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Clifford Algebras, Nonlinear Dynamical Systems and Isochronous Tori

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ABSTRACT

Nonlinear dynamical systems which have a lifting to Clifford algebras are studied. The resulting Clifford systems are shown to be solvable easily by Lie series methods. The main application is to generalise to higher dimensional systems recent results in isochronous systems.
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1. Introduction

Clifford algebras have been used for many years in the general theory of Dirac operators [6],[7] which are, of course, partial differential equations. Recently, however, a number of applications for Clifford algebras to ordinary differential equations have also been found [1]. The main advantage in dealing with these so-called Clifford systems is that vectors in $\mathbb{R}^n$ are given a product structure and so algebraic methods resembling those used for one-dimensional systems can be used. In the present paper we shall consider the characterisation of those systems of equations which have a Clifford extension. We will then show that the algebraic structure defined on $\mathbb{R}^n$ enables us to obtain the Lie series in a simple way and hence explicit solutions of a given Clifford system. The main application here will be to generalise some of the recent results of [4, 5] on the existence of two-dimensional isochronous centres to the case of Clifford systems. This will lead to systems with isochronous tori in $\mathbb{H}$, for example. We shall use the same technique as in [5] consisting of finding a Darboux linearisation to reduce the system to a standard linear one of the form

$$\dot{X} = iX \ , \ X \in \mathfrak{A}$$

where $\mathfrak{A}$ is a given (real) Clifford algebra and i is embedded as an imaginary unit in $\mathfrak{A}$. We shall show that such systems define invariant tori in 2^n-dimensional space.

In section 2 we shall briefly outline the definition and results from elementary Clifford algebra theory which we shall need and in section 3 we shall characterise those systems in $\mathbb{R}^n$ which have a lifting to a Clifford system. Section 4 will consider the Lie series solution of a Clifford system and we shall show that any function $f : [0, \infty) \to \mathfrak{A}$ which is analytic is the solution of some Clifford system. In section 5 we shall consider local centres for systems of the form $\dot{X} = iP(X)$ where P is a polynomial and in section 6 the Darboux theory will be generalised to Clifford systems of the form $\dot{X} = iP(X, \bar{X})$ in order to find isochronous tori.

2. Clifford Algebras

In this section we give a brief account of the main aspects of Clifford algebras which we shall need. All proofs can be found in [6].

Throughout this section (V, Q) denotes a quadratic space, i.e. a vector space V with a quadratic form $Q : V \longrightarrow \mathbb{F}$.
(2.1) **Definition** A pair $(\mathfrak{A}, \nu)$ is a **Clifford algebra** if $\mathfrak{A}$ is generated (as an algebra) by the set $\{\nu(v) : v \in V\} \cup \{\lambda : \lambda \in \mathbb{F}\}$ and we have the Clifford relation

$$(\nu(v))^2 = -Q(v). \ \square$$

If $\{e_i\}_{i=1}^{\dim V}$ is an orthonormal basis of V then $\mathfrak{A}$ is spanned by the products $e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k}$ where $\alpha = \{\alpha_1, \cdots, \alpha_k\} \subseteq \{1, 2, \cdots, \dim V\} \triangleq N_V$. The dimension of $\mathfrak{A}$ (as a vector space over $\mathbb{F}$) is $\leq 2^{\dim V}$. If $\dim \mathfrak{A} = 2^{\dim V}$, $\mathfrak{A}$ is called a **universal Clifford algebra**.

The standard construction of a universal Clifford algebra is via the quotient of the tensor algebra $T(V) = \sum_{k=0}^{\infty} V \otimes \cdots \otimes V$ (where the k^{th} term has k factors and the first two terms are $\mathbb{F}$ and V) by the two-sided ideal I_Q generated by $\{v \otimes v + Q(v)1 : v \in V\}$. Then

$$\mathfrak{A} = T(V)/I_Q$$

is the universal Clifford algebra of dimension $2^{\dim V}$.

Three operations are defined on a Clifford algebra by linearly extending their definitions on the basis elements e_α:

(i)	$e'_\alpha = (-1)^{\lvert\alpha\rvert} e_\alpha , \ \alpha \in N_V$	(principle automorphism)
(ii)	$e^*_\alpha = (-1)^{\frac{1}{2}\lvert\alpha\rvert(\lvert\alpha\rvert-1)} e_\alpha$	(principle anti-automorphism)
(iii)	$\bar{e}_\alpha = (e^*_\alpha)' = (e'_\alpha)^* = (-1)^{\frac{1}{2}\lvert\alpha\rvert(\lvert\alpha\rvert+1)} e_\alpha$	(conjugation)

where $|\alpha|$ is the cardinality of α. The **norm** on a Clifford algebra $\mathfrak{A}$ is defined by the map $\Delta : \mathfrak{A} \longrightarrow \mathfrak{A}$ where

$$\Delta(x) = \bar{x}x \ , \ x \in \mathfrak{A} \ .$$

The set

$$\Gamma = \Gamma(V, Q) = \{w_1 \cdots w_k : w_j \in \mathbb{F} \oplus V \ , \ \Delta(w_j) \neq 0\}$$

is the **Clifford group**; it is a group contained in $\mathfrak{A}$ which is closed under the three operations above.

In this paper we shall be mainly interested in the 'Euclidean' Clifford algebras $\mathfrak{A}_{0,n}$ (or simply $\mathfrak{A}_n$) with quadratic form $Q_n(u) = u_1^2 + \cdots + u_n^2$. The first three are given by

$$\mathfrak{A}_0 \doteq \mathbb{R} \ , \ \mathfrak{A}_1 \doteq \mathbb{C} \ , \ \mathfrak{A}_2 \doteq \mathbb{H} \ .$$

If we define the matrices

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \ E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} , \ E_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \ E_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

then the embeddings $\nu_0 : \mathbb{R}^0 \longrightarrow \mathfrak{A}_0 = \mathbb{R}$, $\nu_1 : \mathbb{R} \longrightarrow \mathfrak{A}_1 = \mathbb{C}$, $\nu_2 : \mathbb{R}^2 \longrightarrow \mathfrak{A}_2 = \mathbb{H}$ are given by

$$\begin{aligned} \nu_0(0) &= 0 \\ \nu_1(y) &= yE_2 \\ \nu_2((x_1, x_2)) &= x_1E_1 + x_2E_2. \end{aligned}$$

Of course, in $\mathfrak{A}_1$, E_1 can be identified with i. The higher dimensional algebras $\mathfrak{A}_n$, $n > 2$ can be obtained from

(2.2) **Theorem** $\mathfrak{A}_n$ can be realised as the subalgebra

$$\left\{ \begin{bmatrix} z & \varsigma \\ -\varsigma' & z' \end{bmatrix} : z, \varsigma \in \mathfrak{A}_{n-1} \right\}$$

of the matrix algebra $M(2, \mathfrak{A}_{n-1})$ of 2×2 matrices with values in $\mathfrak{A}_{n-1}$. Morever, the three operators above are given by

$$\begin{bmatrix} z & \varsigma \\ -\varsigma' & z' \end{bmatrix}' = \begin{bmatrix} z & -\varsigma \\ \varsigma' & z' \end{bmatrix}, \begin{bmatrix} z & \varsigma \\ -\varsigma' & z' \end{bmatrix}^* = \begin{bmatrix} \bar{z} & \bar{\varsigma}' \\ -\bar{\varsigma} & \bar{z}' \end{bmatrix}$$

$$\overline{\begin{bmatrix} z & \varsigma \\ -\varsigma' & z' \end{bmatrix}} = \begin{bmatrix} \bar{\bar{z}} & -\varsigma^* \\ \bar{\varsigma} & z^* \end{bmatrix}. \square$$

