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Resonant response functions for nonlinear oscillators with polynomial type nonlinearities

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Abstract

In this paper we consider the steady-state response of forced, damped, weakly nonlinear oscillators with polynomial type nonlinearities. In particular we define general expressions that can be used to compute *resonant response functions* which define the steady state constant amplitude oscillatory response at the primary resonance and the associated harmonics. The resonant response functions are derived using a normal form transformation which is carried out directly on the second order nonlinear oscillator. The example of a forced Van der Pol oscillator with an additional cubic stiffness nonlinearity is used to demonstrate how the general analysis can be applied.

Keywords: weakly nonlinear, response function, normal form

1. Introduction

In this paper we consider nonlinear oscillators where the nonlinear terms come from polynomial type terms involving velocity $\dot{x}$ and displacement x. These type of systems arise naturally in models of many physical phenomena, and as a result, modelling the response of mechanical and structural systems

with these type of nonlinearities is of interest in a range of applications. For example the escape equation, the Duffing oscillator and the Van Der Pol equation are three of the most well known and intensively studied equations of this type — see [1, 2, 3] and references therein.

In this paper we use normal form transformations to derive a generalized resonance response function (RRF) for the class of nonlinear oscillators being considered. We demonstrate how different forms of the polynomial nonlinearities lead to different contributions in the RRF, and thus to the response of the system.

The main analytical tool used to derive the RRFs in this paper is a normal form transformation is applied directly to the second order nonlinear oscillator without the need for the usual preparatory transformation into first order form [4]. The reason for using normal forms over other similar methods such as multiple scales [5] is that we wish to exploit the properties of the Lie bracket [6]. Normal form transformations are a powerful technique for studying the response of nonlinear oscillators, which has it's origins in the work of Poincaré [7]. The techniques have a long history of development and application, and are particularly useful for identifying resonant interactions in oscillators — for example see [8, 9, 10, 11, 12, 13] and the related approach of using nonlinear normal modes [14, 15, 16, 17, 18]. In addition, a comprehensive overview of normal form theory and related techniques can be found in [19, 20, 21, 22], and a survey of recent developments is given by Stolovitch [23].

The main advantage in using the approach of [4, 24] over the more usual first order normal form, is that it exactly separates the responses at each

2

resonance from the harmonic components of the response. This allows the resulting expressions to be projected onto a basis of complex exponential functions and derivation of RRFs without needing to use harmonic balance type approximations. The technique is demonstrated on an example system which has both damping and stiffness polynomial type nonlinearities.

2. Deriving resonant response functions (RRFs)

In this paper we will consider oscillators of the form

$$\ddot{x} + \omega_n^2 x + N_x(x, \dot{x}, \mathbf{r}) = \mathbf{P_x} \mathbf{r} \qquad (1)$$

where N_x is a function containing both nonlinear and damping terms, ω_n is the undamped natural frequency, $\mathbf{P_x} \mathbf{r}$ represents the harmonic forcing term in which $\mathbf{P_x}$ is the forcing (divided by mass) vector $\left[\frac{P}{2} \; \frac{P}{2}\right]$, $\mathbf{r} = \{r_p, r_m\}^T = \{e^{i\Omega t}, e^{-i\Omega t}\}^T$ and Ω is the forcing frequency. The forcing term is expressed in this way because later in the analysis, exponential trial solutions will be used to obtain nonlinear *resonant response functions* which we denote RRFs.

In this analysis the nonlinear and damping terms are grouped together as we are assuming that both these terms are of a similarly small magnitude — order ε^1 (we use ε as a bookkeeping device to establish the relative magnitudes of the various terms). For this type of single-degree-of-freedom weakly nonlinear system we will take the usual assumption that the case of most interest is when the forcing frequency, Ω, is close to the linear resonance frequency, ω_n, corresponding to when response amplitudes are highest. As the forcing is near-resonant, $\Omega \approx \omega_n$, the response frequency for the system, defined as ω_r, is taken to match the forcing frequency, $\omega_r = \Omega$.

3

A near-identity nonlinear transform will be applied of the form

$$x = u + h(u, \dot{u}, \mathbf{r}), \tag{2}$$

leading to transformed dynamic equation with a simplified form given by

$$\ddot{u} + \omega_n^2 u + N_u(u, \dot{u}, \mathbf{r}) = \mathbf{P_u r}, \tag{3}$$

where N_u contains only the resonant nonlinear terms.

