le 1$, the general summand in the double sum (3.15), multiplied by $\alpha_1^2\alpha_2$, is bounded by $O(1)|a_k|\,k^{-2}m^{-2}$, which is a term of a convergent series due to the assumption $A^+(2)<\infty$. Hence, by Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty} \alpha_1^2\alpha_2 E_z(\xi_1) = \sum_{k,m=1}^{\infty} k a_k m\mu(m) \lim_{n\to\infty} \frac{\alpha_1^2\alpha_2\,\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2(1-\mathrm{e}^{-km\alpha_2})}$$

$$= \sum_{k=1}^{\infty}\frac{a_k}{k^2}\sum_{m=1}^{\infty}\frac{\mu(m)}{m^2} = \frac{A(2)}{\zeta(2)} \equiv \kappa^3, \tag{3.17}$$

according to the notation (3.10). Note that the identity

$$\sum_{m=1}^{\infty}\frac{\mu(m)}{m^s} = \frac{1}{\zeta(s)}, \tag{3.18}$$

used in (3.17) for $s=2$, readily follows by the Möbius inversion formula (3.7) with $F^\sharp(h) = h^{-s}$ and $F(h)=\sum_{m=1}^{\infty}(hm)^{-s} = h^{-s}\zeta(s)$ (cf. [11, §17.5, Theorem 287, p. 326]).

To complete the proof, it remains to notice that the limit (3.17) is equivalent to the first of the asymptotic conditions (3.1) due to the scaling relation $\alpha_1^2\alpha_2 = n_1^{-1}\kappa^3$ (see (3.11)). □

Assumption 3.1. Throughout the rest of the paper, we assume that $A^+(2)<\infty$ and the parameters z_1, z_2 are chosen according to the formulas (3.2), (3.9), (3.10). In particular, the measure Q_z becomes dependent on $n=(n_1,n_2)$, as well as the corresponding expected values.

3.2. The "expected" limit shape

Given $n=(n_1,n_2)\in\mathbb{Z}_+^2$ ($n_1,n_2>0$) and the ratio $\tau = n_2/n_1$ (see (3.10)), for a polygonal line $\varGamma\in\varPi$ and $t\in[0,\infty]$ let us denote by $\varGamma(t)\equiv\varGamma(t;\tau)$ *the piece of* $\varGamma$ *where the slope does not exceed* $t\tau$. In case all edges of $\varGamma$ have the slope bigger than $t\tau$, we set $\varGamma(t) := \varGamma_\emptyset$ (the trivial polygonal line, see Section 2.1.1).

Remark 3.2. The definition of $\varGamma(t)$ implies that under the scaling $\mathfrak{s}_n$ (see (1.3)) the scaled piece $\tilde{\varGamma}_n(t) := \mathfrak{s}_n(\varGamma(t))$ has the slope not bigger than t.

Consider the corresponding subset of $\mathcal{X}$ (see (2.1)),

$$\mathcal{X}(t)\equiv\mathcal{X}(t;\tau) := \{x=(x_1,x_2)\in\mathcal{X}\colon\ x_2/x_1\le t\tau\}, \qquad t\in[0,\infty]. \tag{3.19}$$

According to the association $\varPi\ni\varGamma\leftrightarrow\nu\in\Phi_0$ described in Section 2.1.1, for each $t\in[0,\infty]$ the piece $\varGamma(t)$ of $\varGamma$ is determined by a truncated configuration $\{\nu(x),\ x\in\mathcal{X}(t)\}$, hence its right endpoint $\xi(t)=(\xi_1(t),\xi_2(t))$ is given by

$$\xi(t) = \sum_{x\in\mathcal{X}(t)} x\nu(x), \qquad t\in[0,\infty]. \tag{3.20}$$

In particular, $\mathcal{X}(\infty)=\mathcal{X},\ \xi(\infty)=\xi$ (see (2.3)). Similarly to (3.3), we have

$$E_z[\xi(t)] = \sum_{k=1}^{\infty} k a_k \sum_{x\in\mathcal{X}(t)} x\,\mathrm{e}^{-k\langle\alpha,x\rangle}, \qquad t\in[0,\infty]. \tag{3.21}$$

Recall that the vector-function $g^*(t) = (g_1^*(t), g_2^*(t))$ is defined in (1.8).

Theorem 3.3. *Under Assumption* 3.1*, for each* $t \in [0, \infty]$

$$\lim_{n\to\infty} n_j^{-1} E_z[\xi_j(t)] = g_j^*(t) \qquad (j = 1, 2). \tag{3.22}$$

Proof. Let $j = 1$ (the case $j = 2$ is considered in a similar manner). Theorem 3.2 implies that the claim (3.22) holds for $t = \infty$ (with $\xi_1(\infty) = \xi_1$). Thus, noting from (1.8) that $g_1^*(\infty) = 1$ and $1 - g_1^*(t) = (1+t)^{-2}$, we can rewrite (3.22) (with $j = 1$) in the form

$$\lim_{n\to\infty} n_1^{-1} E_z[\xi_1 - \xi_1(t)] = (1+t)^{-2}. \tag{3.23}$$

Now, like in the proof of Theorem 3.2 (cf. (3.3), (3.13) and (3.15)), from (3.21) we have

$$\begin{aligned} E_z[\xi_1 - \xi_1(t)] &= \sum_{k=1}^{\infty} k a_k \sum_{x \in \mathcal{X} \setminus \mathcal{X}(t)} x_1 \mathrm{e}^{-k\alpha_1 x_1} \mathrm{e}^{-k\alpha_2 x_2} \\ &= \sum_{k,m=1}^{\infty} k a_k m \mu(m) \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1} \sum_{x_2=\hat{x}_2+1}^{\infty} \mathrm{e}^{-km\alpha_2 x_2} \\ &= \sum_{k,m=1}^{\infty} \frac{k a_k m \mu(m)}{1 - \mathrm{e}^{-km\alpha_2}} \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km(\alpha_1 x_1 + \alpha_2(\hat{x}_2+1))}, \end{aligned} \tag{3.24}$$

where $\hat{x}_2 = \hat{x}_2(t) := \lfloor t\tau x_1 \rfloor$, so that

$$0 < \hat{x}_2 + 1 - t\tau x_1 \le 1. \tag{3.25}$$

It is natural to expect that the internal sum in (3.24) may be well approximated by replacing $\hat{x}_2 + 1$ with $t\tau x_1$ and thus reducing it to $S_2(km(\alpha_1 + \alpha_2 t\tau))$ (see the notation (2.25) with $q = 2$). More precisely, recalling that $\alpha_2 \tau = \alpha_1$ (see (3.11)), we obtain the representation

$$\sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km(\alpha_1 x_1 + \alpha_2(\hat{x}_2+1))} = S_2(km\alpha_1(1+t)) - R_n(t; km), \tag{3.26}$$

with

$$R_n(t; km) := \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1(1+t)} \left(1 - \mathrm{e}^{-km\alpha_2(\hat{x}_2+1-t\tau x_1)}\right). \tag{3.27}$$

By the expression (2.26) for $S_2(\cdot)$ we have

$$0 \le S_2(km\alpha_1(1+t)) = \frac{\mathrm{e}^{-km\alpha_1(1+t)}}{(1 - \mathrm{e}^{-km\alpha_1(1+t)})^2} \le \frac{\mathrm{e}^{-km\alpha_1}}{(1 - \mathrm{e}^{-km\alpha_1})^2}. \tag{3.28}$$

On the other hand, applying the upper inequality (3.25) under the second exponent in (3.27) and replacing $1 + t$ by 1 under the first exponent, we obtain the estimates

$$\begin{aligned} 0 \le R_n(t; km) &\le (1 - \mathrm{e}^{-km\alpha_2}) \sum_{x_1=1}^{\infty} x_1 \mathrm{e}^{-km\alpha_1 x_1} \\ &= \frac{(1 - \mathrm{e}^{-km\alpha_2})\, \mathrm{e}^{-km\alpha_1}}{(1 - \mathrm{e}^{-km\alpha_1})^2} && (3.29) \\ &\le \frac{\mathrm{e}^{-km\alpha_1}}{(1 - \mathrm{e}^{-km\alpha_1})^2}. && (3.30) \end{aligned}$$

On substituting (3.26) back into (3.24), from the bounds (3.28) and (3.30) it is evident that we can repeat the arguments used in the proof of Theorem 3.2 (see (3.16)) and thus pass to the limit in (3.24) by Lebesgue's dominated convergence theorem, giving

$$\lim_{n\to\infty} \alpha_1^2\alpha_2 E_z[\xi_1 - \xi_1(t)] = \sum_{k,m=1}^{\infty} k a_k m\mu(m) \lim_{n\to\infty} \frac{\alpha_1^2\alpha_2 \big(S_2(km\alpha_1(1+t)) - R_n(t;km)\big)}{1-\mathrm{e}^{-km\alpha_2}}. \tag{3.31}$$

By virtue of the equality in (3.28) we easily find

$$\lim_{n\to\infty} \frac{\alpha_1^2\alpha_2 S_2(km\alpha_1(1+t))}{1-\mathrm{e}^{-km\alpha_2}} = \lim_{n\to\infty} \frac{\alpha_1^2\alpha_2\,\mathrm{e}^{-km\alpha_1(1+t)}}{(1-\mathrm{e}^{-km\alpha_2})(1-\mathrm{e}^{-km\alpha_1(1+t)})^2} = \frac{1}{k^3m^3(1+t)^2}. \tag{3.32}$$

Furthermore, the estimate (3.29) implies

$$\lim_{n\to\infty} \frac{\alpha_1^2\alpha_2 R_n(t;km)}{1-\mathrm{e}^{-km\alpha_2}} \le \lim_{n\to\infty} \frac{\alpha_1^2\alpha_2\,\mathrm{e}^{-km\alpha_1}}{(1-\mathrm{e}^{-km\alpha_1})^2} = 0. \tag{3.33}$$

Hence, substituting (3.32) and (3.33) into (3.31), we obtain (cf. (3.17))

$$\lim_{n\to\infty} \alpha_1^2\alpha_2 E_z[\xi_1 - \xi_1(t)] = \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2}\,(1+t)^{-2} = \kappa^3(1+t)^{-2}. \tag{3.34}$$

Finally, recalling that $\alpha_1^2\alpha_2 = n_1^{-1}\kappa^3$ (see (3.11)), the limit (3.34) is reduced to (3.23). □

3.2.1. Enhancement: uniform convergence. There is a stronger version of Theorem 3.3.

Theorem 3.4. *The convergence in* (3.22) *is uniform in* $t \in [0,\infty]$, *that is,*

$$\lim_{n\to\infty} \sup_{0\le t\le\infty} \big|n_j^{-1}E_z[\xi_j(t)] - g_j^*(t)\big| = 0 \qquad (j=1,2).$$

We use the following simple criterion of uniform convergence proved in [7, Lemma 4.3].

Lemma 3.5. *Let* $\{f_n(t)\}$ *be a sequence of nondecreasing functions on a finite interval* $[a,b]$, *such that, for each* $t \in [a,b]$, $\lim_{n\to\infty} f_n(t) = f(t)$, *where* $f(t)$ *is a continuous* (*nondecreasing*) *function on* $[a,b]$. *Then the convergence* $f_n(t) \to f(t)$ *as* $n \to \infty$ *is uniform on* $[a,b]$.

Proof of Theorem 3.4. Suppose that $j=1$ (the case $j=2$ is handled similarly). Note that for each $n=(n_1,n_2)$ (with $n_1>0$) the function

$$f_n(t) := n_1^{-1}E_z[\xi_1(t)] = \frac{1}{n_1}\sum_{x\in\mathcal{X}(t)} x_1 E_z[\nu(x)], \qquad t\in[0,\infty],$$

is nondecreasing in t, in view of the definition (3.19) of the sets $\mathcal{X}(t)$. Therefore, by Lemma 3.5 the convergence in (3.22) is uniform on any finite interval $[0,t_0]$.