(2.3) **Remark** The basis of $\mathfrak{A}_n$ is generated by the matrices

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} 0 & f_j \\ f_j & 0 \end{bmatrix}, 1 \le j \le n, \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

where $\{f_1, \cdots, f_n\}$ is the image of a basis of $\mathbb{R}^{n-1}$ in $\mathfrak{A}_{n-1}$. $\square$

In the cases of $\mathfrak{A}_1 \hat{=} \mathbb{C}$, $\mathfrak{A}_2 \hat{=} \mathbb{H}$ we can evaluate the three principle operators as follows:

$$\begin{aligned} \bar{z} &= x - iy \text{ (standard complex conjugate)} \\ z' &= \bar{z} \\ z^* &= z \end{aligned}$$

for $z \in \mathbb{C}$, and if $h = a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3 = a_0 + a_1i + a_2j + a_3k \in \mathfrak{A}_2 \hat{=} \mathbb{H}$ then

$$\begin{aligned} h' &= a_0 - a_1i - a_2j + a_3k \\ h^* &= a_0 + a_1i + a_2j - a_3k \\ \bar{h} &= a_0 - a_1i - a_2j - a_3k \end{aligned}$$

We can regard $\mathfrak{A}_n$ as being obtained from by $\mathfrak{A}_{n-1}$ by adding an **imaginary unit** e_n (such that $e_n^2 = -1$). Then

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} \oplus e_n\mathfrak{A}_{n-1}.$$

Thus, for example,

$$\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}, \ \mathbb{H} = \mathbb{C} \oplus j\mathbb{C}.$$

Returning to the Clifford norm $\Delta(x) = \bar{x}x$, we note that $\bar{x}$ does not commute with x, in general, although it does for $x \in \Gamma$, the Clifford group, since there,

$$\Delta(x) = \Delta(\overline{x}).$$

This will be important later when we look for invariant algebraic curves of the form

$$\bar{x}x = 0.$$

3. Lifting Scalar Analytic Differential Equations to Clifford Algebras

We are interested in scalar analytic differential equations of the form

$$\dot{z} = f(z)\ ,\ z(0) = z_0 \in \mathbb{R} \tag{3.1}$$

where f is analytic, i.e.

$$f(z) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0) z^i$$

and the characterisation of nonlinear n-dimensional systems of the form

$$\dot{x} = F(x)\ ,\ x(0) = x_0 \in V(\subseteq \mathbb{R}^n) \tag{3.2}$$

which can be written

$$\dot{X} = f(X) = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0) X^i \tag{3.3}$$

for $X \in \mathfrak{A}$ where $\mathfrak{A}$ is a universal Clifford algebra over $V(\subseteq \mathbb{R}^n)$ with quadratic form Q.

In the notation of section 2, let

$$e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k}\ ,\ \alpha = \{\alpha_1, \cdots, \alpha_k\}$$

be a basis of $\mathfrak{A}$. Then any $X \in \mathfrak{A}$ can be written

$$X = \sum_{\alpha \subseteq N_V} X_\alpha e_\alpha \tag{3.4}$$

where $N_V = \{1, 2, \cdots, \dim(V)\}$. It will be convenient to enumerate the basis; let

$$\mu : 2^{N_V} \to \{1, 2, \cdots, 2^{\dim(V)}\}$$

be a bijection from the power set of N_V to the first $2^{\dim(V)}$ natural numbers (excluding 0). Then we can write (3.4) in the form

$$\begin{aligned} X &= \sum_{k=1}^{2^{\dim(V)}} X_{\mu^{-1}(k)} e_{\mu^{-1}(k)} \\ &= \sum_{k=1}^{2^{\dim(V)}} \bar{X}_k E_k \end{aligned}$$

where

$$\bar{X}_k = X_{\mu^{-1}(k)}\ ,\ E_k = e_{\mu^{-1}(k)}\ ;$$

we assume that $E_1 = 1$.

(3.5) **Lemma** The product $E_{\ell_1} E_{\ell_2} \cdots E_{\ell_i}$, for any integers $1 \le \ell_j \le K \triangleq 2^{\dim(V)}$ is of the form

$$\alpha(\ell_1, \cdots, \ell_i) E_{\rho(\ell_1, \cdots, \ell_i)}$$

for some real number $\alpha = \alpha(\ell_1, \cdots, \ell_i)$ and some integer ρ with $1 \le \rho(\ell_1, \cdots, \ell_i) \le K$.

Proof By the fundamental property of basis vectors of V we have

$$e_j e_k + e_k e_j = -2Q(e_j)\delta_{jk} ,$$

i.e.

$$e_j e_k = -e_k e_j \ (j \neq k) \ , \ e_j^2 = -Q(e_j). \tag{3.6}$$

Since each E_{ℓ_j} is a product of e_j's the result follows from (3.6). □
Remark It is clear that $\alpha = \pm 1$, since $Q(e_j) = \pm 1$.
(3.7) **Lemma** If $X = \sum_{k=1}^{K} X_k E_k$, in terms of the basis $\{E_k\}_{1\leq k\leq K}$ of $\mathfrak{A}$, then

$$X^i = \sum_{p=1}^{K} \sum_{\substack{\ell_1,\cdots,\ell_i \\ \rho(\ell_1,\cdots,\ell_i)=p}} X_{\ell_1} X_{\ell_2} \cdots X_{\ell_i} \alpha(\ell_1, \cdots, \ell_i) E_p.$$

Proof We have

$$\begin{aligned} X^i &= \left(\sum_{k=1}^{K} X_k E_k \right)^i \\ &= \sum_{\ell_1=1}^{K} \cdots \sum_{\ell_i=1}^{K} X_{\ell_1} E_{\ell_1} X_{\ell_2} E_{\ell_2} \cdots X_{\ell_i} E_{\ell_i} \\ &= \sum_{\ell_1=1}^{K} \cdots \sum_{\ell_i=1}^{K} X_{\ell_1} X_{\ell_2} \cdots X_{\ell_i} E_{\ell_1} \cdots E_{\ell_i} \\ &= \sum_{p=1}^{K} \sum_{\substack{\ell_1,\cdots,\ell_i \\ \rho(\ell_1,\cdots,\ell_i)=p}} X_{\ell_1} X_{\ell_2} \cdots X_{\ell_i} \alpha(\ell_1, \cdots, \ell_i) E_p \end{aligned}$$

by lemma 3.5. □

This leads immediately to the classification of lifted systems of equations:
(3.7) **Theorem** A system

$$\dot{X} = f(X) \ , \ X \in \mathfrak{A}$$

of differential equations is a lifting of an analytic scalar differential equations if and only if

$$\dot{X}_p = \phi_0 \delta_{p1} 1 + \sum_{i=1}^{\infty} \phi_i \sum_{\substack{\ell_1,\cdots,\ell_i \\ \rho(\ell_1,\cdots,\ell_i)=p}} \alpha(\ell_1, \cdots, \ell_i) X_{\ell_1} X_{\ell_2} \cdots X_{\ell_i}$$

for some constants ϕ_i, $1 \leq i < \infty$, where 1 is the unit of $\mathfrak{A}$.
Proof We simply equate coefficients of the basis $\{E_p\}$ in the expression

$$\begin{aligned} \dot{X} &= \dot{X}_1 E_1 + \cdots + \dot{X}_K E_K = \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0) X^i \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} f^{(i)}(0) \sum_{p=1}^{K} \sum_{\substack{\ell_1,\cdots,\ell_i \\ \rho(\ell_1,\cdots,\ell_i)=p}} \alpha(\ell_1, \cdots, \ell_i) X_{\ell_1} X_{\ell_2} \cdots X_{\ell_i} E_p \end{aligned}$$

and set

$$\phi_i = \frac{1}{i!} f^{(i)}(0). \ \square$$

(3.8) **Example** We shall find all the two-dimensional systems which are defined on the Clifford algebra $\mathfrak{A}_{1,0} = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} : x, y \in \mathbb{R} \right\}$. A basis of this algebra is

