Monograph:

Harrison, R.F. and Banks, S.S. (1998) A New Non-Linear Design Method For Active Vehicle Suspension Systems. Research Report. ACSE Research Report 700 . Department of Automatic Control and Systems Engineering

A NEW NON-LINEAR DESIGN METHOD FOR ACTIVE VEHICLE SUSPENSION SYSTEMS

Robert F Harrison and Stephen P Banks

Department of Automatic Control and Systems Engineering, The University of Sheffield, Sheffield, UK

Research Report No 700

Abstract: A novel non-linear design method based on linear quadratic optimal control theory is presented that applies both to linear and (a wide class of) non-linear systems. The method is easy to apply and results in a globally stabilising, near-optimal solution that can be implemented in real-time. The key feature of the design method is the introduction of state-dependence in the weight matrices of the usual linear quadratic cost function, leading to a *non-linear design method*, even for linear dynamics. To demonstrate the method, a simple linear suspension model is used, in conjunction with a non-linear state penalty, which better reflects the *engineering* objectives of active vehicle vibration suppression. Non-linear dynamics can equally well be accommodated. A number of simulations is conducted and compared, favourably, with a passively mounted vehicle. These preliminary results indicate the potential of the method.

Keywords: Non-linear control systems; quadratic optimal regulators; active vehicle suspension; Riccati equations; global stability.

1. INTRODUCTION

Linear quadratic optimal control theory is a highly developed approach for the synthesis of linear optimal control laws and has been applied widely in studies on active vehicle suspension systems. While the approach is attractive in that it is possible to penalise different variables so as to trade-off between, say, ride comfort and handling, or comfort and suspension travel, the way these variables are treated is essentially fixed – no provision is made to allow the suspension to distinguish between a smooth road and a rough one. Evidently, while comfort might be a prime objective under normal circumstances, on rough surfaces the suspension should be stiffened to avoid hitting its limits, hence incurring damage. Although, in principle, time-varying weighting parameters are allowed in the linear quadratic approach, lack of prior knowledge of the road surface, and the anti-causal calculation for the solution makes the introduction of these difficult. The required amplitude dependence can never, therefore, be achieved through the linear quadratic approach.

In this paper we make use of a new result that generalises the theory to non-linear systems to provide a *non-linear design* method that overcomes the shortcomings mentioned above. The method applies to systems having linear or, a broad class of, non-linear dynamics. In brief, it turns out that the infinite time-horizon linear quadratic regulator problem, when solved afresh at every state, leads to a globally stabilising, near-optimal control policy (Banks and Mhanna, 1992). Thus, for admissible system dynamics, the weighting parameters can be made to be functions of the state variables and the desired amplitude dependence obtained. Thus, the design stage allows for the introduction of non-linearity in the weighting matrices, even for linear dynamics, leading to a more "intelligent" control strategy.

In contrast to the finite-time linear quadratic optimal control problem, which must be implemented off-line, our method is causal, but has considerable computational overhead. However, by using a solution to the Riccati equation based upon the matrix sign function (Gardiner and Laub, 1986), it is possible to derive a parallel algorithm (Gardiner and Laub, 1991) suitable for real-time implementation.

In (Lin and Kanellakopoulos, 1997) a different approach to this problem is proposed using the "backstepping" design method (Kanellakopoulos *et al,* 1992) and a non-linear filter to achieve the desired behaviour. Although a direct comparison is not possible owing to the essential differences between the two methods, we illustrate our approach on the simplified (linear) model described there.

The remainder of the paper is organised as follows. In §2 the linear quadratic regulator is first set out, and the generalised results are stated. In §3 the passive and active suspension models of (Lin and Kanellakopoulos, 1997) are presented and the choice of design parameters is discussed. The results of a series of experiments are described in §4 and conclusions are drawn in §5.

