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On the Mathematical Modelling of a Compressible Viscoelastic Fluid

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Abstract

The fundamental principles associated with the development of mathematical models for compressible viscoelastic fluids are described using methodology and examples, that are sometimes conflicting, from the literature. The modelling of compressibility introduces two additional issues that need to be addressed over those that are required for incompressible fluids. The first issue is concerned with the role of variable density in the derivation of viscoelastic constitutive equations. The second, and perhaps more controversial issue, is concerned with the definition of pressure in a compressible setting, where it is seen to be dependent on bulk viscosity. A heuristic derivation of a compressible version of the Upper Convected Maxwell (UCM) model, starting from first principles in elasticity continuum mechanics, is presented, which suggests a dependence between dynamic viscosity and bulk viscosity, thereby addressing both issues.

1 Introduction

A survey of the literature reveals that there has been substantial progress in the development of numerical techniques for predicting the flow of incompressible viscoelastic fluids (see **?**, for example). However, once the incompressibility constraint is relaxed, a corresponding survey unveils very few contributions. Nevertheless, for many engineering applications, the incompressibility assumption is not realistic. The assumption is generally made to simplify the model or to reduce the computational overhead of the associated numerical simulations rather than because the corresponding equations would be difficult to formulate.

The formulation of properly structured equations with an optimum package of constitutive information falls within the remit of nonequilibrium thermodynamics. Modern thermodynamics is equipped with the appropriate tools for an elegant treatment of both reversible and irreversible dynamics. Until relatively recently, the formulation of equations in continuum mechanics using the Poisson bracket formalism was limited to conservative systems. The extension to non-conservative systems or irreversible dynamics came in 1984 with an extended bracket description of continuum systems (**???**). The extended bracket formulation was obtained by supplementing the Poisson bracket with a new bracket, called the 'dissipative bracket'. It is an example of what is known as a two-generator formalism since the nonequilibrium dynamics is driven by two potentials, viz. the total internal energy and the entropy of the system. The two-generator bracket formalism is the basis of the GENERIC formalism (**??**), which among other things provides a systematic means for constructing the dissipative bracket using the degeneracy relations that arise as a consequence of the energy dissipative invariants. On the other hand, **?** adopted a more macroscopic approach in the sense that they were concerned with variable fields defined in three-dimensional space rather than an extended space that also included structural coordinates. In their approach, the dissipative bracket was taken to be the most general description that could be used to represent dissipative dynamics under the conservation of energy constraint.

This paper is concerned with the mathematical modelling of compressible viscoelastic fluids. There are two important issues that need to be addressed over and above those that are required for a description of incompressible fluids: modifications to the rheological equation of state to account for compressibility and the role of pressure in compressible models. In this paper we provide a critical review of some of the models that have been proposed in the literature and also present several derivations of a compressible UCM model. The first derivation is based on the general framework for nonequilibrium dynamics developed by **?**. Three alternative derivations of this model are also presented. The first of these has its foundation in the principles underlying continuum mechanics and elasticity. The second derivation uses a density dependent integral form for the Cauchy stress to obtain a generalisation of the Maxwell model. The final derivation uses modern differential geometry and an analysis of convected derivatives in a compressible context.

? worked within a weakly compressible context, where density is not expected to vary significantly i.e. it assumes an essentially constant value, and showed that a term containing the divergence of velocity must be included in the constitutive relation for the polymeric contribution to the extra stress in order to obtain a compressible version of the incompressible Upper Convected Maxwell (UCM) model. This is achieved by deriving the constitutive equation from the elastic dumbbell model in kinetic theory and observing that particle number can vary with density in the

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compressible setting. When a compressible viscoelastic model is derived heuristically from fundamental rational fluid mechanics this model can be recovered if certain assumptions are made. For example, we note that the UCM model has as a limiting case the Newtonian model, but we know that the compressible Newtonian model has two free parameters that serve to determine its viscous behaviour. However, the compressible UCM of **?** has only one free parameter in the limiting Newtonian case. The extra parameter is commonly denoted by the term bulk viscosity, κ, and is intimately related to the new role that pressure plays in compressible fluids.

Pressure within an incompressible context is a Lagrange multiplier field which serves to ensure local volume conservation. The definition of pressure, p, in terms of the Cauchy stress, $\boldsymbol{\sigma} \equiv -p\mathbf{I} + \mathbf{T}$, is commonly reserved for the *static* pressure and $\mathbf{T}$ is referred to as the *extra* stress. On the other hand

$$\tilde{p} \equiv -\frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) \equiv p - \frac{1}{3}\mathrm{tr}(\mathbf{T}) \tag{1}$$

is commonly referred to as the *augmented* pressure (see **?**, for example). In general flows, $\tilde{p}$ will differ from p, but within an incompressible context, which one of the two quantities is adopted is irrelevant. Absorbing the isotropic part of $\mathbf{T}$ into $\tilde{p}$ and then rewriting the momentum equation in terms of $\tilde{p}$ and the divergence of a trace-free tensor, $\mathbf{T}^D \equiv \mathbf{T} - \frac{1}{3}\mathrm{tr}(\mathbf{T})\mathbf{I}$, leaves the dynamics unchanged. Therefore, in an incompressible setting either p or $\tilde{p}$ can represent the Lagrange multiplier field. However, this is not to say that advantage cannot be taken of the equivalence between the two formulations in the discretisation scheme. For example, **?** showed that, when performing numerical simulations, higher Weissenberg numbers are achievable in incompressible fluids if care is taken in the formulation to absorb the isotropic part of the polymeric stress into the pressure term, i.e. the formulation with $\tilde{p}$ is more stable numerically.

The compressibility of a fluid changes this situation radically because the mass conservation equation no longer enters the equations as a constraint, but as a dynamic equation which is coupled to the momentum equation through the velocity, $\mathbf{u}$, and via the equation of state, which usually specifies pressure as a function of density (and more generally, temperature or entropy). The question then is: Do we use the augmented pressure or static pressure in the equation of state? **?** adopt a weakly compressible framework so the issue is perhaps not as pertinent there. **?** use the static pressure, but in later papers adopt the use of augmented pressure in the equation of state (see **?**, **?**), presumably recognising that augmented pressure, $\tilde{p}$, has more physical significance than static pressure, p, within a dynamic setting. Of course, in an isotropic fluid at equilibrium in a region containing a free surface, the two pressure fields are equal since the isotropic part of the extra stress vanishes.

In this paper a simple argument is established to show that using augmented pressure in an equation of state only involving the state variable density, ρ, (for simplicity we consider only isothermal fluids) leads to a counterintuitive conclusion, namely that, with this choice, bulk viscosity plays no role in the dynamics of a Newtonian fluid. This modelling deficiency for a relatively simple fluid casts doubt on this approach to the equation of state for more complex models such as UCM and Oldroyd B, for example, which generate isotropic stresses that would effectively be ignored with only the trace-free part of the extra stress, $\mathbf{T}^D$, contributing to the equation of motion.

In its most general form we derive the Upper Convected Maxwell constitutive equation for a compressible fluid (see equation (**??**)). In its most natural special case we obtain for the extra stress, $\mathbf{T}$

$$\lambda\left[\overset{\triangledown}{\mathbf{T}} + (\nabla\cdot\mathbf{u})\mathbf{T}\right] + \mathbf{T} = \mu\left(2\mathbf{D} + \left[\frac{\rho}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\rho} - 1\right](\nabla\cdot\mathbf{u})\mathbf{I}\right)$$

where μ is the dynamic viscosity. For constant kinematic viscosity, η_0 ($\eta_0 = \mu/\rho$) the constitutive relation becomes

$$\lambda\overset{\triangledown}{\mathbf{S}} + \mathbf{S} = 2\eta_0\mathbf{D}$$

where the kinematic stress tensor is defined, $\mathbf{S} \equiv \mathbf{T}/\rho$ (other notation is defined later in the paper).

