Uncertainty and Analyticity

Vladimir V. Kisil

Abstract. We describe a connection between minimal uncertainty states and holomorphy-type conditions on the images of the respective wavelet transforms. The most familiar example is the Fock–Segal–Bargmann transform generated by the Gaussian, however, this also occurs under more general assumptions.

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1. Introduction

There are two and a half main examples of reproducing kernel spaces of analytic function. One is the Fock–Segal–Bargmann (FSB) space and others (one and a half)—the Bergman and Hardy spaces. The first space is generated by the Heisenberg group [2, § 1.6; 5, § 7.3], two others—by the group SU(1, 1) [5, § 4.2] (this explains our way of counting).

Those spaces have the following properties, which make their study particularly pleasant and fruitful:

i. There is a group, which acts transitively on functions' domain.
ii. There is a reproducing kernel.
iii. The space consists of holomorphic functions.

Furthermore, for FSB space there is the following property:

iv. The reproducing kernel is generated by a function, which minimises the uncertainty for coordinate and momentum observables.

It is known, that a transformation group is responsible for the appearance of the reproducing kernel [1, Thm. 8.1.3]. This paper shows that the last two properties are equivalent and connected to the group as well.

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2. The Uncertainty Relation

In quantum mechanics [2, § 1.1], an observable (self-adjoint operator on a Hilbert space $\mathcal{H}$) A produces the expectation value $\bar{A}$ on a state (a unit vector) $\phi \in \mathcal{H}$ by $\bar{A} = \langle A\phi, \phi\rangle$. Then, the dispersion is evaluated as follow:

$$\Delta_\phi^2(A) = \left\langle (A-\bar{A})^2\phi, \phi\right\rangle = \left\langle (A-\bar{A})\phi, (A-\bar{A})\phi\right\rangle = \left\|(A-\bar{A})\phi\right\|^2. \quad (1)$$

The next theorem links obstructions of exact simultaneous measurements with non-commutativity of observables.

Theorem 1 (The Uncertainty relation). *If A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$, then*

$$\|(A-a)u\|\,\|(B-b)u\| \geq \tfrac{1}{2}\left|\langle (AB-BA)u,u\rangle\right|, \quad (2)$$

for any $u \in \mathcal{H}$ from the domains of AB and BA and a, $b \in \mathbb{R}$. Equality holds precisely when u is a solution of $((A-a)+\mathrm{i}r(B-b))u = 0$ for some real r.

Proof. The proof is well-known [2, § 1.3], but it is short, instructive and relevant for the following discussion, thus we include it in full. We start from simple algebraic transformations:

$$\begin{aligned}\langle (AB-BA)u,u\rangle &= \langle ((A-a)(B-b)-(B-b)(A-a))u,u\rangle\\ &= \langle (B-b)u,(A-a)u\rangle - \langle ((A-a))u,(B-b)u\rangle\\ &= 2\mathrm{i}\Im\,\langle (B-b)u,(A-a)u\rangle \end{aligned} \quad (3)$$

Then by the Cauchy–Schwartz inequality:

$$\tfrac{1}{2}\langle (AB-BA)u,u\rangle \leq \left|\langle (B-b)u,(A-a)u\rangle\right| \leq \|(B-b)u\|\,\|(A-a)u\|.$$

The equality holds if and only if $(B-b)u$ and $(A-a)u$ are proportional by a *purely imaginary* scalar. □

The famous application of the above theorem is the following fundamental relation in quantum mechanics. Recall [2, § 1.2], that the one-dimensional Heisenberg group $\mathbb{H}^1$ consists of points $(s,x,y) \in \mathbb{R}^3$, with the group law:

$$(s,x,y)*(s',x',y') = (s+s'+\tfrac{1}{2}(xy'-x'y), x+x', y+y'). \quad (4)$$

This is a nilpotent step two Lie group. By the Stone–von Neumann theorem [2, § 1.5], any infinite-dimensional unitary irreducible representation of $\mathbb{H}^1$ is unitary equivalent to the Schrödinger representation $\rho_\hbar$ in $\mathcal{L}_2(\mathbb{R})$ parametrised by the Planck constant $\hbar \in \mathbb{R}\setminus\{0\}$. A physically consistent form of $\rho_\hbar$ is [6, (3.5)]:

$$[\rho_\hbar(s,x,y)f\,](q) = e^{-2\pi\mathrm{i}\hbar(s+xy/2)-2\pi\mathrm{i}xq}\,f(q+\hbar y). \quad (5)$$

