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A Machine Learning Approach to Nonlinear Modal Analysis

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Abstract

Although linear modal analysis has proved itself to be the method of choice for the analysis of linear dynamic structures, extension to nonlinear structures has proved to be a problem. A number of competing viewpoints on nonlinear modal analysis have emerged, each of which preserves a subset of the properties of the original linear theory. From the geometrical point of view, one can argue that the invariant manifold approach of Shaw and Pierre is the most natural generalisation. However, the Shaw-Pierre approach is rather demanding technically, depending as it does on the construction of a polynomial mapping between spaces, which maps physical coordinates into invariant manifolds spanned by independent subsets of variables. The objective of the current paper is to demonstrate a data-based approach to the Shaw-Pierre method which exploits the idea of independence to optimise the parametric form of the mapping. The approach can also be regarded as a generalisation of the Principal Orthogonal Decomposition (POD).

1 Introduction

Modal Analysis is arguably *the* framework for structural dynamic testing of linear structures. While theoretical precursors existed before, the field really flourished in the 1970s with the advent of laboratory-based digital FFT analysers. The philosophy and main theoretical basis of the framework was encapsulated in the classic book [1] quite early and essentially stands to this day. The overall idea is to characterise structural dynamic systems in terms of a number of structural invariants: natural frequencies, dampings, modeshapes, FRFs etc. which can be computed from measured excitation and response data from the structure of interest.

While linear modal analysis has arguably reached its final form - although meaningful work remains to done in terms of parameter estimation etc. - no comprehensive nonlinear theory has yet emerged. Various forms of nonlinear modal analysis have been proposed over the years, but each can only preserve a subset of the desirable properties on the linear. Good surveys of the state of the art at recent waypoints can be found in [2, 3].

Most of the proposed theories of nonlinear modal analysis proposed so far depend on demanding algebra or detailed and intensive numerical computation based on equations of motion. The current paper proposes an approach based only on measured data from the system of interest and adopts a viewpoint, inspired by machine learning, of learning a transformation into 'normal modes' from the data.

The layout of the paper is as follows. Section 2 covers the main features of linear modal analysis, while 3 discusses how the approach breaks down for nonlinear systems. Section 4 gives a very condensed survey of some of the main approaches to nonlinear modal analysis taken in the past and takes some of the ideas presented as motivation for a new approach based on measured data. An example of the approach is presented in 5 and the paper concludes with some overall discussion.

2 Linear Systems

One begins with the standard equation of motion for a Multi-Degree-of-Freedom (MDOF) linear system,

$$[m]\{\ddot{y}\} + [c]\{\dot{y}\} + [k]\{y\} = \{x(t)\} \tag{1}$$

where $\{x(t)\}$ is an excitation force, $\{y\}$ is the corresponding displacement response of the system and $[m]$, $[c]$ and $[k]$ are, respectively, the mass, damping and stiffness matrices of the system. Throughout this paper, square brackets will denote matrices, curly brackets will denote vectors and overdots will indicated differentiation with respect to time.

Linear modal analysis is predicated on the existence of a linear transformation of the responses,

$$[\Psi]\{u\} = \{y\} \tag{2}$$

such that, the equations of motion in the transformed coordinates,

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = [\Psi]^T\{x(t)\} = \{q\} \tag{3}$$

has diagonal mass, damping and stiffness matrices: $[M]$, $[C]$ and $[K]$. The matrix $[\Psi]$ is referred to as the *modal matrix*. The modal matrix encodes patterns of movement or coherent motions of the structure. (It is well-known that matters are actually a little more complicated than this, but the reader can consult basic texts for the details [1, 4]). It immediately follows from (3) that all the degrees of freedom in the transformed coordinate system are uncoupled and each response $u_i(t)$ satisfies a Single-Degree-of-Freedom (SDOF) equation of motion,

$$m_i\ddot{u}_i + c_i\dot{u}_i + k_iu_i = p_i \tag{4}$$

where the m_i are referred to as *modal masses* etc. Corresponding to each of the new degrees of freedom are the standard natural frequencies and damping ratios,

$$\omega_{ni} = \sqrt{\frac{k_i}{m_i}}, \qquad \zeta_i = \frac{c_i}{2\sqrt{m_ik_i}} \tag{5}$$