The purpose of the transformation is to put the dynamic equation, Eq. (1), into a simplified form which can be solved exactly by projecting onto a basis of oscillatory exponential functions. This eliminates the need for a harmonic balance type approximation when deriving the RRFs, as will be described later. We use the harmonic response $u = u_p + u_m$, where $u_p = \frac{U}{2} e^{\mathbf{i}(\omega_r t - \phi)}$ and $u_m = \frac{U}{2} e^{-\mathbf{i}(\omega_r t - \phi)}$ such that $u = U \cos(\omega_r t - \phi)$ where the response amplitude, U, is real. Using the fact that $\Omega = \omega_r$, the RRF relating the displacement amplitude U to the input forcing amplitude P can then be calculated.

The transformation method used here is a normal form transformation method for systems of second order oscillators as described by [4]. This methods is used because it exactly separates the responses at each resonance from the harmonic components of the response. A short description of the method is included in the Appendix, and further details (including its application to multi-degree-of-freedom systems) can be found in [4, 24]. The normal form method centres around finding suitable $N_u(u, \dot{u}, \mathbf{r})$ and $h(u, \dot{u}, \mathbf{r})$ functions for the given nonlinearity in the original oscillator equation $N_x(u, \dot{u}, \mathbf{r})$. Note that while N_x is expressed in terms of x and $\dot{x}$ in Eq. (1), when considering the relationship between the three terms, N_u, h and N_x, it is expressed in

4

terms of u and $\dot{u}$. This is due to a Taylor series expansion during the derivation of this relationship – see Appendix Appendix A for details. Making the substitution $u = u_p + u_m$ into the nonlinear function $N_x(u, \dot{u}, \mathbf{r})$ results in terms of the type u_p, u_m, u_p^2, u_m^2, ...etc, plus forcing terms r_p and r_m. These nonlinear and forcing terms are collected into a vector denoted $\mathbf{u}^*$, and then N_x, N_u and h are each re-expressed in terms of $\mathbf{u}^*$. The result is that the transformed dynamic equation, Eq. (3), can be expressed as

$$\ddot{u} + \omega_n^2 u + \varepsilon \mathbf{n_u} \mathbf{u}^* = \mathbf{P_u} \mathbf{r}, \qquad (4)$$

to order ε^1, by applying the transformation, Eq. (2), which can be expressed as

$$x = u + \varepsilon \mathbf{h} \mathbf{u}^* \qquad (5)$$

where $\mathbf{n_u}$ and $\mathbf{h}$ are coefficient vectors. We will now describe how these vectors can be derived for oscillators with polynomial nonlinearities.

2.1. RRFs for polynomial nonlinearities

From Eq. (4) the resonant terms of the system are determined by the non-zero coefficients of $\mathbf{n_u}$. From Eq. (5) the system response containing constant offset and higher harmonic components (i.e. those at frequencies greater than ω_r) is determined by the non-zero coefficients in matrix $\mathbf{h}$.

We will consider combined nonlinear and damping terms which can be written as a series of the following form [25],

$$N_x \left(u, \dot{u} \right) = \varepsilon n_{x1} \left(u, \dot{u} \right) = \varepsilon \mathbf{n_x} \mathbf{u}^* = \varepsilon \sum_{i=0}^{I} \sum_{j=0}^{J} \hat{\alpha}_{ij} u^i \dot{u}^j \qquad (6)$$

5

to order ε^1, where $\alpha_{ij} = \varepsilon\hat{\alpha}_{ij}$ are the coefficients of the nonlinear and/or damping terms with i, j, I and J positive integers and we assume no parametric excitation.

Now make the substitution $u = u_p + u_m$ into Eq. (6) to give

$$N_x = \sum_{i=0}^{I} \sum_{j=0}^{J} \alpha_{ij}(\mathbf{i}\omega_r)^j (u_p + u_m)^i (u_p - u_m)^j, \tag{7}$$

where the ε notation has been dropped. Expanding the i, j term in the summation gives

$$N_x = \sum_{i=0}^{I} \sum_{j=0}^{J} N_{ij} : \quad N_{ij} = \alpha_{ij}(\mathbf{i}\omega_r)^j \sum_{k=0}^{i+j} \gamma_k u_p^{i+j-k} u_m^k. \tag{8}$$

It is the $u_p^{i+j-k} u_m^k$ terms that define the terms in $\mathbf{u}^*$ with the corresponding coefficients $\alpha_{ij}(\mathbf{i}\omega_r)^j \gamma_k$ included in $\mathbf{n_x}$, (see Eq. (6)). Here γ_k is the kth coefficient in the polynomial expansion and is given by

$$\gamma_k = \sum_{v} C_i^v C_j^{k-v} (-1)^{k-v} \tag{9}$$

where C is the binomial coefficient, taking $C_0^0 = 1$ and $C_a^{a+b} = C_a^{-b} = 0$ for positive integer values of b. Note also that for non-integer values and negative values of a; $\gamma_a = 0$ is defined.

