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OPTIMAL DAMPING OF A MEMBRANE AND TOPOLOGICAL SHAPE OPTIMIZATION

TONI LASSILA

ABSTRACT. We consider a shape optimization problem of finding the optimal damping set of a two-dimensional membrane such that the energy of the membrane is minimized at some fixed end time. Traditional shape optimization is based on sensitivities of the cost functional with respect to small boundary variations of the shapes. We use an iterative shape optimization scheme based on level set methods and the gradient descent algorithm to solve the problem and present numerical results. The methods presented allow for certain topological changes in the optimized shapes. These changes can be realized in the presence of a force term in the level set equation. It is also observed that the gradient descent algorithm on the manifold of shapes does not require an exact line search to converge and that it is sufficient to perform heuristic line searches that do not evaluate the cost functional being minimized.

1. INTRODUCTION

Shape optimization can be seen as part of the field of optimal control. Typically we have a system governed by a partial differential equation whose solution u_Ω depends on some variable geometric shape Ω. The problem is to minimize a given cost functional $J(u_\Omega)$ over the set $\mathscr{S}$ of all admissible geometric shapes with piecewise smooth boundary. Such problems arise for example from the optimal design of structures such as bridges, where we attempt to minimize compliance of the structure due to known loads given certain material constraints.

By considering the variation of the cost functional under small transformations of the boundaries of shapes we can define derivatives with respect to shape. This allows us to derive necessary optimality conditions for the shape optimization problem. The most popular frameworks are the speed method and the perturbation of identity method presented for example in [7] and [22].

Shape optimization problems are typically solved numerically. A widely used approach in the engineering fields has been to discretize the underlying problem and shape using a finite element mesh, derive the sensitivities of the cost functional to small boundary variations of the shape, and then adjust iteratively the mesh points near the boundary of the shape. See [8] or [18] for an introduction to practical shape optimization methods for engineering applications using finite element approximations.

Recently more interest has been given to methods which represent the shape Ω globally as the level set of a continuous function ϕ. A smooth transformation of the boundary of the

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shape is then described with a transport equation for ϕ. These are called level set methods and were made popular by Sethian and Osher. See [5] for a survey of level set methods applied to shape optimization problems. In this paper we briefly cover the basic framework for shape optimization using implicit functions and level set methods to represent smooth transformations of shapes.

To demonstrate numerical methods for shape optimization we consider a shape optimization problem of finding the optimal damping set for a membrane with fixed boundary, modeled by the two-dimensional wave equation. We add a fixed damping factor which affects a subset of the membrane and causes decay in the energy of the vibration. The objective is to find the shape of the damping set that minimizes the energy at some fixed end time given the initial position of the membrane and the damping factor applied. This problem was previously studied in [14] and solved numerically using finite differences on regular grids. In this paper we use instead finite elements with irregular triangular meshes that conform to the boundary of the damping set when solving the wave equations. The results lead to some insight into shape optimization problems where the correct topological properties of the shape are not known beforehand.

2. Shape optimization

2.1. **Basic framework of shapes.** Let $D \subset \mathbb{R}^n$ be a domain of interest and $\mathscr{S}$ a family of open subsets of D with piecewise smooth compact boundaries. Elements of $\mathscr{S}$ are called shapes. A shape functional $J : \mathscr{S} \to \mathbb{R}$ is invariant with respect to homeomorphisms that preserve the shapes i.e. for all shapes $\Omega \in \mathscr{S}$ and homeomorphisms g of D we have that

$$g(\Omega) = \Omega \quad \Rightarrow \quad J(g(\Omega)) = J(\Omega).$$

The shape optimization problem is to find an optimal shape $\Omega^* \in \mathscr{S}$ s.t.

$$J(\Omega^*) \hookrightarrow \min!.$$

The existence of solutions for such optimization problems depends on the chosen family of shapes $\mathscr{S}$ as well as the properties of J. If the shape functional J is lower semicontinuous in the L^p topology of the characteristic functions χ_Ω then typically a sufficient condition for the existence of an optimal solution is that the family of shapes $\mathscr{S}$ fulfills the uniform cone condition or the stronger condition that all shapes have uniformly Lipschitz boundaries. We refer the reader to the monograph [7] for in-depth coverage of the theory of of smooth geometric shapes as well as the classical theory of shape optimization.

2.2. **Boundary variation formulation.** To perform optimization in the family of shapes we would like to define the concept of derivative with respect to shape. The following approach is called the speed method for shape derivatives. Let $\vec{\psi}_s$ be a one-parameter family of smooth transformations $\vec{\psi}_s : D \to D$, for $s \geq 0$, s.t. $\vec{\psi}_0 = I$. Then for a given $\Omega \in \mathscr{S}$ we define the shape derivative of J with respect to the flow $\vec{\psi}$ at the shape Ω as the Gâteaux-derivative

$$dJ_S(\Omega; \vec{\psi}) := \lim_{s \to 0^+} \frac{J(\vec{\psi}_s(\Omega)) - J(\Omega)}{s}, \tag{1}$$

provided that the limit exists. From now on we consider only flows $\vec{\psi}$ of some Lipschitz vector field $\vec{v} : D \to \mathbb{R}^n$ s.t. the action of $\vec{\psi}$ on a point $\vec{x}_0 \in D$ is given by

$$\vec{x}(0) = \vec{x}_0, \quad \vec{x}'(s) = \vec{v}(\vec{x}(s)), \quad \vec{\psi}_s(\vec{x}_0) = \vec{x}(s). \tag{2}$$