For large t, by the triangle inequality we get

$$|n_1^{-1}E_z[\xi_1(t)] - g_1^*(t)| \le |n_1^{-1}E_z(\xi_1) - 1| + |g_1^*(t)-1| + n_1^{-1}E_z[\xi_1-\xi_1(t)] \tag{3.35}$$

(in the last term, $\xi_1 \geq \xi_1(t)$ for all $t \geq 0$). We know that $\lim_{n\to\infty} n_1^{-1} E_z(\xi_1) = 1$ by Theorem 3.3 and $\lim_{t\to\infty} g_1^*(t) = 1$ (see (1.8)); thus, in view of (3.35) it remains to show that for any $\varepsilon > 0$ there is a $t_0 = t_0(\varepsilon)$ such that, for all large enough $n = (n_1, n_2)$ and all $t \geq t_0$,

$$n_1^{-1} E_z[\xi_1 - \xi_1(t)] \leq \varepsilon. \tag{3.36}$$

To this end, from the formulas (3.24) and (3.26) we have

$$0 \leq E_z[\xi_1 - \xi_1(t)] \leq \sum_{k,m=1}^{\infty} \frac{k|a_k|m}{1 - \mathrm{e}^{-km\alpha_2}} \bigl(S_2(km\alpha_1(1+t)) + R_n(t; km)\bigr). \tag{3.37}$$

For the part of the sum (3.37) with $S_2(km\alpha_1(1+t))$, on substituting the equality (3.28) and adapting the estimate (3.16) derived in the proof of Theorem 3.2 we obtain for all $k, m \geq 1$ and $t > 0$

$$\frac{\alpha_1^2 \alpha_2 S_2(km\alpha_1(1+t))}{1 - \mathrm{e}^{-km\alpha_2}} = \frac{\alpha_1^2 \alpha_2 \,\mathrm{e}^{-km\alpha_1(1+t)}}{(1 - \mathrm{e}^{-km\alpha_2})(1 - \mathrm{e}^{-km\alpha_1(1+t)})^2} \leq \frac{C_1(\tau_*/2) C_2(1/2)}{(km)^3 (1+t)^2}.$$

Therefore, recalling that $\alpha_1^2 \alpha_2 = n_1^{-1} \kappa^3$ (see (3.11)) and using the condition $A^+(2) < \infty$, we have uniformly in t (and for all n)

$$\frac{1}{n_1} \sum_{k,m=1}^{\infty} \frac{k|a_k|m}{1 - \mathrm{e}^{-km\alpha_2}} S_2(km\alpha_1(1+t)) = \frac{O(1)}{(1+t)^2} \sum_{k,m=1}^{\infty} \frac{|a_k|}{k^2 m^2} = \frac{O(1)}{(1+t)^2} \leq \frac{\varepsilon}{2}, \tag{3.38}$$

provided t is large enough.

On the other hand, by the dominated convergence argument (cf. (3.31)) and due to the bound (3.29) leading to the limit (3.33), the contribution from $R_n(t; km)$ to the sum (3.37) is asymptotically negligible, uniformly in t, which implies that for all n large enough,

$$\frac{1}{n_1} \sum_{k,m=1}^{\infty} \frac{k|a_k|m}{1 - \mathrm{e}^{-km\alpha_2}} R_n(t; km) \leq \frac{\varepsilon}{2}. \tag{3.39}$$

Thus, substituting the estimates (3.38) and (3.39) into (3.37) yields (3.36) as desired, which completes the proof of the theorem. ☐

3.3. Refined asymptotics of the expectation

We need to sharpen the asymptotic estimate $E_z(\xi) - n = o(|n|)$ provided by Theorem 3.2.

Theorem 3.6. *Under the condition $A^+(1) < \infty$, we have $E_z(\xi) - n = O(|n|^{2/3})$ as $n \to \infty$.*

For the proof of Theorem 3.6, some preparations are required.

3.3.1. Integral approximation of sums. Let a function $f\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ be continuous and integrable on $\mathbb{R}_+^2$, together with its partial derivatives up to the second order. Set (cf. (3.6))

$$F(h) := \sum_{x \in \mathbb{Z}_+^2} f(hx), \qquad h > 0. \tag{3.40}$$

Adapting the well-known Euler–Maclaurin formula (see, e.g., [9, §12.2]) to the double summation in (3.40), one can verify (see more details in [7, §5.1]) that the above conditions on the function $f(x)$ ensure the absolute convergence of the double series (3.40) for any $h > 0$ and, moreover, $F(h)$ has the following asymptotics at the origin,

$$\lim_{h\downarrow 0} h^2 F(h) = \int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x < \infty. \tag{3.41}$$

In particular, (3.41) implies that

$$F(h) = O(h^{-2}), \qquad h \downarrow 0. \tag{3.42}$$

Furthermore, assume that for some $\beta > 2$

$$F(h) = O(h^{-\beta}), \qquad h \to +\infty, \tag{3.43}$$

and consider the Mellin transform of $F(h)$ (see, e.g., [21, Ch. VI, §9]),

$$\widehat{F}(s) := \int_0^\infty h^{s-1} F(h)\,\mathrm{d}h \qquad (s \in \mathbb{C}). \tag{3.44}$$

The estimates (3.42), (3.43) ensure that the function $\widehat{F}(s)$ is well defined (and analytic) if $2 < \Re(s) < \beta$. Moreover, $\widehat{F}(s)$ can be analytically continued into the strip $1 < \Re(s) < 2$. More precisely, consider the function

$$\varDelta_f(h) := F(h) - h^{-2} \int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x, \qquad h > 0, \tag{3.45}$$

that is, the error in the approximation of the function $F(h)$ by the corresponding integral (cf. (3.41)). The following lemma was proved in [7, Lemma 5.2].

Lemma 3.7. *Under the above conditions, the function $\widehat{F}(s)$ defined in* (3.44) *is meromorphic in the strip $1 < \Re(s) < \beta$, with a single* (*simple*) *pole at $s = 2$. Moreover, $\widehat{F}(s)$ satisfies the identity*

$$\widehat{F}(s) = \int_0^\infty h^{s-1} \varDelta_f(h)\,\mathrm{d}h, \qquad 1 < \Re(s) < 2. \tag{3.46}$$

Remark 3.3. The identity (3.46) is a two-dimensional analog of the Müntz formula for univariate functions (see [18, §2.11, pp. 28–29]).

In turn, by the inversion formula for the Mellin transform (see [21, Theorem 9a, pp. 246–247]), from (3.46) it follows that, for any $c \in (1, 2)$,

$$\varDelta_f(h) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} h^{-s} \widehat{F}(s)\,\mathrm{d}s \tag{3.47}$$

(see [7, Lemma 5.3] for more details).

3.3.2. Proof of Theorem 3.6. Our argumentation follows the same lines as in the proof of a similar result in [7, §5.2] for the special case of the coefficients (1.7) (with $\rho = 1$), but adapted to a more general context based on the cumulant expansions. To be specific, let us consider the coordinate ξ_1 of the random vector $\xi = (\xi_1, \xi_2)$ (for ξ_2 the proof is similar).

Step 1. According to (3.15) we have

$$E_z(\xi_1) = \sum_{k,m=1}^{\infty} a_k \mu(m) F(km), \tag{3.48}$$

where $F(h)$ is given by (3.14). Note that the corresponding function $f(x) = x_1 \mathrm{e}^{-\langle \alpha, x\rangle}$ (see (3.12)) has the property

$$\int_{\mathbb{R}_+^2} f(x)\,\mathrm{d}x = \int_0^\infty x_1 \mathrm{e}^{-\alpha_1 x_1}\,\mathrm{d}x_1 \int_0^\infty \mathrm{e}^{-\alpha_2 x_2}\,\mathrm{d}x_2 = \frac{1}{\alpha_1^2 \alpha_2}. \tag{3.49}$$

Moreover, by virtue of the relation $\alpha_1^2\alpha_2 = \kappa^3/n_1$ (see (3.11)) we have (cf. (3.17))

$$\frac{1}{\alpha_1^2\alpha_2} \sum_{k,m=1}^{\infty} \frac{a_k \mu(m)}{k^2 m^2} = \frac{n_1}{\kappa^3} \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} \equiv n_1. \tag{3.50}$$

Thus, subtracting (3.50) from (3.48) and substituting (3.49) we obtain the representation

$$E_z(\xi_1) - n_1 = \sum_{k,m=1}^{\infty} a_k \mu(m)\, \Delta_f(km), \tag{3.51}$$

where $\Delta_f(h)$ is defined in (3.45).

Step 2. Recalling the notation $\tau = n_2/n_1$ and the relation $\alpha_2 = \alpha_1/\tau$ (see (3.10) and (3.11)), the Mellin transform (3.44) of the function $F(h)$ may be represented in the form

$$\widehat{F}(s) = \alpha_1^{-s-1} \tilde{F}(s), \tag{3.52}$$

where

$$\tilde{F}(s) = \int_0^\infty \frac{y^s \mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2 (1-\mathrm{e}^{-y/\tau})}\,\mathrm{d}y, \qquad \Re(s) > 2. \tag{3.53}$$

Clearly, the functions $f(x)$, $F(h)$ satisfy all the hypotheses of Section 3.3.1, including the asymptotics (3.42) and (3.43), with any $\beta > 2$. Hence, by Lemma 3.7 the function $\widehat{F}(s)$ is regular for $1 < \Re(s) < 2$, and the formula (3.47) together with (3.52) yields

$$\Delta_f(h) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} h^{-s} \alpha_1^{-s-1} \tilde{F}(s)\,\mathrm{d}s, \qquad 1 < c < 2. \tag{3.54}$$

Thus, substituting the representation (3.54) (with $h = km$) into (3.51) we get

$$E_z(\xi_1) - n_1 = \frac{1}{2\pi\mathrm{i}} \sum_{k,m=1}^{\infty} a_k \mu(m) \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\tilde{F}(s)}{\alpha_1^{s+1}(km)^s}\,\mathrm{d}s, \qquad 1 < c < 2. \tag{3.55}$$

Step 3. Aiming to mollify the singularity of the integrand in (3.53) at zero, set

$$\phi(y) := \frac{y\,\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2} \left(\frac{1}{1-\mathrm{e}^{-y/\tau}} - \frac{\tau}{y} - \frac{1}{2} \right), \qquad y > 0, \tag{3.56}$$

and consider the regularized integral

$$\mathcal{I}(s) := \int_0^\infty y^{s-1}\phi(y)\,\mathrm{d}y, \tag{3.57}$$

so that (3.53) is rewritten in the form

$$\tilde{F}(s) = \mathcal{I}(s) + \tau\int_0^\infty \frac{y^{s-1}\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y + \frac{1}{2}\int_0^\infty \frac{y^{s}\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y. \tag{3.58}$$

The integrals in (3.58) are easily evaluated: if $\Re(s) > 2$ then

$$\begin{aligned}\int_0^\infty \frac{y^{s-1}\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y &= \int_0^\infty y^{s-1}\sum_{k=1}^\infty k\,\mathrm{e}^{-ky}\,\mathrm{d}y = \sum_{k=1}^\infty k\int_0^\infty y^{s-1}\mathrm{e}^{-ky}\,\mathrm{d}y \\ &= \sum_{k=1}^\infty \frac{1}{k^{s-1}}\int_0^\infty u^{s-1}\mathrm{e}^{-u}\,\mathrm{d}u = \zeta(s-1)\,\Gamma(s),\end{aligned} \tag{3.59}$$

and likewise

$$\int_0^\infty \frac{y^{s}\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2}\,\mathrm{d}y = \zeta(s)\,\Gamma(s+1). \tag{3.60}$$

Thus, substituting (3.59) and (3.60) into (3.58) we get

$$\tilde{F}(s) = \mathcal{I}(s) + \tau\zeta(s-1)\,\Gamma(s) + \tfrac{1}{2}\zeta(s)\,\Gamma(s+1), \qquad \Re(s) > 2. \tag{3.61}$$

Step 4. The representation (3.61) renders an explicit analytic continuation of $\widehat{F}(s)$ into the strip $0 < \Re(s) < 2$ (cf. Lemma 3.7). To show this, let us first investigate the integral (3.57).

Lemma 3.8. *The function* $\phi(y)$ *defined in* (3.56) *has the following asymptotic expansions*

$$\phi(y) = \tfrac{1}{12}\tau^{-1} + O(y^2), \qquad y \to 0, \tag{3.62}$$

$$\phi(y) = \tfrac{1}{2}y\,\mathrm{e}^{-y}\,(1 + o(1)), \qquad y \to \infty, \tag{3.63}$$

which can be formally differentiated to produce the corresponding expansions of $\phi'(y)$, $\phi''(y)$.