The pressure using the Helmholtz free density given in **?** is shown, for the UCM, to be given by

$$p = p_0(\rho) + \tfrac{1}{2}\mathrm{tr}(\mathbf{T})$$

for an isothermal fluid, where $p_0(\rho)$ is some equation of state. This gives, in the absence of body forces, the momentum equation

$$\begin{aligned}\rho\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} &= -\nabla p + \nabla\cdot\mathbf{T}\\ &= -c^2\nabla\rho - \tfrac{1}{2}\nabla\mathrm{tr}(\mathbf{T}) + \nabla\cdot\mathbf{T}\end{aligned}$$

where the speed of sound at constant volume is $c^2 = \mathrm{d}p/\,\mathrm{d}\rho$.

This paper is organised as follows. In section 2, examples of compressible viscoelastic models in the literature are used to motivate the modifications to viscoelastic models that are necessary due to fluid compressibility. Section 3 looks more closely at the implications of these models. In particular the problem of the definition of pressure where a resolution of the difficulty is suggested by the work of **?**. Section 4 looks more closely at viscoelastic models from a rational mechanical perspective and first principles. At the end of this section an heuristic explanation is given for one of the major modifications introduced by fluid compressibility, which shows the modification is simply due to a coordinate transformation of equations derived from a hyperelastic function. Section 5 presents a compressible viscoelastic fluid model starting from an integral model and in section 6, the same conclusion is reached from an analysis of convected derivatives within a compressible context. Finally a summary is provided in section 7.

2 Review of Current Compressible Models

The governing equations for fluid flow comprise the mathematical statements of momentum and mass conservation together with a constitutive equation and an equation of state. The conservation of momentum is

$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = \nabla \cdot \boldsymbol{\sigma} \equiv -\nabla p + \nabla \cdot \mathbf{T} \tag{2}$$

where the Cauchy stress tensor is expressed as $\boldsymbol{\sigma} = -p\mathbf{I} + \mathbf{T}$. Conservation of mass is given by

$$\frac{\mathrm{D}\,\rho}{\mathrm{D}t} + \rho\nabla \cdot \mathbf{u} = 0 \tag{3}$$

where the density, ρ, is a field variable for compressible fluids. Hence, we require a constitutive equations for the extra stress, $\mathbf{T}$, and an equation of state for the pressure, p, in order to close the system of equations. In order to satisfy isotropy, $\mathbf{T}$ must be symmetric. However, it must also satisfy the constraints of objectivity, which is activated by the use of so called objective derivatives, namely upper and lower convective derivatives of the field variables. The upper convective derivative of a general symmetric second rank tensor, $\mathbf{A}$, is defined to be (see **?**, for example):

$$\overset{\triangledown}{\mathbf{A}} \equiv \frac{\mathrm{D}\,\mathbf{A}}{\mathrm{D}t} - \nabla\mathbf{u} \cdot \mathbf{A} - \mathbf{A} \cdot \nabla\mathbf{u}^T \tag{4}$$

The objectivity of this derivative is true and remains unaltered for compressible fluids. However, if stress is viewed as a tensor *density* then some modification to the above expression is necessary. We return to this issue in Section 6.2. It is interesting to note that this derivative reduces to the material derivative when acting on scalar fields, but has yet another form when operating on vector fields, which is rarely seen in the literature. In other fields of mathematical physics all these objective derivatives are manifestations of the *Lie* derivative, which operates on tensors of arbitrary rank and type including, for example, dual vector fields which leads to the definition of the lower convective derivative (see **?**, for example).

2.1 Model A (?)

The generalisation of incompressible fluid models to compressible fluids was initially treated by considering weakly compressible fluids. For example, **?** investigated the properties of the following equations:

$$\rho_0 \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \mathbf{T}^D, \tag{5}$$

and

$$\frac{\mathrm{D}\,p}{\mathrm{D}t} + \beta\nabla \cdot \mathbf{u} = 0, \tag{6}$$

with constitutive relation

$$\lambda \overset{\triangledown}{\mathbf{T}} + \mathbf{T}(1 + \lambda\nabla \cdot \mathbf{u}) = 2\mu\mathbf{D}. \tag{7}$$

where $\mathbf{D} \equiv \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$.

2.2 Model B (?)

For an extension to an Oldroyd B model **?** considered an extension of this model to fluids with a solvent component and obtained a compressible Oldroyd B model. The equation of motion is

$$\rho_0 \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot (\mathbf{T}_s + \mathbf{T}_e) \tag{8}$$

where $\mathbf{T}_s$ and $\mathbf{T}_e$ are the solvent and polymeric contributions to the extra-stress tensor. The constitutive equation for the solvent contribution is

$$\mathbf{T}_s \equiv 2\mu_s \mathbf{D} + \mu_s \left(\frac{\kappa}{\mu_s} - \frac{2}{3} \right) (\nabla \cdot \mathbf{u})\mathbf{I}. \tag{9}$$

where μ_s is the dynamic solvent viscosity and κ is the bulk viscosity. The constitutive relation for the polymeric contribution, $\mathbf{T}_e$, is the UCM model

$$\lambda \overset{\triangledown}{\mathbf{T}}_e + \mathbf{T}_e = 2\mu_e \mathbf{D}$$

where μ_e is the dynamic polymeric viscosity. This is a generalisation of the Oldroyd B model. The continuity equation is given by (**??**) and the equation of state is

$$\tilde{p} = B \left[\left(\frac{\rho}{\rho_0} \right)^m - 1 \right] + p_a \tag{10}$$

where B and m are given constants. **?** choose $m = 4$ and B may be related to sound speed at a given pressure. The reference pressure, p_a, can be taken as atmospheric pressure in which case the speed of sound, c, is defined by $c^2 \equiv p'(\rho) = c_0^2 (\rho/\rho_0)^{m-1}$ where c_0 is the speed of sound at atmospheric pressure given by $c_0^2 = mB/\rho_0$. Thus, in this model, augmented pressure has the form

$$\tilde{p} = p - \frac{1}{3}\mathrm{tr}(\mathbf{T}_s + \mathbf{T}_e)$$

Thus the momentum equation (**??**) can be written

$$\rho_0 \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = -\nabla\tilde{p} + \nabla \cdot (\mathbf{T}_s + \mathbf{T}_e)^D$$

where $\tilde{p}$ is given by (**??**). Note that, κ and any isotropic parts of $\mathbf{T}_e$ are ignored by this model.

2.3 Model C (?)

? using extended thermodynamics gives the model for a Maxwell fluid as follows

$$\begin{aligned} \rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} &= \nabla \cdot \boldsymbol{\sigma} \\ \frac{\mathrm{D}\,\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} &= 0 \\ \lambda \left[\overset{\triangledown}{\mathbf{T}} + \tfrac{2}{3}(\mathbf{T} : \mathbf{D})\mathbf{I} \right] + \mathbf{T} &= 2\mu_e \mathbf{D}^D \\ \mathbf{T} &\equiv \boldsymbol{\sigma}^D \\ -p &\equiv \tfrac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}) \\ \rho &= \rho(\boldsymbol{\sigma}) \end{aligned} \tag{11}$$

where we have removed all temperature dependence. Extended thermodynamics constructs the field equations using balance equations of mass, momentum, momentum flux and energy flux. These are derived by taking moments of the phase density or particle number density, f, which satisfies Boltzmann's equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} + \mathbf{b} \cdot \frac{\partial f}{\partial \mathbf{v}} = C[f, f].$$

In this equation, $C[f, f]$ is known as the collision operator, $\mathbf{x}$ and $\mathbf{v}$ are the particle's position and velocity respectively and $\mathbf{b}$ is a body force independent of $\mathbf{v}$. The fluid density, for example, is given by

$$\rho(t, \mathbf{x}) = \int m f(t, \mathbf{v}, \mathbf{x})\,\mathrm{d}\mathbf{v}$$

where m is the mass of the particle. In simple terms, the actual particle position and velocity in a fluid is replaced by a density, f, and changes in its motion arise solely through the collision operator for which there are various approximate forms. In this way a continuum model is derived from a large number of interacting particles.