Using the normal form technique the resonant terms can be identified from Eq. (8), see Appendix Appendix B.1, allowing the transformed dynamic equation to be derived as

$$\ddot{u} + \omega_n^2 u + \sum_{i=0}^{I} \sum_{j=0}^{J} N_{ij,res} = \mathbf{P_u r}, \tag{10}$$

where

even $i+j$: $\qquad\qquad\qquad\qquad N_{ij,res} = 0$

odd $i+j$: $\quad N_{ij,res} = \alpha_{ij}(\mathbf{i}\omega_r)^j \left(\gamma_k u_p^{k+1} u_m^k + \gamma_{k+1} u_p^k u_m^{k+1} \right), \quad k = \frac{i+j-1}{2}$ (11)

6

and the subscript *res* indicates that only the resonant terms are included
(the other terms will be represented in the nonlinear near-identity transform
– discussed later in this section).

To proceed we note that $\gamma_k = (-1)^j \gamma_{k+1}$ for the case where $k = (i + j - 1)/2$, see Appendix Appendix C, and that

$$u_p^{k+1} u_m^k + u_p^k u_m^{k+1} = \left(\frac{U}{2}\right)^{2k} u, \quad u_p^{k+1} u_m^k - u_p^k u_m^{k+1} = \left(\frac{U}{2}\right)^{2k} \frac{1}{\mathbf{i}\omega_{\mathbf{r}}} \dot{u} \quad (12)$$

obtained via the relationship $u = u_p + u_m$ with $u_p = \frac{U}{2} e^{\mathbf{i}(\omega_r t - \phi)}$ and $u_m = \frac{U}{2} e^{-\mathbf{i}(\omega_r t - \phi)}$. Using these relationships and Eq. (11) allows Eq. (10) to be written as

$$\ddot{u} + D(U)\dot{u} + \left[\omega_n^2 + K(U)\right] u = \mathbf{P_u r}, \quad (13)$$

where $D(U)$ and $K(U)$ are damping and stiffness like terms respectively. They may be written as

$$
\begin{aligned}
D(U) &= \sum_{i=0,e}^{I} \sum_{j=0,o}^{J} \alpha_{ij} (\mathbf{i}\omega_r)^{j-1} \gamma_{(i+j-1)/2} \left(\frac{U}{2}\right)^{(i+j-1)}, \\
K(U) &= \sum_{i=0,o}^{I} \sum_{j=0,e}^{J} \alpha_{ij} (\mathbf{i}\omega_r)^{j} \gamma_{(i+j-1)/2} \left(\frac{U}{2}\right)^{(i+j-1)}
\end{aligned}
\quad (14)
$$

where the summation subscript o and e indicate that only the odd or even terms, respectively, of the variable are considered.

Now consider the case where there is non-zero near-resonant forcing such that $\Omega = \omega_r$. Following the near-identity transform, the resonant terms remained in the dynamic equation while the non resonant terms were included in the transform, Eq. (5). Note that $D(U)$ and $K(U)$ are functions of response amplitude U but not time (see Eq. (12)). Therefore, the resonant response

7

may be written as an RRF of the form

$$\frac{U}{P} = \frac{1}{\sqrt{\{K(U) + \omega_n^2 - \omega_r^2\}^2 + \{D(U)\omega_r\}^2}}, \tag{15}$$

where $\mathbf{P_u} = \mathbf{P_x} = \begin{bmatrix} \frac{P}{2} & \frac{P}{2} \end{bmatrix}$, has been used (see Eq. (A3)). This is the resonance response function (RRF) of the system which shows the relationship between the frequency and the response amplitude, hence allowing the response amplitude U to be computed. Due to the form of K and D it can be seen that the even terms in $N_x(x, \dot{x})$ have no effect on the resonant response, whereas the odd terms can contribute either in the form of apparent damping or of stiffness. At the same time, Eq. (15) reflects that the nonlinear system can also adopt a indirect superposition to get the final dynamic equation in u form, for each of the nonlinearities adds its own contribution to the resonant response function of the system.