Elements of the Lie algebra $\mathfrak{h}_1$, corresponding to the infinitesimal generators X and Y of one-parameters subgroups $(0,t/(2\pi),0)$ and $(0,0,t)$ in $\mathbb{H}^1$, are represented in (5) by the (unbounded) operators M and D on $\mathcal{L}_2(\mathbb{R})$:

$$M = -\mathrm{i}q, \quad D = \hbar\tfrac{d}{dq}, \quad \text{with the commutator} \quad [M,D] = \mathrm{i}\hbar I. \quad (6)$$

In the Schrödinger model of quantum mechanics, $f(q) \in \mathcal{L}_2(\mathbb{R})$ is interpreted as a wave function (a state) of a particle, with M and D are the observables of its coordinate and momentum.

Corollary 2 (Heisenberg–Kennard uncertainty relation). *For the coordinate M and momentum D observables we have the* Heisenberg–Kennard uncertainty *relation:*

$$\Delta_\phi(M) \cdot \Delta_\phi(D) \geq \frac{h}{2}. \tag{7}$$

The equality holds if and only if $\phi(q) = e^{-cq^2}$, $c \in \mathbb{R}_+$ is the vacuum state in the Schrödinger model.

Proof. The relation follows from the commutator $[M, D] = \mathrm{i}\hbar I$, which, in turn, is the representation of the Lie algebra $\mathfrak{h}_1$ of the Heisenberg group. The minimal uncertainty state in the Schrodinger representation is a solution of the differential equation: $(M - \mathrm{i}rD)\phi = 0$ for some $r \in \mathbb{R}$, or, explicitly:

$$(M - \mathrm{i}rD)\phi = -\mathrm{i}\left(q + r\hbar\frac{d}{dq}\right)\phi(q) = 0. \tag{8}$$

The solution is the Gaussian $\phi(q) = e^{-cq^2}$, $c = \frac{1}{2r\hbar}$. For $c > 0$, this function is in the state space $\mathcal{L}_2(\mathbb{R})$. □

It is common to say that the Gaussian $\phi(q) = e^{-cq^2}$ represents the ground state, which minimises the uncertainty of coordinate and momentum.

3. Wavelet transform and analyticity

3.1. Induced wavelet transform

The following object is common in quantum mechanics [4], signal processing, harmonic analysis [8], operator theory [7,9] and many other areas [5]. Therefore, it has various names [1]: coherent states, wavelets, matrix coefficients, etc. In the most fundamental situation [1, Ch. 8], we start from an irreducible unitary representation ρ of a Lie group G in a Hilbert space $\mathcal{H}$. For a vector $f \in \mathcal{H}$ (called mother wavelet, vacuum state, etc.), we define the map $\mathcal{W}_f$ from $\mathcal{H}$ to a space of functions on G by:

$$[\mathcal{W}_f v](g) = \tilde{v}(g) := \langle v, \rho(g)f \rangle . \tag{9}$$

Under the above assumptions, $\tilde{v}(g)$ is a bounded continuous function on G. The map $\mathcal{W}_f$ intertwines $\rho(g)$ with the left shifts on G:

$$\mathcal{W}_f \circ \rho(g) = \Lambda(g) \circ \mathcal{W}_f, \qquad \text{where } \Lambda(g) : \tilde{v}(g') \mapsto \tilde{v}(g^{-1}g'). \tag{10}$$

Thus, the image $\mathcal{W}_f\mathcal{H}$ is invariant under the left shifts on G. If ρ is square integrable and f is admissible [1, § 8.1], then $\tilde{v}(g)$ is square-integrable with respect to the Haar measure on G. At this point, none of admissible vectors has an advantage over others.

It is common [5, § 5.1], that there exists a closed subgroup $H \subset G$ and a respective $f \in \mathcal{H}$ such that $\rho(h)f = \chi(h)f$ for some character χ of H. In this case, it is enough to know values of $\tilde{v}(\mathsf{s}(x))$, for any continuous section s from the homogeneous space $X = G/H$ to G. The map $v \mapsto \tilde{v}(x) = \tilde{v}(\mathsf{s}(x))$ intertwines ρ with the representation ρ_χ in a certain function space on X induced by the character χ of H [3, § 13.2]. We call the map $\mathcal{W}_f : v \mapsto \tilde{v}(x)$ the *induced wavelet transform* [5, § 5.1].