2. THE DESIGN METHOD

2.1 Linear quadratic regulator

The linear quadratic optimal regulation problem is expressed as follows. Find the control policy, **u**, that minimises the cost functional:

$$J = \int_0^\infty (\mathbf{x}'Q\mathbf{x} + \mathbf{u}'R\mathbf{u})dt \qquad (1)$$

subject to the linear time invariant dynamics:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \qquad (2)$$

where **x** is an n-vector of system states, **u** is an m-vector of control variables, A and B are matrices of appropriate dimension and the superscript, t, indicates transposition. The matrices Q and R are positive semi-definite and definite, respectively, and are used to penalise particular states according to the engineering objective and the control effort.

It is well known (e.g. Friedland, 1987) that the control policy which solves the above optimisation problem is a linear combination of the system states and is given by:

$$\mathbf{u} = K\mathbf{x} \qquad (3)$$

where K is in turn given by:

$$K = -R^{-1}B'P \qquad (4)$$

and P is the positive definite solution of the algebraic matrix Riccati equation:

$$0 = PA + A'P - PBR^{-1}B'P + Q \qquad (5)$$

A unique, positive definite solution to the above exists if the pair (A,B) is stabilizable and (A,Γ) is detectable, with $Q = \Gamma'\Gamma$.

2.2 Non-linear quadratic regulator

The extension of the above to non-linear systems looks identical, except that, instead of performing a single optimisation and applying the resulting gain-matrix for all time, the optimisation has to be carried out at every time step. Consider a non-linear dynamical system that can be expressed in the form:

$$\dot{\mathbf{x}} = A(\mathbf{x})\mathbf{x} + B(\mathbf{x})\mathbf{u} \qquad (6)$$

where the Jacobians of $A(\mathbf{x})$ and $B(\mathbf{x})$ are subject to some bounded growth conditions (Lipscihtz) and $A(\mathbf{0}) = 0, B(\mathbf{0}) = 0$, then at each point, $\bar{\mathbf{x}}$, on the state trajectory, a linear system is defined with fixed A and B. In (Banks and Mhanna, 1992) it is shown that solving the infinite-time linear quadratic optimal control problem, pointwise on the state trajectory, results in a globally stabilising, near-optimal quadratic control policy for systems described by equation (6). Thus, by choosing the **u** that minimises the usual quadratic cost function at every time step, we have a globally optimal control policy for a very wide class of non-linear system. Evidently, $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}})$ and Q are subject, pointwise, to the same conditions as for the linear case. It is clear that the proposed solution is identical to the one obtained from equations (2, 3 and 4) when the dynamics are linear.

As an aside, the dual situation follows directly from the reasoning in (Banks and Mhanna, 1992) and thus state estimation is possible via a non-linear observer although this aspect is not addressed here.

Because the control synthesis takes place pointwise, the designer is now free to select Q and R in ways that are more directly applicable to the control *engineering* objectives. In particular, these can be made functions of the instantaneous state variables, i.e.

$$J = \int_0^\infty (\mathbf{x}'Q(\bar{\mathbf{x}})\mathbf{x} + \mathbf{u}'R(\bar{\mathbf{x}})\mathbf{u})dt \qquad (7)$$

subject to the needs for the solution of the Riccati equation and the invertibility of R. Ensuring that $A(\bar{\mathbf{x}}), B(\bar{\mathbf{x}}), R(\bar{\mathbf{x}})$ and $Q(\bar{\mathbf{x}})$ satisfy these requirements a priori is difficult in general, however, for polynomial functions which are not identically zero, the required properties will be lost only at isolated points and will not, therefore, persist.

3. SUSPENSION MODEL

Because we wish to emphasise the use of the non-linear optimal control method for *design*, i.e. how the integration of mathematical synthesis and engineering objectives can be achieved, we adopt a linear model of a vehicle suspension. The two-degree-of-freedom, quarter-car model of figure 1 has been widely studied in the literature, and it represents an active element operating in parallel with passive linear elements – a spring, k_1, and a damper, c_1.