According to the Hadamard-Zolesio structure theorem (see [7], Chapter 8, Theorem 3.5) the shape derivative defined by (1) has support only on a subset of the boundary $\partial\Omega$ and thus depends only on the component of $\vec{v}$ normal to the boundary. In the case where it is bounded and linear, we can find a unique scalar function $\nabla_S J \in L^2(\partial\Omega;\mathbb{R})$ s.t.

$$d_S J(\Omega;\vec{\psi}) = \int_{\partial\Omega} \nabla_S J(\vec{v}\cdot\vec{n})\,dS. \tag{3}$$

If the shape functional J is shape differentiable in the sense of (3), we have the necessary optimality condition $\nabla_S J(\Omega^*) = 0$ for an optimal shape Ω^* without the presence of constraints.

2.3. **Topological variation formulation.** As defined above, the speed method of shape optimization works with diffeomorphic maps of Ω_0, the initial guess for the damping set. For shape optimization problems where the optimal shape can consist of more than one component it has been thought that in order to find the optimal shape it is either necessary to know beforehand the topological properties of the optimal shape, or to use a special class of methods that fall under the so called topological optimization. The topological derivative of a shape functional $J(\Omega)$ can be defined as in [20] as the limit of

$$d_T J(\Omega;\vec{x}^0) = \lim_{\rho\to 0^+} \frac{J(\Omega\cup B(\vec{x}^0,\rho)) - J(\Omega)}{\mu(B(\vec{x}^0,\rho))}, \tag{4}$$

where $B(\vec{x},\rho)$ is an open ball of radius ρ centered at $\vec{x}\notin\partial\Omega$ and $\mu(B(\vec{x},\rho))$ its measure. This derivative, when it exists, gives us an idea where in D we should add new components of Ω.

An interesting recent development is the attempt to combine the methods of boundary variation and topological optimization. A theoretical approach was given in [21] where a so-called domain differential was defined as

$$DJ(\Omega;\vec{\psi},\vec{x}^0)(\rho,s) = \mu(B(\vec{x}^0,\rho))\cdot d_T J(\Omega;\vec{x}^0) + s\cdot d_S J(\Omega;\vec{\psi}). \tag{5}$$

In [6] the authors derive topological gradients as a subset of shape gradients, but the algorithm given was more akin to the typical bubble method where an initial guess is obtained using topological derivatives and then a pure shape optimization method is used.

A common method of combining topological and boundary variation optimization is to perform regularly in between boundary variation level set iterations a step where small parts of the shape Ω are removed in locations given by the topological derivative. This introduces the creation of holes into the process, which in practice gives good convergence properties in the optimal support problem for the linear elasticity model. See for example [2], [1], and [24]. A similar step that adds new material away from the shape can also be adopted.

In [4] the authors combined boundary variation and topological level set methods. In practice this gave better topological convergence than was achieved using pure boundary variations when the optimal shape contained a hole.

In [3] a method was given where the topological gradient alone was used to write a level set equation, which was then made stable by using an orthogonal projection technique. This was the method we chose to compare against the traditional boundary variation methods. To proceed we introduce next the concept of level set methods for practical shape optimization.

3. Level set methods and implicit functions

Inherent in the problem of shape optimization is that the shape functional J encodes the representation of the abstract shapes within itself. For theoretical analysis this is sufficient, but for more practical methods we need an explicit way of representing the shapes under consideration.

Recently an approach to describe shapes using implicit functions and their level sets has gained popularity. Let Ω be a given subset of $\mathbb{R}^n$ with piecewise C^k boundary for $k \geq 1$. Then there exist continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\Omega = \{\vec{x} : \phi(x) < 0\}, \quad \partial\Omega = \{\vec{x} : \phi(x) = 0\}.$$

Such functions are called implicit functions or level set functions. We can choose ϕ in such a way that apart from the corners of $\partial\Omega$ it is everywhere Lipschitz and at least C^k in some small neighborhood of $\partial\Omega$ (the latter claim follows from the implicit function theorem). An example of such an implicit function is easy to exhibit, namely we consider the signed distance function

$$\phi_d(\vec{x}) = \begin{cases} -\mathrm{dist}(\vec{x}, \partial\Omega), & \vec{x} \in \Omega \\ +\mathrm{dist}(\vec{x}, \partial\Omega), & \vec{x} \in \Omega^c \end{cases}. \tag{6}$$

From here on we identify every piecewise C^k shape Ω by some representative implicit function ϕ that is C^k smooth near the boundary $\partial\Omega$. It turns out that the existence of a locally C^k implicit function is an equivalent definition of a C^k smooth shape for sets with compact boundary. This allows us to consider the shape functional as a function of the implicit function ϕ and not the actual set Ω.

Consider a given piecewise C^k shape Ω and its image under the flow of $\vec{\psi}$ of some velocity field $\vec{v}$ with flow $\vec{\psi}$. Denote the image of the shape by the flow as $\vec{\psi}_s(\Omega) = \Omega_s$ and let $\phi(\vec{x}, s)$ be an implicit function for Ω_s. We get for each $\vec{x}_0 \in \partial\Omega_0$

$$\phi(\vec{\psi}_s(\vec{x}_0), s) = 0,$$

and differentiating with respect to s gives together with (2)

$$\phi_s(\vec{x}, s) + \vec{v}(\vec{x}) \cdot \nabla\phi(\vec{x}, s) = 0.$$

Using the fact that for smooth shapes Ω the boundary outer normal is given by $\vec{n} = \nabla\phi / |\nabla\phi|$ we get the form

$$\phi_s(\vec{x}, s) + (\vec{v}(\vec{x}) \cdot \vec{n})|\nabla\phi(\vec{x}, s)| = 0 \tag{7}$$

where again only the component of the velocity field normal to the boundary is significant. Equation (7) is called a level set equation. It transports the level sets of ϕ advectively along the flow $\vec{\psi}$.