It is reasonable to refer to quantities like these as *modal invariants* because they are independent of the level of excitation. It is well-known that there is a dual frequency-domain representation of (4),

$$U_i(\omega) = G_i(\omega)P_i(\omega) \tag{6}$$

where $U_i(\omega)$ (resp. $P_i(\omega)$) is the Fourier transform of $u_i(t)$ (resp. $p_i(t)$) and $G_i(\omega)$ is a modal Frequency Response Function (FRF),defined by,

$$G_i(\omega) = \frac{1}{-m\omega^2 + ic_i\omega + k} \tag{7}$$

The overall structural FRF matrix $[H]$ defined by,

$$\{Y(\omega)\} = [H(\omega)]\{X(\omega)\} \tag{8}$$

can be recovered as a linear sum of the modal FRFs, i.e.

$$H_{ij}(\omega) = \sum_{k=1}^{N} \frac{\psi_{ik}\psi_{jk}}{-m_k\omega^2 + ic_k omega + k_k} \tag{9}$$

and the H_{ij} are also invariant under changes of excitation level. Equation (9) is a manifestation of the *Principle of Superposition* for linear systems [4]. From this theory, one can see that modal analysis allows the decomposition of an N-DOF system into N independent SDOF systems; the dynamics of the individual SDOF systems then allow the reconstruction of the general dynamics via superposition. If only a single mode, or u_i, is excited, all the physical degrees of freedom y_i will be in linear proportion to that mode and the structure will thus execute a coherent motion; this fact can serve as an alternative definition of a mode.

3 Nonlinear Systems

Unfortunately, most real structures have nonlinear dynamics e.g. nonlinear equations of motion [4]. The effect of structural nonlinearity on modal analysis is very destructive. Generally, all of the quantities previously described as modal *invariants* become dependent on the amplitude of excitation (energy). Furthermore, decoupling of the system into SDOF systems (certainly by linear transformation) is lost. One generally loses superposition.

However, the engineer still needs to characterise the structure and faces essentially three alternatives [5].

1. Retain the philosophy and basic theory of modal analysis but learn how to characterise nonlinear systems in terms of the particular ways in which amplitude invariance is lost. This alternative encompasses *linearisation*.
2. Retain the philosophy of modal analysis but extend the theory to encompass objects which *are* amplitude invariants of nonlinear systems.
3. Discard the philosophy and seek theories that address the nonlinearity directly.

Although practicality will often drive one to accept (1), this cannot be regarded as a permanent solution to the issues presented by nonlinearity. It is the belief of the current authors that one needs to proceed by developing a mixture of (2) and (3). *Nonlinear modal analysis* - the subject of the current paper - falls within the ideas of alternatives (2) and (3).

4 Nonlinear Modal Analysis

Some would argue that *nonlinear modal analysis* does not even make sense as a term Murdock argues [6]:

> "The phrase "mode interactions" is an oxymoron. The original context for the concept of modes was a diagonalizable linear system with pure imaginary eigenvalues. In diagonalized form, such a system reduces to a collection of linear oscillators. Therefore, in the original coordinates the solutions are linear combinations of independent oscillations, or modes, each with its own frequency. The very notion of mode is dependent on the two facts that the system is linear and that the modes do not interact."

However, this is arguably somewhat pessimistic and based on a limiting semantics. In general one might argue that engineers are not necessarily thinking of the mathematical basis when they refer to a mode; one might further argue that engineers are generally working with two main ideas regarding what a 'mode' is:

1. A coherent motion of the structure (it may be global or local).
2. A decomposition into lower-dimensional dynamical systems the motions of which correspond to 'modes'. Superposition may or may not be possible.

In general, it is not possible to keep all the properties of a linear normal mode (LNM) when passing to a nonlinear theory. Based on the two ideas above of what a mode should be, the origins of nonlinear modal analysis are centred around two main concepts for a nonlinear normal mode (NNM). Adopting definition (a) led to the idea of a *Rosenberg normal mode* [7]; definition (b) led to the idea of a *Shaw/Pierre normal mode* [8].

4.1 Rosenberg Normal Modes

Based on idea (a) the coherent motion concept - Rosenberg observed [7]:

- For linear systems, normal solutions are periodic with all coordinate motions sharing the same period.
- The ratios of displacements of given masses are constant for all time i.e. $u_i = c_i u_1$, where the c_i are constants.