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ E_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Any product $E_{\ell_1} \cdots E_{\ell_i}$ is of the form E_2^j for some power j. But

$$E_2^j = \begin{cases} E_2 & \text{if } j \text{ is odd} \\ E_1 & \text{if } j \text{ is even} \end{cases}$$

Hence

$$\rho(\ell_1, \cdots, \ell_i) = \begin{cases} 1 & \text{if } \ell_i = 2 \text{ for an even number of } j\text{'s} \\ 2 & \text{if } \ell_i = 2 \text{ for an odd number of } j\text{'s} \end{cases}$$

and

$$\alpha(\ell_1, \cdots, \ell_i) = 1, \text{ for all } \ell_1, \cdots, \ell_i.$$

It follows from theorem 3.7 that all systems of the required type are of the form:

$$\begin{aligned} \dot{x}_1 &= \sum_{\substack{i=0 \\ i_1+i_2=i \\ i_2 \text{ even}}}^{\infty} \phi_i x_1^{i_1} x_2^{i_2} \\ \dot{x}_2 &= \sum_{\substack{i=0 \\ i_1+i_2=i \\ i_2 \text{ odd}}}^{\infty} \phi_i x_1^{i_1} x_2^{i_2} \end{aligned}$$

The simplest nonlinear example of this system is

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2^2 \\ \dot{x}_2 &= 2x_1 x_2 \end{aligned}$$

which can be written

$$\dot{X} = X^2$$

where

$$X = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}.$$

Another example is the system

$$\dot{X} = \sin(X).$$

Now, if $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ then $P^{-1}XP = \begin{pmatrix} x_1 + x_2 & 0 \\ 0 & x_1 - x_2 \end{pmatrix}$, so

$$\sin(X) = \frac{1}{2} \begin{pmatrix} \sin(x_1+x_2) + \sin(x_1-x_2) & \sin(x_1+x_2) - \sin(x_1-x_2) \\ \sin(x_1+x_2) - \sin(x_1-x_2) & \sin(x_1+x_2) + \sin(x_1-x_2) \end{pmatrix}$$

so we have the equivalent system

$$\begin{aligned} \dot{x}_1 &= \frac{1}{2}\sin(x_1 + x_2) + \frac{1}{2}\sin(x_1 - x_2) \\ \dot{x}_2 &= \frac{1}{2}\sin(x_1 + x_2) - \frac{1}{2}\sin(x_1 - x_2) \end{aligned}$$

(3.9) **Example** Consider now the Clifford algebra $\mathfrak{A}_{0,2}$. This is realised by the embedding

$$(x_1, x_2) \to x_1 \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \; E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \; E_4 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

(Of course, $\mathfrak{A}_{0,2}$ is isomorphic to $\mathbb{H}$.) Note that

$$E_2^2 = E_3^2 = E_4^2 = -I$$

and

$$E_j E_k = E_\ell$$

if $\{j, k, \ell\}$ is a cyclic permutation of $\{2, 3, 4\}$. Hence we see that

$$\rho(\ell_1, \cdots, \ell_i) = \begin{cases} 1 & \text{if } n_2, n_3, n_4 \text{ even or } n_2, n_3, n_4 \text{ odd} \\ 2 & \text{if either } n_2 \text{ odd or } n_3, n_4 \text{ odd} \\ 3 & \text{if either } n_3 \text{ odd or } n_2, n_4 \text{ odd} \\ 4 & \text{if either } n_4 \text{ odd or } n_2, n_3 \text{ odd} \end{cases}$$

where n_j is the number of $\ell_1, \cdots, \ell_i$ equal to j, $1 \leq j \leq 4$. Moreover, we have

$$\alpha(\ell_1, \cdots, \ell_i) = \operatorname{sgn} P(\ell_1, \cdots, \ell_i) \cdot N(\ell_1, \cdots, \ell_i)$$

where $P(\ell_1, \cdots, \ell_i)$ is the permutation which puts $E_{\ell_1} \cdots E_{\ell_i}$ in the form

$$E_1 \cdots E_1 \underbrace{E_2 \cdots E_2}_{n_2} \underbrace{E_3 \cdots E_3}_{n_3} \underbrace{E_4 \cdots E_4}_{n_4}$$

without transposing any E_i with itself and $N(\ell_1, \cdots, \ell_i)$ is given by

$$N(\ell_1, \cdots, \ell_i) = \begin{cases} (-1)^{[n_2/2]+[n_3/2]+[n_4/2]} & \begin{array}{l} \text{if } n_2, n_3, n_4 \text{ even} \\ \text{or } n_2, n_3 \text{ even } n_4 \text{ odd} \\ \text{or } n_2, n_4 \text{ even } n_3 \text{ odd} \\ \text{or } n_3, n_4 \text{ even } n_2 \text{ odd} \\ \text{or } n_2 \text{ even } n_3, n_4 \text{ odd} \\ \text{or } n_4 \text{ even } n_2, n_2 \text{ odd} \end{array} \\ -(-1)^{[n_2/2]+[n_3/2]+[n_4/2]} & \text{otherwise} \end{cases}$$

For example, the system of equations

$$\begin{aligned}
\dot{x}_1 &= \left(x_1^2 - x_2^2 - x_3^2 - x_4^2\right) x_1 - 2x_1x_2^2 - 2x_1x_3^2 - 2x_1x_4^2 \\
\dot{x}_2 &= 2x_1^2x_2 + \left(x_1^2 - x_2^2 - x_3^2 - x_4^2\right) x_2 \\
\dot{x}_3 &= 2x_1^2x_3 + \left(x_1^2 - x_2^2 - x_3^2 - x_4^2\right) x_3 \\
\dot{x}_4 &= 2x_1^2x_4 + \left(x_1^2 - x_2^2 - x_3^2 - x_4^2\right) x_4
\end{aligned}$$

can be written in the form

$$\dot{X} = X^3$$

on $\mathfrak{A}_{0,2}$, where

$$X = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

4. The General Solution of Analytic Clifford Systems

In this section we shall obtain general series solution for analytic Clifford systems. To do this first consider a simple scalar equation

$$\dot{x} = f(x) \ , \ x(0) = x_0 \in \mathbb{R} \tag{4.1}$$

defined on $\mathbb{R}$, where f is analytic. The solution is given by the Lie series

$$x(t) = \sum_{i=0}^{\infty} t^i \frac{(L_f)^i}{i!} x \Big|_{x=x_0} \tag{4.2}$$

for sufficiently small x_0, where L_f is the Lie derivative with respect to f. (See [2].) To generalise this to Clifford systems we note that we can derive (4.2) in the following way: define

$$\begin{aligned}
\phi_1 &= x \\
\phi_2 &= \dot{\phi}_1 = \frac{\partial \phi_1}{\partial x} f = (L_f)x \\
\phi_3 &= \dot{\phi}_2 = (L_f)^2 x \\
&\cdots
\end{aligned}$$

Then, with $\Phi = (\phi_1, \phi_2, \cdots)$ we have

$$\dot{\Phi}(t) = A\Phi(t) \ , \ \Phi(0) = \Phi_0 \tag{4.3}$$

where A is the left-shift operator

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$\Phi_0 = \left(x_0, \, (L_f)x\big|_{x=x_0}, \, (L_f)^2 x\Big|_{x=x_0}, \cdots \right)^T .$$

The solution of (4.2) is

$$\Phi(t) = e^{At}\Phi(0)$$

and $\phi_1 = x$ is given by the first element of $e^{At}\Phi(0)$, which is easily seen to be the same as (4.2). Note, however, that the group property

$$e^{At_1}\left(e^{At_2}\Phi(0)\right) = e^{A(t_1+t_2)}\Phi(0) \tag{4.4}$$

is only valid locally, near $t_1, t_2 = 0$. Hence, one may call the map

$$\psi_t \to e^{At}$$

a local representation of the one-dimensional transformation group ψ_t, where

$$\psi_t(x_0) = x(t; x_0)$$

on a space of operators defined on the linear subspace of the linear space of all sequences $(s_1, s_2, \cdots)$ such that the power series $\sum \frac{t^i}{i!} s_i$ converges for some $t > 0$.