Signed distance functions (6) are a subclass of implicit functions that have the special property that $|\nabla\phi| = 1$ almost everywhere. Such functions have nice computational properties thanks to the unit scaling of the gradient. Usually the level set equation (7) does not preserve signed distance functions so possible numerical issues might arise as the gradient $\nabla\phi$ grows during the evolution of the equation. These problems can be rectified by regularly rescaling ϕ so that it becomes a signed distance function without moving the zero-level set. A common idea used for example in [23] is to let ϕ_0 be the unscaled version of our implicit function ϕ and regularly solve the equation

$$\phi_s(\vec{x}, s) + \mathrm{sgn}(\phi_0(\vec{x}))(|\nabla\phi(\vec{x}, s)| - 1) = 0, \quad \phi(\vec{x}, 0) = \phi_0(\vec{x}) \tag{8}$$

for a short interval. This equation quickly rescales ϕ to be closer to a signed distance function while leaving the zero-level set intact. We shall see that this reinitialization not only makes things numerically more robust, but has also other effects when dealing with topology changes in shape optimization.

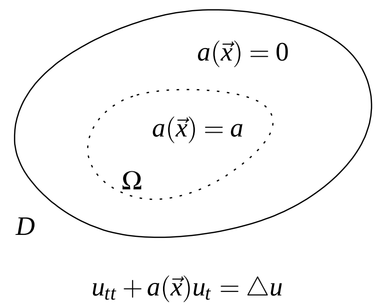

FIGURE 1. Geometry for the damped membrane.

4. OPTIMAL WAVE DAMPING PROBLEM

The following problem was studied in [14], where the author used finite difference methods based on rectangular meshes to solve both the wave equations and the level set equation. We refer the reader to that work for some of the details such as the derivation of the shape gradient.

Let $D \subset \mathbb{R}^2$ be a plane domain with piecewise smooth boundary and consider the two-dimensional wave equation with Dirichlet boundary conditions. This equation models the vibrations of an ideal membrane that is fixed at the edges. Consider additionally that in some subset $\Omega \subset D$ we apply a fixed damping factor $a > 0$. The geometry of the problem is shown in Figure 1. The resulting equation for the displacement of the membrane is then

$$\left\{ \begin{array}{rcl} u_{tt} - \Delta u + a(\vec{x})u_t &=& 0, \quad (\vec{x},t) \in D \times (0,T) \\ u &=& 0, \quad (\vec{x},t) \in \partial D \times [0,T] \\ u(\vec{x},0) &=& u_0(\vec{x}), \quad u_t(\vec{x},0) = u_1(\vec{x}) \end{array} \right. \tag{9}$$

with initial data (u_0, u_1) for the membrane. The damping coefficient is defined here to be piecewise constant:

$$a(\vec{x}) := \left\{ \begin{array}{ll} a, & \vec{x} \in \Omega \\ 0, & \vec{x} \notin \Omega. \end{array} \right. .$$

We refer to Ω as the damping set of the membrane. The energy of the membrane is known to be

$$J(\Omega, a, t) = \tfrac{1}{2} \int_D \left[|u_t(t)|^2 + |\nabla u(t)|^2 \right] dx. \tag{10}$$

The objective is to minimize the total energy of the membrane at some fixed end time T:

$$\min_{\Omega \in \mathscr{S}} \quad J(\Omega, a, T).$$

This is a shape optimization problem whose solution depends additionally on the chosen constants a and T.

In [15] it was proved that a relaxed formulation replacing the characteristic damping χ_Ω with a function in $L^\infty(D,[0,1])$ results in a problem that has a unique solution which corresponds to an optimal solution of the original problem at least for small damping factors a. It was also pointed out that there exists a limit damping factor after which an overdamping phenomenon occurs and leads to the non-existence of optimal solutions with finitely many components. The one-dimensional case was analyzed in [9]. In this work we choose a moderate so as to avoid any problems relating to overdamping, and T large enough so that observability problems relating to the finite propagation speed of waves do not occur.

Without any kind of constraint on the damping set we will obtain the trivial solution $\Omega^* = D$, so in addition we introduce the area constraint

$$A(\Omega) = A_0 \text{ (fixed).} \tag{11}$$

This constraint was handled in [14] by using a penalty coefficient $\lambda > 0$ and augmenting a quadratic penalty term in the energy. We proceed in the same fashion. The augmented shape functional of energy is therefore

$$\widetilde{J}(\Omega, a, T) := \frac{\lambda}{2}\left[A(\Omega) - A_0\right]^2 + \frac{1}{2}\int_D \left[|u_t(T)|^2 + |\nabla u(T)|^2\right] dx. \tag{12}$$