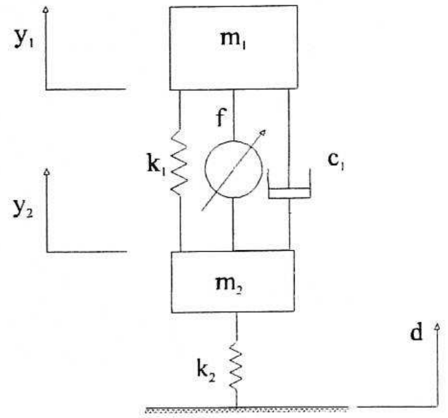

Fig. 1. Schematic of the two-degree-of-freedom, quarter-car model.

The motions of the body and wheel (sprung, m_1, and unsprung, m_2, masses, respectively) are denoted by y_1 and y_2 respectively, while the deviation of the road surface from some datum is denoted by d. The tyre is represented by a linear spring, k_2, with no damping, for simplicity. We assume that the control force, f, can be applied directly as a result of the control signal, with negligible actuator dynamics. Again this is chosen for simplicity, so as not to obscure the main point of the paper.

The equations of motion for the quarter car are given by:

$$\ddot{y}_1 = -\frac{k_1}{m_1}(y_1 - y_2) - \frac{c_1}{m_1}(\dot{y}_1 - \dot{y}_2) + \frac{1}{m_1}f \quad (8a)$$

$$\begin{aligned}\ddot{y}_2 &= \frac{k_1}{m_2}(y_1 - y_2) + \frac{c_1}{m_2}(\dot{y}_1 - \dot{y}_2) \\ &\quad - \frac{k_2}{m_2}(y_2 - d) - \frac{1}{m_2}f\end{aligned} \quad (8b)$$

We choose state variables thus: $x_1 = y_1, x_2 = \dot{y}_1, x_3 = y_2, x_4 = \dot{y}_2$, and identify the control signal, $\mathbf{u}$, with the force, f, leading to the form of equation (2). Note that there is a direct feed-forward path between the control force and sprung mass acceleration – one of the primary indicators of ride quality. A more realistic model would, of course, incorporate actuator dynamics.

3.1 Design objectives

For the purposes of this paper let us suppose that our primary objective is to minimise passenger discomfort. We do this by attempting to reduce the accelerations to which the passenger is subject – vertical only, in this simple case. Thus a candidate for the cost function is $\ddot{y}_1 = C_a\mathbf{x} + D_a\mathbf{u}$, where C_a is the second row of A, and D_a is the second element of B. However, ride comfort can only take precedence when safety and integrity are not compromised. Thus it is necessary to penalise some measure which embodies these ideas, usually via the "rattlespace deflection", $y_1 - y_2 = C_r\mathbf{x}$, with $C_r = [1 \quad 0 \quad -1 \quad 0]$. In the conventional linear quadratic approach we construct a cost function thus:

$$\begin{aligned}J &= \int_0^\infty \left(q_a\ddot{y}_1^2 + q_r(y_1 - y_2)^2 + ru^2\right)dt \\ &= \int_0^\infty \left(\begin{aligned}&\mathbf{x}^t\left(q_aC_a^tC_a + q_rC_r^tC_r\right)\mathbf{x} \\ &+2\mathbf{x}^t q_aC_a^tD_au + \left(q_aD_a^2 + r\right)u^2\end{aligned}\right)dt\end{aligned} \quad (9)$$

Letting $N = q_aC_a^tD_a$ and $R = q_aD_a^2 + r$ we accommodate the cross-term in the usual way, thus $Q \leftarrow Q - NR^{-1}N^t$, $A \leftarrow A - BR^{-1}N^t$ (Friedland, 1987), with the original $Q = q_aC_aC_a^t + q_rC_rC_r^t$. The parameters q_a, q_r are used to control the trade-off between ride and handling.