Proof. By Taylor's expansion it is easy to check that, as $y \to 0$,

$$\frac{1}{1-\mathrm{e}^{-y/\tau}} - \frac{\tau}{y} - \frac{1}{2} = \frac{y}{12\tau}\left(1 + O(y^2)\right),$$

$$\frac{y\,\mathrm{e}^{-y}}{(1-\mathrm{e}^{-y})^2} = \frac{1}{y}\left(1 + O(y^2)\right),$$

and (3.62) follows on substituting this into (3.56). Since differentiation of Taylor expansions is legitimate, from (3.62) we also get $\phi'(y) = O(y)$ and $\phi''(y) = O(1)$, as $y \to 0$.

The asymptotics (3.63) follow immediately from (3.56), and it is also straightforward to see that the main asymptotic contribution to the derivatives of $\phi(y)$, as $y \to \infty$, is furnished by the term $y\,\mathrm{e}^{-y}$, so that $\phi'(y) \sim \frac{1}{2}y\,\mathrm{e}^{-y}$ and $\phi''(y) \sim \frac{1}{2}y\,\mathrm{e}^{-y}$ as $y \to \infty$. □

In view of (3.62) and (3.63), the integral (3.57) is absolutely convergent if $\Re(s) > 0$, and therefore the function $\mathcal{I}(s)$ is regular in the corresponding half-plane.

Returning to the representation (3.61), note that the gamma function $\Gamma(s)$ is analytic for $\Re(s) > 0$ (see, e.g., [17, §4.41, p. 148]), whereas the Riemann zeta function $\zeta(s)$ is meromorphic in the complex plane $\mathbb{C}$ with a single (simple) pole at point $s = 1$ (see, e.g., [17, §4.43, p. 152]). Thus, the right-hand side of (3.61) is meromorphic in the half-plane $\Re(s) > 0$, with the simple poles at $s = 1$ and $s = 2$.

Step 5. Setting $s = \sigma + \mathrm{i}t$, let us estimate the function $\tilde{F}(s)$ as $t \to \infty$. First of all, integrating by parts (twice) in (3.57) and using the asymptotic formulas (3.62), (3.63) for the function $\phi(y)$ and its derivatives, we obtain

$$\mathcal{I}(s) = \frac{1}{s(s+1)} \int_0^\infty y^{s+1} \phi''(y)\,\mathrm{d}y = O(t^{-2}), \qquad t \to \infty, \tag{3.64}$$

uniformly in $0 < c_1 \le \sigma \le c_2 < \infty$. The gamma function in such a strip satisfies the uniform estimate (see [17, §4.42, p. 151])

$$\Gamma(s) = O\big(|t|^{\sigma-1/2}\,\mathrm{e}^{-\pi|t|/2}\big), \qquad t \to \infty. \tag{3.65}$$

We also have the following uniform bounds on the growth of the Riemann zeta function as $t \to \infty$ (see [12], Theorem 1.9, p. 25),

$$\zeta(s) = \begin{cases} O(\ln|t|), & 1 \le \sigma \le 2, \\ O\big(|t|^{(1-\sigma)/2} \ln|t|\big), & 0 \le \sigma \le 1. \end{cases} \tag{3.66}$$

Therefore, substituting the estimates (3.64), (3.65) and (3.66) into (3.61) and comparing the resulting contributions, it is easy to check that for $1 \le c \le 2$, uniformly in $n \in \mathbb{Z}_+^2$,

$$\tilde{F}(c + \mathrm{i}t) = O(t^{-2}), \qquad t \to \infty. \tag{3.67}$$

Step 6. Interchanging the order of summation and integration in (3.55) gives

$$\begin{aligned} E_z(\xi_1) - n_1 &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\tilde{F}(s)}{\alpha_1^{s+1}} \sum_{k=1}^\infty \frac{a_k}{k^s} \sum_{m=1}^\infty \frac{\mu(m)}{m^s}\,\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\tilde{F}(s)\,A(s)}{\alpha_1^{s+1}\zeta(s)}\,\mathrm{d}s, \end{aligned} \tag{3.68}$$

where we used the notation (2.22) and the formula (3.18). This computation is justified by virtue of the absolute convergence, since for $1 < c < 2$

$$\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{|\tilde{F}(s)|}{\alpha_1^{c+1}} \sum_{k=1}^\infty \frac{|a_k|}{k^c} \sum_{m=1}^\infty \frac{1}{m^c}\,\mathrm{d}|s| = \frac{A^+(c)\,\zeta(c)}{\alpha_1^{c+1}} \int_{-\infty}^\infty |\tilde{F}(c+\mathrm{i}t)|\,\mathrm{d}t < \infty, \tag{3.69}$$

where $A^+(c) \le A^+(1) < \infty$ due to the hypothesis of Theorem 3.6, whereas the last integral in (3.69) is finite thanks to the bound (3.67).

Thus, on substituting (3.61) into (3.68) we have

$$E_z(\xi_1) - n_1 = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} A(s)\,\alpha_1^{-s-1}\,\varPsi(s)\,\mathrm{d}s \qquad (1 < c < 2), \tag{3.70}$$

where

$$\varPsi(s) := \frac{\tilde{F}(s)}{\zeta(s)} = \frac{\mathcal{I}(s) + \tau\zeta(s-1)\Gamma(s)}{\zeta(s)} + \frac{1}{2}\Gamma(s+1). \tag{3.71}$$

Step 7. Since $\zeta(s) \neq 0$ for $\Re(s) \geq 1$, the function $\Psi(s)$ defined by the expression (3.71) is analytic in the half-plane $\Re(s) > 1$; moreover, it can be extended by continuity to the line $\Re(s) = 1$, where the singularity at $s = 1$ (due to the pole of $\zeta(s)$ in the denominator) can be removed by setting $\Psi(1) := \lim_{s\to 1} \Psi(s) = \frac{1}{2}\Gamma(2) = \frac{1}{2}$.

Let us show that the integration contour $\Re(s) = c > 1$ in (3.70) can be moved to $\Re(s) = 1$. By the Cauchy theorem, it suffices to check that

$$\lim_{T\to\pm\infty} \int_{1+\mathrm{i}T}^{c+\mathrm{i}T} A(s)\,\alpha_1^{-s-1}\,\Psi(s)\,\mathrm{d}s = 0. \tag{3.72}$$

To this end, note that for $1 \leq \sigma \leq c$

$$|A(\sigma + \mathrm{i}T)| \leq A^+(\sigma) \leq A^+(1) < \infty$$

(see (2.22)) and $\alpha_1^{-\sigma-1} \leq \alpha_1^{-c-1}$ (since $\alpha_1 \to 0$, we may assume that $\alpha_1 < 1$). From (3.67) we also have $|\tilde{F}(\sigma + \mathrm{i}T)| = O(T^{-2})$ as $T \to \infty$; furthermore, it is known (see [18, Eq. (3.11.8), p. 60]) that $\zeta(\sigma + \mathrm{i}T)^{-1} = O(\ln |T|)$ as $T \to \infty$, uniformly in $\sigma \geq 1$. Hence, substituting these estimates into (3.71) we obtain, uniformly in $1 \leq \sigma \leq c$ and $n \in \mathbb{Z}_+^2$,

$$\Psi(\sigma + \mathrm{i}T) = \frac{\tilde{F}(\sigma + \mathrm{i}T)}{\zeta(\sigma + \mathrm{i}T)} = O\bigl(T^{-2}\ln|T|\bigr) \to 0, \qquad T \to \infty. \tag{3.73}$$

As a result, the limit (3.72) follows. Thus, the representation (3.70) takes the form

$$E_z(\xi_1) - n_1 = \frac{1}{2\pi\mathrm{i}} \int_{1-\mathrm{i}\infty}^{1+\mathrm{i}\infty} A(s)\,\alpha_1^{-s-1}\,\Psi(s)\,\mathrm{d}s. \tag{3.74}$$

Step 8. Finally, the formula (3.74) yields the bound

$$|E_z(\xi_1) - n_1| = \frac{A^+(1)}{2\pi\alpha_1^2} \int_{-\infty}^{\infty} |\Psi(1 + \mathrm{i}t)|\,\mathrm{d}t = O(|n|^{2/3}), \tag{3.75}$$

since $\alpha_1 \asymp |n|^{-1/3}$ according to (3.2) and the integral in (3.75) is finite thanks to the bound (3.73). This completes the proof of Theorem 3.6.

4. Asymptotics of higher-order moments

Throughout this section, we again assume that $A^+(2) < \infty$, except in Section 4.3 where a stronger condition $A^+(1)$ is required.

4.1. The variance–covariance of ξ

Denote $\mu_z := E_z(\xi)$ and let $K_z := \mathrm{Cov}_z(\xi, \xi) = E_z(\xi - \mu_z)^\top(\xi - \mu_z)$ be the covariance matrix of the random vector $\xi = \sum_{x\in\mathcal{X}} x\nu(x)$. Recalling that the random variables $\nu(x)$ are independent for different $x \in \mathcal{X}$ and using (2.19) with $q = 2$, the elements $K_z(i, j) = \mathrm{Cov}_z(\xi_i, \xi_j)$ of the matrix K_z are given by

$$K_z(i, j) = \sum_{x\in\mathcal{X}} x_i x_j \,\mathrm{Var}_z[\nu(x)] = \sum_{x\in\mathcal{X}} x_i x_j \sum_{k=1}^{\infty} k^2 a_k\, z^{kx}, \qquad i, j \in \{1, 2\}. \tag{4.1}$$

4.1.1. Asymptotics of the covariance matrix.

Theorem 4.1. *As $n \to \infty$,*

$$K_z(i,j) \sim B_{ij}\,(n_1 n_2)^{2/3}, \qquad i,j \in \{1,2\}, \tag{4.2}$$

where the matrix $B = (B_{ij})$ is given by

$$B = \kappa^{-1} \begin{pmatrix} 2\tau^{-1} & 1 \\ 1 & 2\tau \end{pmatrix}. \tag{4.3}$$

Proof. The calculations below follow the lines of the proof of Theorem 3.2, so we only sketch the proof. Let us first consider the element $K_z(1,1)$. Substituting the parameterization $z = \mathrm{e}^{-\alpha}$ (see (3.2)) into (4.1), we obtain (cf. (3.3))

$$K_z(1,1) = \sum_{x \in \mathcal{X}} x_1^2 \sum_{k=1}^{\infty} k^2 a_k \, \mathrm{e}^{-k\langle \alpha, x\rangle}. \tag{4.4}$$

Using the Möbius inversion formula (3.7), similarly to (3.15) the right-hand side of (4.4) can be rewritten in the form

$$\begin{aligned} K_z(1,1) &= \sum_{m=1}^{\infty} m^2 \mu(m) \sum_{k=1}^{\infty} k^2 a_k \sum_{x \in \mathbb{Z}_+^2} x_1^2 \mathrm{e}^{-km\langle \alpha, x\rangle} \\ &= \sum_{k,m=1}^{\infty} m^2 \mu(m)\, k^2 a_k \sum_{x_1=1}^{\infty} x_1^2 \mathrm{e}^{-km\alpha_1 x_1} \sum_{x_2=0}^{\infty} \mathrm{e}^{-km\alpha_2 x_2} \\ &= \sum_{k,m=1}^{\infty} m^2 \mu(m)\, k^2 a_k \, \frac{\mathrm{e}^{-km\alpha_1}(1+\mathrm{e}^{-km\alpha_1})}{(1-\mathrm{e}^{-km\alpha_1})^3 (1-\mathrm{e}^{-km\alpha_2})}, \end{aligned} \tag{4.5}$$

where we used the expressions (2.26) for $S_1(\cdot)$ and $S_3(\cdot)$. Similarly to the estimate (3.16), by virtue of (2.30) the general term in the sum (4.5) is bounded by $O(\alpha_1^{-3}\alpha_2^{-1})\,|a_k| k^{-2} m^{-2}$, uniformly in k, m, and furthermore,

$$\sum_{k,m=1}^{\infty} \frac{|a_k|}{k^2 m^2} = A^+(2)\,\zeta(2) < \infty.$$

Therefore, by the dominated convergence argument, from (4.5) we obtain, similarly to (3.16),

$$\lim_{n\to\infty} \alpha_1^3 \alpha_2 K_z(1,1) = 2 \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \frac{2A(2)}{\zeta(2)} = 2\kappa^3. \tag{4.6}$$

To reduce this limit to (4.2), observe using the scaling relations (3.11) that

$$\alpha_1^3 \alpha_2 = \frac{\alpha_1}{\alpha_2}\, \alpha_1^2 \alpha_2^2 = \tau \kappa^4 (n_1 n_2)^{-2/3}, \tag{4.7}$$

and from (4.6) we get

$$\lim_{n\to\infty} (n_1 n_2)^{-2/3} K_z(1,1) = \tau^{-1}\kappa^{-4} \lim_{n\to\infty} \alpha_1^3 \alpha_2 K_z(1,1) = 2\tau^{-1}\kappa^{-1} = B_{11}, \tag{4.8}$$

as required (cf. (4.2), (4.3)).