The shape optimization problem we consider is

$$\min_{\Omega \in \mathscr{S}} \quad \widetilde{J}(\Omega, a, T). \tag{13}$$

It was shown in [14] that the shape functional (12) is shape differentiable in the sense (3) and has the shape gradient defined on $\partial\Omega$

$$\nabla_S \widetilde{J}(x) = \left[A(\Omega) - A_0\right] + \int_0^T u_t(\vec{x}, t) p(\vec{x}, t)\, dt, \tag{14}$$

where n is the outward pointing unit normal of $\partial\Omega$, $u(\vec{x}, t)$ is the solution of equation (9) and $p(\vec{x}, t)$ is the solution of the adjoint equation

$$\left\{ \begin{array}{rcll} p_{tt} - \Delta p - a(\vec{x}) p_t & = & 0, & (\vec{x}, t) \in D \times (0, T) \\ p & = & 0, & (\vec{x}, t) \in \partial D \times [0, T] \\ p(\vec{x}, T) & = & -u_t(\vec{x}, T), & \\ p_t(\vec{x}, T) & = & -a(\vec{x}) u_t(\vec{x}, T) - \triangle u(\vec{x}, T) & \end{array} \right. . \tag{15}$$

In practice, finding an optimal damping set Ω^* requires numerical methods.

In addition to the boundary variation shape derivative we also know the topological derivative (4). For the optimal damping problem is in fact closely related to the shape derivative given by the boundary variation. Similarly to the linear elasticity problem ([4]), the topological derivative differs from the boundary variation derivative only up to the sign of ϕ, i.e. we should add material outside Ω where $\nabla_S \widetilde{J} < 0$ and create a hole inside Ω where $\nabla_S \widetilde{J} > 0$.

5. Iterative method for shape optimization

The simplest derivative based method for shape optimization is the gradient descent method. Let Ω_0 be a given initial guess of the optimal shape and ϕ_0 its implicit function. If the shape gradient $\nabla_S J$ is known at Ω_0, we can let $\vec{v} = -\nabla_S J$ on the boundary of $\partial\Omega$ and extend $\vec{v}$ smoothly to the rest of D. For (14) with smooth initial data this is straightforward. Substituting into (7) we get the level set equation for gradient descent

$$\phi_s(\vec{x}, s) - \nabla_S \widetilde{J}(\vec{x}) |\nabla \phi(\vec{x}, s)| = 0, \quad \phi(\vec{x}, 0) = \phi_0(\vec{x}). \tag{16}$$

For small enough s we have $J(\Omega_0) > J(\Omega_s)$. Iterated steps of equation (16) are equivalent to gradient descent on an infinite-dimensional manifold.

As previously noted, equation (16) is a hyperbolic advection equation. Based on this, numerical methods for its resolution have been devised using upwind discretization schemes. The most popular discretization methods are those of Lax-Friedrichs, and Godunov. We refer the reader to [11] and [16] for in depth treatment of numerical methods for hyperbolic conservation laws in general as well as the special case of level set equations. In this work the level set equation was solved using the method of Godunov. We refer to this as the boundary variation level set iteration (or BVLS).

We also tested the method of [3] where the level set equation was written with a force term containing the topological gradient

$$\phi_s(\vec{x},s) = \nabla_T J - \langle \nabla_T J, \phi(\vec{x},s) \rangle \frac{\phi(\vec{x},s)}{|\phi(\vec{x},s)|^2}, \tag{17}$$

where $\nabla_T J \in L^2(D,\mathbb{R})$ is the topological gradient. Thus the update ϕ_s in (17) is simply an L^2-orthogonal projection of $\nabla_T J$ to the orthogonal complement of the implicit function ϕ, which has the feature that ϕ remains at all times in the unit ball of $L^2(D)$. We refer to this iteration as the topological level set iteration (or TLS).

6. Implementation

Evaluation of the shape gradient (14) requires first the solution of the two wave equations (9) and (15). We performed this using an unstructured triangular mesh with piecewise linear elements and the Elmer FEM package (see [12]). The mesh was adapted at each iteration to the boundary of the damping set. The time integration was performed using the Newmark-Bossak scheme. From the solutions of the wave equations a first-order approximation for the shape gradient (14) was computed. The level set equation (16) for gradient descent was then solved on a regular rectangular mesh using the Level Set Toolkit for Matlab (see [13]). We will examine the computational cost of the various stages of the iteration later.

Optimization methods based on descent directions usually require that we perform a line search to find a step size $s \geq 0$ that solves the one-dimensional optimization problem

$$\min_{s\geq 0} \quad \widetilde{J}(\vec{\psi}_s(\Omega)),$$

where $\vec{\psi}$ is the flow in the direction of the negative shape gradient $-\nabla_S \widetilde{J}$. These line searches can be performed either exactly or approximately, sometimes even heuristically. In the level set based gradient descent method it suffices to find a step size such that the energy $\widetilde{J}$ decreases on each iteration.

In practice it becomes quickly clear that accurate evaluation of the energy (12) requires a very fine mesh with elements of good quality to be used when solving for u. We attribute this problem to the term $|\nabla u|$, which is known to converge only like $O(h)$ for piecewise linear basis functions on triangular meshes (see [10]). In addition, poor quality elements with malformed simplices can cause the error of the term $|\nabla u|$ to increase without bound as was shown in [19], which sets stringent quality requirements for the mesh generator used.

The aforementioned issues might have been rectified by moving to higher order elements or using methods specifically designed for hyperbolic problems. However, the former would have increased the computational effort while the latter methods are usually restricted to working on structured rectangular meshes. Our objective was to improve on the computational results in [14] by using unstructured finite element meshes, which capture the shape of the damping set as accurately as possible near its boundary without spending too much computational effort away from the boundary.