The problem that the group property (4.4) does not hold for all time is due to the fact that the solution of (4.1) may have a singularity when t is extended into the complex plane. For example, suppose that

$$x(t) = \frac{1}{1+t^2} \tag{4.5}$$

is the solution of (4.1). This has poles at $t = \pm i$ and so the Lie series (4.2) will not extend beyond $t = 1$, on the real time axis. We next prove that any analytic function h of t such as (4.5) (with $h'(0) \neq 0$) is the solution of an analytic differential equation. In fact if $h(t)$ is analytic for $0 \leq t < \tau$, we have

$$h(t) = \sum_{i=0}^{\infty} h^{(i)}(0)\frac{t^i}{i!}$$

and so if we choose the Lie derivatives of f so that

$$(L_f)^i x\Big|_{x=h(0)} = h^{(i)}(0) \tag{4.6}$$

we may solve for the ordinary derivatives of f recursively, since

$$(L_f)^i x\Big|_{x=x_0} = f^i(x_0)\frac{d^i f}{dx^i}(x_0) + \alpha_i$$

where α_i depends on f and its derivatives up to order $i-1$. Clearly, α_i is given recursively by

$$\alpha_{i+1} = L_f \alpha_i + i f^i \frac{df}{dx}\frac{d^i f}{dx^i}$$

and so

$$\frac{d^i f}{dx^i}(x_0) = \left((L_f)^i x\Big|_{x=x_0} - \alpha_i \right) / f^i(x_0) \tag{4.7}$$

provided $f(x_0) = \frac{dh}{dt}(0) \neq 0$. (See also [3].) We then define f in terms of its Taylor series:

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(x_0)}{i!}(x - x_0)^i$$

where $x_0 = h(0)$. We have therefore proved
(4.8) **Lemma** Given any scalar analytic function f, the Clifford system

$$\dot{X} = f(X) \ , \ X(0) = X_0 \in \mathfrak{A} \tag{4.9}$$

has a solution, given for sufficiently small t, by the Lie series

$$X(t) = \sum_{i=0}^{\infty} t^i \frac{(L_f)^i}{i!} X \Big|_{X=X_0} .$$

Morever, this series is analytic in X_0. Conversely, given any locally analytic function $\xi : \mathfrak{A} \times \mathbb{R} \to \mathfrak{A}$ (i.e. jointly analytic, so that

$$\xi(X_0, t) = \sum_{i=0}^{\infty} h_i(X_0) \frac{t^i}{i!}$$

where each h_i is analytic at X_0) then there exists a Clifford differential system of the form (4.9) for which ξ represents the solution through X_0. Moreover, we can find f by an inductive procedure. □
(4.10) **Example** Consider the simple scalar system

$$\dot{x} = x^2 \ , \ x(0) = x_0$$

which has the solution

$$x(t) = \frac{x_0}{1 + t x_0} = \sum_{i=0}^{\infty} t^i x_0^{i+1}$$

and the series is valid for $t < 1/|x_0|$. Hence the solution of the Clifford system

$$\dot{X} = X^2 \ , \ X(0) = X_0 \in \mathfrak{A}$$

is given by

$$X(t) = \sum_{i=0}^{\infty} t^i X_0^{i+1}$$

for $t < 1/\|X_0\|$, where $\|X_0\|$ is the norm of X_0 (i.e. the standard matrix norm in a given matrix representation of $\mathfrak{A}$). As a specific example, the system

$$\frac{d}{dt}\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}^2 \ , \ X(0) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

on $\mathfrak{A}_{1,0}$ has the solution

$$x_1(t) = -x_2(t) = \frac{1}{1 - 2t}$$

for $t < 1/2$.

5. Local Solutions and Centres

Consider a differential equation

$$\dot{X} = f(X)\ ,\ X(0) = X_0 \in \mathfrak{A} \tag{5.1}$$

defined on the universal Clifford algebra $\mathfrak{A}$, where $f : \mathbb{R} \to \mathbb{R}$ is an analytic function. We shall be particularly interested in the case where f is a polynomial:

$$f(y) = a_0 + a_1 y + \cdots + a_m y^m.$$

We shall first discuss the equilibrium points of (5.1). Let

$$f(y) = \prod_{i=1}^{m} (y - \lambda_i)$$

be the factorisation of f so that

$$f(X) = \prod_{i=1}^{m} (X - \lambda_i I) = 0$$

implies that at least one $X - \lambda_i I$ is noninvertible. Hence, using any matrix representation of $\mathfrak{A}$ we have

(5.2) **Lemma** The equilibrium points of (5.1) are given by the set

$$E = \{X \in A : \det(X - \lambda_i I) = 0 \text{ for some } i, \text{ and then } \prod_{i=1}^{m} (X - \lambda_i I) = 0\}. \ \square$$

(5.3) **Remark** We shall see in the example below that the condition $\det(X - \lambda_i I) = 0$ for some i is necessary but not sufficient for the existence of equilibrium points. Hence, when we have solved the equations

$$\det(X - \lambda_i I) = 0,\ 1 \le i \le m$$

we must check which solutions satisfy

$$\prod_{i=1}^{m} (X - \lambda_i I) = 0. \ \square$$

(5.4) **Example** Consider the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 3 \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}^2 \tag{5.5}$$

The eigenvalues are given by the scalar equation $2 + 3\lambda + \lambda^2 = 0$, i.e. $\lambda = -2$ or -1. Hence we must have

$$\det\left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 \text{ or } \det\left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

giving

$$(x_1+2)^2 - x_2^2 = 0 \text{ or } (x_1+1)^2 - x_2^2 = 0$$

so that

$$x_1 = -2 \pm x_2 \text{ or } x_1 = -1 \pm x_2 \tag{5.6}$$

Now the right hand side of (5.5) is

$$\left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and substituting (5.6) gives

$$\begin{pmatrix} 2x_2^2 \mp x_2 & 2x_2^2 \mp x_2 \\ 2x_2^2 \mp x_2 & 2x_2^2 \mp x_2 \end{pmatrix} \text{ or } \begin{pmatrix} 2x_2^2 \pm x_2 & 2x_2^2 \pm x_2 \\ 2x_2^2 \pm x_2 & 2x_2^2 \pm x_2 \end{pmatrix}$$

so that the right hand side of (5.5) is zero when $x_2 = 0$ or $\pm 1/2$. Hence the equilibrium points are

$$(-2,0), (-1,0), (-\frac{3}{2}, \frac{1}{2}), (-\frac{3}{2}, -\frac{1}{2}).$$

Note that the solutions $(-\frac{5}{2}, -\frac{1}{2}), (-\frac{1}{2}, \frac{1}{2})$ of (5.3) are not equilibrium points as pointed out in the above remark. Of course, the right hand side of (5.2) has another factorisation:

$$\left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} - \begin{pmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{pmatrix}\right)\left(\begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix} - \begin{pmatrix} -3/2 & -1/2 \\ -1/2 & -3/2 \end{pmatrix}\right).$$

Next we consider the local solutions of Clifford systems of the form

$$\begin{aligned} \dot{X} &= a_0 + a_1 X + \cdots + a_m X^m \\ &= \prod_{i=1}^{m} (X - \lambda_i I)\ ,\ X \in \mathfrak{A} \end{aligned} \tag{5.7}$$

where λ_i are the roots of the corresponding scalar polynomial. First consider the scalar equation

$$\dot{x} = a_0 + a_1 x + \cdots + a_m x^m\ ,\ x \in \mathbb{R}. \tag{5.8}$$

The following result gives a simple representation for the solution of this equation in a neighbourhood of each equilibrium point λ_i.