The element $K_z(2,2)$ is analyzed in a similar fashion. Finally, for $K_z(1,2)$ we obtain, similarly as in (4.5) and (4.6),

$$\begin{aligned} K_z(1,2) &= \sum_{x\in\mathcal{X}} x_1 x_2 \sum_{k=1}^{\infty} k^2 a_k \,\mathrm{e}^{-k\langle\alpha,x\rangle} \\ &= \sum_{k,m=1}^{\infty} k^2 a_k\, m^2 \mu(m) \sum_{x_1=1}^{\infty} x_1 \,\mathrm{e}^{-km\alpha_1 x_1} \sum_{x_2=1}^{\infty} x_2 \,\mathrm{e}^{-km\alpha_2 x_2} \\ &= \sum_{k,m=1}^{\infty} k^2 a_k\, m^2 \mu(m)\, \frac{\mathrm{e}^{-km\alpha_1}\,\mathrm{e}^{-km\alpha_2}}{(1-\mathrm{e}^{-km\alpha_1})^2(1-\mathrm{e}^{-km\alpha_2})^2} \\ &\sim \alpha_1^{-2}\alpha_2^{-2} \sum_{k=1}^{\infty} \frac{a_k}{k^2} \sum_{m=1}^{\infty} \frac{\mu(m)}{m^2} = \alpha_1^{-2}\alpha_2^{-2}\kappa^3 \qquad (n\to\infty). \end{aligned}$$

Hence, using the identity $\alpha_1^2\alpha_2^2 = \kappa^4 (n_1 n_2)^{-2/3}$ (cf. (4.7)), it follows as in (4.8) that

$$\lim_{n\to\infty} (n_1 n_2)^{-2/3} K_z(1,2) = \kappa^{-4} \lim_{n\to\infty} \alpha_1^2\alpha_2^2 K_z(1,2) = \kappa^{-1} = B_{12},$$

according to the notation (4.3). Thus, the proof of the theorem is complete. □

4.1.2. The norm of the covariance matrix. The next lemma is an immediate corollary of Theorem 4.1.

Lemma 4.2. *As $n\to\infty$,*

$$\det K_z \sim 3\kappa^{-2}(n_1 n_2)^{4/3} \asymp |n|^{8/3}. \tag{4.9}$$

This result implies that the matrix K_z is non-degenerate, at least asymptotically as $n\to\infty$. In fact, from (4.1) it is easy to deduce (e.g., using the Cauchy–Schwarz inequality together with the characterization of the equality case) that K_z is positive definite; in particular, $\det K_z > 0$ and hence K_z is invertible. Let $V_z := K_z^{-1/2}$ be the (unique) square root of the matrix K_z^{-1}, that is, a symmetric, positive definite matrix such that $V_z^2 = K_z^{-1}$.

We need some general facts about the matrix norm $\|\cdot\|$, which we state as a lemma (see [7, §7.2, p. 2301] for simple proofs and bibliographic comments).

Lemma 4.3. (a) *If A is a real matrix then $\|A^{\top}A\| = \|A\|^2$.*

(b) *If $A=(a_{ij})$ is a real $d\times d$ matrix, then*

$$\frac{1}{d}\sum_{i,j=1}^{d} a_{ij}^2 \le \|A\|^2 \le \sum_{i,j=1}^{d} a_{ij}^2 .$$

(c) *Let A be a real symmetric 2×2 matrix with $\det A\ne 0$. Then*

$$\|A^{-1}\| = \frac{\|A\|}{|\det A|}.$$

Let us now estimate the norm of the matrices K_z and $V_z = K_z^{-1/2}$.

Lemma 4.4. *As $n \to \infty$, one has*

$$\|K_z\| \asymp |n|^{4/3}, \qquad \|V_z\| \asymp |n|^{-2/3}. \tag{4.10}$$

Proof. Lemma 4.3(b) and Theorem 4.1 imply

$$\|K_z\|^2 \asymp \sum_{i,j=1}^{2} K_z(i,j)^2 \asymp (n_1 n_2)^{4/3} \asymp |n|^{8/3} \qquad (n \to \infty),$$

which proves the first estimate in (4.10). Furthermore, using parts (a) and (c) of Lemma 4.3, we have

$$\|V_z\|^2 = \|V_z^2\| = \|K_z^{-1}\| = \frac{\|K_z\|}{\det K_z} \asymp \frac{|n|^{4/3}}{|n|^{8/3}} = |n|^{-4/3},$$

according to the known asymptotics of $\det K_z$ and $\|K_z\|$ (see (4.9) and (4.10), respectively). Hence, the second estimate in (4.10) follows, and the proof of the lemma is complete. □

4.2. The cumulants of ξ_j

By the parameterization $z = \mathrm{e}^{-\alpha}$ (see (3.2)), the expansion (2.21) takes the form

$$\varkappa_q[\xi_j] = \sum_{x \in \mathcal{X}} x_j^q \sum_{k=1}^{\infty} k^q a_k \,\mathrm{e}^{-k\langle \alpha, x\rangle}, \qquad q \in \mathbb{N}. \tag{4.11}$$

Lemma 4.5. *For each $q \in \mathbb{N}$, as $n \to \infty$,*

$$\varkappa_q[\xi_j] \sim \frac{q!\,\kappa^3}{\alpha_1^{q+1}\alpha_2} \asymp |n|^{(q+2)/3}, \qquad n \to \infty. \tag{4.12}$$

Proof. Let $j = 1$ (the case $j = 2$ is handled in a similar fashion). Using the Möbius inversion formula (3.7), similarly to (3.15) the right-hand side of (4.11) (with $j = 1$) can be rewritten as

$$\begin{aligned}
\varkappa_q[\xi_1] &= \sum_{k,m=1}^{\infty} k^q a_k\, m^q \mu(m) \sum_{x \in \mathbb{Z}_+^2} x_1^q \mathrm{e}^{-km\langle \alpha, x\rangle} \\
&= \sum_{k,m=1}^{\infty} k^q a_k\, m^q \mu(m) \sum_{x_1=1}^{\infty} x_1^q \mathrm{e}^{-km\alpha_1 x_1} \sum_{x_2=0}^{\infty} \mathrm{e}^{-km\alpha_2 x_2} \\
&= \sum_{k,m=1}^{\infty} k^q a_k\, m^q \mu(m)\, S_{q+1}(km\alpha_1)\, \frac{1}{1 - \mathrm{e}^{-km\alpha_2}},
\end{aligned} \tag{4.13}$$

where we used the notation (2.25). Applying Lemma 2.7(a) and then the estimate (2.30) (cf. (3.16)), we obtain

$$\begin{aligned}
\frac{\alpha_1^{q+1}\alpha_2\, S_{q+1}(km\alpha_1)}{1 - \mathrm{e}^{-km\alpha_2}} &\le \frac{\alpha_1^{q+1}\alpha_2\, \bar{c}_{q+1}\, \mathrm{e}^{-km\alpha_1}}{(1 - \mathrm{e}^{-km\alpha_1})^{q+1}(1 - \mathrm{e}^{-km\alpha_2})} \\
&\le \frac{\bar{c}_{q+1}\, C_{q+1}(1/2)\, C_1(\tau_*/2)}{(km)^{q+2}}.
\end{aligned}$$

Consequently, the general summand in the double series (4.13) multiplied by $\alpha_1^{q+1}\alpha_2$ is bounded, uniformly in $k, m \ge 1$, by $O(1)\,|a_k|k^{-2}m^{-2}$, which is a term of a convergent series owing to the condition $A^+(2) < \infty$. Hence, applying Lebesgue's dominated convergence theorem and using Lemma 2.7(b), we obtain

$$\lim_{n\to\infty} \alpha_1^{q+1}\alpha_2\,\varkappa_q[\xi_1] = q!\sum_{k,m=1}^{\infty} \frac{\mu(m)\,a_k}{k^2m^2} = q!\,\frac{A(2)}{\zeta(2)} = q!\,\kappa^3. \tag{4.14}$$

Finally, according to (3.2) we have $\alpha_1^{q+1}\alpha_2 \asymp |n|^{-(q+2)/3}$, and hence (4.14) implies (4.12). $\square$

In Section 5.4 we will require an asymptotic bound for the *sixth-order central moment* of ξ_j, which is established next.

Lemma 4.6. *Set* $\xi_j^0 := \xi_j - E_z(\xi_j)$ $(j = 1, 2)$. *Then*

$$E_z\bigl[(\xi_j^0)^6\bigr] \asymp |n|^4, \qquad n \to \infty. \tag{4.15}$$

Proof. Using the expression of the sixth central moment via the cumulants (see (2.18) with $X = \xi_j$ and $q = 6$), we have

$$E_z\bigl[(\xi_j^0)^6\bigr] = \varkappa_6[\xi_j] + 15\varkappa_4[\xi_j]\varkappa_2[\xi_j] + 10(\varkappa_3[\xi_j])^2 + 15(\varkappa_2[\xi_j])^3. \tag{4.16}$$

Applying Lemma 4.5 to the cumulants involved in (4.16), it is easy to check that the main asymptotic term is given by $(\varkappa_2[\xi_j])^3 \asymp |n|^4$, which proves the relation (4.15). $\square$

4.3. The Lyapunov coefficient

Let us introduce the *Lyapunov coefficient* (of the third order)

$$L_z := \|V_z\|^3 \sum_{x\in\mathcal{X}} |x|^3 \mu_3[\nu(x)], \tag{4.17}$$

where $\mu_3[\nu(x)]$ is the third-order absolute central moment of $\nu(x)$,

$$\mu_3[\nu(x)] := E_z\bigl[|\nu^0(x)|^3\bigr], \qquad \nu^0(x) := \nu(x) - E_z[\nu(x)]. \tag{4.18}$$

The next asymptotic estimate will play an important role in the proof of the local limit theorem in Section 5.3 below.

Lemma 4.7. *Suppose that* $A^+(1) < \infty$. *Then*

$$L_z \asymp |n|^{-1/3}, \qquad n \to \infty. \tag{4.19}$$

Proof. In view of the definition (4.17) and the asymptotics $\|V_z\| \asymp |n|^{-2/3}$ (see (4.10)), for the proof of (4.19) it suffices to show that

$$M_3 := \sum_{x\in\mathcal{X}} |x|^3 \mu_3[\nu(x)] \asymp |n|^{5/3}, \qquad n \to \infty. \tag{4.20}$$

Starting with a *lower bound* for M_3, observe using the relation (2.18) with $q = 3$ that

$$\mu_3[\nu(x)] \ge E_z[\nu^0(x)^3] = \varkappa_3[\nu(x)]. \tag{4.21}$$

Hence, using the formula (2.21) and Lemma 4.5 (with $q = 3$), from (4.20) we get

$$M_3 \geq \sum_{x\in\mathcal{X}} |x|^3 \varkappa_3[\nu(x)] \geq \sum_{x\in\mathcal{X}} x_1^3 \varkappa_3[\nu(x)] = \varkappa_3[\xi_1] \asymp |n|^{5/3}, \qquad n \to \infty, \tag{4.22}$$

which is in agreement with the claim (4.20).