To derive a non-Newtonian model the usual hierarchy for a Newtonian fluid – where the flux in the n^{th} tensorial equation acts as a source in the $(n+1)^{\text{th}}$ tensorial equation – is generalised by adding a further set of tensors. Identification of these tensors with quantities such as density, energy flux, stress etc. followed by decomposition into non-convective and convective parts lead to the system of equations (**??**).

2.4 Model D (?)

? postulate a thermodynamically consistent formulation for non-isothermal polymeric fluids. They achieve this by extending the Poisson bracket of Hamiltonian systems to include an operator associated with dissipation. The equation reads

$$\frac{\partial x}{\partial t} = L\frac{\delta E}{\delta x} + M\frac{\delta S}{\delta x} \tag{12}$$

where $\delta/\delta x$ denotes functional derivatives, E the total energy, S the total entropy and x is an eleven dimensional vector field, $x = [\rho, \mathbf{u}, \epsilon, \mathbf{c}]$, incorporating the density, velocity, internal energy and conformation tensor (2nd rank), respectively. In equation (**??**), L and M are matrices associated with the reversible and irreversible dynamics of the flow, respectively. For isothermal polymeric fluids they obtain the usual conservation equations for mass

$$\frac{\mathrm{D}\,\rho}{\mathrm{D}t} + \rho\nabla\cdot\mathbf{u} = 0 \tag{13}$$

and momentum

$$\rho\,\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla\cdot\mathbf{T} + \rho\mathbf{b} \tag{14}$$

are derived and a more complex than usual equation for energy[1]. The constitutive equation implied for an isothermal fluid reduces to

$$\lambda\overset{\triangledown}{\mathbf{T}} + \mathbf{T}(1+\lambda\nabla\cdot\mathbf{u}) = 2\mu\mathbf{D}. \tag{15}$$

3 Model discussion

The models given in Section **??** highlight different issues when moving from incompressible to compressible complex fluids: Models A and D introduce an extra term in the constitutive equation; Models B, C and D raise the issue of pressure; Model C is derived from extended thermodynamics; Model D, derived from non-equilibrium thermodynamics, implies that the conformation tensor is the more natural object to use in modelling compressible flows.

3.1 Model A, weak compressibility

The constant β in (**??**) is referred to as the compressibility factor. The term in brackets in the constitutive equation (**??**) was part of the subject for discussion in the paper of **?** and was introduced by way of a reinterpretation of the dumbbell model. These equations are approximations for weakly compressible fluids derived by considering the approximation for density

$$\rho = \rho_0(1+\epsilon)$$

where $\epsilon = (p - p_0)/\beta$. The continuity equation (**??**) now becomes

$$\frac{\rho_0}{\beta}\frac{\mathrm{D}\,p}{\mathrm{D}t} + \rho_0(1+\epsilon)\nabla\cdot\mathbf{u} = 0$$

so that

$$\frac{\mathrm{D}\,p}{\mathrm{D}t} + \beta(1+\epsilon)\nabla\cdot\mathbf{u} = 0$$

If we now assume that $\epsilon \ll 1$ we arrive at (**??**). Similarly, we can ignore the term $\rho_0\epsilon\,\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t}$ in the momentum equation to arrive at (**??**).

3.2 Model B, choice of pressure in the equation of state

With reference to Model B (see Section **??**), we have for general κ, that neither $\mathbf{T}_s$ nor $\mathbf{T}_e$ are trace-free and thus the augmented pressure, $\tilde{p}$, contains (all) the isotropic content of the Cauchy stress. Of particular note is that the equation of state is adapted to relate density to the augmented pressure, $\tilde{p}$, or conversely, the isotropic content of the total stress is directly related to density. If, on the other hand, the equation of state is written using the static pressure,

[1] This paper addresses isothermal flow only thus avoiding the energy equation

p, the model is changed. Thus the choice of an appropriate equation of state is one of the fundamental issues that needs to be resolved when extending existing incompressible viscoelastic models to compressible fluids. To explore the significance of this, consider two isothermal equations of state:

$$p = f(\rho) \tag{16}$$

and

$$\tilde{p} = f(\rho) \tag{17}$$

Using the first equation of state (**??**), we obtain

$$\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = -c^2\nabla\rho + \nabla\cdot\mathbf{T} \tag{18}$$

where $c^2 \equiv f'(\rho)$, and $\mathbf{T} \equiv \mathbf{T}_s + \mathbf{T}_e = \mathbf{T}^D - P\mathbf{I}$, where $-P\mathbf{I}$ is the isotropic part of $\mathbf{T}$. Thus, we may write the model as

$$\begin{aligned} \rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} &= -c^2\nabla\rho + \nabla\cdot\mathbf{T}^D - \nabla P, \\ \frac{\mathrm{D}\,\rho}{\mathrm{D}t} &= -\rho\nabla\cdot\mathbf{u}. \end{aligned} \tag{19}$$

On the other hand, if the equation of state (**??**) is used, we obtain

$$\begin{aligned} \rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} &= -c^2\nabla\rho + \nabla\cdot\mathbf{T}^D, \\ \frac{\mathrm{D}\,\rho}{\mathrm{D}t} &= -\rho\nabla\cdot\mathbf{u}. \end{aligned} \tag{20}$$

Ignoring the effects of viscoelasticity for the moment, and writing

$$\mathbf{T} = \mu\left[\nabla\mathbf{u} + \nabla\mathbf{u}^T - \frac{2}{3}(\nabla\cdot\mathbf{u})\right]\mathbf{I} + \kappa(\nabla\cdot\mathbf{u})\mathbf{I} \tag{21}$$

implies $P \equiv -\kappa(\nabla\cdot\mathbf{u})$. Then the use of (**??**) leading to (**??**) is seen to ignore the contribution from bulk viscosity, κ. In general, it would appear that *any* isotropic term generated by the constitutive relation is ignored. Another important consequence is that lift/drag calculations are also unaffected by changes to κ.

These observations strongly indicate that the model used in **?** is insufficiently general and that there are necessarily isotropic stresses that do not (directly) affect density. This can be made as a modelling assumption and is implicit in the use of (**??**) and (**??**). This view is supported by, for example, **?** and **?**. **?** appears to imply that, p, is associated with equations of state and not $\tilde{p}$, by referring to p as *the thermodynamic pressure*, giving an equation of state in the general form $p = p(\rho, \Theta)$, where Θ is temperature, and by defining the Cauchy stress for a Newtonian fluid as

$$\boldsymbol{\sigma} = (-p + \kappa\nabla\cdot\mathbf{u})\mathbf{I} + 2\mu\left[\mathbf{D} - \frac{1}{3}(\nabla\cdot\mathbf{u})\mathbf{I}\right] \tag{22}$$

Note that from statistical mechanical principles, the viscosities in this model are proportional to density implying constant *kinematic and bulk* viscosities. Constant kinematic viscosity is also convenient numerically (see **?**).

