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On Feedback Control of Chaos

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Abstract: *This paper investigates the problem of stabilising one of the high order periodic orbits of a chaotic oscillator using linear and nonlinear feedback controllers. An algorithm to extract the unstable periodic orbit that provides the reference trajectory is proposed. The advantages and disadvantages of each control structure are analysed using numerical simulation.*

Keywords: *Chaos, nonlinear control ,feedback, stabilisation, unstable periodic orbit, controller, differential equations, oscillator*

1. Introduction

There has been an increasing interest in recent years in the study of controlling chaotic nonlinear systems in the physics, mathematics and engineering community. A good review on the state-of-the-art development and current research in this area can be obtained from the survey article [1].

Since the early attempts at controlling chaos much has changed and the attitude towards chaos itself has been greatly modified. At the beginning the major research effort was spent on eliminating chaotic behaviour from nonlinear systems. Nowadays it has been pointed out that, under certain conditions, chaotic behaviour may be useful [2].

For example, in communications it has been suggested that chaotic systems can be used for secure communication [3]. Chaotic lasers, chaotic diode resonators and Chua electronic circuits have been controlled and applications have been found in the medical sciences and process engineering.

It has been suggested that brain and heart research may benefit from further developments in controlling chaotic systems. For several types of chemical reactors which can exhibit chaotic motion, such as fluidised bed reactors, by stabilising one of the high-order unstable periodic orbits which develop during the chaotic regime very good mixing of the reactants can be achieved while avoiding at the same time the risk of an explosion. In power electronics chaos has been detected in simple DC buck converters so controlling and suppressing chaos in this area may also be beneficial. Hence, controlling a chaotic system has become a very important goal and is the subject of much on-going research.

In this paper it is shown that existing techniques used to control nonlinear systems can be successfully applied to control chaotic oscillators. In particular a conventional linear controller and two nonlinear controllers are designed and used to stabilise one of the high-order unstable periodic orbits embedded within the chaotic attractor of a well known nonlinear oscillator. Based on numerical simulations the advantages and disadvantages presented by each implementation are discussed. A practical algorithm to extract the unstable periodic orbit from data is also introduced.

2. Feedback Control of the Chaotic Van der Pol Oscillator

The control engineering approach to controlling chaotic systems is based almost entirely on using

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conventional feedback controllers to suppress chaos, to stabilise the system while tracking a reference periodic orbit or to synchronise two chaotic systems while rejecting uncertain disturbances.

Usually, conventional feedback controllers are designed for non-chaotic systems, in particular most of them are linear feedback controllers designed for linear systems following a well established control methodology. In contrast, deriving a control algorithm that ensures that the chaotic system trajectory follows one of its unstable periodic orbits may not be trivial at all.

However, this does not mean that for example the chaotic system's sensitivity to initial condition makes it uncontrollable by means of conventional feedback controllers, as some have speculated. Indeed it turns out that conventional feedback control of chaotic systems are generally difficult yet it is not impossible.

The chaotic system considered here is the modified Van der Pol oscillator with periodic forcing which can be described by the following differential equation

$$\frac{d^2y}{dt^2} = \frac{dy}{dt}(a - by^2) - cy^3 + A\cos t \qquad (1)$$

where $a=0$, $b=0.1$, $c=1$, $A=10$. Equation (1) without the forcing term was used to model a vacuum tube circuit originally studied by Van der Pol. If the system is acted on by a periodic forcing term as in equation (1) various nonlinear phenomena can occur including hysteresis or jump effects, subharmonic or superharmonic vibrations and, for the given choice of parameters, chaos.

2.1 Unstable Periodic Orbit Extraction

There are situations when it is desired to stabilise one of the high-order periodic orbits of the system. For example, in a fluidised bed reactor, better mixing and better heat transfer can be achieved when the particles in the reactor move chaotically i.e. the pressure fluctuations inside the reactor are chaotic. However, if the reactor is uncontrolled it is possible for the pressure inside the reactor to build up to dangerous levels that can lead to an explosion.

In such situations, a possible solution is to control the system to one of the high-order unstable periodic orbits, ensuring similar mixing performance for the reactants while precluding any risk of explosion.