Let us now obtain a suitable *upper bound* on M_3. First, using the elementary inequality

$$|x|^3 = (x_1^2 + x_2^2)^{3/2} \leq \sqrt{2}\,(x_1^3 + x_2^3)$$

(which follows from Hölder's inequality for the function $y = x^{3/2}$), we have

$$M_3 \leq \sqrt{2} \sum_{x\in\mathcal{X}} (x_1^3 + x_2^3)\,\mu_3[\nu(x)]. \tag{4.23}$$

To estimate the moment $\mu_3[\nu(x)]$ (see (4.18)), observe that for any $u, v \geq 0$

$$|u - v|^3 = (u - v)^2 |u - v| \leq (u - v)^2 (u + v) = (u - v)^3 + 2v(u - v)^2. \tag{4.24}$$

Setting in (4.24) $u = \nu(x),\ v = E_z[\nu(x)]$ and taking the expectation, we get the inequality

$$\begin{aligned} \mu_3[\nu(x)] &\leq E_z[\nu^0(x)^3] + 2E_z[\nu(x)] \cdot E_z[\nu^0(x)^2] \\ &= \varkappa_3[\nu(x)] + 2\varkappa_1[\nu(x)]\,\varkappa_2[\nu(x)], \end{aligned}$$

according to the identities (2.17), (2.18) applied to $\nu(x)$. Note that the term $\varkappa_3[\nu(x)]$ here is the same as in (4.21), so upon the substitution into (4.23) it gives the contribution of the order of $|n|^{5/3}$ into the upper bound for M_3.

Next, using the expansion (2.19) with $q = 1$ and $q = 2$, we obtain

$$\begin{aligned} \sum_{x\in\mathcal{X}} x_1^3 \varkappa_1[\nu(x)]\,\varkappa_2[\nu(x)] &= \sum_{x\in\mathcal{X}} x_1^3 \sum_{k=1}^{\infty} k a_k \,\mathrm{e}^{-k\langle\alpha,x\rangle} \sum_{\ell=1}^{\infty} \ell^2 a_\ell \,\mathrm{e}^{-\ell\langle\alpha,x\rangle} \\ &\leq \sum_{k,\ell\geq 1} k|a_k|\,\ell^2|a_\ell| \sum_{x\in\mathbb{Z}_+^2} x_1^3 \,\mathrm{e}^{-(k+\ell)\langle\alpha,x\rangle}. \end{aligned} \tag{4.25}$$

Using the notation (2.25) and the bounds of Lemmas 2.7 and 2.8, the internal sum in (4.25) can be estimated, uniformly in $k, \ell \geq 1$, as follows (cf. (3.16))

$$\begin{aligned} \sum_{x\in\mathbb{Z}_+^2} x_1^3 \,\mathrm{e}^{-(k+\ell)\langle\alpha,x\rangle} &= \sum_{x_1=1}^{\infty} x_1^3 \,\mathrm{e}^{-(k+\ell)\alpha_1 x_1} \sum_{x_2=0}^{\infty} \mathrm{e}^{-(k+\ell)\alpha_2 x_2} \\ &= S_4((k+\ell)\alpha_1) \cdot \frac{1}{1 - \mathrm{e}^{-(k+\ell)\alpha_2}} \\ &\leq \frac{\bar{c}_4 \,\mathrm{e}^{-(k+\ell)\alpha_1}}{(1 - \mathrm{e}^{-(k+\ell)\alpha_1})^4\,(1 - \mathrm{e}^{-(k+\ell)\alpha_2})} \\ &= \frac{O(1)}{(k+\ell)^5 \alpha_1^4 \alpha_2} = \frac{O(|n|^{5/3})}{(k+\ell)^5}, \end{aligned} \tag{4.26}$$

in view of the asymptotics $\alpha_1 \asymp \alpha_2 \asymp |n|^{-1/3}$ (see (3.2)). The analogous sum with x_2^3 in place of x_1^3 in (4.25) is estimated similarly, so combining (4.23) and (4.26) we get

$$M_3 = O(|n|^{5/3}) \sum_{k,\ell\geq 1} \frac{k|a_k|\,\ell^2|a_\ell|}{(k+\ell)^5}. \tag{4.27}$$

Furthermore, by the elementary inequality

$$(k+\ell)^5 = (k+\ell)^2(k+\ell)^3 \ge k^2\ell^3$$

the (double) series on the right-hand side of (4.27) is bounded by

$$\sum_{k,\ell\ge 1} \frac{k|a_k|\,\ell^2|a_\ell|}{k^2\ell^3} = \sum_{k=1}^{\infty}\frac{|a_k|}{k}\sum_{\ell=1}^{\infty}\frac{|a_\ell|}{\ell} = A^+(1)^2 < \infty,$$

according to the lemma's hypothesis. Thus, returning to (4.27) we see that $M_3 = O(|n|^{5/3})$, and together with the lower bound (4.22) this completes the proof of (4.20). □

5. A local limit theorem and the limit shape

5.1. Statement of the theorem

The role of the local limit theorem in our approach is to yield the asymptotics of the probability $Q_z\{\xi = n\} \equiv Q_z(\varPi_n)$ appearing in the representation of the measure P_n as a conditional distribution, $P_n(\cdot) = Q_z(\cdot\,|\varPi_n) = Q_z(\cdot)/Q_z(\varPi_n)$ (see Section 2.1).

To prove such a theorem (see Theorem 5.1 below), we will require a technical condition on the generating function $\beta(u)$ as follows.

Assumption 5.1. There exists a constant $\delta_* > 0$ such that for any $\theta \in (0,1)$ the function $u \mapsto \ln\beta(u)$ ($u \in \mathbb{C}$) satisfies the inequality

$$\ln\beta(\theta) - \Re(\ln\beta(\theta\,\mathrm{e}^{\mathrm{i}t})) \ge \delta_*\,\theta\,(1-\cos t), \qquad t \in \mathbb{R}. \tag{5.1}$$

Remark 5.1. In terms of the coefficients $\{a_k\}$ in the power series expansion of the function $\ln\beta(u)$ (see (2.13)), the left-hand side of (5.1) is expressed as $\sum_{k=1}^{\infty} a_k\theta^k(1-\cos kt)$. Consequently, if $a_1 > 0$ and $a_k \ge 0$ for all $k \ge 2$ then the inequality (5.1) is satisfied, with $\delta_* = a_1 > 0$.

As before, we denote $\mu_z = E_z(\xi)$, $K_z = \mathrm{Cov}_z(\xi,\xi)$, $V_z = K_z^{-1/2}$ (see Section 4.1). Consider the probability density function of a two-dimensional normal distribution $\mathcal{N}(\mu_z, K_z)$ (i.e., with mean μ_z and covariance matrix K_z), given by

$$f_{\mu_z,K_z}(x) = \frac{1}{2\pi\sqrt{\det K_z}}\exp\left(-\tfrac{1}{2}|(x-\mu_z)V_z|^2\right), \qquad x \in \mathbb{R}^2. \tag{5.2}$$

Theorem 5.1. *Assume that* $A^+(1) < \infty$ *and suppose that Assumption* 5.1 *holds. Then, uniformly in* $x \in \mathbb{Z}_+^2$,

$$Q_z\{\xi = x\} = f_{\mu_z,K_z}(x) + O(|n|^{-5/3}), \qquad n \to \infty. \tag{5.3}$$

Corollary 5.2. *Under the conditions of Theorem* 5.1

$$Q_z\{\xi = n\} \asymp |n|^{-4/3}, \qquad n \to \infty. \tag{5.4}$$

With the asymptotic results of Sections 3.3 and 4.1 at hand, it is not difficult to deduce the corollary from the theorem.

Proof of Corollary 5.2. According to Theorem 3.6, we have $\mu_z = n + O(|n|^{2/3})$. Together with the asymptotics of $\|V_z\|$ (see (4.10)) this implies

$$|(n-\mu_z)V_z| \le |n-\mu_z| \cdot \|V_z\| = O(|n|^{2/3}) \cdot |n|^{-2/3} = O(1).$$

Hence, with the help of Lemma 4.2 we get

$$f_{\mu_z,K_z}(n) = \frac{1}{2\pi\sqrt{\det K_z}}\, \mathrm{e}^{-|(n-\mu_z)V_z|^2/2} \asymp \frac{1}{\sqrt{\det K_z}} \asymp |n|^{-4/3},$$

and (5.4) now readily follows from (5.3). □

5.2. Estimates of the characteristic functions

Before proving Theorem 5.1, we have to make some technical preparations. Recall from Section 2.1 that, with respect to the measure Q_z, the random variables $\{\nu(x)\}_{x\in\mathcal{X}}$ are independent and have the characteristic functions (2.14). Hence, the characteristic function $\varphi_\xi(\lambda) := E_z(\mathrm{e}^{\mathrm{i}\langle\lambda,\,\xi\rangle})$ of the vector sum $\xi = \sum_{x\in\mathcal{X}} x\nu(x)$ is given by

$$\varphi_\xi(\lambda) = \prod_{x\in\mathcal{X}} \varphi_{\nu(x)}(\langle\lambda, x\rangle) = \prod_{x\in\mathcal{X}} \frac{\beta(z^x \mathrm{e}^{\mathrm{i}\langle\lambda,x\rangle})}{\beta(z^x)}, \qquad \lambda\in\mathbb{R}^2. \tag{5.5}$$

The next lemma provides a useful estimate (proved in [7, Lemma 7.12]) for the characteristic function $\varphi_{\xi^0}(\lambda) = \mathrm{e}^{-\langle\lambda,\mu_z\rangle}\varphi_\xi(\lambda)$ of the centered random vector $\xi^0 := \xi - \mu_z$. Recall that the Lyapunov ratio L_z is defined in (4.17), and that $V_z = K_z^{-1/2}$ (see Section 4.1).

Lemma 5.3. *If $y\in\mathbb{R}^2$ is such that $|y| \le L_z^{-1}$ then*

$$\left|\varphi_{\xi^0}(yV_z) - \mathrm{e}^{-|y|^2/2}\right| \le 16 L_z |y|^3\, \mathrm{e}^{-|y|^2/6}.$$

Under Assumption 5.1, $\varphi_\xi(\lambda)$ admits a simple global bound (cf. [7, Lemma 7.13]).

Lemma 5.4. *Suppose that Assumption* 5.1 *is satisfied* (*with $\delta_* > 0$*). Then*

$$|\varphi_\xi(\lambda)| \le \exp\{-\delta_* J_\alpha(\lambda)\}, \qquad \lambda\in\mathbb{R}^2, \tag{5.6}$$

where

$$J_\alpha(\lambda) := \sum_{x\in\mathcal{X}} \mathrm{e}^{-\langle\alpha,x\rangle}\bigl(1-\cos\langle\lambda,x\rangle\bigr), \qquad \lambda\in\mathbb{R}^2. \tag{5.7}$$

Proof. From (5.5) we have

$$|\varphi_\xi(\lambda)| = \exp\Biggl\{\sum_{x\in\mathcal{X}} \ln\bigl|\varphi_{\nu(x)}(\langle\lambda,x\rangle)\bigr|\Biggr\}. \tag{5.8}$$

Furthermore, using (2.15) and Assumption 5.1 (with $\theta = z^x$, see (5.1)), we have

$$\begin{aligned}\ln|\varphi_{\nu(x)}(t)| = \Re(\ln\varphi_{\nu(x)}(t)) &= \Re(\ln\beta(z^x\mathrm{e}^{\mathrm{i}t})) - \ln\beta(z^x)\\ &\le -\delta_* z^x(1-\cos t), \qquad t\in\mathbb{R}.\end{aligned}$$

Utilizing this estimate under the sum in (5.8) (with $t = \langle\lambda,x\rangle$) and substituting $z^x = \mathrm{e}^{-\langle\alpha,x\rangle}$ (see the notation (3.2)), we arrive at the inequality (5.6). □