This approach is also implicit in **?**, where the Cauchy stress is given by

$$\boldsymbol{\sigma} = 2\mu_1\mathbf{D} + \mu_2(\nabla\cdot\mathbf{u})\mathbf{I} - p\mathbf{I} \tag{23}$$

where μ_1 is the dynamic viscosity, μ_2 is the second coefficient of viscosity and $p = p(\rho, \Theta)$. The bulk viscosity is defined by $\kappa \equiv \frac{2}{3}\mu_1 + \mu_2 \geq 0$. Once again $\mathrm{tr}(\boldsymbol{\sigma} + p\mathbf{I}) \neq 0$ except for the special case of zero bulk viscosity, $\mu_2 = -(2/3)\mu_1$. The implication is, again, that there can be a part of isotropic stress that has no directly corresponding density. The authors go on to point out that for isotropic fluids, the Cauchy stress, which includes p is, in general, a function of the three principal invariants, I_1, I_2, I_3, of $\mathbf{D}$, i.e.

$$\boldsymbol{\sigma} = h_0(I_1, I_2, I_3)\mathbf{I} + h_1(I_1, I_2, I_3)\mathbf{D} + h_3(I_1, I_2, I_3)\mathbf{D}^2 \tag{24}$$

The implication here is that h_0 is the thermodynamic pressure, but **?** do not clarify this point.

This issue is explored in some detail by **?**, who distinguishes between three types of pressure: the thermodynamic pressure, π; the mean pressure, $\tilde{p}$; and an arbitrary pressure, p. The thermodynamic pressure is related directly to the thermodynamic variables, ρ and Θ. The "mean" pressure is defined as the augmented pressure above. For incompressible fluids, $\pi = p$ and may be chosen to be also equal to $\tilde{p}$.

Since this discussion inevitably involves the energy conservation equation, we state it here for a general fluid using the definition $\dot{\gamma} \equiv 2\mathbf{D}$

$$\rho \frac{\mathrm{D}\, e}{\mathrm{D}t} = \boldsymbol{\sigma} : \dot{\boldsymbol{\gamma}} - \nabla \cdot \mathbf{q} \tag{25}$$

where for an incompressible fluid, $\boldsymbol{\sigma}$ can be replaced by a trace-free extra stress. In terms of entropy, η, and using the relation

$$\rho\Theta \frac{\mathrm{D}\, \eta}{\mathrm{D}t} = \rho \frac{\mathrm{D}\, e}{\mathrm{D}t} + \pi \nabla \cdot \mathbf{u} \tag{26}$$

equation (**??**) can be written in the form

$$\begin{aligned} \rho\Theta \frac{\mathrm{D}\, \eta}{\mathrm{D}t} &= \pi \nabla \cdot \mathbf{u} + \boldsymbol{\sigma} : \dot{\boldsymbol{\gamma}} - \nabla \cdot \mathbf{q} \\ &= (\pi - p) \nabla \cdot \mathbf{u} + \mathbf{T} : \dot{\boldsymbol{\gamma}} - \nabla \cdot \mathbf{q} \end{aligned} \tag{27}$$

where Truesdell suggests setting $\pi = p$ so that we can associate the extra stress, $\mathbf{T}$ (not necessarily trace-free), with entropy generation. Thus Truesdell gives a reasonable criterion to associate with the choice of thermodynamic pressure. Following Truesdell and setting $\pi = p$ and also making the choice, $p = \tilde{p}$, gives for entropy production

$$\rho\Theta \frac{\mathrm{D}\, \eta}{\mathrm{D}t} = \mathbf{T}^D : \dot{\boldsymbol{\gamma}} - \nabla \cdot \mathbf{q} \tag{28}$$

and only the trace-free part of the Cauchy stress, $\mathbf{T}^D$, contributes to entropy production. Clearly bulk viscosity must be associated with entropy generation in common with dynamic viscosity and so should not be absorbed into the thermodynamic equation of state.

On the other hand, the stresses associated with viscoelasticity are not so easily addressed. For a perfectly elastic body, the elastic stress is recoverable and so for time scales much less than the relaxation time for a Maxwell model, for example, one might expect a significant part of the stress not to be associated with entropy change and conversely for longer time scales. This will be examined later in this paper.

3.3 Model C, trace-free evolution of viscoelastic stress

In the final equation of (**??**), the equation of state is given as a general relation between the Cauchy stress and density. The natural choice being the first invariant of $\boldsymbol{\sigma}$ — the augmented pressure, p.

The third equation in (**??**) is an evolution equation for deviatoric stress, with an extra isotropic term, $\frac{2}{3}\mathbf{T} : \mathbf{D}$, added for consistency. We show this by proving that $\mathbf{T}$ remains trace free given a trace free initial condition. Using the property

$$\mathrm{tr}(\frac{\mathrm{D}\, \mathbf{T}}{\mathrm{D}t}) = \frac{\mathrm{D}\, \mathrm{tr}(\mathbf{T})}{\mathrm{D}t} = \mathbf{0}$$

gives

$$\mathrm{tr}(\overset{\triangledown}{\mathbf{T}}) = -u_{a,b} T_{a,b} - u_{b,a} T_{a,b} \equiv -2\mathbf{D} : \mathbf{T}. \tag{29}$$

We also have $\mathrm{tr}(\frac{2}{3}(\mathbf{T} : \mathbf{D})\mathbf{I}) = 2\mathbf{D} : \mathbf{T}$, which together with the definitions of $\mathbf{T}^D$ and $\mathbf{D}^D$, proves that $\mathbf{T}$ evolves trace-free. The entropy production is given once again by (**??**) implying, like the previous model, that there is no contribution to entropy from volume change.

3.4 Model D, the dissipative bracket and resolution of the pressure problem

? provide a framework for modelling thermodynamically consistent models for flow systems. In their framework, the governing evolution equation for an arbitrary functional F is given as

$$\frac{dF}{dt} = \{F, H\} + [F, H], \tag{30}$$

where $\{\cdot, \cdot\}$ and $[\cdot, \cdot]$ are the Poisson and dissipative brackets, respectively. The Poisson bracket operates on F and the total (extended) internal energy or Hamiltonian of the system. When the entropy density is one of the variables H is the Helmholtz free energy of the system.

The Poisson bracket is a bilinear, antisymmetric operator and it satisfies the Jacobi identity. The main drawback of systems developed solely using the Poisson bracket is that they are conservative since $\{H, H\} = 0$ due to antisymmetry and thus they are applicable for a treatment of reversible dynamics only.

The extended bracket description of continuum systems obtained by augmenting the Poisson bracket with the dissipative bracket is due to **?**, **?** and **?**. Although the Poisson bracket still plays a dominant role in this extended bracket description, the addition of the dissipative bracket allows for the description of nonconservative dissipative dynamics. In this description the entropic or irreversible terms are clearly delineated from the non-entropic terms. The dissipative bracket is bilinear, symmetric and positive definite

By specifying H and applying the chain rule to the arbitrary functional F and equating its functional derivative of each of the variables in turn the relation (**??**) gives a system of partial differential equations for the conformation tensor $\mathbf{C}$, the momentum $\mathbf{p}$, the density ρ and the entropy s relating the partial time derivative of each of the variables in terms of variables and gradients.