It is well known [4] that any chaotic attractor consists of an infinite number of unstable periodic orbits as $t \rightarrow \infty$ and that the trajectory of the system comes arbitrary close to such periodic orbits during its evolution.

In order to unveil one of these unstable periodic orbits, the model estimated for the modified Van der Pol oscillator was integrated with an integration step

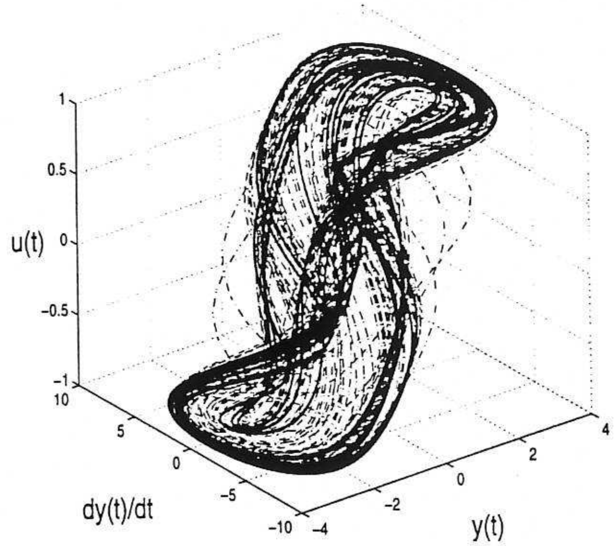


Figure 1. Chaotic attractor (dash-dot) and Unstable Periodic orbit (cont)

$dt=\pi/150$ to generate 20,000 data points.

The idea is to use the simulated data sequence to determine the pairs of time instants $T_{per}(i) = \left(t_{in}(i), t_{fin}(i)\right)$ such that the trajectory of the system between these time instants is nearly periodic that is

$$|x(t_{fin}(i), x_0) - x(t_{in}(i), x_0)| < \varepsilon \qquad (2)$$

where $x(t_{fin}(i), x_0)$ and $x(t_{in}(i), x_0)$ are the vectors of state variables at moment $t_{fin}(i)$ and $t_{in}(i)$ respectively, x_0 represents the vector of initial conditions and ε is a small positive value representing the radius of the ball (or neighbourhood) in the state space which contains both ends of the nearly periodic trajectory.

For this example $\varepsilon = 0.05$ and the state variable vector was defined as $x = \left[y, \frac{dy}{dt}, u\right]$

Using a software routine to search for recurrent points several periodic trajectories were identified. One such trajectories was used to generate the reference signal, shown in figure (1), which is a periodic extension of the original periodic orbit.

This reference signal $(\tilde{y}, \frac{d\tilde{y}}{dt}, \tilde{u})$ will be used in the following sections to implement different control strategies.

2.2 A conventional linear controller

The problem considered here is to control the periodically driven Van der Pol oscillator to the unstable periodic orbit unveiled in the previous section. For the purpose of designing a suitable controller the model of the system was assumed unknown and had to be identified directly from a set of noisy observations recorded from the system. A direct identification procedure similar to that introduced in [5] but based on a polynomial model structure was employed here.

The estimation procedure produced the following set of parameters *a=0.1072, b=0.104, c=1.0036, A=9.9203*. Even though the estimated parameters are very close to the original system's parameters it would be interesting to investigate the influence of the estimation errors on the control performance.

By introducing $y_1 = y$ and $y_2 = \frac{dy}{dt}$ equation (1) can be rewritten as

$$\frac{dy_1}{dt} = y_2$$
$$\frac{dy_2}{dt} = y_2(a - by_1^2) - cy_1^3 + A\cos t \qquad (3)$$