For example, if $G = \mathbb{H}^1$, $H = \{(s, 0, 0) \in \mathbb{H}^1 : \ s \in \mathbb{R}\}$ and its character $\chi_\hbar(s, 0, 0) = e^{2\pi i\hbar s}$, then any vector $f \in \mathcal{L}_2(\mathbb{R})$ satisfies $\rho_\hbar(s, 0, 0)f = \chi_\hbar(s)f$ for the representation (5). Thus, we still do not have a reason to prefer any admissible vector to others.

3.2. Right shifts and analyticity

To discover some preferable mother wavelets, we use the following a general result from [5, § 5]. Let G be a locally compact group and ρ be its representation in a Hilbert space $\mathcal{H}$. Let $[\mathcal{W}_f v](g) = \langle v, \rho(g)f\rangle$ be the wavelet transform defined by a vacuum state $f \in \mathcal{H}$. Then, the right shift $R(g)$: $[\mathcal{W}_f v](g') \mapsto [\mathcal{W}_f v](g'g)$ for $g \in G$ coincides with the wavelet transform $[\mathcal{W}_{f_g} v](g') = \langle v, \rho(g')f_g\rangle$ defined by the vacuum state $f_g = \rho(g)f$. In other words, the covariant transform intertwines right shifts on the group G with the associated action ρ on vacuum states, cf. (10):

$$R(g) \circ \mathcal{W}_f = \mathcal{W}_{\rho(g)f}. \tag{11}$$

Although, the above observation is almost trivial, applications of the following corollary are not.

Corollary 3 (Analyticity of the wavelet transform, [5, § 5]**).** *Let G be a group and dg be a measure on G. Let ρ be a unitary representation of G, which can be extended by integration to a vector space V of functions or distributions on G. Let a mother wavelet $f \in \mathcal{H}$ satisfy the equation*

$$\int_G a(g)\, \rho(g)f\, dg = 0,$$

for a fixed distribution $a(g) \in V$. Then any wavelet transform $\tilde{v}(g) = \langle v, \rho(g)f\rangle$ obeys the condition:

$$D\tilde{v} = 0, \qquad \textit{where} \quad D = \int_G \bar{a}(g)\, R(g)\, dg, \tag{12}$$

with R being the right regular representation of G.

Some applications (including discrete one) produced by the $ax+b$ group can be found in [8, § 6]. We turn to the Heisenberg group now.

Example 4 (Gaussian and FSB transform). The Gaussian $\phi(x) = e^{-cq^2/2}$ is a null-solution of the operator $\hbar cM - iD$. For the centre $Z = \{(s, 0, 0) : \ s \in \mathbb{R}\} \subset \mathbb{H}^1$, we define the section $\mathsf{s} : \mathbb{H}^1/Z \to \mathbb{H}^1$ by $\mathsf{s}(x, y) = (0, x, y)$. Then, the corresponding induced wavelet transform is:

$$\tilde{v}(x, y) = \langle v, \rho(\mathsf{s}(x, y))f\rangle = \int_{\mathbb{R}} v(q)\, e^{\pi i\hbar xy - 2\pi ixq}\, e^{-c(q+\hbar y)^2/2}\, dq. \tag{13}$$

The infinitesimal generators X and Y of one-parameters subgroups $(0, t/(2\pi), 0)$ and $(0, 0, t)$ are represented through the right shift in (4) by

$$R_*(X) = -\tfrac{1}{4\pi} y\partial_s + \tfrac{1}{2\pi}\partial_x, \quad R_*(Y) = \tfrac{1}{2} x\partial_s + \partial_y.$$

For the representation induced by the character $\chi_\hbar(s, 0, 0) = e^{2\pi \mathrm{i}\hbar s}$ we have $\partial_s = 2\pi\mathrm{i}\hbar I$. Cor. 3 ensures that the operator

$$\hbar c \cdot R_*(X) + \mathrm{i} \cdot R_*(Y) = -\frac{\hbar}{2}(2\pi x + \mathrm{i}\hbar c y) + \frac{\hbar c}{2\pi}\partial_x + \mathrm{i}\partial_y \tag{14}$$

annihilate any $\tilde{v}(x, y)$ from (13). The integral (13) is known as Fock–Segal–Bargmann (FSB) transform and in the most common case the values $\hbar = 1$ and $c = 2\pi$ are used. For these, operator (14) becomes $-\pi(x + \mathrm{i}y) + (\partial_x + \mathrm{i}\partial_y) = -\pi z + 2\partial_{\bar{z}}$ with $z = x + \mathrm{i}y$. Then the function $V(z) = e^{\pi z\bar{z}/2}\, \tilde{v}(z) = e^{\pi(x^2+y^2)/2}\, \tilde{v}(x, y)$ satisfies the Cauchy–Riemann equation $\partial_{\bar{z}} V(z) = 0$.