(5.9) **Lemma** Near λ_i, we can write the solution of (5.8) in the form

$$x \cong \lambda_i + C\frac{e^{t/\beta_i}}{\left(\prod_{j\neq i}(\lambda_i - \lambda_j)^{\beta_j}\right)^{1/\beta_i}}$$

where

$$\beta_i = \prod_{\substack{j=1 \\ j\neq i}}^{m} (\lambda_i - \lambda_j).$$

Proof The proof is straightforward computation:

$$\frac{dx}{\prod_{i=1}^{m}(x - \lambda_i)} = dt$$

i.e.

$$dx \sum_{i=1}^{m} \left(\frac{\prod_{\substack{j=1\\ j\neq i}}^{m} (\lambda_i - \lambda_j)}{x - \lambda_i} \right) = dt$$

i.e.

$$\sum_{i=1}^{m} \beta_i d \log(x - \lambda_i) = dt$$

and so

$$\prod_{i=1}^{m} (x - \lambda_i)^{\beta_i} = C' e^t.$$

For x near λ_i we have

$$(x - \lambda_i)^{\beta_i} \prod_{\substack{j=1\\ j\neq i}}^{m} (\lambda_i - \lambda_j)^{\beta_j} \cong C' e^t$$

and the result follows. □

(5.10) **Corollary** The solution of the Clifford system (5.7) near the equilibrium point $\bar{X}_i$ (associated with λ_i) is given by

$$X \cong \bar{X}_i + C \left(\prod_{\substack{j=1\\ j\neq i}}^{m} (\bar{X}_i - \bar{X}_j)^{-\beta_j/\beta_i} \right) e^{t/\beta_i}$$

for some $C \in \mathfrak{A}$ where the powers are taken in the usual way as logs of matrices. □

(5.11) **Remark** From this result we expect that if β_i is pure imaginary, then the equation has an isochronous centre near $\bar{X}_i$. This can be seen in the following example:

(5.12) **Example** (See also [1].) Consider the equation

$$\dot{X} = iX + X^3 \tag{5.13}$$

on $\mathfrak{A}_{0,1} \cong \mathbb{C}$, i.e.

$$\frac{d}{dt}(x_1 + ix_2) = i(x_1 + ix_2) + (x_1 + ix_2)^3$$

where $X = (x_1 + ix_2)$. This equation is equivalent to the two equations

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1^3 - 3x_1 x_2^2 \\ \dot{x}_2 &= x_1 + 3x_1^2 x_2 - x_2^3 \end{aligned}$$

The equilibrium points are given by

$$X(X + Ie^{i\frac{3\pi}{4}})(X - Ie^{i\frac{3\pi}{4}}) = 0$$

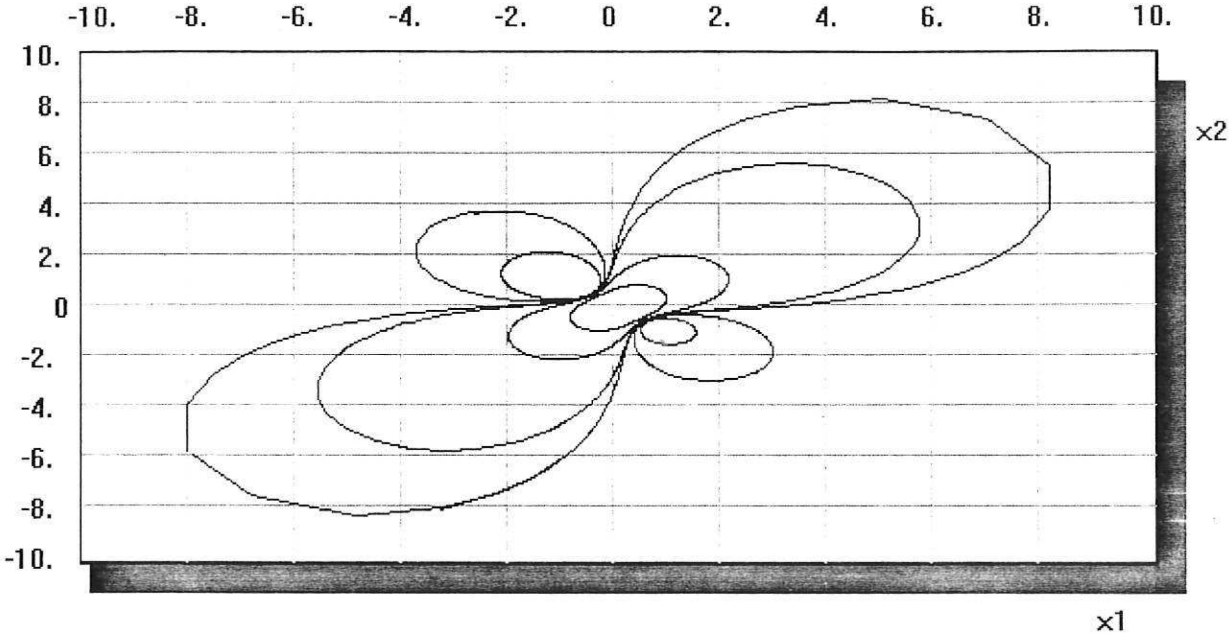
i.e.

$$x_1 + ix_2 = 0,\ e^{i\frac{3\pi}{4}} \text{ or } -e^{i\frac{3\pi}{4}}$$

so that we get the points $(0,0), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Here,

$$\begin{aligned} \beta_1 &= -e^{i\frac{3\pi}{4}} e^{i\frac{3\pi}{4}} = -e^{i\frac{3\pi}{2}} = i \\ \beta_2 &= e^{i\frac{3\pi}{4}} (e^{i\frac{3\pi}{4}} + e^{i\frac{3\pi}{4}}) = -2i \\ \beta_3 &= -e^{i\frac{3\pi}{4}} (-e^{i\frac{3\pi}{4}} - e^{i\frac{3\pi}{4}}) = -2i \end{aligned}$$

and we get three isochronous centres with periods $2\pi, 4\pi, 4\pi$:



System with three isochronous centres

6. Darboux Theory and Isochronous Tori

In this final section we shall consider the existence of isochronous orbits in systems of the form

$$\dot{X} = p(X, \bar{X}) \ , \ X \in \mathfrak{A}$$

where p is a polynomial, and $\mathfrak{A}$ is some Clifford algebra. In order to do this we shall generalise some results from [4, 5] on the Darboux theory of differential equations. We shall first consider the linear equation

$$\dot{X} = iX \ , \ X(0) = X_0 \in \mathfrak{A}_n \tag{6.1}$$

on the Euclidean Clifford algebra $\mathfrak{A}_n$. Note that, here, i refers to an imaginary unit in $\mathfrak{A}_n$ (the image of the usual complex i in $\mathfrak{A}_n$) and so does not commute with X in general. The solution is

$$X(t) = e^{it} X_0$$

and so it clearly represents an isochronous centre with period 2π. We seek to determine all systems in $\mathbb{R}^n$ which can be lifted to an equation of this form. If $E_1, \cdots, E_{2^n}$ is a basis of $\mathfrak{A}_n$, as in section 3, then we can write

$$X = \sum_{k=1}^{2^n} x_k E_k$$

and we can identify E_1 with 1 and E_2 with i, E_3 with j (the quaternion) etc. Note that, from lemma 3.1, for any two basis elements E_i, E_j we have $E_i E_j = \pm E_{k(i,j)}$ where $E_{k(i,j)} = E_k$ ($= -E_{k(j,i)}$) is another basis element. This gives