When the arguments of the functional F are ρ, s, $\mathbf{p}$ and $\mathbf{C}$, the Poisson bracket is given by equation (5.5-11) in the monograph of **?**. Unlike the Poisson bracket, the dissipative bracket has additional tensor coefficients connected with the averaging of the fast part of the time derivative of the macroscopic variables over a relatively large time scale. It can also, as we will see, include independent variables. In this sense the dissipative bracket (necessarily) is not as elegant a construction as the Poisson bracket. In equation (8.1-5) of **?** there are four tensor coefficients of fourth rank: three (Λ, Q and L) are associated with a general incompressible, isothermal, viscoelastic fluid while the fourth (B) is associated with transport affinity/flux. More precisely, Λ is associated with relaxation time(s), Q with viscosity, and L with 'non-affine motion in the system'. **?** ignore the fourth tensor coefficient B. Familiar models are recovered through the specification of these tensor coefficients. For example, the Newtonian fluid is recovered by specifying Q alone; the UCM model is recovered by specifying Λ alone; and the Oldroyd B model is recovered by specifying both Q and Λ.

The dissipative bracket for the UCM model can be specified by

$$[F, H] = -\int_\Omega \frac{2}{\mu}\frac{\delta F}{\delta \mathbf{C}} : \left(\mathbf{C} \cdot \frac{\delta H}{\delta \mathbf{C}}\right) \mathrm{d}^3\mathbf{x} \tag{31}$$

where the viscosity $\mu = G\lambda$, G is the 'spring' constant and λ is a relaxation time. The Hamiltonian is defined by

$$H = \int (K + E - s\Theta)\, \mathrm{d}^3\mathbf{x}$$

where K is the kinetic energy, E is the elastic energy, s is the entropy and Θ is the temperature. For a Maxwell system, the Hamiltonian is given by

$$H = \int_\Omega \left[\frac{1}{2\rho}\mathbf{p}\cdot\mathbf{p} + \frac{1}{2}G\mathrm{tr}(\mathbf{C}) - \frac{1}{2}G\ln\det(\mathbf{C})\right] \mathrm{d}^3\mathbf{x}$$

from which we derive

$$\frac{\delta H}{\delta \mathbf{C}} = \frac{1}{2}G(\mathbf{I} - \mathbf{C}^{-1}).$$

Therefore, the dissipative bracket for a Maxwell system is

$$[F, H] = -\frac{1}{\lambda}\int_\Omega \frac{\delta F}{\delta \mathbf{C}} : (\mathbf{C} - \mathbf{I})\, \mathrm{d}^3\mathbf{x}$$

The Poisson bracket is

$$\{F, H\} = \int_\Omega \frac{\delta F}{\delta \mathbf{C}} : \overset{\nabla}{\mathbf{C}}\, \mathrm{d}^3\mathbf{x}$$

giving the relation for conformation tensor

$$\lambda \overset{\nabla}{\mathbf{C}} + \mathbf{C} = \mathbf{I}. \tag{32}$$

The Poisson bracket shows the momentum balance to include a term $\nabla \cdot \mathbf{T}$ where

$$\mathbf{T} \equiv G(\mathbf{C} - \mathbf{I})$$

If we now modify H to include density dependence

$$H = \int_\Omega \left[\frac{1}{2\rho}\mathbf{p}\cdot\mathbf{p} + \frac{1}{2}G\mathrm{tr}(\rho\mathbf{C}) - \frac{1}{2}G\ln\det(\rho\mathbf{C})\right] \mathrm{d}^3\mathbf{x}$$

and include a density term in the denominator of the dissipation bracket, (??), we find (??) is unchanged but the relationship between the extra-stress and conformation tensors becomes

$$\mathbf{T} \equiv G\rho(\mathbf{C} - \mathbf{I})$$

This results in the constitutive equation for a compressible UCM fluid

$$\lambda(\overset{\nabla}{\mathbf{T}} + \nabla \cdot \mathbf{u}\mathbf{T}) + \mathbf{T} = \rho\mu\dot{\boldsymbol{\gamma}}$$

where, in terms of kinematic stress $\mathbf{S} \equiv \mathbf{T}/\rho \equiv G(\mathbf{C} - \mathbf{I})$, we have simply

$$\lambda \overset{\nabla}{\mathbf{S}} + \mathbf{S} = \mu\dot{\boldsymbol{\gamma}}.$$

The pressure is defined by

$$p = \rho\frac{\partial a}{\partial \rho} + \mathbf{C} : \frac{\partial a}{\partial \mathbf{C}} - a \tag{33}$$

where $a = a_0(\rho, \Theta) + \frac{1}{2}\rho G(\mathrm{tr}(\mathbf{C}) - \ln|\mathbf{C}|)$ is the Helmholtz free energy density[2]. We find that

$$p = \rho\frac{\partial a_0}{\partial \rho} + a - a_0 + \tfrac{1}{2}\rho G \mathrm{tr}(\mathbf{C} - \mathbf{I}) - a = p_0(\rho) + \tfrac{1}{2}\mathrm{tr}(\mathbf{T})$$

where $p_0(\rho) \equiv \rho\dfrac{\partial a_0}{\partial \rho} - a_0$. This implies the momentum equation

$$\rho\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \mathbf{T} = -\nabla p_0 - \tfrac{1}{2}\nabla \mathrm{tr}(\mathbf{T}) + \nabla \cdot \mathbf{T}.$$

4 Elasticity, Viscoelasticity and the Principle of Objectivity

In order to find a generalisation of viscoelasticity to compressible fluids it is necessary to revisit basic elasticity, the principle of objectivity and, for clarity, to use the notation of linear maps rather than matrix algebra in which to frame the dynamics. The concept and notation of pull back and push forward maps within the context of elasticity may be found in the book of ?.

In this section we use the symbol t as the current time and $\tau \le t$ as a reference past time. With the natural choice of a Cartesian frame, these relations, being linear, can be defined using matrix algebra. To define the notation, consider material coordinates, X^A, that coincide with spatial coordinates, x^a, in the fluid at time, τ, then the flow of the material is represented by the map $\phi_{\tau,t}$

$$\phi_{\tau,t} : (t, X^A) \mapsto [x^a = x(t, X^A)].$$

For example, in 2D we have

$$\phi_{\tau,t} : (t, X^A) \mapsto [x = x(t, X, Y), y = y(t, X, Y)]$$

This induces the linear pullback map, $(\phi_{\tau,t})^*$, from one material point of the flow at time t to its previous position at time τ. A function value at such a point does not change under the map (only its point of application) so that using comoving coordinates, X, Y we find $(\phi^*_{\tau,t}f)(t, X, Y) = f(t, X, Y)$. However, in Cartesian coordinates $(\phi^*_{\tau,t}f)(t, x, y) \ne f(t, x, y)$. The mapping $\phi_{\tau,t}$ is, in general, nonlinear but the pullback map, $(\phi_{\tau,t})^*$, is linear on dual vectors and can be represented using the deformation tensor, $\mathbf{F}$, as follows. Acting on the Cartesian basis dual vectors (or 1-forms), $\mathrm{d}x^a$

$$\phi^*_{\tau,t}\,\mathrm{d}x^a = \frac{\partial x^a}{\partial X^A}\,\mathrm{d}X^A \equiv F^a{}_A\,\mathrm{d}X^A$$

[2]Note that in equation (9.1-13) of ? the Helmholtz free energy is defined by $a = a_0(\rho, \Theta) + \frac{1}{2}G(\mathrm{tr}(\mathbf{C}) - \rho\ln|\mathbf{C}|)$ which leads to an expression for pressure

$$p = \rho\frac{\partial a_0}{\partial \rho} - a_0 - \tfrac{3}{2}G\rho$$

which is independent of stress. This is likely to be a misprint since the constant number density n is replaced by a term proportional to density, ρ in the equations referred to.