This example shows, that the Gaussian is a preferred vacuum state (as producing analytic functions through FSB transform) exactly for the same reason as being the minimal uncertainty state: the both are derived from the identity $(\hbar c M + \mathrm{i}D)e^{-cq^2/2} = 0$.

3.3. Uncertainty and analyticity

The main result of this paper is a generalisation of the previous observation, which bridges together Cor. 3 and Thm. 1. Let G, H, ρ and $\mathcal{H}$ be as before. Assume, that the homogeneous space $X = G/H$ has a (quasi-)invariant measure $d\mu(x)$ [3, § 13.2]. Then, for a function (or a suitable distribution) k on X we can define the integrated representation:

$$\rho(k) = \int_X k(x)\rho(\mathsf{s}(x))\, d\mu(x), \tag{15}$$

which is (possibly, unbounded) operators on (possibly, dense subspace of) $\mathcal{H}$. In particular, $R(k)$ denotes the integrated right shifts, for $H = \{e\}$.

Theorem 5. *Let k_1 and k_2 be two distributions on X with the respective integrated representations $\rho(k_1)$ and $\rho(k_2)$. The following are equivalent:*

i. *A vector $f \in \mathcal{H}$ satisfies the identity*

$$\Delta_f(\rho(k_1)) \cdot \Delta_f(\rho(k_2)) = |\langle[\rho(k_1), \rho(k_1)]f, f\rangle| \,.$$

ii. *The image of the wavelet transform $\mathcal{W}_f : v \mapsto \tilde{v}(g) = \langle v, \rho(g)f\rangle$ consists of functions satisfying the equation $R(k_1 + \mathrm{i}rk_2)\tilde{v} = 0$ for some $r \in \mathbb{R}$, where R is the integrated form* (15) *of the right regular representation on G.*

Proof. This is an immediate consequence of a combination of Thm. 1 and Cor. 3. □

Example 4 is a particular case of this theorem with $k_1(x, y) = \delta'_x(x, y)$ and $k_2(x, y) = \delta'_y(x, y)$ (partial derivatives of the delta function), which represent vectors X and Y from the Lie algebra $\mathfrak{h}_1$. The next example will be of this type as well.

3.4. Hardy space

Let $\mathrm{SU}(1,1)$ be the group of 2×2 complex matrices of the form $\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}$ with the unit determinant $|\alpha|^2-|\beta|^2=1$. A standard basis in the Lie algebra $\mathfrak{su}_{1,1}$ is

$$A=\frac{1}{2}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\quad B=\frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad Z=\begin{pmatrix}\mathrm{i} & 0\\ 0 & -\mathrm{i}\end{pmatrix}.$$

The respective one-dimensional subgroups consist of matrices:

$$e^{tA}=\begin{pmatrix}\cosh\frac{t}{2} & -\mathrm{i}\sinh\frac{t}{2}\\ \mathrm{i}\sinh\frac{t}{2} & \cosh\frac{t}{2}\end{pmatrix},\ e^{tB}=\begin{pmatrix}\cosh\frac{t}{2} & \sinh\frac{t}{2}\\ \sinh\frac{t}{2} & \cosh\frac{t}{2}\end{pmatrix},\ e^{tZ}=\begin{pmatrix}e^{\mathrm{i}t} & 0\\ 0 & e^{-\mathrm{i}t}\end{pmatrix}.$$

The last subgroup—the maximal compact subgroup of $\mathrm{SU}(1,1)$—is usually denoted by K. The commutators of the $\mathfrak{su}_{1,1}$ basis elements are

$$[Z,A]=2B,\qquad [Z,B]=-2A,\qquad [A,B]=-\frac{1}{2}Z. \tag{16}$$

Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ with the rotation-invariant measure. The mock discrete representation of $\mathrm{SU}(1,1)$ [10, § VI.6] acts on $\mathcal{L}_2(\mathbb{T})$ by unitary transformations

$$[\rho_1(g)f](z)=\frac{1}{(\bar\beta z+\bar\alpha)}\,f\left(\frac{\alpha z+\beta}{\bar\beta z+\bar\alpha}\right),\qquad g^{-1}=\begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}. \tag{17}$$