Denoting $(\tilde{y}_1, \tilde{y}_2) = \left(\tilde{y}, \frac{d\tilde{y}}{dt}\right)$ the reference unstable trajectory that is targeted (observe that the reference input $\tilde{u}$ is not used at this stage) in the sense that for any given *e>0* there exists an T_e such that

$$|y_1 - \tilde{y}_1| < e$$
$$|y_2 - \tilde{y}_2| < e \qquad (4)$$

for all $t > T_e$

For this purpose consider initially the conventional feedback controller of the form

$$\begin{bmatrix} v \\ w \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} y_1 - \tilde{y}_1 \\ y_2 - \tilde{y}_2 \end{bmatrix} \qquad (5)$$

which yields the following controlled Van der Pol equation

$$\frac{dy_1}{dt} = f_c(y_1, y_2) = -k_{11}y_1 + (1 - k_{12})y_2 + k_{11}\tilde{y}_1 + k_{12}\tilde{y}_2$$
$$\frac{dy_2}{dt} = g_c(y_1, y_2) = -k_{21}y_1 - cy_1^3 + [(a - by_1^2) - k_{22}]y_2$$
$$+k_{21}\tilde{y}_1 + k_{22}\tilde{y}_2 \qquad (6)$$

The problem of determining a suitable controller described by equation (5) consists of determining the feedback gain matrix

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \qquad (7)$$

which ensures that the controlled system is stable. By linearising equation (3) around the controlled trajectory the following characteristic equation can be formed

$$\det[J_c - sI] =$$
$$= \det\begin{bmatrix} -k_{11} - s & 1 - k_{12} \\ -k_{21} - 3c\tilde{y}_1^2 - 2b\tilde{y}_1\tilde{y}_2 & -k_{22} + (a - b\tilde{y}_1^2) - s \end{bmatrix}$$
$$= s^2 + s(k_{11} + k_{22} + b\tilde{y}_1 - a) + k_{11}(k_{22} + b\tilde{y}_1 - a)$$
$$+(1 - k_{12})(k_{21} + 3c\tilde{y}_1^2 + 2b\tilde{y}_1\tilde{y}_2) = 0 \qquad (8)$$

It follows that the gain matrix can be determined by requiring that this equation has all its roots located in the open left-hand side of the complex plane. This can lead however to more than one solution. For this reason here it is assumed that $k_{11} = k_{12} = 0$. Then it follows that the gains k_{21} and k_{22} have to be chosen to satisfy the following inequalities

$$k_{22} > -b\tilde{y}_1^2 + a$$
$$k_{21} > -3c\tilde{y}_1^2 - 2b\tilde{y}_1\tilde{y}_1 \qquad (9)$$

It should be noted that for $k_{11} = k_{12} = 0$ the corresponding controlled linearised Van der Pol equation obtained from (6) is completely controllable, so that the controlled Van der Pol oscillator is locally controllable by a conventional feedback of the "canonical form"

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} w = -\begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_{21}\ k_{22}] \begin{bmatrix} y_1 - \tilde{y}_1 \\ y_2 - \tilde{y}_2 \end{bmatrix} \qquad (10)$$

The computer simulations shown in figures (2) and (3a,b) illustrate that the control is quite efficient.

It should be pointed out that the conventional concept of controllability only means that the system trajectory can be brought from any initial position to any desired target position of the state vector by a suitably designed control input when both the initial control position and the desired periodic orbit are located inside the controllability region. For large values of the control gain k_{21}, k_{22} called high gain, the controlled Van der Pol equation becomes approximately