The choice of two different types of meshes (irregular for the wave equations vs. regular for the level set equation) required some interpolation to be performed when moving back and forth between the meshes. At each iteration we started with a discrete level set function ϕ^k defined on the regular mesh. From this data we computed some initial boundary points by looking at the values at any two adjacent mesh points. If they had different signs, we performed first order interpolation to find the approximate position of the boundary in between. These interpolated points were used to initialize the irregular mesh. In addition, we added points from a regular mesh with a given parameter h, provided that the points to be added were not closer than $h/2$ distance from the initial boundary points. Finally, the mesh was balanced using DISTMESH ([17]) while fixing the initial boundary points to obtain the irregular triangulated mesh for use with the FEM solver. After solving the wave equations we used ready-made library methods of interpolating from a triangular mesh to a regular rectangular mesh, followed by some correction steps.

When choosing the pseudo-time step length for the level set iteration we used the observation that for large enough penalty terms λ the area constraint (11) forces the area $A(\Omega_k)$ to oscillate around A_0. The idea is then to use the area constraint to decide a suitable step size as follows:

(1) Let $\bar{\mu}$ be some default step size and $\varepsilon > 0$ some tolerance for the area constraint.
(2) Find smallest nonnegative integer j s.t. the solution of (16) for $\Omega_k \to \Omega_{k+1}$ with step size
$$\mu = 2^{-j}\bar{\mu}$$
gives a new damping set Ω_{k+1} s.t. the area constraint is violated at most
$$|A(\Omega_{k+1}) - A_0| < \varepsilon.$$

In practice the gradient descent method is known to be rather forgiving with regards to step size selection rules, and in the cases studied we have nice convergence of the shapes to (at least) local optimums. We use an initial step size of $\bar{\mu} = 10^{-7}$ and a tolerance of $\varepsilon = 0.2$.

To prevent numerical instability due to the increasing or decreasing gradient $\nabla\phi$ we regularly solve the reinitialization equation (8) to reset ϕ into a signed distance function.

7. RESULTS

7.1. **Boundary variation level set iteration.** We verified our solver by first comparing it to the results obtained in [14] by solving a simple problem on the unit square $D = (0,1) \times (0,1)$ with smooth initial data:

$$\begin{cases} u(\vec{x},0) &= 100\sin(\pi x_1)\sin(\pi x_2) \\ u_t(\vec{x},0) &= 0 \end{cases} . \tag{18}$$

The problem parameters were fixed at $T = 1$ and $a = 10$ to obtain a well-posed problem. The results for the initial guess of one disc that is slightly off-center are shown in Figure 2. We verify that the solution is the same as found and supported by theoretical analysis in [14]. Due to the different approaches taken to discretizing the wave equations (finite elements vs. finite differences), our solver used only the mesh parameter $h_{US} = 0.05$ for the unstructured triangular mesh used to solve the wave equations and $h_{SR} = 0.01$ for the structured rectangular mesh used for the level set iteration compared to Munch's $h \approx 0.006$ used for both methods, but obtained the same accuracy of solution.

To demonstrate that the BVLS iteration allows topology changes, we studied equation (9) in an L-shaped domain
$$D = \big[(0,1) \times (0,1)\big] \setminus \big[[\tfrac{1}{2},1) \times [\tfrac{1}{2},1)\big].$$
The initial position of the membrane corresponds to the second eigenmode of the undamped problem
$$\begin{cases} u(\vec{x},0) &= 100\sin(2\pi x_1)\sin(2\pi x_2) \\ u_t(\vec{x},0) &= 0 \end{cases}$$
and the membrane is initially motionless.

Physical intuition says that the optimal damping set consists of three separate components located around the extremal points of the initial position of the membrane $u(\vec{x},0)$. To

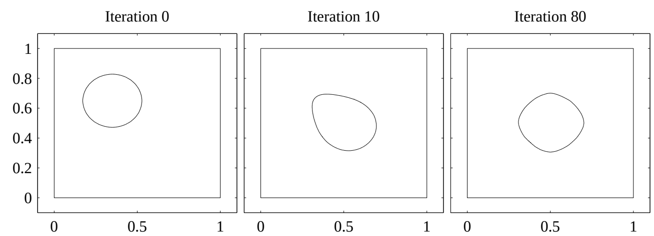

FIGURE 2. Evolution of the damping set in a unit square with the initial data (18) for the membrane using the BVLS iteration. As in [14], the damping set converges to a symmetric optimal solution.

test this hypothesis we solved the problem numerically using two distinct initial guesses for the damping set Ω:

- CASE A: BVLS when initial guess Ω_0 is two discs.
- CASE B: BVLS when initial guess Ω_0 is one disc.

Neither initial guess for the damping set possesses the correct number of connected components that we would expect from the true solution. Therefore, any numerical shape optimization method must be able to handle changes in topology in order to find the optimal damping set.

It is known that even with BVLS we can observe certain types of topological changes in the deformed shapes. Consider CASE A, where the initial guess is two discs located roughly symmetrically. Figure 4 shows the evolution of the damping set. We observe that the flow given by the negative shape gradient tears the other disc into two, and the resulting three components converge towards the extremal points of the initial position of the membrane. In this case the initial guess was close enough for the level set method to find the correct solution. The final energy of the solution was $\widetilde{J} = 2179$.