Figure 1: Illustration of the mapping from time τ to current time, t, along flow lines of $\mathbf{u}$

and since at time, τ, the comoving basis coincides with the Cartesian frame

$$\phi^*_{\tau,t}\,\mathrm{d}x^a \equiv F^a{}_A\,\mathrm{d}x^A,$$

which means that mapping a dual vector, $\phi^*_{\tau,t}\mathbf{v}$ becomes a transformation of components given by $\mathbf{Fv}$, which being both coordinate and basis dependent is difficult to apply more generally than the direct use of $\phi^*_{\tau,t}\mathbf{v}$. On tensors of the form $\mathbf{A} = A_{ab}\,\mathrm{d}x^a \otimes \mathrm{d}x^b$, the matrix relation for a Cartesian frame is

$$\phi^*_{\tau,t}\mathbf{A} = \mathbf{F}^T\mathbf{AF}$$

It now follows from the Euclidean metric, $\mathbf{g} = \mathbf{I}$ in Cartesian components (or any orthonormal frame) that

$$\phi^*_{\tau,t}\mathbf{g} = \mathbf{F}^T\mathbf{F}$$

This is the conventional notation for $\mathbf{C}(\tau,t) \equiv \phi^*_{\tau,t}\mathbf{g}$. To evaluate such an object normally assumes knowledge of the map $\phi_{\tau,t}$, i.e. the functions, $x^a(X^A)$, but it is also possible to find $\mathbf{C}$ given the velocity field, $\mathbf{u}$, by solving a differential equation.

The mapping, $\phi_{\tau,t}$ is continuous and one to one, i.e. it is a diffeomorphism. Therefore it has an inverse, $\phi^{-1}_{\tau,t}$, which maps objects at the current time, t, to previous times, τ, and induces a push forward map, $(\phi_{\tau,t})^{-1}_*$ that acts in the same direction. Hence, constitutive models using $\mathbf{C}^{-1}(\tau,t) \equiv (\phi^{-1}_{\tau,t})_*\mathbf{g}$, for example, also fail the principle of objectivity.

The mapping, $\phi_{\tau,t}$ induces four linear maps, which act on a second rank tensor, $\mathbf{A}$, in the following way:

$$\begin{aligned}
(\phi_{\tau,t})^*\mathbf{A} &= \mathbf{F}^T\mathbf{AF} & \mathbb{E}^3_t \mapsto \mathbb{E}^3_\tau \quad (34)\\
(\phi_{\tau,t})_*\mathbf{A} &= \mathbf{FAF}^T & \mathbb{E}^3_\tau \mapsto \mathbb{E}^3_t \quad (35)\\
(\phi^{-1}_{\tau,t})^*\mathbf{A} &= (\mathbf{F}^{-1})^T\mathbf{AF}^{-1} & \mathbb{E}^3_\tau \mapsto \mathbb{E}^3_t \quad (36)\\
(\phi^{-1}_{\tau,t})_*\mathbf{A} &= \mathbf{F}^{-1}\mathbf{A}(\mathbf{F}^{-1})^T & \mathbb{E}^3_t \mapsto \mathbb{E}^3_\tau \quad (37)
\end{aligned}$$

where $\mathbb{E}^3_t$ indicates space at time, t. Note that $(\phi_{\tau,t})^*$ maps in the opposite direction to $\phi_{\tau,t}$ itself, whereas $(\phi_{\tau,t})_*$ maps in the same direction. Only the mappings $(\phi_{\tau,t})_*$ and $(\phi^{-1}_{\tau,t})^*$ are acceptable under the principle of objectivity.

The map $\phi_{\tau,t}$ is illustrated in Fig. **??**, which shows the effect of the mapping on a square at time τ being mapped to the current time, t, by the flow (shown in space and time). A stationary fluid will map the square at τ to the dotted square at time t. So, roughly speaking, elasticity involves the comparison of fluid in the mapped square at time t to that of the fluid in the dotted (undistorted) square.

Using $(\phi^{-1}_{\tau,t})^*$and $(\phi_{\tau,t})_*$ gives, for example, permitted constitutive relations

$$\mathbf{T} = \mu(\mathbf{g} - \mathbf{B}^{-1}), \tag{38}$$

and

$$\mathbf{T} = \nu(\mathbf{B} - \mathbf{g}), \tag{39}$$

where $\mathbf{B} \equiv (\phi_{\tau,t})_*\mathbf{g}$ and $\mathbf{B}^{-1} \equiv (\phi^{-1}_{\tau,t})^*\mathbf{g}$, with constants, μ, ν. In practice, the relation (**??**), is used rarely because $\mathbf{B}^{-1}$ has the property that a stretched body decreases the magnitude of $\mathbf{B}^{-1}$. On the other hand, the relation (**??**) uses $\mathbf{B}$, which increases as the body is stretched.

4.1 Maxwell model

It is useful to note that $\mathbf{B}(\tau,t) = \mathbf{F}^T\mathbf{F}$ can be written in the form

$$\begin{aligned}
\mathbf{B}(\tau,t) &\equiv (\phi_{\tau,t})_*\mathbf{g}\\
&= \sum_A F^a{}_A F^b{}_A \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b}\\
&\equiv \sum_A \frac{\partial x^a}{\partial X^A}\frac{\partial x^b}{\partial X^A}\frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b}\\
&= \sum_A \frac{\partial}{\partial X^A} \otimes \frac{\partial}{\partial X^A},
\end{aligned}$$

i.e. the deformation field has unit matrix components when written in a comoving (material) frame, $\dfrac{\partial}{\partial X^A}$. This has the immediate consequence that $\mathbf{B}(\tau, t)$ has the property

$$\overset{\triangledown}{\mathbf{B}} = \mathbf{0} \tag{40}$$

since the convected derivative of a material basis, $\partial/\partial X^A$ (or the dual vector basis, $\mathrm{d}X^A$), vanishes. This takes advantage of the fact that

$$(\phi_{\tau,t})_* \frac{\partial}{\partial X^A} = \frac{\partial}{\partial X^A}.$$

for comoving cordinates. The Maxwell model can be written

$$\mathbf{T} = \frac{\mu}{\lambda^2} \int_{-\infty}^{t} \exp\left(\frac{\tau - t}{\lambda}\right) (\mathbf{B}(\tau, t) - \mathbf{g})\, \mathrm{d}\tau$$

which can be differentiated explicitly to give the differential form of the UCM model

$$\lambda \overset{\triangledown}{\mathbf{T}} + \mathbf{T} = \mu \dot{\boldsymbol{\gamma}}.$$

Similarly, the LCM model

$$\lambda \overset{\triangle}{\mathbf{T}} + \mathbf{T} = \mu \dot{\boldsymbol{\gamma}}$$

is derived from differentiating

$$\mathbf{T} = -\frac{\mu}{\lambda^2} \int_{-\infty}^{t} \exp\left(\frac{\tau - t}{\lambda}\right) (\mathbf{B}^{-1}(\tau, t) - \mathbf{g})\, \mathrm{d}\tau.$$

Since $\mathbf{T} = T_{ab}\, \mathrm{d}x^a \otimes \mathrm{d}x^b$ is physically equivalent to $\mathbf{T} = T^{ab} \dfrac{\partial}{\partial x^a} \otimes \dfrac{\partial}{\partial x^b}$, the two models are fundamentally different.