Based on a number of assumptions: symmetric systems, conservative systems, no internal resonance, Rosenberg essentially defined a mode as a periodic motion of the system where the second property above was generalised to $u_i = f_i(u_1)$. All masses still move with the same period and all pass through equilibrium at the same time. There is no general principle for reconstruction from these modes although Rosenberg did give special cases where it was possible. He required that the definition reduced to the correct one for linear systems and provided a simple generalisation. Although the Rosenberg idea has generated a great deal of progress [9, 10], the remainder of this paper will concentrate on the Shaw-Pierre concept.

4.2 Shaw-Pierre Normal Modes

The Shaw-Pierre concept of a nonlinear normal mode is based on idea (b) decomposition in terms of lower-dimensional dynamical systems [8]. They observed that linear modal analysis can be reformulated in terms of invariant subspaces if motion is initiated with a subset of LNMs present (even one) only these modes persist. For nonlinear systems, the NNMs are defined in terms of *invariant manifolds*; if motion is initiated on such a manifold it stays there for all time.

There are a number of advantages to the Shaw-Pierre formulation over that of Rosenberg; perhaps foremost is that non-conservative systems are naturally accommodated. In the approach, the equations of motion are cast in first order form so that displacements and velocities are on equal footing. Suppose $\{y\} = (y_1, \dots, y_n)$ is the vector of displacements and $\{z\} = (z_1, \dots, z_N)$ is the vector of velocities, then the equations of motion are expressed as,

$$\{\dot{y}\} = \{z\} \tag{10}$$

$$\{\dot{z}\} = \{f(\{y\}, \{z\})\} \tag{11}$$

Then the NNM definition generalises the Rosenberg ansatz to say that all coordinates in an NNM are functions of a single displacement/velocity pair (u, v),

$$\begin{pmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ \vdots \\ y_N \\ z_N \end{pmatrix} = \begin{pmatrix} u \\ v \\ Y_2(u,v) \\ Z_2(u,v) \\ \vdots \\ Y_N(u,v) \\ Z_N(u,v) \end{pmatrix} \tag{12}$$

Substituting the anzatz into the equations of motion generates a system of partial differential equations,

$$\frac{\partial Y_i(u,v)}{\partial u} v + \frac{\partial Z_i(u,v)}{\partial v} u f_1(u, v, Y_2(u,v), Z_2(u,v), \dots, Y_N(u,v), Z_N(u,v)) = Z_i(u,v) \tag{13}$$

$$\begin{aligned} &\frac{\partial Z_i(u,v)}{\partial u} v + \frac{\partial Y_i(u,v)}{\partial v} u f_1(u, v, Y_2(u,v), Z_2(u,v), \dots, Y_N(u,v), Z_N(u,v)) \\ &\quad = f_i(u, v, Y_2(u,v), Z_2(u,v), \dots, Y_N(u,v), Z_N(u,v)) \end{aligned} \tag{14}$$

Unfortunately, these are at least as difficult to solve as the original equations of motion; however, one can adopt a power series solution,

$$y_k = Y_k(u,v) = \sum_i \sum_j a_{kij} u^i v^j \tag{15}$$

$$z_k = Z_k(u,v) = \sum_i \sum_j b_{kij} u^i v^j \tag{16}$$

and can then solve a system of algebraic equations for the coefficients a_{kij} and b_{kij}. Shaw and Pierre also showed that the modes can be approximately recombined to the physical motions i.e. approximate superposition. A polynomial expansion of the form above will be one of the main ingredients in the new NNM procedure proposed in this paper; the other major ingredient will come from consideration of the *Principal Orthogonal Decomposition.*

4.3 The Principal Orthogonal Decomposition

All the methods presented so far are based on the equations of motion of the system of interest. However, there also exists a decomposition-motivated approach based on modes adapted to sampled time data: the Principal Orthogonal Decomposition (POD) [11]. The method is essentially Principal Component Analysis (PCA) from the discipline of multivariate statistics. Just as in linear modal analysis, one adopts a linear transformation,

$$\{y\} = [\phi]\{u\} \tag{17}$$

The difference between linear modal analysis and the POD is in the transformation matrix. $[\phi]$ is constructed as the matrix that diagonalises the covariance matrix $[\Sigma_u]$ of the transformed variable $\{u\}$, i.e.