The respective derived representation ρ_{1*} of the $\mathfrak{su}_{1,1}$ basis is:

$$\rho_{1*}^{A}=\tfrac{\mathrm{i}}{2}(z+(z^2+1)\partial_z),\quad \rho_{1*}^{B}=\tfrac{1}{2}(z+(z^2-1)\partial_z),\quad \rho_{1*}^{Z}=-\mathrm{i}I-2\mathrm{i}z\partial_z.$$

Thus, $\rho_{1*}^{B+\mathrm{i}A}=-\partial_z$ and the function $f_+(z)\equiv 1$ satisfies $\rho_{1*}^{B+\mathrm{i}A}f_+=0$. Recalling the commutator $[A,B]=-\frac{1}{2}Z$ we note that $\rho_1(e^{tZ})f_+=e^{\mathrm{i}t}f_+$. Therefore, there is the following identity for dispersions on this state:

$$\Delta_{f_+}(\rho_{1*}^{A})\cdot\Delta_{f_+}(\rho_{1*}^{B})=\tfrac{1}{2},$$

with the minimal value of uncertainty among all eigenvectors of the operator $\rho_1(e^{tZ})$.

Furthermore, the vacuum state f_+ generates the induced wavelet transform for the subgroup $K=\{e^{tZ}\mid t\in\mathbb{R}\}$. We identify $\mathrm{SU}(1,1)/K$ with the open unit disk $D=\{w\in\mathbb{C}\mid |w|<1\}$ [5, § 5.5; 9]. The map $\mathsf{s}:\mathrm{SU}(1,1)/K\rightarrow\mathrm{SU}(1,1)$ is defined as $\mathsf{s}(w)=\frac{1}{\sqrt{1-|w|^2}}\begin{pmatrix}1 & w\\ \bar w & 1\end{pmatrix}$. Then, the induced wavelet transform is:

$$\begin{aligned}\tilde v(w)=\langle v,\rho_1(\mathsf{s}(w))f_+\rangle &= \frac{1}{2\pi\sqrt{1-|w|^2}}\int_{\mathbb{T}}\frac{v(e^{\mathrm{i}\theta})\,d\theta}{1-we^{-\mathrm{i}\theta}}\\ &= \frac{1}{2\pi\mathrm{i}\sqrt{1-|w|^2}}\int_{\mathbb{T}}\frac{v(e^{\mathrm{i}\theta})\,de^{\mathrm{i}\theta}}{e^{\mathrm{i}\theta}-w}.\end{aligned}$$

Clearly, this is the Cauchy integral up to the factor $\frac{1}{\sqrt{1-|w|^2}}$, which presents the conformal metric on the unit disk. Similarly, we can consider the operator

$\rho_{1*}^{B-\mathrm{i}A} = z + z^2\partial_z$ and the function $f_-(z) = \frac{1}{z}$ simultaneously solving the equations $\rho_{1*}^{B-\mathrm{i}A} f_- = 0$ and $\rho_1(e^{tZ})f_- = e^{-\mathrm{i}t} f_-$. It produces the integral with the conjugated Cauchy kernel.

Finally, we can calculate the operator (12) annihilating the image of the wavelet transform. In the coordinates $(w,t) \in (\mathrm{SU}(1,1)/K) \times K$, the restriction to the induced subrepresentation is, cf. [10, § IX.5]:

$$\mathfrak{L}^{B-\mathrm{i}A} = e^{2\mathrm{i}t}(-\frac{1}{2}w + (1-|w|^2)\partial_{\bar{w}}).$$

Furthermore, if $\mathfrak{L}^{B-\mathrm{i}A}\tilde{v}(w) = 0$, then $\partial_{\bar{w}}(\sqrt{1-w\bar{w}} \cdot \tilde{v}(w)) = 0$. That is, $V(w) = \sqrt{1-w\bar{w}} \cdot \tilde{v}(w)$ is a holomorphic function on the unit disk.

Similarly, we can treat representations of $\mathrm{SU}(1,1)$ in the space of square integrable functions on the unit disk. The irreducible components of this representation are isometrically isomorphic [5, § 4–5] to the weighted Bergman spaces of (purely poly-)analytic functions on the unit, cf. [11].

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Vladimir V. Kisil
School of Mathematics, University of Leeds, Leeds LS2 9JT, England
Web: `http://www.maths.leeds.ac.uk/~kisilv/`
e-mail: `kisilv@maths.leeds.ac.uk`