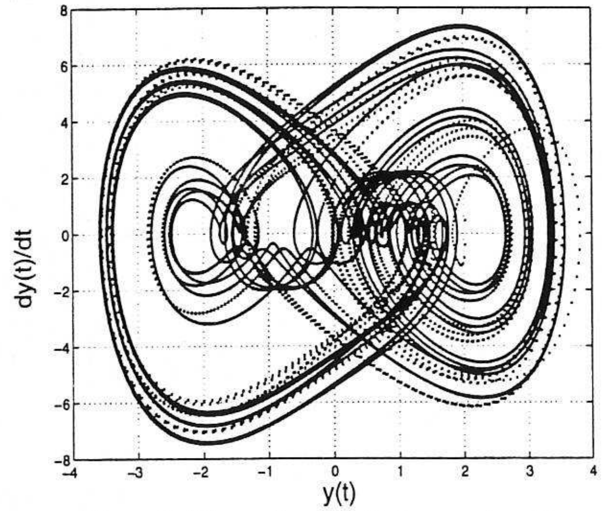
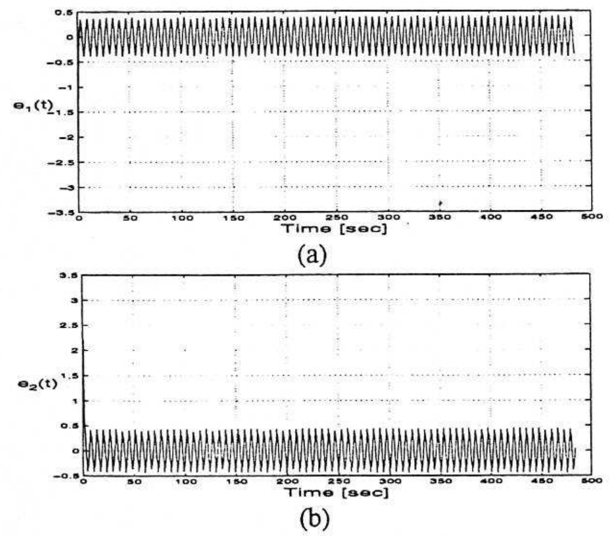
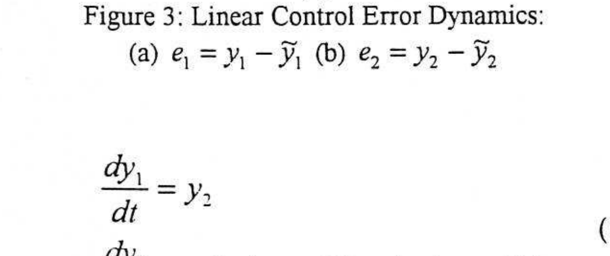


Figure 2: Linear Control: Reference (cont.) Controlled Trajectory (dot)



(a)



(b)

Figure 3: Linear Control Error Dynamics: (a) $e_1 = y_1 - \tilde{y}_1$ (b) $e_2 = y_2 - \tilde{y}_2$

$$\begin{aligned} \frac{dy_1}{dt} &= y_2 \\ \frac{dy_2}{dt} &= -k_{21}(y_1 - \tilde{y}_1) - k_{22}(y_2 - \tilde{y}_2) \end{aligned} \tag{11}$$

which has a particular solution $(y_1, y_2) = (\tilde{y}_1, \tilde{y}_2)$. This implies that for very large values of the control gain the feedback controller should have a much better effect in achieving the desired goal.

For small values of the gain vector however the oscillatory term *Acos(t)* may dominate the designed feedback control input. Hence in the simulation results the gain vector was assigned relatively large values $[k_{21} \quad k_{22}]$=[25 25]. In practice by increasing the control gain the control error can be further reduced.

2.2 A feedback controller with feedforward

The control problem under investigation can be reformulated by observing that the reference trajectory which represents the control goal is in fact a solution of the same Van der Pol differential equation (1) namely

$$\begin{aligned} \frac{d\tilde{y}_1}{dt} &= \tilde{y}_2 \\ \frac{d\tilde{y}_2}{dt} &= \tilde{y}_2(a - b\tilde{y}_1^{\,2}) - c\tilde{y}_1^{\,3} + A\cos t \end{aligned} \tag{12}$$

By subtracting (12) from (3) and denoting $e = \begin{bmatrix} y_1 - \tilde{y}_1 \\ y_2 - \tilde{y}_2 \end{bmatrix}$ it is possible to derive the following error equation

$$\frac{de}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix} e + \begin{bmatrix} 0 \\ b\tilde{y}_1^2\tilde{y}_2 + c\tilde{y}_1^3 - by_1^2y_2 - cy_1^3 \end{bmatrix} + \begin{bmatrix} 0 \\ A\cos t - A\tilde{u} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t) \tag{13}$$

where *w(t)* is the feedback control $w(t) = \phi(y(t), \tilde{y}(t), \tilde{u}(t), t)$ in order to achieve the control goal specified in equation (4) (here y and $\tilde{y}$ denote the vectors $[y_1 \quad y_2]^T$ and $[\tilde{y}_1 \quad \tilde{y}_2]^T$ respectively) This problem is equivalent to that of rendering the origin a global attractor for the error system (13).

The main task is to choose an appropriate function *w(t)* in order to achieve the desired control by linearising the system involved via a feedback plus feedforward action.