$$[\Sigma_u] = E[(\{u\} - \{\overline{u}\})(\{u\} - \{\overline{u}\})^T] \tag{18}$$

where E denotes the expectation operator and overbars denote mean values. The first mode is then linear combination of the $y_i$vwith maximal power; the second mode is the orthogonal combination with next greatest power etc. The combinations are termed the *Principal Orthogonal Modes* (POMs). The important point for the current paper is that the POMs are *statistically independent*, they do not influence each other in any way; this idea will adopted in the current paper as a definition of orthogonality or *normality* for NNMs in general. The reason for statistical independence is that the u_i in the transformed basis are uncorrelated because of the diagonalisation of $[\Sigma_u]$ i.e.

$$E[(u_i - \overline{u}_i)(u_j - \overline{u}_j] = 0 \quad i \neq j \tag{19}$$

4.4 A New Data-Based Approach

The relevant background is now in place to allow the proposal of a new approach to NNMs. Like the POD, it is a data-based approach; however, unlike the POD it can be applied to nonlinear systems. The idea is to take a transformation like the Shaw-Pierre transformation - a truncated multinomial - but in this case,

$$u_k = \phi_i^{-1}(y_1, z_1, \ldots, y_N, z_N) = \sum_i a_{ki} y_i + \sum_i \sum_j a_{kij} y_i y_j + \sum_i \sum_j b_{kij} y_i z_j + \sum_i \sum_j c_{kij} z_i z_j + \ldots \tag{20}$$

so that the obtained u_i are statistically independent i.e. are unpredictable from each other. Adopting a machine learning approach, one can *learn* the undetermined coefficients from measured data, freeing the decomposition from complicated analysis. Motivated by the POD example, one can see that the problem can be framed as an optimisation problem; in the case of the POD, the answer can be obtained by varying the transformation coefficients in order to minimise the off-diagnonal elements of the covariance matrix $[\Sigma_u]$ or the second order correlations as given in equation (19). Unfortunately, zeroing the second-order correlations is not sufficient to give independence for nonlinear systems; one would need to remove all higher-order correlations between different u_i also. Thankfully, this is not an unsurmountable problem, one simply adds the correlations up to a given order in the objective function to be minimised. The process can now be illustrated through a simplified example.

5 A Simplified Example

The system of interest will be a nonlinear two-DOF lumped parameter system as illustrated in Figure 1.

with equations of motion,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} =$$

$$\begin{bmatrix} -c_1 + c_2 & c_2 \\ c_2 & -c_2 \end{bmatrix} \begin{Bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{Bmatrix} + \begin{bmatrix} -k_1 + k_2 & k_2 \\ k_2 & -k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} + \begin{Bmatrix} -k_3 y_1^3 \\ 0 \end{Bmatrix} + \begin{Bmatrix} x_1 \\ 0 \end{Bmatrix} \tag{21}$$

The model parameters adopted were: $m = 0.1$, $c_1 = 0.005$, $c_2 = 0.01$, $k_1 = 50$, $k_2 = 100$, $k_3 = 10^{-4}$. (Note that this means the damping is proportional, so the underlying system truly uncouples.) Data were

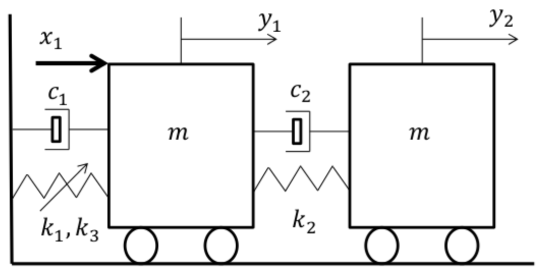

Fig. 1: 2DOF system adopted here for the case study.

simulated using a fixed-step 4^{th}-order Runge-Kutta algorithm and the excitation x_1 was chosen to be a Gaussian white noise sequence with zero mean and unit variance. In order to generate the power spectral densities (PSDs) which follow, 100 independent realisations were taken with 4000 points each; the magnitude of each FRF was then generated by averaging over the relevant spectra from the realisations. In all cases, the frequency shown is the normalised frequency f_n such that the Nyquist frequency is given by $f_n = 0.5$.

For the first illustration, the u_i were computed using the standard modal transformation of the underlying linear system. Figure 2 shows the results. Both modes are present in the PSDs for the transformed coordinates; the system is clearly not uncoupled by standard linear modal analysis, as one would expect.