Details of the convergence of the iteration are presented in Table 1. We measured the total time spent on each iteration for solving the two wave equations (”Wave time”), solving the level set equation (”LS time”), performing the interpolation between the two meshes (”Interp time”) plus the number of pseudo-time step halvings that were needed to find a good pseudo-time step. Every pseudo-time step consists furthermore of several shorter pseudo-time steps, the lengths of which are dictated by the CFL condition of the discretized equation. There is also a certain computational cost related to performing the actual meshing.

Closer study of CASE A reveals that the gradient descent iteration stalls near iteration 15. One component is near bifurcation, but the shape gradient vanishes at the bifurcation point and thus no progress or change in topology is made. Before iteration 20 we performed the reinitialization process given by equation (8). Immediately afterwards the component underwent bifurcation, changing the topology, and allowing the iteration proceeded. This effect can be explained and is not related to the reinitialization as such, but rather the specific equation used to perform the reinitialization. The reinitialization equation can be written as

$$\phi_s(\vec{x},s) + \operatorname{sgn}(\phi_0(\vec{x}))|\nabla\phi(\vec{x},s)| = \operatorname{sgn}(\phi_0(\vec{x}))$$

so that in addition to the advection term we have a force term of magnitude $\operatorname{sgn}(\phi_0)$. Near the bifurcation point at iteration 20 (middle panel in Figure 4) the gradient $|\nabla\phi|$ is very

TABLE 1. Iteration convergence and computational cost of CASE A with the boundary-evolution level set method.

Iter	Final energy	Wave time (s)	LS time (s)	Interp time(s)	P-time steps
1	38728	18.3	24.2	4.1	2
5	17430	17.5	5.2	3.9	3
10	3735	16.7	2.0	4.1	4
15	4971	16.5	5.2	3.9	14
20	5001	16.4	7.7	4.1	23
25	2713	18.4	1.3	3.9	1
30	2169	20.1	1.0	3.9	1
35	2296	17.4	1.5	4.0	1
40	2179	18.5	1.1	4.1	1

small, so that the reinitialization equation is locally

$$\phi_s(\vec{x}, s) = \operatorname{sgn}(\phi_0(\vec{x})). \tag{19}$$

In the vicinity of the bifurcation point we have $\phi_0 \geq 0$ and so equation (19) tends to increase the values of ϕ, causing the component to finally bifurcate into two. The idea is shown in Figure 3. In fact, any positive force term would suffice to push the boundary of the shape over the threshold so that the change of topology is realized.

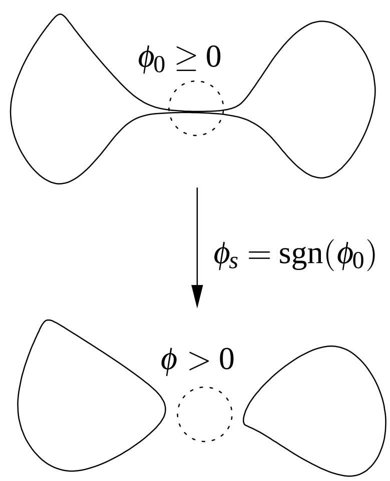

FIGURE 3. Bifurcation of a shape under equation (19).

In CASE B, we had only one disc in the initial guess. The resulting evolution is given in Figure 5. This time the disc is elongated to cover two extremal points but remains as one piece. No new component is created near the third extremal point. This solution is only a local optimum, which we observe by noting that the final energy of the solution was $\widetilde{J} = 20267$.

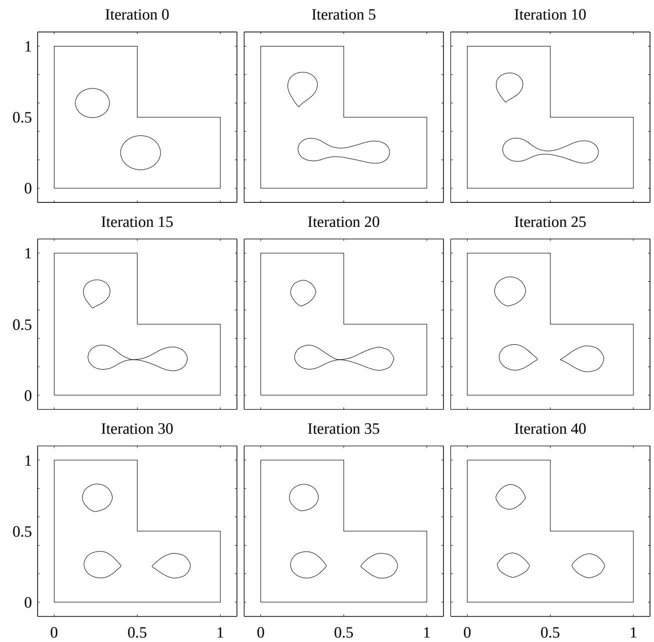

FIGURE 4. Evolution of the damping set for CASE A with an initial guess of two discs using the BVLS iteration. One disc undergoes a bifurcation into two and the correct optimal damping set is found.

It should be again stated that in the case of Lipschitz-continuous velocity fields $\vec{v}$ acting on a given shape Ω, the deformations $\Omega(0) \to \Omega(s)$ given by equation (7) are diffeomorphisms. This means that the speed method of shape optimization does not allow for topological changes of the shape such as bifurcation of one component into two or the merging of two components into one. Furthermore, the advective nature of equation (7) prevents new components of $\partial\Omega$ from emerging away from the existing boundary. As we have seen, the first limitation is not present in the BVLS iteration, but the second limitation remains.