5.3. Proof of Theorem 5.1

By definition, the characteristic function of the random vector ξ is given by the Fourier series

$$\varphi_\xi(\lambda) = \sum_{x\in\mathbb{Z}_+^2} Q_z\{\xi = x\}\,\mathrm{e}^{\mathrm{i}\langle\lambda,m\rangle}, \qquad \lambda\in\mathbb{R}^2,$$

hence the Fourier coefficients are expressed as

$$Q_z\{\xi = x\} = \frac{1}{4\pi^2}\int_{T^2} \mathrm{e}^{-\mathrm{i}\langle\lambda,x\rangle}\,\varphi_\xi(\lambda)\,\mathrm{d}\lambda, \qquad x\in\mathbb{Z}_+^2, \tag{5.9}$$

where $T^2 := \{\lambda = (\lambda_1,\lambda_2)\in\mathbb{R}^2\colon |\lambda_1|\le\pi,\ |\lambda_2|\le\pi\}$. On the other hand, the characteristic function corresponding to the normal probability density $f_{\mu_z,K_z}(\cdot)$ (see (5.2)) is given by

$$\varphi_{\mu_z,K_z}(\lambda) = \mathrm{e}^{\mathrm{i}\langle\lambda,\mu_z\rangle - |\lambda V_z^{-1}|^2/2}, \qquad \lambda\in\mathbb{R}^2,$$

so by the Fourier inversion formula

$$f_{\mu_z,K_z}(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \mathrm{e}^{-\mathrm{i}\langle\lambda,x\rangle}\,\mathrm{e}^{\mathrm{i}\langle\lambda,\mu_z\rangle - |\lambda V_z^{-1}|^2/2}\,\mathrm{d}\lambda, \qquad x\in\mathbb{Z}_+^2. \tag{5.10}$$

Denote $D_z := \{\lambda\in\mathbb{R}^2\colon |\lambda V_z^{-1}| > L_z^{-1}\}$. If $\lambda\in D_z^c := \mathbb{R}^2\setminus D_z$ then, on account of the asymptotics of $\|V_z\|$ and L_z (see (4.10) and (4.19), respectively), we get

$$|\lambda| = |\lambda V_z^{-1} V_z| \le |\lambda V_z^{-1}|\cdot\|V_z\| \le L_z^{-1}\|V_z\| = O(|n|^{-1/3}) = o(1),$$

which implies that $D_z^c\subset T^2$ for all $n=(n_1,n_2)$ large enough. Hence, subtracting (5.10) from (5.9) it is easy to see that, uniformly in $x\in\mathbb{Z}_+^2$,

$$\bigl|Q_z\{\xi = x\} - f_{\mu_z,K_z}(x)\bigr| \le I_1 + I_2 + I_3, \tag{5.11}$$

where

$$I_1 := \frac{1}{4\pi^2}\int_{D_z^c} \bigl|\varphi_\xi(\lambda) - \mathrm{e}^{\mathrm{i}\langle\lambda,\mu_z\rangle - |\lambda V_z^{-1}|^2/2}\bigr|\,\mathrm{d}\lambda, \tag{5.12}$$

$$I_2 := \frac{1}{4\pi^2}\int_{D_z} \mathrm{e}^{-|\lambda V_z^{-1}|^2/2}\,\mathrm{d}\lambda, \tag{5.13}$$

$$I_3 := \frac{1}{4\pi^2}\int_{T^2\cap D_z} |\varphi_\xi(\lambda)|\,\mathrm{d}\lambda. \tag{5.14}$$

By the substitution $\lambda = yV_z$, the integral (5.12) is reduced to

$$\begin{aligned} I_1 &= \frac{\det V_z}{4\pi^2}\int_{|y|\le L_z^{-1}} \bigl|\varphi_\xi(yV_z) - \mathrm{e}^{\mathrm{i}\langle yV_z,\mu_z\rangle - |y|^2/2}\bigr|\,\mathrm{d}y \\ &= \frac{1}{4\pi^2\sqrt{\det K_z}}\int_{|y|\le L_z^{-1}} \bigl|\varphi_{\xi^0}(yV_z) - \mathrm{e}^{-|y|^2/2}\bigr|\,\mathrm{d}y \\ &= O(|n|^{-4/3})\,L_z\int_{\mathbb{R}^2} |y|^3\,\mathrm{e}^{-|y|^2/6}\,\mathrm{d}y = O(|n|^{-5/3}), \end{aligned} \tag{5.15}$$

on account of Lemmas 4.2, 4.7 and 5.3. Similarly, using the change of variables $\lambda = yV_z$ in the integral (5.13) and passing to the polar coordinates, by Lemmas 4.2 and 4.7 we get

$$\begin{aligned} I_2 &= \frac{\det V_z}{4\pi^2} \int_{|y|>L_z^{-1}} \mathrm{e}^{-|y|^2/2}\, \mathrm{d}y \\ &= \frac{\det V_z}{2\pi} \int_{L_z^{-1}}^{\infty} r\, \mathrm{e}^{-r^2/2}\, \mathrm{d}r = O(|n|^{-4/3})\, \mathrm{e}^{-L_z^{-2}/2} = o(|n|^{-5/3}). \end{aligned} \tag{5.16}$$

Finally, let us turn to the integral (5.14). Note that if $\lambda \in D_z$ (i.e., $|\lambda V_z^{-1}| > L_z^{-1}$), then $|\lambda| > \eta|\alpha|$ for a small enough constant $\eta > 0$, and hence $\max\{|\lambda_1|/\alpha_1, |\lambda_2|/\alpha_2\} > \eta$; for otherwise, from (3.2) and Lemmas 4.4 and 4.7 it would follow

$$1 < L_z|\lambda V_z^{-1}| \le L_z \eta |\alpha| \cdot \|K_z\|^{1/2} = O(\eta) \to 0 \quad \text{as } \eta \downarrow 0,$$

which is a contradiction. Thus, also using Lemma 5.4 to estimate the integrand in (5.14), we get the bound

$$I_3 \le \frac{1}{4\pi^2} \sum_{j=1}^{2} \int_{T^2} \mathbb{1}_{\{|\lambda_j|>\eta\alpha_j\}}(\lambda)\, \mathrm{e}^{-\delta_* J_\alpha(\lambda)}\, \mathrm{d}\lambda, \tag{5.17}$$

where $\mathbb{1}_B(\lambda)$ is the indicator of a set $B \subset \mathbb{R}^2$. To estimate the first integral in (5.17) (i.e., with $j = 1$), let us keep in the summation (5.7) only the pairs of the form $x = (x_1, 1)$, $x_1 \in \mathbb{Z}_+$, giving a lower bound

$$\begin{aligned} J_\alpha(\lambda) \ge \sum_{x_1=0}^{\infty} \mathrm{e}^{-\alpha_1 x_1} \bigl(1 - \Re(\mathrm{e}^{\mathrm{i}(\lambda_1 x_1 + \lambda_2)})\bigr) &= \frac{1}{1-\mathrm{e}^{-\alpha_1}} - \Re\left(\frac{\mathrm{e}^{\mathrm{i}\lambda_2}}{1-\mathrm{e}^{-\alpha_1+\mathrm{i}\lambda_1}}\right) \\ &\ge \frac{1}{1-\mathrm{e}^{-\alpha_1}} - \frac{1}{|1-\mathrm{e}^{-\alpha_1+\mathrm{i}\lambda_1}|}, \end{aligned} \tag{5.18}$$

because $\Re(s) \le |s|$ for any $s \in \mathbb{C}$. Since $\eta\alpha_1 \le |\lambda_1| \le \pi$, we have

$$|1-\mathrm{e}^{-\alpha_1+\mathrm{i}\lambda_1}| \ge |1-\mathrm{e}^{-\alpha_1+\mathrm{i}\eta\alpha_1}| \sim \alpha_1\sqrt{1+\eta^2} \qquad (\alpha_1 \to 0).$$

Substituting this estimate into (5.18), we conclude that $J_\alpha(\lambda)$ is asymptotically bounded below by $C(\eta)\,\alpha_1^{-1} \asymp |n|^{1/3}$ (with $C(\eta) := 1-(1+\eta^2)^{-1/2} > 0$), uniformly in λ such that $\eta\alpha_1 \le |\lambda_1| \le \pi$. Thus, the first integral in (5.17) is bounded by

$$O(1)\exp\bigl(-\mathrm{const}\cdot |n|^{1/3}\bigr) = o(|n|^{-5/3}).$$

The second integral in (5.17) (with $j = 2$) is estimated in a similar fashion by reducing the summation in (5.7) to that over the pairs $x = (1, x_2)$ only.

As a result, we get that $I_3 = o(|n|^{-5/3})$. Substituting this estimate, together with (5.15) and (5.16), into (5.11) we obtain (5.3), and the proof of Theorem 5.1 is complete.

5.4. Proof of the limit shape results

With all preparations at hand, we are finally in a position to prove the uniform convergence of the scaled polygonal paths $\tilde{\xi}_n(\cdot) := (n_1^{-1}\xi_1(\cdot), n_2^{-1}\xi_2(\cdot))$ to the limit $g^*(\cdot) = (g_1^*(\cdot), g_2^*(\cdot))$ *in probability* with respect to both measures Q_z and P_n. Note that Theorems 5.5 and 5.6 below can be easily reformulated using the tangential distance $d_{\mathcal{T}}(\tilde{\Gamma}_n, \gamma^*)$ defined in (1.9) (cf. Theorem 1.1 which is stated in these terms).

Let us first establish the universality of the limit shape under the measure Q_z.

Theorem 5.5. *Under Assumption* 3.1*, for each* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} Q_z\left\{\sup_{0\le t\le\infty}\left|n_j^{-1}\xi_j(t) - g_j^*(t)\right| \le \varepsilon\right\} = 1 \qquad (j = 1, 2).$$

Proof. By Theorems 3.3 and 3.4, the expectation of the random process $n_j^{-1}\xi_j(t)$ uniformly converges to $g_j^*(t)$ as $n \to \infty$. Therefore, we only need to check that, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} Q_z\left\{\sup_{0\le t\le\infty} n_j^{-1}\left|\xi_j(t) - E_z[\xi_j(t)]\right| > \varepsilon\right\} = 0. \tag{5.19}$$

Note that the random process $\xi_j^0(t) := \xi_j(t) - E_z[\xi_j(t)]$ has independent increments and zero mean, hence it is a martingale with respect to the filtration $\mathcal{F}_t = \sigma\{\nu(x),\, x \in \mathcal{X}(t)\}$, $t \in [0, \infty]$. From the definition of $\xi_j(t)$ (see (3.20)), it is also clear that $\xi_j^0(\cdot)$ is *càdlàg* (i.e., its paths are everywhere right-continuous and have left limits). Therefore, applying the Doob–Kolmogorov submartingale inequality (see, e.g., [23, Theorem 6.14, p. 99]) and using Theorem 4.1, we obtain

$$Q_z\left\{\sup_{0\le t\le\infty} |\xi_j^0(t)| > \varepsilon n_j\right\} \le \frac{\mathrm{Var}_z(\xi_j(\infty))}{\varepsilon^2 n_j^2} \asymp |n|^{-2/3} \to 0, \qquad n \to \infty.$$

Hence, the limit (5.19) follows. □

Let us now prove our main result about the universality of the limit shape under the measure P_n (cf. Theorem 1.1).