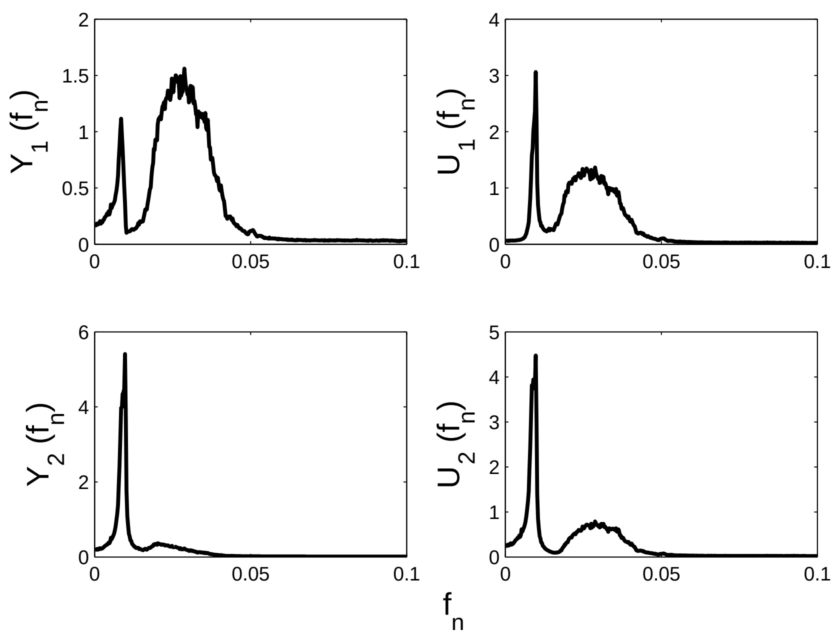

Fig. 2: PSDs for physical and transformed variables: standard linear modal analysis

For the second illustration, the optimisation approach proposed in the last section was adopted. Because the optimisation problem is a continuous one, the Self-Adaptive Differential Evolution (SADE) algorithm was used here, as it has proved powerful for other structural dynamic problems in the recent past [12]. In order to reproduce the results of the (linear) POD, the objective function was chosen as the sum of second-order correlations between the u_j. Because the POD is constrained to generate an orthogonal transformation matrix; a penalty term was added to the objective function to enforce the constraint. In all the applications of SADE, 10 runs were made with the coefficents initialised randomly between -10 and 10. This is usual as SADE is a stochastic algorithm; one makes several runs with different initialisations in order to reduce the possibility of hitting local minima. As SADE is an evolutionary algorithm, the probability of hitting a local minimum is already reduced. As a further simplication, the expansion only used the displacement variables y_i; this would be sufficient for the POD of a linear system.

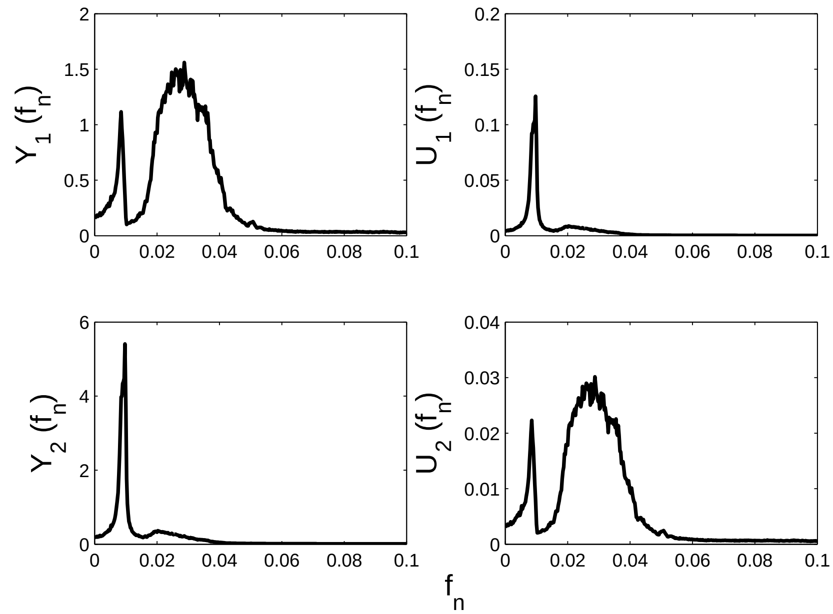

Fig. 3: PSDs for physical and transformed variables: SADE with linear transformation

The results of the POD computation are shown in Figure 3; the system is not uncoupled. Although the first 'mode' is separated out more effectively than when the modal matrix is used, the second 'mode' is actually worse.