7.2. **Topological level set iteration.** By using the observation that for the optimal damping problem the shape and topological gradients are equivalent up to sign, we also attempted to solve the problem using the method proposed by Amstutz and Andra and equation (17). We used again an initial guess of one disc. This will be referred to as CASE C.

The resulting evolution is shown in Figure 6 and the convergence and computational cost in Table 2. The iteration converges much better in topology and finds the correct topology after only a few steps even when the initial topology is far from correct. In addition, the topological level set iteration (17) is much simpler to perform as it has no spatial derivatives to approximate and in fact has been reduced to simple pseudo-time stepping. Since the implicit function no longer approximates a signed distance function we also don't need the reinitialization equation (8).

As mentioned previously we used a coarser mesh than in [14]. The purpose of choosing a fine mesh in [14] was likely related to using initial guesses consisting of a large number of small discs to accelerate convergence to the correct topology of the shape. However, since we have seen that properly constructed topological level set methods are able to to

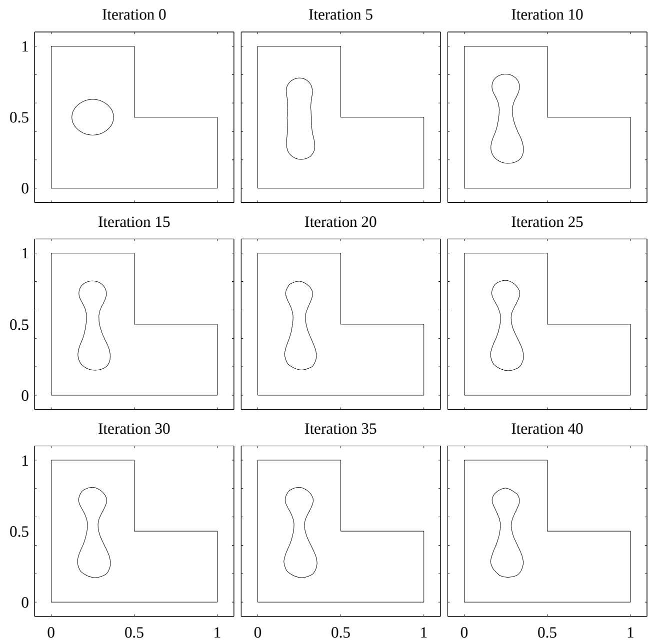

FIGURE 5. Evolution of the damping set for CASE B with an initial guess of only one disc using the BVLS iteration. The method is unable to discover the correct topological properties of the optimal damping set and gets stuck in a local optimum.

TABLE 2. Iteration convergence and computational cost of CASE C with the topological level set method of Amstutz and Andra.

Iter	Final energy	Wave time (s)	LS time (s)	Interp time(s)	P-time steps
1	60547	21.7	1.19	4.2	4
2	31570	19.7	0.22	3.8	2
3	27426	21.6	0.31	4.3	3
4	22785	18.5	0.19	4.2	2
5	18136	22.6	0.41	4.2	4
6	5507	20.9	0.37	4.2	4
7	4523	18.9	0.44	4.1	5
8	4153	24.9	0.14	4.0	1
9	5662	17.2	0.26	4.0	3
10	3649	16.7	0.21	4.0	2

discover the correct solution even when the initial guess is not close to the actual topology, it seems unnecessary to use extremely complicated initial guesses. A practical way would be to first use a coarse mesh with the TLS iteration to find the correct topology of the shape, and then switch to BVLS with a fine mesh to find the accurate boundary of the shape.

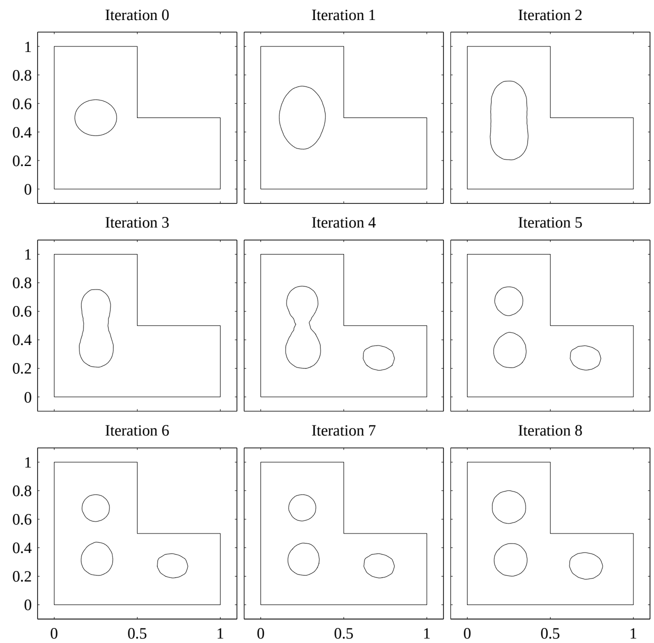

FIGURE 6. Evolution of the damping set for CASE C with an initial guess of only one disc using the TLS iteration. This method has much better topological convergence properties.

8. CONCLUSIONS

We have studied a problem in numerical shape optimization related to finding the optimal damping set for a two-dimensional membrane. The membrane was L-shaped and the initial data was chosen such that the optimal damping set Ω^* consists of three separate components. Two level set iteration based methods for shape optimization were presented. Depending on the initial guess of the damping set and the iteration method chosen, the method either converged or got stuck in a local optimum.