(6.2) **Lemma** The equation (6.1) is equivalent to the system

$$\begin{aligned} \dot{x}_i &= \mp x_j \\ \dot{x}_j &= \pm x_i \end{aligned} \tag{6.3}$$

where $j = k(2, i)$, $1 \le i, j \le 2^n$.
Proof We can write (6.1) in the form

$$\begin{aligned} \dot{x}_1 E_1 + \dot{x}_2 E_2 + \cdots + \dot{x}_{2^n} E_{2^n} &= i(x_1 E_1 + x_2 E_2 + \cdots + x_{2^n} E_{2^n}) \\ &= (x_1 E_2 E_1 + x_2 E_2^2 + \cdots + x_{2^n} E_2 E_{2^n}) \\ &= x_1 E_{k(2,1)} - x_2 \pm x_3 E_{k(2,3)} \pm \cdots \pm x_{2^n} E_{k(2,2^n)} \\ &= x_1 E_2 - x_2 \pm x_3 E_{k(2,3)} \pm \cdots \pm x_{2^n} E_{k(2,2^n)} \end{aligned}$$

and equating coefficients gives (6.3), since if $j = k(2, i)$ then $E_2 E_i = E_j$ and so $E_2 E_j = -E_i$ and hence $i = k(2, j)$.□
(6.4) **Remark** The equation (6.1) has solutions on a 2^n-dimensional torus. □
(6.5) **Example** On $\mathfrak{A}_2 = \mathbb{H}$ we have

$$\dot{X} = iX$$

is equivalent to

$$\dot{x}_1 + \dot{x}_2 i + \dot{x}_3 j + \dot{x}_4 k = i(x_1 + x_2 i + x_3 j + x_4 k)$$

which gives the equations

$$\begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= x_1 \\ \dot{x}_3 &= -x_4 \\ \dot{x}_4 &= x_3 \; . \; \square \end{aligned}$$

(6.6) **Remark** We can use multipliers other than i in $\mathfrak{A}_n$ for the equation (6.1). For example, if $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$, then if

$$F_i = \frac{1}{2} e_i e_{n-i+1} \; (1 \le i \le [\frac{1}{2} n])$$

we have

$$F_i^2 = -\frac{1}{4} \; , \; F_i F_j = F_j F_i \; (i \ne j)$$

and so

$$\exp(tF_i) = \cos\frac{t}{2} + 2F_i \sin\frac{t}{2}.$$

Hence the system

$$\dot{X} = F_i X \tag{6.7}$$

has solution

$$X(t) = \exp(tF_i) X_0$$

which has period 4π. The equation (6.7) is

$$\dot{x}_1 E_1 + \dot{x}_2 E_2 + \cdots + \dot{x}_{2^n} E_{2^n} = \frac{1}{2} e_i e_{n-i+1} (x_1 E_1 + x_2 E_2 + \cdots + x_{2^n} E_{2^n}).$$

Now, $e_i e_{n-i+1} E_k$ is $\pm E_\ell$ for some ℓ and so

$$E_k = \mp e_i e_{n-i+1} E_\ell$$

so that we get the equations

$$\begin{aligned} \dot{x}_\ell &= \pm \frac{1}{2} x_k \\ \dot{x}_k &= \mp \frac{1}{2} x_\ell \,. \end{aligned}$$

We now consider a system of the form

$$\dot{X} = iX + \sum_{i+j=2}^{K} \alpha_{ij} X^i \bar{X}^j \,,\ X(0) = X_0 \in \mathfrak{A} \tag{6.8}$$

where we assume $X(t)$ and $\bar{X}(t)$ commute. (as seen in section 2, this will certainly be the case if $X(t) \in \Gamma$, the Clifford group of $\mathfrak{A}$, although this is not necessary.)The definition 2.1 in [5] can be generalised as follows:

(6.9) **Definition** (1) An **invariant algebraic hypersurface** for the equation (6.8) is a hypersurface in $\mathbb{C}^{2^n}$ given by an equation $F(X, \bar{X}) = 0$, with $F(X, \bar{X}) \in \mathbb{C}[X, \bar{X}]$ such that there exists $K(X, \bar{X}) \in \mathbb{C}[X, \bar{X}]$ satisfying

$$DF \triangleq F_X \dot{X} + F_{\bar{X}} \dot{\bar{X}} = F(X, \bar{X}) K(X, \bar{X}).$$

(2) A **Darboux factor** is a polynomial $F(X, \bar{X})$ such that $F(X, \bar{X}) = 0$ is an invariant algebraic hypersurface.

(3) A **generalised Darboux factor** is a Darboux factor or an analytic Darboux factor of the form $\exp(G(X, \bar{X}))$ where $G(X, \bar{X}) \in \mathbb{C}[X, \bar{X}]$.

In either case, $K(X, \bar{X})$ is called the **cofactor** of F.

(4) If $DF(X, \bar{X}) \equiv 0$ for a nonconstant function F, it is called a **first integral** of the system (6.8).

(5) The system (6.8) is **Darboux linearisable** if there exists a **Darboux function** Z of the form

$$Z = \prod_{j=0}^{k} F_j^{\alpha_j} \,,\ \alpha_j \in \mathbb{C}$$

where $F_j(X, \bar{X}) \in \mathbb{C}[X, \bar{X}]$ is a (generalised) Darboux factor for all j, regular at the origin, which transforms (6.8) into the system (6.1).

For Clifford systems we only get a sufficient condition for Darboux linearisability:

(6.10) **Theorem** The system (6.8) is Darboux linearisable if there exist invariant algebraic hypersurfaces $F_0(X, \bar{X}) = X + o(X, \bar{X}) = 0$ and $F_j(X, \bar{X}) = 0$, $1 \le j \le k$ such that F_j and DF_j belong to the centre of $\mathfrak{A}$ for $1 \le j \le k$ and for which

$$K_0 + \sum_{j=1}^{k} \alpha_j K_j = i$$

where K_j is the cofactor of F_j, $0 \leq j \leq k$. A linearising change of coordinates is given by

$$Z = F_0 \prod_{j=1}^{k} F_j^{\alpha_j}.$$

(Here, i refers to the imaginary unit of $\mathfrak{A}$, so that for $\mathbb{H}=\mathfrak{A}_{0,2}$ we have $i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ where the i inside the matrix is the ordinary complex value. Note that powers of commuting elements can be defined in the usual way by logs.)