4.2 Elasticity and compressibility

This section suggests that in the elastic limit a viscoelastic model must have a certain form. In material coordinates, a purely elastic model is given by

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{F}}$$

where $\mathbf{P}$ is known as the Piola tensor and the hyperelastic function used is

$$W = \tfrac{1}{2}\alpha \mathrm{tr}(\mathbf{B}) = \tfrac{1}{2}\alpha \frac{\partial x_i}{\partial X^J} \frac{\partial x_i}{\partial X^J} \tag{41}$$

giving

$$\mathbf{P} = \alpha \mathbf{F} = \alpha \frac{\partial x^i}{\partial X_J} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial X^J}$$

and the momentum equation becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = \mathrm{DIV}(\mathbf{P}) = \alpha \frac{\partial}{\partial \mathbf{X}} \cdot \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \alpha \frac{\partial^2 \mathbf{x}}{\partial X^A \partial X^A}.$$

The tensor relation between the Cauchy stress $\boldsymbol{\sigma}$ and Piola stress is

$$\mathbf{P} = J\boldsymbol{\sigma}$$

where $J \equiv |\mathbf{F}|$ is the Jacobian of the transformation (in matrix notation the relation is $\mathbf{P}\mathbf{F}^T = J\boldsymbol{\sigma}$). The Jacobian relates density at some previous time, τ, to density at the current time, t, by

$$J = \frac{\rho(\tau)}{\rho(t)},$$

so that

$$\boldsymbol{\sigma} = \frac{\rho(t)}{\rho(\tau)}\mathbf{P}. \tag{42}$$

Now $\mathbf{P}$ in this case (for W defined by (**??**)) is equivalent to $\alpha\mathbf{B}$, since in explicit tensor form

$$\mathbf{P} = \alpha\mathbf{F} = \alpha\frac{\partial x^i}{\partial X^A}\frac{\partial}{\partial x^i}\otimes\frac{\partial}{\partial X^A} = \alpha\frac{\partial}{\partial X^A}\otimes\frac{\partial}{\partial X^A} = \alpha\mathbf{B}$$

Thus, taking the upper-convective derivative of $\boldsymbol{\sigma}$ gives

$$\overset{\nabla}{\boldsymbol{\sigma}} = \alpha\mathbf{B}\frac{1}{\rho(\tau)}\frac{\mathrm{D}\,\rho}{\mathrm{D}t} = \boldsymbol{\sigma}\frac{1}{\rho}\frac{\mathrm{D}\,\rho}{\mathrm{D}t} = -\nabla\cdot\mathbf{u}\boldsymbol{\sigma} \tag{43}$$

by the equation of continuity. In the momentum equations, a purely elastic model is thus

$$\rho\frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} = \nabla\cdot\boldsymbol{\sigma}$$

with $\boldsymbol{\sigma}$ satisfying (**??**).

5 An integral model route to the compressible Maxwell model

In this section, an alternative derivation of the compressible Maxwell model is presented based on a generalisation of the integral form of the constitutive equation. In particular, we consider a density dependent integral equation for the extra-stress tensor. This general model not only has the incompressible Maxwell model as a special case but also all other Maxwell models reviewed in this paper. We do not explore more general models (e.g. Oldroyd B) since these are readily found by extending the Maxwell model in an appropriate fashion.

Let us assume the following integral equation for extra-stress $\mathbf{T}$ at time t

$$\mathbf{T}(t) = \frac{\eta(\rho(t))}{\lambda^2}\int_{-\infty}^{t}\exp\left(\frac{\tau-t}{\lambda}\right)\left(\nu(\rho(\tau))\mathbf{B} - \nu(\rho(t))\mathbf{I}\right)\mathrm{d}\tau \tag{44}$$

where η and ν are model dependent functions (to be discussed). Taking the upper-convected derivative of (**??**) gives

$$\overset{\nabla}{\mathbf{T}} = -\frac{\mathbf{T}}{\lambda} + \frac{\dot\eta}{\eta}\mathbf{T} + \frac{\eta\nu}{\lambda}\left(\dot{\boldsymbol{\gamma}} - \frac{\dot\nu}{\nu}\mathbf{I}\right)$$

or

$$\overset{\nabla}{\mathbf{T}} = -\frac{\mathbf{T}}{\lambda} + \frac{\rho}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\rho}\frac{\dot\rho}{\rho}\mathbf{T} + \frac{\eta\nu}{\lambda}\left(\dot{\boldsymbol{\gamma}} - \frac{\rho}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}\frac{\dot\rho}{\rho}\mathbf{I}\right)$$

Finally, using mass conservation $\dot\rho/\rho = -\nabla\cdot\mathbf{u}$, yields the differential form of the constitutive equation (**??**)

$$\lambda\overset{\nabla}{\mathbf{T}} + \mathbf{T}\left(1 + \lambda\frac{\rho}{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}\rho}\nabla\cdot\mathbf{u}\right) = \eta\nu\left(\dot{\boldsymbol{\gamma}} + \frac{\rho}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}(\nabla\cdot\mathbf{u})\mathbf{I}\right) \tag{45}$$

For bulk viscosity $\kappa \geq 0$ and assuming $\mu, \eta \geq 0$, we have the condition

$$\mathrm{tr}\left(\dot{\boldsymbol{\gamma}} + \frac{\rho}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}(\nabla\cdot\mathbf{u})\mathbf{I}\right) \geq 0.$$

That is

$$\frac{\rho}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\rho} \geq -\frac{2}{3}.$$

Equation (**??**) is the most general form of the constitutive equation for which the bulk and dynamic viscosities are independent.

If we make the simplifying assumption that $\eta = \eta_0\rho/\rho_0$, then the constitutive equation (**??**) assumes the form

$$\lambda\overset{\nabla}{\mathbf{T}} + (1 + \lambda\nabla\cdot\mathbf{u})\mathbf{T} = \mu\left[\dot{\boldsymbol{\gamma}} + n(\nabla\cdot\mathbf{u})\,\mathbf{I}\right] \tag{46}$$

where the dynamic viscosity $\mu = \eta_0\nu\rho/\rho_0$ and $n = \frac{\rho}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\rho}$. In this case the bulk and dynamic viscosities can no longer be chosen independently. For example, a particular choice of shear viscosity law $\eta = \eta(\rho)$ (e.g Barus), completely determines the bulk viscosity, $\kappa \equiv \eta(n + 2/3)$ and vice versa. For example, choosing $n = -2/3$ (i.e. vanishing κ) so that the right hand side is trace-free gives the model

$$\lambda \overset{\nabla}{\mathbf{T}} + (1 + \lambda \nabla \cdot \mathbf{u})\mathbf{T} = \eta \left[\dot{\boldsymbol{\gamma}} - \frac{2}{3}(\nabla \cdot \mathbf{u})\,\mathbf{I} \right] \tag{47}$$

and $\mu = \mu_0(\rho/\rho_0)^{1/3}$. On the other hand choosing $n = 0$ gives the relation $\mu = \mu_0\rho/\rho_0$. This is the simplest case and leads to an evolution equation for the conformation tensor, $\mathbf{c} = \mathbf{T}/\rho$ (only applying to isothermal models) of the form

$$\lambda \overset{\nabla}{\mathbf{c}} + \mathbf{c} = \eta_0 \dot{\boldsymbol{\gamma}} \tag{48}$$

which is identical in form to the evolution equation for $\boldsymbol{\sigma}$ in the incompressible Maxwell model. Kinetic theory (**?**) suggests this term to be determined from $\mu = \mu_0\rho/\rho_0$ and so, with this assumption, (**??**) connects bulk viscosity to dynamic viscosity, e.g. choosing n constant leads to power law forms for shear viscosity, $\mu = \mu_0(\rho/\rho_0)^{n+1}$ (and thus $\kappa = (n + 2/3)\eta_0(\rho/\rho_0)^{n+1}$). The three most natural choices would appear to be $n = -2/3$, $n = 1$ and $n = 0$. The next section suggests the choice $n = 0$, but the adoption of (**??**) allows for other choices including a Barus law, for example, of the form

$$\eta \propto \exp(A\rho)$$

for some constant $A > 0$, which may be applied to both η and ν in (**??**) and (**??**). However, these forms belong more to generalisations of White-Metzner models (see **?**) and are beyond the scope of this paper.