The final illustration is based on a SADE optimisation minimising all correlations up to third order and using a cubic transformation. Again, to simplify the computation, the terms with velocities were disregarded. As in the last example a penalty function was used to impose orthogonality on the coefficient vectors obtained; however, it is not immediately clear what constraints should be used for the nonlinear problem as the transformation is nonlinear. Orthogonality has the advantage that it reduces to the correct constraint in the linear case. The results of the computation are shown in Figure 4.

The results show a major improvement on the linear results: modal and POM. There is much better uncoupling into what appear to be individual modes. However, these results must be regarded as preliminary for a number of reasons. The main issue with the current result is as follows. Across the 10 SADE runs performed, the transformations varied in their ability to uncouple the DOFs; the results shown above are the best uncoupling conditions selected *by eye* across the runs, they did not correspond to the lowest objective function. This may not be a matter for undue concern at the moment for three reasons. Firstly, it is not immediately clear what constraints should be imposed on the coefficient vectors for the transformations; even if orthogonality is one necessary constraint, there may be others. Secondly, the constraint was imposed using a Lagrange multiplier in the objective function; the value of the multiplier was set arbitrarily to unity; in the usual practice of machine learning the value of the weighting should usually be set in a principled manner like cross-validation on an independent data set. Finally, the transformation here was restricted to use displacement variables only even though the system simulated had damping, so velocities should probably have been included. These matters certainly require further investigation.

6 Conclusions

The current paper proposes a new approach to nonlinear modal analysis based on a generalisation of the Principal Orthogonal Decomposition (POD) and also motivated by the Shaw-Pierre approach to nonlinear

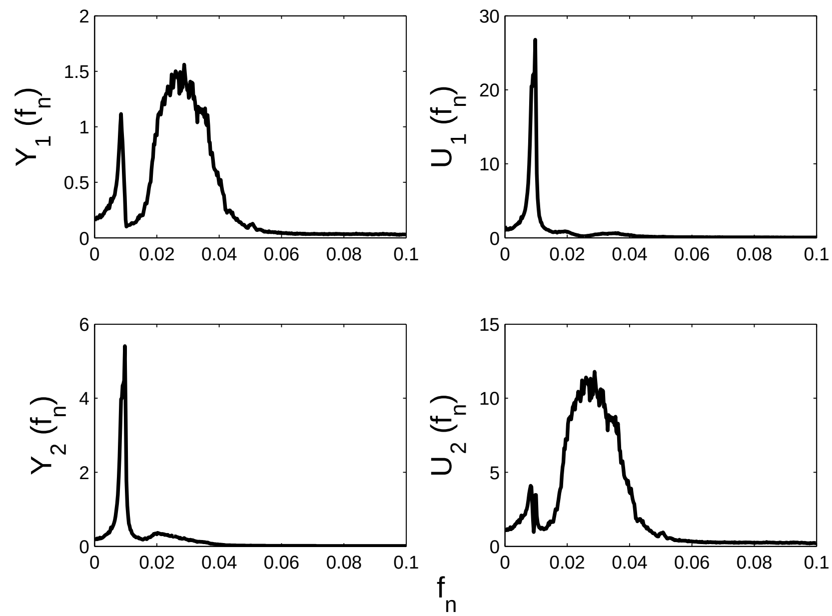

Fig. 4: PSDs for physical and transformed variables: SADE with cubic transformation

modal analysis. The approach is based on optimising a nonlinear transformation from the physical coordinates to a frame in which the coordinate variables are statistically independent. Independence is used here as the measure or 'normality' or orthogonality of the derived 'modes'. Because independence is central to the approach taken here, it currently only makes sense for random data; however, that issue bears further investigation. An advantage of the approach taken here is that complicated algebraic analysis is not needed; in fact even the details of the equations of motion are not needed; this makes the method particularly suited to experimental investigation of nonlinear systems in principle, only measured responses are needed. The results presented here are very preliminary, much remains to be done before a robust and principled methodology emerges. The foremost issue to be dealt with concerns the objective function used for optimisation. The main idea of the approach here is to secure statistical independence of the transformed variables or modes and this cannot be accomplished by considering a finite number of correlation statistics. Other statistics which do measure total independence are available, but are so computationally demanding that they do not easily sit within the optimisation framework which needs fast evaluation of the objective function. This issue is the subject of further work.

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