Proof The proof is similar to that in [5], given the commutativity of the F_j. □

To illustrate the theory we note that in [5] the Darboux linearisation method is used to find cubic isochronous centres, i.e. systems of the form

$$\begin{aligned} \dot{x} &= -M\frac{\partial F}{\partial y} \\ \dot{y} &= M\frac{\partial F}{\partial x} \end{aligned} \tag{6.11}$$

where

$$F(x,y) = \frac{1}{2}\left(x^2 + y^2\right)(1 + x + Ay)^{2a}(1 + x + By)^{2b}$$

and

$$M(x,y) = (1 + x + Ay)^{1-2a}(1 + x + By)^{1-2b}\ .$$

In order to generalise this to Clifford systems defined on $\mathfrak{A}_{0,2} = \mathbb{H}$, first note that (6.11) can be written in the form

$$\dot{z} = 2iM\frac{\partial F}{\partial \bar{z}}. \tag{6.12}$$

We could also look for cubic isochronous tori, but the calculations are considerable, so we illustrate the theory by finding quadratic systems in $\mathfrak{A}_{0,2} = \mathbb{H}$. Note, however, that higher-order isochronous tori can be found in a similar manner. Thus, in comparison with (6.11) we look for systems of the form

$$\dot{X} = i[X(I + AX + \bar{X}\bar{A}) + 2aX\bar{X}\bar{A}], \tag{6.13}$$

where we have used

$$\begin{aligned} F(X) &= \frac{1}{2}X\bar{X}(I + AX + \bar{X}\bar{A})^{2a} \\ \frac{\partial F}{\partial \bar{X}} &= \frac{1}{2}X(I + AX + \bar{X}\bar{A})^{2a} + aX\bar{X}\bar{A}(I + AX + \bar{X}\bar{A})^{2a-1} \end{aligned}$$

and

$$M = (I + AX + \bar{X}\bar{A})^{1-2a}.$$

Note that $(I + AX + \bar{X}\bar{A})$ commutes with anything in $\mathfrak{A}_{0,2}$ (i.e. it is in the centre). We have
(6.14) **Theorem** The system (6.13) is a quadratic isochronous torus if

$$iAX + i\bar{X}\bar{A} + 2ia\bar{X}\bar{A} + \alpha(AiX - \bar{X}i\bar{A}) = 0$$

with linearising change of coordinates

$$Z = X(I + AX + \bar{X}\bar{A})^{\alpha}.$$

Proof We must find the cofactors of X and $(I + AX + \bar{X}\bar{A})$. For X, we have

$$\dot{X} = 2i\left(\frac{1}{2}X(I + AX + \bar{X}\bar{A}) + aX\bar{X}\bar{A}\right)$$

so the cofactor of X is

$$i(I + AX + \bar{X}\bar{A}) + 2ia\bar{X}\bar{A}.$$

For $(I + AX + \bar{X}\bar{A})$, we have

$$\begin{aligned}
\frac{d}{dt}(I + AX + \bar{X}\bar{A}) &= A\dot{X} + \dot{\bar{X}}\,\bar{A} \\
&= A2i\left(\frac{1}{2}X(I + AX + \bar{X}\bar{A}) + aX\bar{X}\bar{A}\right) - 2\left(\frac{1}{2}(I + AX + \bar{X}\bar{A})\bar{X} + aAX\bar{X}\right)i\bar{A} \\
&= (AiX - \bar{X}i\bar{A})(I + AX + \bar{X}\bar{A})
\end{aligned}$$

since $X\bar{X}\bar{A} = \bar{A}X\bar{X} = \bar{A}\bar{X}X$. The result follows since

$$K_0 + \alpha K_1 = i + iAX + i\bar{X}\bar{A} + 2ia\bar{X}\bar{A} + \alpha(AiX - \bar{X}i\bar{A}). \;\square$$

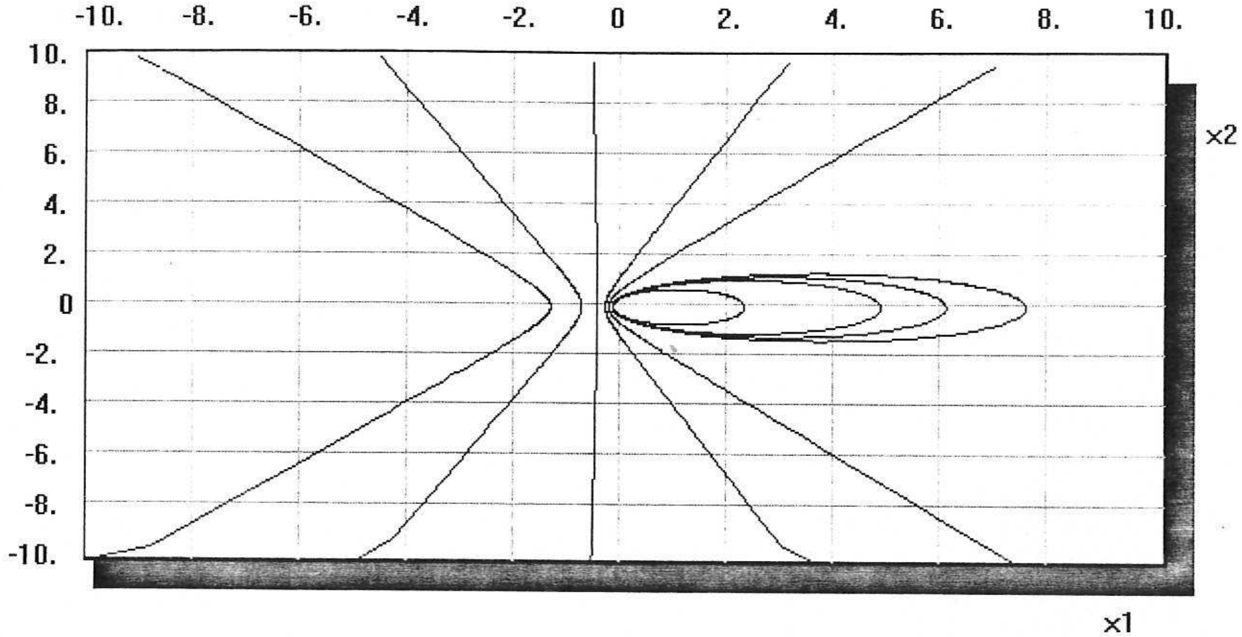
(6.15) **Example** As a simple example take $\alpha = a = -1$, $A = I$. Then, since