To illustrate the *non-linear design* procedure we introduce state-dependence into q_r : thus $q_r = 200\psi(y_1 - y_2, .055, .001)$ with

$$\psi(\xi, \theta, \delta) = \begin{cases}((\xi - \theta)/\delta)^p, \xi > \theta \\ 0, \quad |\xi| \le \theta \\ ((\xi + \theta)/\delta)^p, \xi < -\theta\end{cases} \quad (10)$$

where $\theta \ge 0$ defines a deadzone, $\delta > 0$, the distance within which ψ first reaches unity, and p=5. The rationale for this functional form is as follows. The primary objective is to reduce body acceleration hence we choose a constant q_a (=1000). Although this could also be allowed to be state-dependent, we choose not to make it so because of technical difficulties in ensuring the necessary detectability conditions that arise from the inclusion of the cross-term in equation 9. This, in turn, arises from the specific choice of model and will be discussed later. The secondary objective, which can over-ride the first for safety reasons, is to reduce overly large excursions in the suspension strut. Thus, for a rattlespace of ±0.08m, a deadzone of ±0.055m is allowed before control action is taken. The non-linearity increasing to unity within the next 0.001m of travel and dominating the cost function very rapidly as travel approaches the limits.

We have been guided here by the function chosen in (Lin and Kanellakopoulos, 1997), however, there the non-linearity is applied to the strut closure in a very different way.

4. RESULTS

We use the passive system (equation 8 with $f(t) = 0$ for all t) as a reference and compare its behaviour with that of the *non-linearly controlled* vehicle model for a variety of road surface profiles

$$d(t) = \begin{cases} a(1 - \cos 8\pi t), & 0 \le t \le 0.25 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

as suggested in (Lin and Kanellakopoulos, 1997). Figures 2–10 display the relevant results. In each, the dashed line indicates the passive behaviour and the solid curve indicates the controlled behaviour. For $a = 0.025m$, figures 2–4 show the sprung-mass acceleration, the rattlespace deflection and the control signal, respectively, as functions of time. Likewise for figures 5–7 ($a = 0.038m$) and figures 8–10 ($a = 0.055m$). In all simulations, $r = 0.0001$ and all other parameters are as given in (Lin and Kanellakopoulos, 1997). Simulations are carried out in Matlab™ using Euler's method as the integration routine with a step length of 0.001s.

For a small bump ($a = 0.025m$) we expect little or no effect from the rattlespace weighting and that the linear situation should obtain with q_a=1000, q_r=0. This is indeed the case. Here, acceleration excursions are reduced (figure 2) and the reduction in rattlespace deflection (figure 3) arises indirectly from penalising the acceleration. The control signal that achieves these reductions is shown in figure 4.

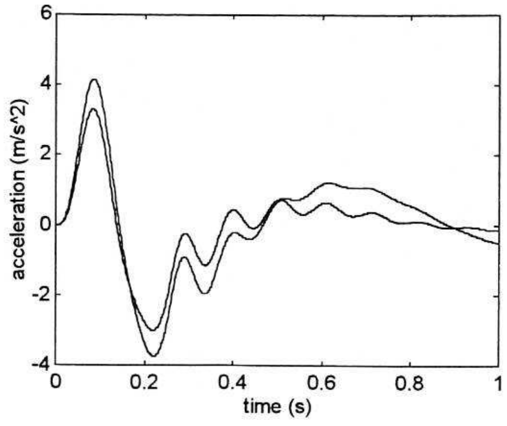

Fig. 2 Sprung-mass acceleration for $a = 0.025m$.

As the severity of the disturbance increases we expect to see the state-dependency come into play. For $a = 0.038m$ we see a sudden large reversal in the control signal (figure 7) as the relative displacement approaches the limits of travel.