In this example, it is relatively easy to see that if the linear part of equation (13) is controllable then (4) is ensured by choosing

$$\begin{aligned} w(t) &= (L - BK)e + l(y,t) + h(\tilde{y},t) - A\cos t + A\tilde{u} \\ &= (L - BK)e + by_1^2y_2 + cy_1^3 - b\tilde{y}_1^2\tilde{y}_2 - c\tilde{y}_1^3 \\ &\quad - A\cos t + A\tilde{u} \end{aligned} \tag{14}$$

The compensation of the periodic terms in the error equation (12) means that the control is more effective than when using a purely linear controller.

The results of the simulations carried out with the same choice of control gain as in the previous example, namely K=[*5 5*], are illustrated in figures (6) and (7a,b).

From these figures it is apparent that the performance of the controller is similar, in terms of error dynamics and accuracy of the control, to that achieved with the previous nonlinear controller. The new controller however no longer includes the expensive nonlinear action provided by the terms $l(y,t)$ and $h(\tilde{y},t)$.

A simple explanation of the fact that the performance of the controller was not affected by omitting two nonlinear compensation terms can be given if these nonlinear terms are rewritten by making the substitution $y(t) = \tilde{y}(t) + e(t)$

$$\begin{aligned} l(y,t) + h(\tilde{y},t) &= by_1^2 y_2 + cy_1^3 - b\tilde{y}_1^2\tilde{y}_2 - c\tilde{y}_1^3 = \\ &= (b+c)e^3 + \left[(2b+3c)\tilde{y}_1 + b\tilde{y}_2\right]e^2 \\ &+ \left[(b+3c)\tilde{y}_1^2 + 2b\tilde{y}_1\tilde{y}_2\right]e \end{aligned} \quad (16)$$

Noting that the reference signal is bounded, it is easy to see that expression (16) tends to zero as $e \to 0$. In particular for the choice of parameter values a, b and c given in equation (1) the contribution of the nonlinear terms (16) will be very small and will decrease rapidly as e approaches zero.

The only other source of errors which explains the small ripple in figures (7a,b) remains the estimation error of the parameter A in equation (1). In principle, however, the control error due to parameter inaccuracy can be reduced by increasing the control gain.

3. Discussion

This paper has investigated the problem of suppressing and controlling chaos in the modified Van der Pol oscillator. For this purpose an algorithm for unveiling one of the high order periodic orbits embedded within the chaotic attractor has been proposed and tested. This orbit was to provide a reference trajectory for the control system.

A linear and two nonlinear controllers were implemented and simulations were carried out in each case. The simulation results have shown that in the case of the linear controller the existence of the forcing term require the use of high gain controllers to dominate the oscillatory effects. By using a nonlinear controller that compensates all the nonlinearities involved in the error equation (12) it has been shown that better performance and less control effort can be achieved but this at the expense of significantly complicating the control structure.

To alleviate this problem a simplified control structure in which two nonlinear compensation terms have been omitted has been proposed. Numerical simulations and theoretical results have proven that by simplifying the control structure the overall performance of the control was not affected.

Numerical simulations carried out also illustrated that the modelling errors were successfully rejected by the controller.

References

[1] Chen, G. and Dong, X: *From Chaos to order-Perspectives and methodologies in controlling chaotic nonlinear systems.* In: International Journal of Bifurcation and Chaos, vol.3, no. 6, 1993, p.1363-1409

[2] Shinbrot, T., Greboji, C., Otto, E. and Yorke, J. A.: *Using Small Perturbations to Control Chaos* In: Nature, v. 363, 1993, p.411-417

[3] Pecora, L. and Carrol, T.L.:*Driving Systems with Chaotic Signals,* In: Phys. Review, vol.A44, no. 4, 1991 , p.2374-2383

[4] Moon, F.C.: *Chaotic and Fractal Dynamics: An Introduction for Applied Scientist and Engineers* Wiley and Sons, New York, 1992

[5] Coca, D. and Billings, S. A.: *Continuous System Identification for Linear and Nonlinear Systems using Wavelet Decompositions* In: International Journal of Bifurcation and Chaos, vol. 7 no. 1 1997, p.87-96