$$X = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}$$

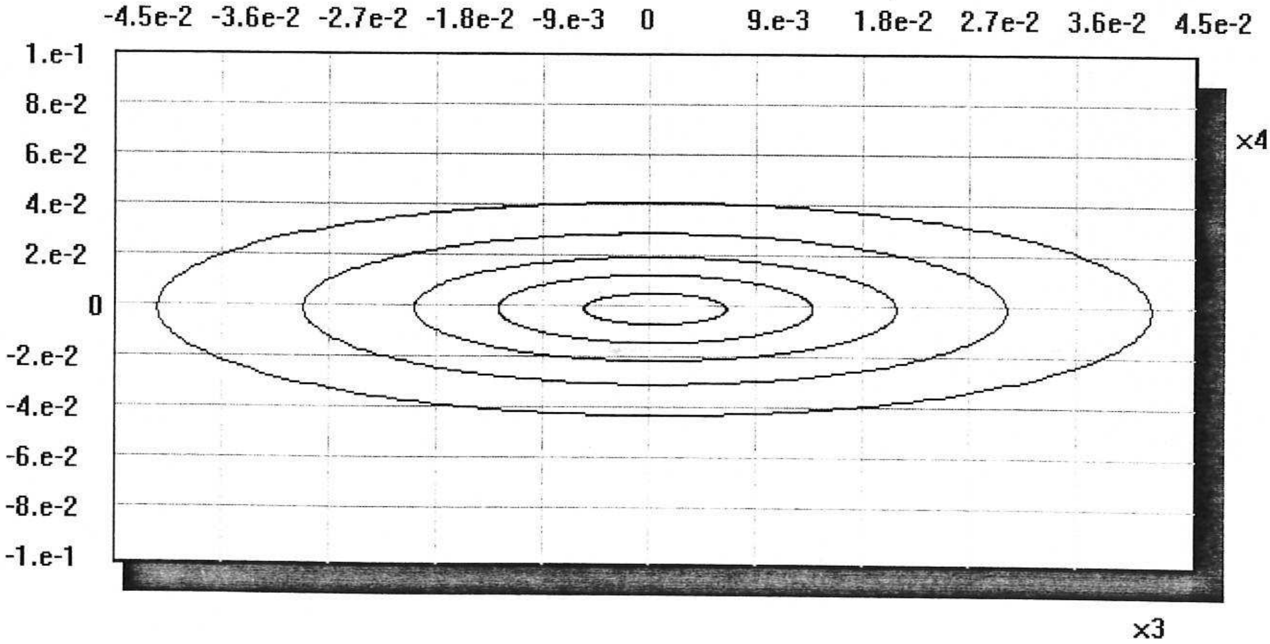
we get the system

$$\begin{aligned}
\dot{x}_1 &= -x_2 - 2x_1x_2 \\
\dot{x}_2 &= x_1 - 2x_2^2 - 2x_3^2 - 2x_4^2 \\
\dot{x}_3 &= -x_4 - 2x_1x_4 \\
\dot{x}_4 &= x_3 + 2x_1x_3
\end{aligned}$$

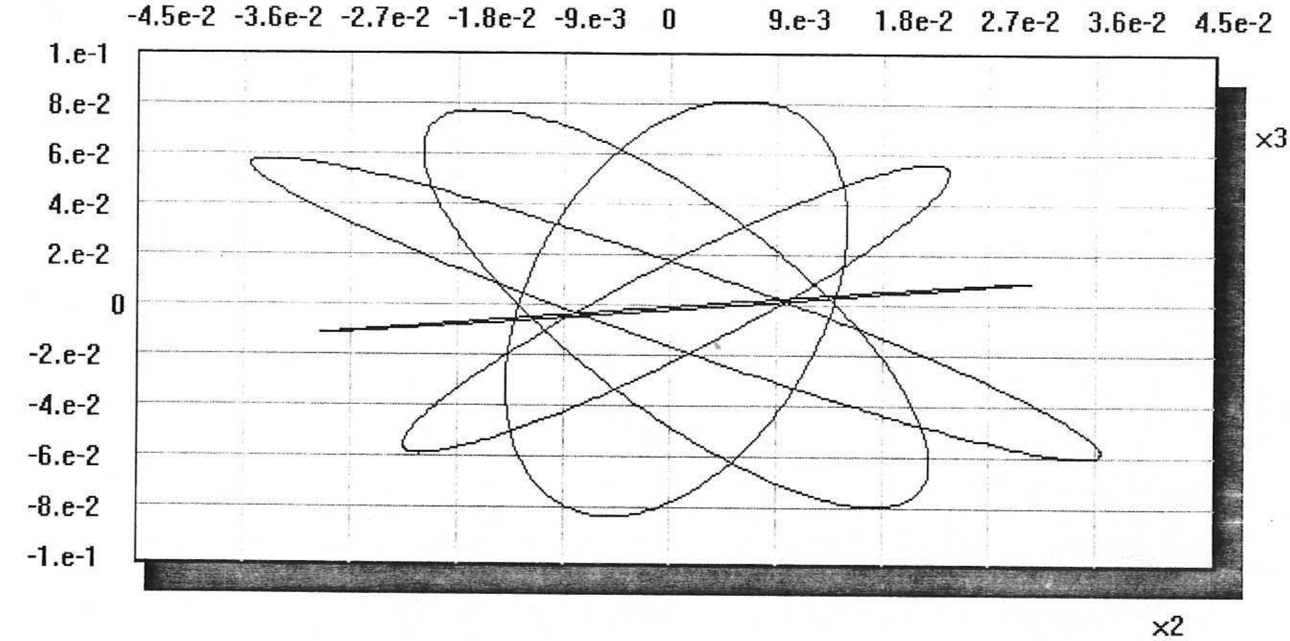
which has typical trajectories shown in the figures below:



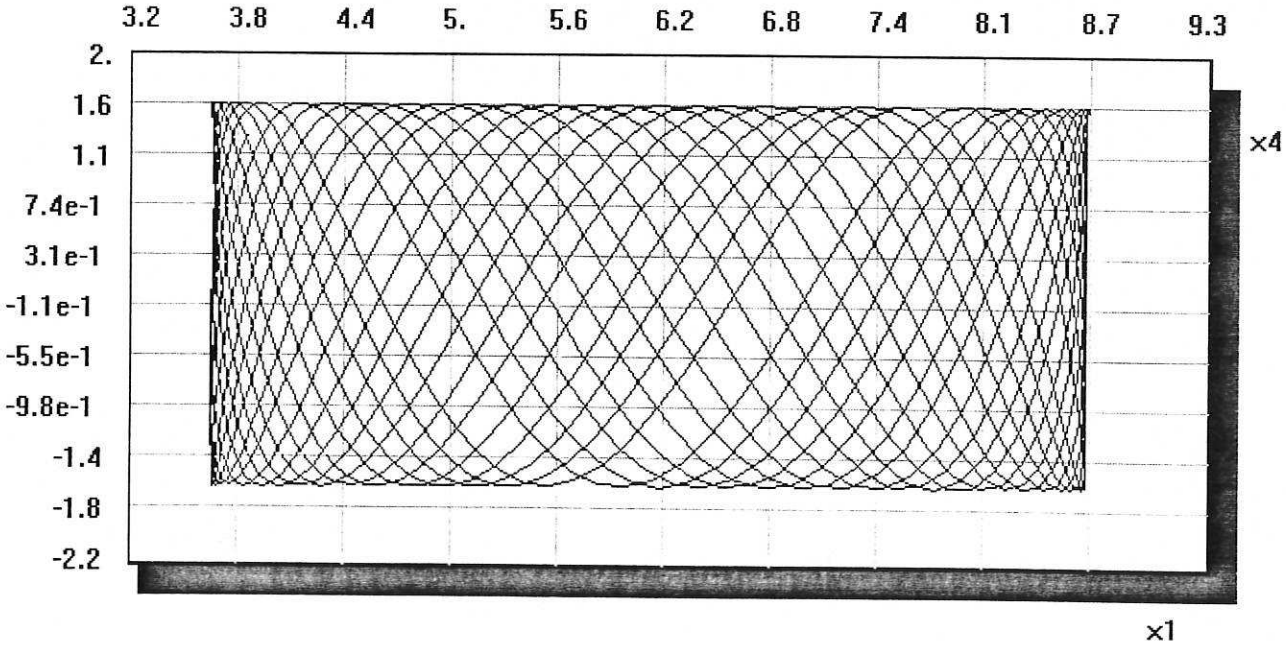
Isochronous torus in x1-x2 plane



Isochronous torus in x3-x4 plane



Isochronous torus in x2-x3 plane



Typical orbit for larger initial values

7. Conclusions

In this paper we have studied the lifting of certain dynamical systems to Clifford algebras, where their properties may be more easily studied. In particular, we have generalised the Lie series to analytic Clifford systems and shown how to obtain local approximations to the solutions. Lifting a system to a Clifford algebra enables one to use the product structure on the algebra which is not present when $\mathbb{R}^n$ is simply regarded as a vector space. This has the effect of making the system appear, in a sense, like a one-dimensional system. The only thing which is no longer present, compared with the one-dimensional case, is commutativity. The last part of the paper is concerned with generalising a number of results in the theory of ischronous centres to Clifford systems and conditions for quadratic isochronous tori have been found. This depends heavily on finding commutative Darboux factors and so classifying all isochronous tori may prove to be difficult.

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