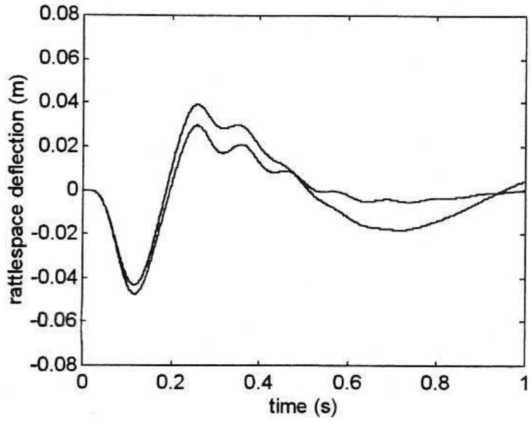

Fig. 3. Rattlespace deflection for $a = 0.025m$.

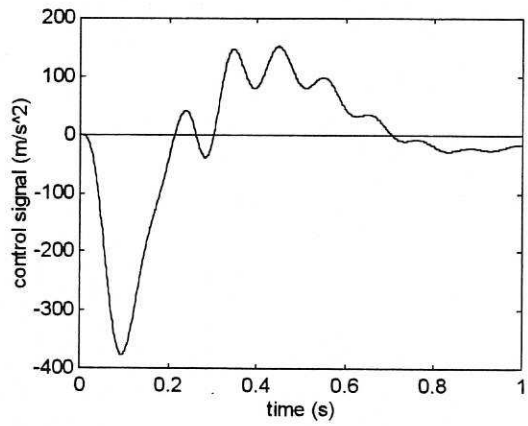

Fig. 4. Control signal for $a = 0.025m$.

Here the rattlespace penalty dominates the cost function. The control signal reverts to the linear situation for t>0.35 approximately. The control signal behaviour is manifested directly in the acceleration signal (figure 5) while seeming to have no discernible effect on rattlespace deflection (figure 6). Such impulsive accelerations would obviously have implications for passenger comfort and for the driver's visual acuity through the mechanism of "jerk" (derivative of acceleration). We shall return to this later.

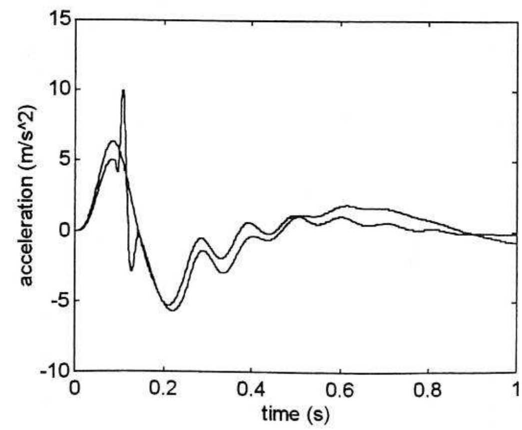

Fig. 5 Sprung-mass acceleration for $a = 0.038m$.

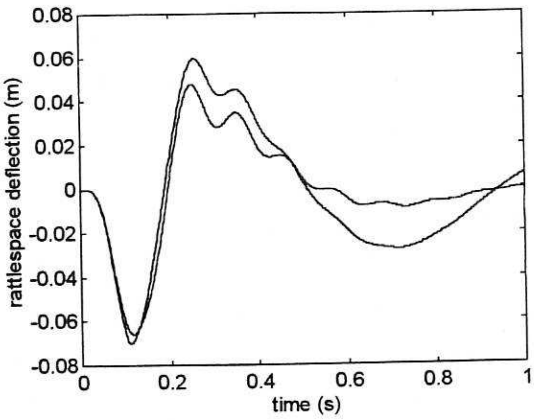

Fig. 6. Rattlespace deflection for $a = 0.038m$.