6 A differential geometric route to the compressible Maxwell model

6.1 The Lie derivative

The Lie derivative is a derivative along flow lines (see for example, **?** for further discussion). It uses transport along flow lines to compare changes in scalar, vector or general tensor fields. The Lie derivative of a tensor field, $\mathbf{A}$ with respect to a flow field, $\mathbf{U}$ is defined

$$\mathcal{L}_{\mathbf{U}}\mathbf{A} = \lim_{\epsilon \to 0} \frac{\mathbf{A} - (\phi_{t-\epsilon,t})_*\mathbf{A}}{\epsilon}.$$

In words, the field $\mathbf{A}$ at time t is compared with the flow transformed field from a previous time $t - \epsilon$ in the limit of small ϵ. For a second rank tensor, $\mathbf{T}$, of rank $(2, 0)$ and vector field $U = \partial_t + \mathbf{u}$, in 3D space + time, it is possible to show that

$$\mathcal{L}_U \mathbf{T} \equiv \overset{\nabla}{\mathbf{T}}.$$

That is, the Lie derivative with respect to a velocity field, $\mathbf{u}$ with time component unity, of a $(2, 0)$ tensor $\mathbf{T}$ (such as stress), is equivalent to the upper convected derivative of $\mathbf{T}$. A complication arises when we formulate a physically equivalent representation of stress using the dual vector basis, call it $\tilde{\mathbf{T}}$, which has rank $(0, 2)$. The Lie derivative of this object is equivalent to the lower convected derivative. One might formally write this

$$\overset{\triangle}{\mathbf{T}} \equiv \widetilde{\mathcal{L}_U \tilde{\mathbf{T}}}.$$

Unlike the upper and lower convected derivatives, the Lie derivative acts on tensors of any rank and type[3].

6.2 Tensor densities

For compressible flow, **?** states that the convected derivative must be modified by introducing a term associated with the Jacobian of the transformation associated with the flow. The Jacobian is associated with density change and, via the mass continuity equation involves $\nabla \cdot \mathbf{u}$. In terms of Lie derivatives this associates the stress with a volume form. In integral form the conservation of momentum can be written

$$\int_V \left(\rho \frac{\mathrm{D}\,\mathbf{u}}{\mathrm{D}t} - \nabla \cdot \boldsymbol{\sigma} \right) \star 1 = 0$$

[3] e.g. for a scalar f the Lie derivative, $\mathcal{L}_U f \equiv \frac{\mathrm{D}\,f}{\mathrm{D}t}$

The symbol $\star 1$ is the volume form [4] and has Lie derivative

$$\mathcal{L}_U \star 1 = (\nabla \cdot \mathbf{u}) \star 1$$

Strictly, the Lie derivative of stress involves the volume form. Thus

$$\mathcal{L}_U(\boldsymbol{\sigma} \star 1) = \overset{\nabla}{\boldsymbol{\sigma}} \star 1 + (\nabla \cdot \mathbf{u})\boldsymbol{\sigma} \star 1$$

This is Oldroyd's formulation using the notation of modern differential geometry.

Applying the Lie derivative to the $(2,0)$ tensor metric space $\mathbf{g}$, with matrix components $\mathbf{I}$, we obtain

$$\mathcal{L}_U \mathbf{g} = \overset{\nabla}{\mathbf{I}} \equiv -(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \equiv -\dot{\boldsymbol{\gamma}}$$

Using this we can write the incompressible Newtonian stress as

$$\mathbf{T} = -\mu \mathcal{L}_U \mathbf{g} = \mu \dot{\boldsymbol{\gamma}}$$

However, for compressible fluids, we must (following Oldroyd) express the stress as a tensor density. This appears to imply

$$\mathbf{T} \star 1 = -\mu \mathcal{L}_U(\mathbf{g} \star 1) = \mu(\dot{\boldsymbol{\gamma}} - (\nabla \cdot \mathbf{u})\mathbf{g}) \star 1$$

so that in matrix form

$$\mathbf{T} = \mu(\dot{\boldsymbol{\gamma}} - (\nabla \cdot \mathbf{u})\mathbf{I}),$$

which has (inadmissible) negative bulk viscosity, $\kappa = -(1/3)\mu$. This can be resolved by including a linear dependence on density in the formulation and writing

$$\begin{aligned} \mathbf{T} \star 1 &= -\mu \mathcal{L}_U(\rho \mathbf{g} \star 1) \\ &= \mu \rho \dot{\boldsymbol{\gamma}} \star 1 + \left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \mathbf{u} \right) \mathbf{g} \star 1, \end{aligned}$$

which on using the conservation of mass gives

$$\mathbf{T} = \mu \rho \dot{\boldsymbol{\gamma}}$$

More generally, for isothermal generalised Newtonian fluids we can express the stress as

$$\mathbf{T} \star 1 = -\mathcal{L}_U(\mu(\rho)\mathbf{g} \star 1)$$

where μ is some function of density, ρ, giving

$$\mathbf{T} = \mu \left(\dot{\boldsymbol{\gamma}} + \left[\frac{\rho}{\mu} \frac{\mathrm{d}\mu}{\mathrm{d}\rho} - 1 \right] (\nabla \cdot \mathbf{u})\mathbf{I} \right)$$

Combining this with the Maxwell model gives

$$\lambda \left[\overset{\nabla}{\mathbf{T}} + (\nabla \cdot \mathbf{u})\mathbf{T} \right] + \mathbf{T} = \mu \left(\dot{\boldsymbol{\gamma}} + \left[\frac{\rho}{\mu} \frac{\mathrm{d}\mu}{\mathrm{d}\rho} - 1 \right] (\nabla \cdot \mathbf{u})\mathbf{I} \right)$$

which matches the special case of the model given in (**??**) when $\mu = \eta\nu$ and $\eta = \eta_0$ (constant). Again, the simplest case ($\mu = \eta_0\rho$, or constant kinematic stress) yields the model (**??**).

7 Summary

After a brief survey of compressible viscoelastic models in use, we have demonstrated a natural way to extend macroscopic viscoelastic incompressible models to compressible fluids. Using the ideas of **?** and by returning to first principles, we have demonstrated the physical significance of the choice of pressure in terms of entropy generation. Within an incompressible context this choice is largely academic, but for compressible fluids the choice leads to different equations of motion, the significance of which increases with the compressibility of the fluid. The essence of the problem is that of the dissipation associated with isotropic strain rate (expansion or contraction). For Newtonian fluids this is governed by bulk viscosity and this should be extended to apply to viscoelastic models. This is addressed in **?** giving equation (**??**) and leads to an extra term in the pressure for the UCM of $\frac{1}{2}\mathrm{tr}(\mathbf{T})$ in addition to the pressure term given by an equation of state. The outcomes of this paper (Section 6) is the suggestion that, for isothermal fluids well modelled by the Maxwell constitutive equation, the bulk viscosity may be inferred once the dependence of dynamic viscosity on density is known. The more general model is given by (**??**).

[4] the volume form, $\star 1 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ in Cartesian coordinates, is a third rank antisymmetric tensor

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