Theorem 5.6. *Let* $A^+(1) < \infty$ *and Assumption* 5.1 *be satisfied. Then for any* $\varepsilon > 0$

$$\lim_{n\to\infty} P_n\left\{\sup_{0\le t\le\infty}\left|n_j^{-1}\xi_j(t) - g_j^*(t)\right| \le \varepsilon\right\} = 1 \qquad (j = 1, 2).$$

Proof. Like in the proof of Theorem 5.5, the claim is reduced to the limit (cf. (5.19))

$$\lim_{n\to\infty} P_n\left\{\sup_{0\le t\le\infty} |\xi_j^0(t)| > \varepsilon n_j\right\} = 0, \tag{5.20}$$

where $\xi_j^0(t) = \xi_j(t) - E_z[\xi_j(t)]$. Using the definition (2.10) we easily get the bound

$$P_n\left\{\sup_{0\le t\le\infty} |\xi_j^0(t)| > \varepsilon n_j\right\} \le \frac{Q_z\left\{\sup_{0\le t\le\infty} |\xi_j^0(t)| > \varepsilon n_j\right\}}{Q_z\{\xi = n\}}. \tag{5.21}$$

Again applying the Doob–Kolmogorov submartingale inequality [23, Theorem 6.14, p. 99] (but now with the sixth moment) and using Lemma 4.6, we obtain

$$Q_z\left\{\sup_{0\le t\le\infty} |\xi_j^0(t)| > \varepsilon n_j\right\} \le \frac{E_z\left[(\xi_j^0)^6\right]}{\varepsilon^6 n_j^6} \asymp |n|^{-2}. \tag{5.22}$$

On the other hand, by Corollary 5.2

$$Q_z\{\xi = n\} \asymp |n|^{-4/3}. \tag{5.23}$$

Combining (5.22) and (5.23), we conclude that the right-hand side of (5.21) is dominated by a quantity of order of $O(|n|^{-2/3}) \to 0$, and so the limit in (5.20) follows. □

6. Examples

Let us now consider a few illustrative examples by specifying the generating function $u \mapsto \beta(u) = \sum_{k=0}^{\infty} b_k u^k$ (see (2.4)). Although the associated multiplicative measures Q_z and P_n are defined primarily in terms of the coefficients $\{b_k\}$ (see (2.8) and (2.12), respectively), explicit expressions for b_k may be complicated, so we will not always attempt to give such expressions. For our purposes, it is more important to focus on the function $u \mapsto \ln \beta(u)$ and its power expansion coefficients $\{a_k\}$, since these are the ingredients that determine the convergence to the limit shape (1.4). In particular, we will have to check the basic condition $A^+(2) < \infty$ (see Assumption 3.1), as well as the refined condition $A^+(1) < \infty$ and Assumption 5.1, both needed for the limit shape result under the measure P_n (see Theorem 5.6).

Remark 6.1. It is worth pointing out that Examples 6.1, 6.2 and 6.3 have direct analogs in the theory of decomposable combinatorial structures, corresponding to the well-known meta-classes of *multisets*, *selections*, and *assemblies*, respectively (see [2] and [1, §2.2]). For further details about this correspondence and, more generally, for an extensive discussion of the combinatorial interpretation of the generating functions described in Examples 6.1–6.6 below, the reader is referred to the recent paper [4, §§ 6.1, 6.2].

Example 6.1. For $r \in (0, \infty)$, $\rho \in (0, 1]$, let Q_z be a measure on the space $\varPi$ determined by the formula (2.5) with coefficients

$$b_k = \binom{r+k-1}{k} \rho^k, \qquad k \in \mathbb{Z}_+.$$

A particular case with $\rho = 1$ was considered in [7] (cf. (1.7)). Note that $b_0 = 1$, in accordance with our convention in Section 2.1, and $b_1 = r\rho$. By the binomial expansion formula, the generating function of the sequence (1.7) is given by

$$\beta(u) = (1 - \rho u)^{-r}, \qquad |u| < \rho^{-1}, \tag{6.1}$$

and formula (2.5) specializes to

$$Q_z\{\nu(x) = k\} = \binom{r+k-1}{k} \rho^k z^{kx} (1 - \rho z^x)^r, \quad k \in \mathbb{Z}_+ \quad (x \in \mathcal{X}), \tag{6.2}$$

which is a negative binomial distribution with parameters r and $p = 1 - \rho z^x$.

If $r = 1$ then $b_k = \rho^k$, $\beta(u) = (1 - \rho u)^{-1}$ and, according to (6.2),

$$Q_z\{\nu(x) = k\} = \rho^k z^{kx} (1 - \rho z^x), \quad k \in \mathbb{Z}_+ \quad (x \in \mathcal{X}).$$

In turn, from formulas (1.6) and (2.12) we get

$$P_n(\varGamma) = \frac{\rho^{N_\varGamma}}{\sum_{\varGamma' \in \varPi_n} \rho^{N_{\varGamma'}}}, \qquad \varGamma \in \varPi_n, \tag{6.3}$$

where $N_\varGamma := \sum_{x \in \mathcal{X}} \nu(x)$ is the total number of integer points on $\varGamma \setminus \{0\}$. Furthermore, if also $\rho = 1$ then (6.3) is reduced to the uniform distribution on $\varPi_n$ (see (2.12)),

$$P_n(\varGamma) = \frac{1}{\#(\varPi_n)}, \qquad \varGamma \in \varPi_n.$$

In the general case, using (6.1) we note that

$$\ln \beta(u) = -r \ln(1-\rho u) = r \sum_{k=1}^{\infty} \frac{\rho^k u^k}{k},$$

and so the coefficients $\{a_k\}$ in the expansion (2.13) are given by

$$a_k = \frac{r\rho^k}{k} > 0, \qquad k \in \mathbb{N} \qquad (0 < \rho \le 1).$$

As pointed out in Remark 5.1, this implies that Assumption 5.1 is satisfied; also, it readily follows that $A^+(\sigma) < \infty$ for any $\sigma > 0$ (and each $\rho \in (0,1]$).

Example 6.2. For $m \in \mathbb{N},\ \rho \in (0,1]$, consider the generating function

$$\beta(u) = (1+\rho u)^m, \qquad |u| < \rho^{-1}, \tag{6.4}$$

with the coefficients in the expansion (2.4) given by

$$b_k = \binom{m}{k} \rho^k = \frac{m(m-1)\cdots(m-k+1)}{k!}\, \rho^k, \qquad k = 0, 1, \ldots, m.$$

In particular, $b_0 = 1$, $b_1 = m\rho$. Accordingly, the formula (2.5) gives a binomial distribution

$$Q_z\{\nu(x) = k\} = \binom{m}{k} \frac{\rho^k z^{kx}}{(1+\rho z^x)^m}, \qquad k = 0, 1, \ldots, m \quad (x \in \mathcal{X}), \tag{6.5}$$

with parameters m and $p = \rho z^x (1+\rho z^x)^{-1}$.

In the special case $m = 1$, the measure Q_z is concentrated on the subspace $\check{\varPi}$ of polygonal lines with "simple" edges, that is, containing no lattice points between the adjacent vertices. Here we have $b_0 = 1$, $b_1 = \rho$, and $b_k = 0\ (k \ge 2)$, so that (6.5) is reduced to

$$Q_z\{\nu(x) = k\} = \frac{\rho^k z^{kx}}{1+\rho z^x}, \qquad k = 0, 1 \quad (x \in \mathcal{X}).$$

Accordingly, the formula (2.12) specifies on the corresponding subspace $\check{\varPi}_n$ the distribution

$$P_n(\varGamma) = \frac{\rho^{N_\varGamma}}{\sum_{\varGamma' \in \check{\varPi}_n} \rho^{N_{\varGamma'}}}, \qquad \varGamma \in \check{\varPi}_n, \tag{6.6}$$

where the number of integer points $N_\varGamma$ coincides here with the number of vertices on $\varGamma \setminus \{0\}$. Furthermore, if also $\rho = 1$ then (6.6) is reduced to the uniform distribution on $\check{\varPi}_n$,

$$P_n(\varGamma) = \frac{1}{\#(\check{\varPi}_n)}, \qquad \varGamma \in \check{\varPi}_n.$$

In the general case, from (6.4) we obtain

$$\ln \beta(u) = m \ln(1+\rho u) = m \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \rho^k}{k}\, u^k, \tag{6.7}$$

hence the coefficients $\{a_k\}$ in the expansion (2.13) are given by

$$a_k = \frac{m(-1)^{k-1}\rho^k}{k}, \qquad k \in \mathbb{N} \qquad (0 < \rho \le 1),$$

and in particular $a_1 = m\rho > 0$. Note that $A^+(\sigma) < \infty$ for any $\sigma > 0$. Finally, let us check that Assumption 5.1 holds. Using (6.7) we obtain, for any $\theta \in (0,1)$ and all $t \in \mathbb{R}$,

$$\begin{aligned}
\ln\beta(\theta) - \Re(\ln\beta(\theta\,\mathrm{e}^{\mathrm{i}t})) &= m\ln(1+\rho\theta) - m\Re\bigl(\ln(1+\rho\theta\,\mathrm{e}^{\mathrm{i}t})\bigr)\\
&= m\ln(1+\rho\theta) - m\ln|1+\rho\theta\,\mathrm{e}^{\mathrm{i}t}|\\
&= -\frac{m}{2}\ln\left(\frac{1+2\rho\theta\cos t+\rho^2\theta^2}{(1+\rho\theta)^2}\right)\\
&\ge -\frac{m}{2}\left(\frac{1+2\rho\theta\cos t+\rho^2\theta^2}{(1+\rho\theta)^2}-1\right)\\
&= \frac{m\rho\theta(1-\cos t)}{(1+\rho\theta)^2} \ge \frac{m\rho}{(1+\rho)^2}\,\theta(1-\cos t).
\end{aligned}$$

Thus, the inequality (5.1) holds with $\delta_* = m\rho/(1+\rho)^2 > 0$.

Example 6.3. For $b \in (0,\infty)$, $\rho \in [0,1]$, consider the generating function

$$\beta(u) = \exp\left(\frac{bu}{1-\rho u}\right) = \exp\left(b\sum_{k=1}^{\infty} u^k\rho^{k-1}\right), \qquad |u| < \rho^{-1}. \tag{6.8}$$

Clearly, the corresponding coefficients b_k in the expansion (2.4) are positive, with $b_0 = 1$, $b_1 = b$, $b_2 = \frac{1}{2}b^2 + b\rho$, etc. More systematically, one can use the well-known Faà di Bruno's formula (see, e.g., [13, Ch. I, §12, p. 34]) to obtain (for $\rho > 0$)

$$b_k = \rho^k\sum_{m=1}^{k}\left(\frac{b}{\rho}\right)^m \sum_{(j_1,\dots,j_k)\in\mathcal{J}_m}\frac{1}{j_1!\cdots j_k!}, \qquad k \in \mathbb{N}, \tag{6.9}$$

where $\mathcal{J}_m$ is the set of all nonnegative integer k-tuples $(j_1,\dots,j_k)$ such that $j_1+\dots+j_k = m$ and $1\cdot j_1 + 2\cdot j_2 + \dots + k\cdot j_k = k$.

Taking the logarithm of (6.8), we see that the coefficients $\{a_k\}$ in (2.13) are given by

$$a_k = b\rho^{k-1} > 0, \qquad k \in \mathbb{N} \qquad (0 < \rho \le 1). \tag{6.10}$$

Therefore, Assumption 5.1 is automatic (see Remark 5.1); moreover, $A^+(\sigma) < \infty$ for any $\sigma > 0$, except for the case $\rho = 1$ where $A^+(\sigma) < \infty$ only for $\sigma > 1$.

In the special case $\rho = 0$, we have $\beta(u) = \mathrm{e}^{bu}$ and the expression (6.9) is reduced to $b_k = b^k/k!$, whereas (6.10) simplifies to $a_1 = b$ and $a_k = 0$ for $k \ge 2$. In this case, the random variables $\nu(x)$ ($x \in \mathcal{X}$) have a Poisson distribution with parameter bz^x,

$$Q_z\{\nu(x) = k\} = \frac{b^k z^{kx}}{k!}\,\mathrm{e}^{-bz^x}, \qquad k \in \mathbb{Z}_+ \qquad (x \in \mathcal{X}),$$

which leads, according to (2.12), to the following distribution on $\mathit{\Pi}_n$

$$P_n(\mathit{\Gamma}) = \left(\sum_{\{k'_x\}\in\mathit{\Pi}_n}\prod_{x\in\mathcal{X}}\frac{b^{k'_x}}{k'_x!}\right)^{-1}\prod_{x\in\mathcal{X}}\frac{b^{k_x}}{k_x!}, \qquad \mathit{\Gamma} \leftrightarrow \{k_x\} \in \mathit{\Pi}_n.$$

Example 6.4. Extending Example 6.3 (for simplicity, with $b = 1$), let us set for $r > 0$ and $\rho \in (0, 1]$

$$\beta(u) := \exp\left(\frac{u}{(1-\rho u)^r}\right), \qquad |u| < \rho^{-1}. \tag{6.11}$$

Taking the logarithm of (6.11) we get the power series expansion (cf. (6.1))

$$\ln \beta(u) = \frac{u}{(1-\rho u)^r} = \sum_{k=1}^{\infty} \binom{r+k-2}{k-1} \rho^{k-1} u^k, \tag{6.12}$$

which has positive coefficients a_k (cf. (1.7)). Hence, Assumption 5.1 is satisfied by virtue of Remark 5.1. To check the condition $A^+(\sigma) < \infty$, observe using Stirling's asymptotic formula for the gamma function (see [9, §12.5, p. 130]) that

$$a_k = \binom{r+k-2}{k-1} \rho^{k-1} = \frac{\Gamma(k+r-1)}{\Gamma(r)\,\Gamma(k)}\,\rho^{k-1} \sim \frac{k^{r-1}}{\Gamma(r)}\,\rho^{k-1} \qquad (k \to \infty),$$

hence $A^+(\sigma) < \infty$ for any $\sigma > 0$ if $\rho < 1$, whereas if $\rho = 1$ then $A^+(\sigma) < \infty$ only for $\sigma > r$.