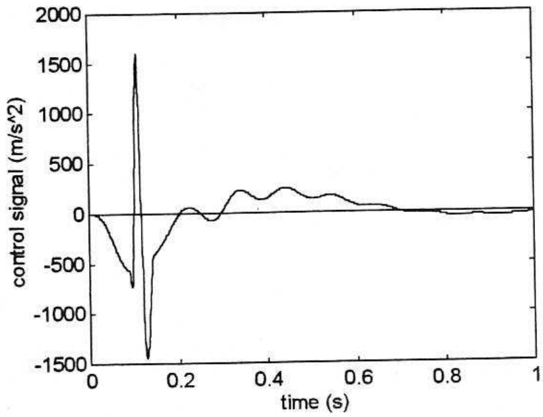

Fig. 7. Control signal for $a = 0.038m$.

For a large disturbance ($a = 0.055m$) we see highly non-linear behaviour in the close-loop system. Now the strut deflection approaches, closely, the limits of travel. Indeed, for this size of bump, the passive system would "bottom out" with implications for safety, comfort and structural integrity. The controlled system is prevented from bottoming out by the application of large control "spikes" (figure 10). Once again these are manifested directly in the acceleration signal (figure 8) with even more severe implications for the passengers

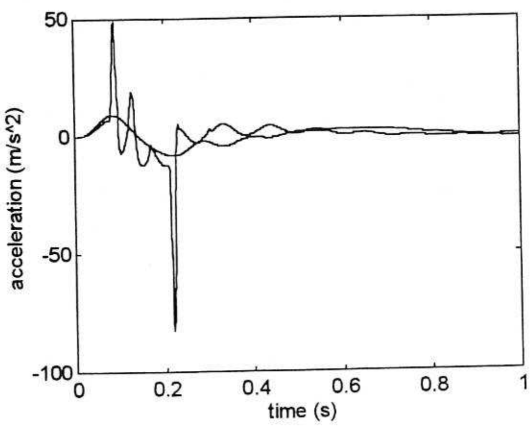

Fig. 8 Sprung-mass acceleration for $a = 0.055m$.

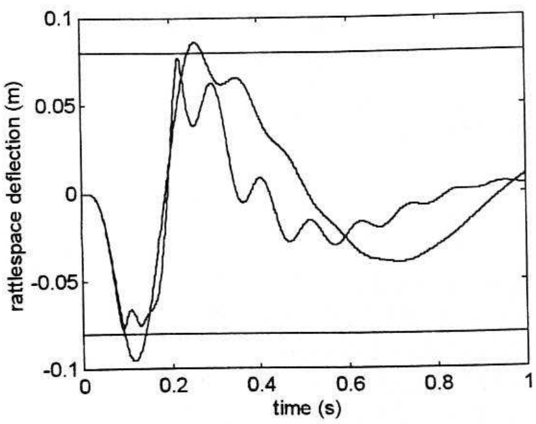

Fig. 9. Rattlespace deflection for $a = 0.055m$.

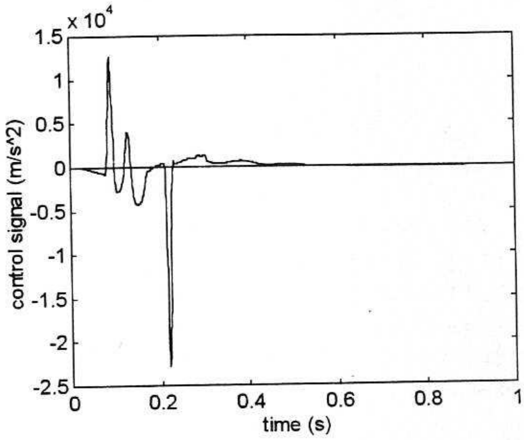

Fig. 10. Control signal for $a = 0.055m$.

Now the effect of the state-dependent weighting is clearly evident in the rattlespace deflection response, which bears little resemblance to the linear behaviour of the passive system.