Using unstructured triangular meshes that are refined near the boundaries of the damping set Ω we were able to obtain similar results to those presented in [14] with fewer mesh points. The problem with unstructured meshes is related to the accurate evaluation of the energy functional $J(\Omega, a, T)$. We discovered a lack of monotonicity of the discrete energy during the course of the gradient descent iteration. Subsequently, we chose to use a heuristic step size selection rule for the descent step that did not directly evaluate the value of the energy functional being minimized. The results were good in the cases studied.

Boundary variation level set methods are known to allow certain changes of topology in the shapes being optimized. However, in our problem it was noticed that the boundary variation level set iteration stalled near a bifurcation point until an otherwise unrelated reinitialization procedure was able to effect the change in topology due to the presence of an implicit force term. The requirement for a force term to be present in the level set equation in order to obtain efficient topology discovering shape optimization methods has been previously documented in literature.

We have also tested a topological level set iteration suggested in [3], which achieved much better convergence for the optimal damping problem even when the initial topology was far from correct. For problems where the topological derivative is closely related with the boundary variation shape derivative and thus relatively easy to compute, it seems preferred to use one of the suggested topological optimization methods.

Despite the theoretical differences between boundary variation and topological derivatives, in practice there are great similarities in the resulting methods. It remains to refine the theory in such a way as to unify the concepts of shape and topological derivatives and explain entirely satisfactorily the results obtained using numerical level set methods.

REFERENCES

[1] G. Allaire, F. de Gournay, F. Jouve, and A.-M. Toader. Structural optimization using topological and shape sensitivity via a level-set method. *Control and Cybernetics*, 34(1):59–80, 2005.
[2] G. Allaire, F. Jouve, and A.-M. Toader. Structural optimization using sensitivity analysis and a level-set method. *Journal of Computation Physics*, 194(1):363–393, 2004.
[3] S. Amstutz and H. Andrä. A new algorithm for topology optimization using a level-set method. *Journal of Computational Physics*, 216:573–588, 2006.
[4] M. Burger, B. Hackl, and W. Ring. Incorporating topological derivatives into level set methods. *Journal of Computational Physics*, 194:344–362, 2004.
[5] M. Burger and S.J. Osher. A survey on level set methods for inverse problems and optimal design. *European Journal of Applied Mathematics*, 16(2):263–301, 2005.
[6] J. Céa, S. Garreau, P. Guillaume, and M. Masmoudi. The shape and topological optimizations connection. *Computer Methods in Applied Mechanics and Engineering*, 188:713–726, 2000.
[7] M.C. Delfour and J.-P. Zolésio. *Shape optimization and shape derivatives*, chapter Tangential calculus and shape derivatives, pages 37–60. Marcel Dekker, 1999.
[8] J. Haslinger and P. Neittaanmäki. *Finite element approximation for optimal shape design - theory and applications*. Wiley & Sons, 1988.
[9] P. Hébrard and A. Henrot. Optimal shape and position of the actuators for the stabilization of a string. *Systems & Control Letters*, 48(3-4):199–209, 2001.
[10] S. Larsson and V. Thomée. *Partial differential equations with numerical methods*. Springer, 2003.
[11] R.J. LeVeque. *Numerical methods for conservation laws*. Birkhauser, 1990.
[12] M. Lyly, J. Ruokolainen, and E. Järvinen. ELMER - A finite element solver for multiphysics. *CSC-report on scientific computing 1999-2000*, pages 156–159, 2000.
[13] I.M. Mitchell. A toolbox of level set methods (version 1.1). 2007. Department of Computer Science, University of British Columbia, Vancouver, BC, Canada, Tech. Rep. TR-2007-11, June 2007.
[14] A. Münch. Optimal internal stabilization of a damped wave equation by a level set approach. 2006. Preprint.
[15] A. Münch, P. Pedregal, and F. Periago. Optimal design of the damping set for the stabilization of the wave equation. *Journal of Differential Equations*, 231:331–358, 2006.
[16] S.J. Osher and R. Fedkiw. *Level set methods and dynamic implicit surfaces*, volume 153 of *Applied Mathematics Sciences*. Springer-Verlag, 2002.
[17] P.-O. Persson and G. Strang. A simple mesh generator in MATLAB. *SIAM Review*, 46(2):329–345, 2004.
[18] O. Pironneau. *Optimal shape design for elliptic systems*. Springer-Verlag, 1984.

[19] J.R. Shewchuk. What is a good linear element? Interpolation, conditioning, and quality measures. 2002. Preprint.

[20] J. Sokolowski and A. Zochowski. On the topological derivative in shape optimization. *SIAM Journal on Control and Optimization*, 37(4):1251–1272, 1999.

[21] J. Sokolowski and A. Zochowski. Optimality conditions for simultaneous topology and shape optimization. *SIAM Journal on Control and Optimization*, 42(4):1198–1221, 2003.

[22] J. Sokolowski and J.-P. Zolésio. *Introduction to shape optimization: shape sensitivity analysis*. Springer, 2003.

[23] M. Sussman, P. Smereka, and S. Osher. An improved level set method for incompressible two-phase flows. *Journal of Computational Physics*, 114:146–159, 1994.

[24] X. Wang, M. Yulin, and M.Y. Wang. Incorporating topological derivatives into level set methods for structural topology optimization. In *Proceedings of the 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, 30 August - 1 September*, 2004.