On substituting (6.12) into Taylor's expansion of the exponential function in (6.11), it is evident that the corresponding coefficients b_k in the power series expansion of $\beta(u)$ are also positive, with $b_0 = b_1 = 1$, $b_2 = r\rho + \frac{1}{2}$, etc.

Example 6.5. Combining the exponential form of Example 6.4 with the generating function from Example 6.2, for $\rho \in [0, 1]$, $m \in \mathbb{N}$ consider

$$\beta(u) := \exp\bigl\{u(1+\rho u)^{m-1}\bigr\}. \tag{6.13}$$

Since $u \mapsto u(1+\rho u)^{m-1}$ is a polynomial of degree m with positive coefficients, it follows that the coefficients $\{b_k\}$ in the power series expansion of the function (6.13) are positive for all $k \in \mathbb{Z}_+$.

From (6.13) by the binomial formula we obtain the expansion

$$\ln \beta(u) = u(1+\rho u)^{m-1} = \sum_{k=1}^{m} \binom{m-1}{k-1} \rho^{k-1} u^k,$$

with the expansion coefficients $a_k > 0$ for $k = 1, \dots, m$ and $a_k = 0$ for $k \ge m+1$. Hence, Assumption 5.1 is satisfied and $A^+(\sigma) < \infty$ for any $\sigma > 0$.

Example 6.6. With $r \in (0, \infty)$, $\rho \in (0, 1]$, consider the generating function

$$\beta(u) = \left(\frac{-\ln(1-\rho u)}{\rho u}\right)^r = \left(1 + \sum_{k=1}^{\infty} \frac{\rho^k u^k}{k+1}\right)^r =: \beta_1(u)^r. \tag{6.14}$$

If $r = m \in \mathbb{N}$ then from (6.14) it is evident that the coefficients $\{b_k\}$ in the power series expansion of $\beta(u)$ are positive for all $k \in \mathbb{Z}_+$; however, for non-integer $r > 0$ this is not so clear, since the binomial expansion of $(1+t)^r$ involves negative terms. Yet the positivity of b_k for $k \ge 0$ holds for *any real* $r > 0$, which will be established below.

Let us first analyze the coefficients $\{a_k\}$ in the power series expansion of $\ln \beta(u) = r \ln \beta_1(u)$ (see (6.14)). Differentiation of the identity $r \ln \beta_1(u) = \sum_{k=1}^{\infty} a_k u^k$ gives

$$r\beta_1'(u) = \beta_1(u) \sum_{k=1}^{\infty} k a_k u^{k-1}. \tag{6.15}$$

Differentiating (6.15) again $k-1$ times ($k \geq 1$), by the Leibniz rule we obtain

$$\beta_1^{(k)}(0) = \frac{1}{r} \sum_{i=0}^{k-1} \binom{k-1}{i} \beta_1^{(k-1-i)}(0)\,(i+1)!\,a_{i+1}, \qquad k \in \mathbb{N}. \tag{6.16}$$

But we know from (6.14) that $\beta_1^{(j)}(0) = \rho^j j!/(j+1)$ ($j \in \mathbb{Z}_+$), and so the recurrence relation (6.16) specializes (after some cancellations) to the equation

$$\frac{k}{k+1} = \frac{1}{r} \sum_{i=0}^{k-1} \frac{\rho^{-i-1}(i+1)}{k-i}\, a_{i+1}. \tag{6.17}$$

Furthermore, denoting for short $\tilde{a}_j := r^{-1}\rho^{-j} j a_j$ ($j \in \mathbb{N}$) we can simplify (6.17) to

$$\frac{k}{k+1} = \sum_{i=0}^{k-1} \frac{\tilde{a}_{i+1}}{k-i}. \tag{6.18}$$

Setting here $k = 1, 2, 3, \dots$ we find successively

$$\tilde{a}_1 = \tfrac{1}{2}, \quad \tilde{a}_2 = \tfrac{5}{12}, \quad \tilde{a}_3 = \tfrac{3}{8}, \quad \tilde{a}_4 = \tfrac{251}{720}, \quad \dots$$

More generally, let us prove that

$$\frac{1}{k(k+1)} \leq \tilde{a}_k \leq \frac{k}{k+1}, \qquad k \in \mathbb{N}. \tag{6.19}$$

Since $\tilde{a}_1 = \frac{1}{2}$, the claim (6.19) is true for $k = 1$. Suppose now that the inequalities (6.19) hold for $\tilde{a}_1, \dots, \tilde{a}_{k-1}$ ($k \geq 2$), which entails that $\tilde{a}_1, \dots, \tilde{a}_{k-1} > 0$. Observe that the recurrence (6.18) (with k replaced by $k-1$) implies

$$\frac{k}{k+1} = \sum_{i=0}^{k-2} \frac{\tilde{a}_{i+1}}{k-i} + \tilde{a}_k \leq \sum_{i=0}^{k-2} \frac{\tilde{a}_{i+1}}{k-1-i} + \tilde{a}_k = \frac{k-1}{k} + \tilde{a}_k,$$

and it follows that

$$\tilde{a}_k \geq \frac{k}{k+1} - \frac{k-1}{k} = \frac{1}{k(k+1)}. \tag{6.20}$$

On the other hand, using that $\tilde{a}_1, \dots, \tilde{a}_{k-1} > 0$, from (6.18) we also get

$$\frac{k}{k+1} = \tilde{a}_k + \sum_{i=0}^{k-2} \frac{\tilde{a}_{i+1}}{k-i} \geq \tilde{a}_k. \tag{6.21}$$

Thus, the inequalities (6.20) and (6.21) prove the claim (6.19) for the $\tilde{a}_k$, and by induction it is valid for all $k \in \mathbb{N}$.

For the original coefficients a_k, the inequalities (6.19) are rewritten as

$$\frac{r\rho^k}{k^2(k+1)} \leq a_k \leq \frac{r\rho^k}{k+1}, \qquad k \in \mathbb{N}, \tag{6.22}$$

and in particular $a_k > 0$ for all $k \in \mathbb{N}$, so that Assumption 5.1 is automatically satisfied due to Remark 5.1. Furthermore, the inequalities (6.22) imply that $A^+(\sigma) < \infty$ for any $\sigma > 0$.

Finally, we can resolve the question of why the formula (6.14) defines a generating function with *nonnegative* coefficients: since Taylor's coefficients of the exponential function are positive, it is evident from the relation $\beta(u) = \exp\left\{\sum_{k=1}^{\infty} a_k u^k\right\}$ that $b_k > 0$ for all $k \in \mathbb{Z}_+$.

Acknowledgments

This work was supported in part by a Leverhulme Research Fellowship. Partial support by the Hausdorff Research Institute for Mathematics (Bonn) is also acknowledged. The author is grateful to Boris Granovsky, Ilya Molchanov, Stanislav Molchanov, Anatoly Vershik and Yuri Yakubovich for helpful discussions, and to the anonymous referees for pointing out an error in the original manuscript and for constructive comments that have helped to improve the presentation.

References

[1] R. Arratia, A.D. Barbour, S. Tavaré, Logarithmic Combinatorial Structures: a Probabilistic Approach, EMS Monographs in Mathematics, European Mathematical Society, Zürich, 2003; MR2032426.

[2] R. Arratia, S. Tavaré, Independent process approximations for random combinatorial structures, Adv. Math. 104 (1994) 90–154; MR1272071.

[3] I. Bárány, The limit shape of convex lattice polygons, Discrete Comput. Geom. 13 (1995) 279–295; MR1318778.

[4] L.V. Bogachev, Unified derivation of the limit shape for multiplicative ensembles of random integer partitions with equiweighted parts, Random Struct. Algorithms, first published online (18 April 2014), 40 pp.; DOI:10.1002/rsa.20540

[5] L.V. Bogachev, S.M. Zarbaliev, On the approximation of convex functions by random polygonal lines (Russian), Dokl. Akad. Nauk 364 (3) (1999) 299–302; English transl.: Approximation of convex functions by random polygonal lines, Doklady Math. 59 (1999) 46–49; MR1706217.

[6] L.V. Bogachev, S.M. Zarbaliev, A proof of the Vershik–Prokhorov conjecture on the universality of the limit shape for a class of random polygonal lines (Russian), Dokl. Akad. Nauk 425 (3) (2009) 299–304; English transl. in Doklady Math. 79 (2009) 197–202; MR2541116.

[7] L.V. Bogachev, S.M. Zarbaliev, Universality of the limit shape of convex lattice polygonal lines, Ann. Probab. 39 (2011) 2271–2317; MR2932669.

[8] L.V. Bogachev, S.M. Zarbaliev, Inverse problem of the limit shape for convex lattice polygonal lines, Preprint (2011), `http://arxiv.org/abs/1110.6636` (last accessed 01.11.2011).

[9] H. Cramér, Mathematical Methods of Statistics, Princeton Mathematical Series, vol. 9, Princeton University Press, Princeton, NJ, 1946; MR0016588.

[10] M.M. Erlihson, B.L. Granovsky, Limit shapes of Gibbs distributions on the set of integer partitions: The expansive case, Ann. Inst. H. Poincaré Probab. Statist. 44 (2008) 915–945; MR2453776.

[11] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, 6th ed., Oxford University Press, Oxford, 2008; MR2445243.

[12] A. Ivić, The Riemann Zeta-Function: The Theory of the Riemann Zeta-Function with Applications, A Wiley-Interscience Publication, Wiley, New York, 1985; MR0792089.

[13] Ch. Jordan, Calculus of Finite Differences, 3rd ed., Chelsea, New York, 1965; MR0183987.

[14] M. Kendall, A. Stuart, The Advanced Theory of Statistics, Volume 1: Distribution Theory, 4th ed., Macmillan, New York, 1977; MR0467977.

[15] Yu.V. Prokhorov, Private communication (1998).

[16] Ya.G. Sinaĭ, A probabilistic approach to the analysis of the statistics of convex polygonal lines (Russian), Funktsional. Anal. i Prilozhen. 28 (2) (1994) 41–48; English transl.: Probabilistic approach to the analysis of statistics for convex polygonal lines, Funct. Anal. Appl. 28 (1994) 108–113; MR1283251.

[17] E.C. Titchmarsh, The Theory of Functions, 2nd ed., Oxford University Press, Oxford, 1952.

[18] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed., Oxford University Press, Oxford, 1986; MR0882550.

[19] A.M. Vershik, The limit form of convex integral polygons and related problems (Russian), Funktsional. Anal. i Prilozhen. 28 (1) (1994) 16–25; English transl.: The limit shape of convex lattice polygons and related topics, Funct. Anal. Appl. 28 (1994) 13–20; MR1275724.

[20] A.M. Vershik, Statistical mechanics of combinatorial partitions, and their limit configurations (Russian), Funktsional. Anal. i Prilozhen. 30 (2) (1996) 19–39; English transl.: Statistical mechanics of combinatorial partitions, and their limit shapes, Funct. Anal. Appl. 30 (1996) 90–105; MR1402079.

[21] D.V. Widder, The Laplace Transform, Princeton Mathematical Series, vol. 6, Princeton University Press, Princeton, NJ, 1941; MR0005923.

[22] Yu. Yakubovich, Ergodicity of multiplicative statistics, J. Combin. Theory Ser. A 119 (2012) 1250–1279; MR2915644.

[23] J. Yeh, Martingales and Stochastic Analysis, Series on Multivariate Analysis, vol. 1, World Scientific, Singapore, 1995; MR1412800.