	$a = 0.025m$	$a = 0.038m$	$a = 0.055m$
Accn. (passive)	2.435	5.625	11.784
Accn. (active)	1.336	4.270	79.343
Rattlespace (passive)	$3.939e^{-4}$	$9.100e^{-4}$	$19.000e^{-4}$
Rattlespace (active)	$2.550e^{-4}$	$6.069e^{-4}$	$10.000e^{-4}$
Control	$1.375e^{4}$	$7.473e^{4}$	$5.181e^{6}$

Table 1. Integrated-square values for the three bump semi-heights.

From Table 1 we can compare the integrated square of the values of the three signals of interest, over the interval zero to one second. We see that for the small and moderate disturbances the active system reduces the integrated-square of the acceleration and of the rattlespace deflection. In the case of the severe bump the rattlespace value is reduced but at the expense of seriously increased integrated square acceleration and almost two orders of magnitude additional integrated square control effort. Recall, however, that the passive system bottoms out and so the figures in the table pertaining to this situation are not reliable.

Clearly, as the state-dependent penalty dominates the cost function, the control policy becomes very aggressive, resulting in a marked deterioration in ride comfort. However, it must be recognised that the model used here is highly simplified, to the point of being unrealistic. In particular, the direct feedforward of the control signal into the acceleration response is not strictly proper and leads to a number of difficulties, both technical, as alluded to earlier, and from the point of view of interpretation of the present results; e.g. the effect of the control signal would be significantly filtered in a physical system. What is highlighted, however, is the need for careful design of the penalty function.

In (Lin and Kanellakopoulos, 1997) qualitatively similar effects are also experienced in that they are able, by virtue of the introduction of non-linearity, to prevent "bottoming" occurring. However, a direct comparison is not possible because detailed behaviour depends strongly on the individual methods and the simulations of (Lin and Kanellakopoulos, 1997) include non-linear actuator dynamics that are ignored here so as not to obscure the simplicity of the design method. It should be reiterated that non-linearities could be accommodated directly, subject to the conditions of §2.2. This is no more difficult than in the linear case, and does not require re-linearisation as might be the case with gain-scheduling.

5. CONCLUSIONS

A new method for the design and synthesis of active vehicle suspension systems is proposed, based on a generalisation of linear quadratic optimal control theory. The method is simple to apply and affords much greater design flexibility than the conventional approach. The resulting controller is non-linear, even for linear dynamics, and can be implemented in real-time.

To illustrate the applicability of the method, a simple linear two-degree-of-freedom quarter-car model has been studied using a rationale suggested in (Lin and Kanellakopoulos, 1997) to design a non-linear penalty function. Preliminary results show that the method has potential and could be tuned to provide desired closed-loop behaviour. However, the shortcomings of the simplified model prevent more concrete use being made of the results.

Some areas for future work are: the selection of more appropriate penalty functions, $Q(\mathbf{x}), R(\mathbf{x})$; the real-time implementation of such a system; the robustness of the method to modelling errors, and quantification of how near to optimal the solution is, and under what conditions.

REFERENCES

Banks, S.P. and K.J. Mhana (1992). Optimal control and stabilisation for non-linear systems, *IMA Journal of Mathematical Control and Information,* **9**, 179–196.

Gardiner, J D. and A.J. Laub (1986). A generalisation of the matrix-sign-function for algebraic Riccati equations, *International Journal of Control,* **44**, 823–832.

Gardiner, J.D. and A.J. Laub (1991). Parallel algorithms for algebraic Riccati equations, *International Journal of Control* **54**, 1317–1333.

Lin, J-S. and I. Kanellakopoulos (1997). Non-linear design of active suspensions, *IEEE Control Systems,* **17**, 45–59.

Kanellakopoulos, I., P.V. Kokotovic and A.S. Morse (1992). A toolkit for linear feedback design, *Systems and Control Letters* **18**, 83–92.

Friedland, B. (1987). *Control system design: an introduction to state-space methods*, McGraw-Hill Book Co., New York.