In addition the phase of the resonant response can be calculated from Eq. (14) as

$$\phi = \arctan\left(\frac{D(U)\omega_r}{K(U) + \omega_n^2 - \omega_r^2}\right) \tag{16}$$

Table 1 lists a selection of example polynomial nonlinear terms for N and their effect on the resonant response.

2.2. Non-resonant response for polynomial nonlinearities

The non-resonant response of the system is captured by h, which to order ε^1 can be written as

$$h = \varepsilon \mathbf{hu}^* = h_{(0)} + \sum_{k=2}^{I+J} h_{(k)} \tag{17}$$

8

Table 1: The polynomial nonlinearities' contributions to the resonance response function, note γ_k is written in the form $\gamma_k(i,j)$ and $k = (i + j - 1)/2$.

Nonlinear terms	γ for resonant terms	Contribution to $K(U)$	Contribution to $D(U)$
$x^2 \dot{x}^0$	none	none	none
$x^2 \dot{x}^1$	$\gamma_k(2,1) = -\gamma_{k+1}(2,1) = 1$	none	$\alpha_{21} U^2/4$
$x^2 \dot{x}^2$	none	none	none
$x^2 \dot{x}^3$	$\gamma_k(2,3) = -\gamma_{k+1}(2,3) = -2$	none	$\alpha_{23} \omega_r^2 U^4/8$
$x^3 \dot{x}^0$	$\gamma_k(3,0) = \gamma_{k+1}(3,0) = 3$	$\alpha_{30} U^2/4$	none
$x^3 \dot{x}^1$	none	none	none
$x^3 \dot{x}^2$	$\gamma_k(3,2) = \gamma_{k+1}(3,2) = -2$	$\alpha_{32} \omega_r^2 U^4/8$	none
$x^3 \dot{x}^3$	none	none	none

where the constant time-invariant response is given by

$$h_{(0)} = \sum_{i=0,e}^{I} \sum_{j=0,e}^{J} \alpha_{ij} (\mathbf{i}\omega_r)^{j-2} \gamma_{(i+j)/2} \left(\frac{U}{2}\right)^{i+j} \tag{18}$$

see Appendix Appendix B.2.1, and the response at the kth higher harmonic is given by

$$h_{(k)} = \sum_{i=0}^{I} \sum_{j=0,e}^{J} 2\alpha_{ij} (\mathbf{i}\omega_r)^{j} \left(\frac{U}{2}\right)^{i+j} \frac{1}{(k^2-1)\omega_r^2} \gamma_{(i+j-k)/2} \cos[k(\omega_r t - \phi)] +$$
$$\sum_{i=0}^{I} \sum_{j=0,o}^{J} 2\alpha_{ij} \mathbf{i}(\mathbf{i}\omega_r)^{j} \left(\frac{U}{2}\right)^{i+j} \frac{1}{(k^2-1)\omega_r^2} \gamma_{(i+j-k)/2} \sin[k(\omega_r t - \phi)] \tag{19}$$

where $k = 2, 3, 4 \ldots$, see Appendix Appendix B.2.2. Table 2, gives the distribution of the harmonics in the system response for some nonlinear terms.

Table 2: The nonlinearities contribution to the constant off-set and the second and third harmonic. Using the abbreviations S_k and C_k for $\sin[k(\omega_r t - \phi)]$ and $\cos[k(\omega_r t - \phi)]$ respectively and writing γ_k in the form $\gamma_k(i,j)$

Nonlinear terms	$h_{(0)}$, Eq. (18)	$h_{(2)}$, Eq. (19)	$h_{(3)}$, Eq. (19)
$x^2\dot{x}^0$	$-\alpha_{20}\omega_r^{-2}U^2/2$	$\alpha_{20}\omega_r^{-2}U^2 C_2/6$	0
	$(\gamma_1(2,0) = 2)$	$(\gamma_0(2,0) = 1)$	$(\gamma_{-0.5}(2,0) = 0)$
$x^2\dot{x}^1$	0	0	$-\alpha_{21}\omega_r^{-1}U^3 S_3/32$
	(odd j)	$(\gamma_{0.5}(2,1) = 0)$	$(\gamma_0(2,1) = 1)$
$x^2\dot{x}^2$	$-\alpha_{22}U^4/8$	0	0
	$(\gamma_2(2,2) = -2)$	$(\gamma_1(2,2) = 0)$	$(\gamma_{0.5}(2,2) = 0)$
$x^2\dot{x}^3$	0	0	$-\alpha_{23}\omega_r U^5 S_3/64$
	(odd j)	$(\gamma_{1.5}(2,3) = 0)$	$(\gamma_1(2,3) = -2)$
$x^3\dot{x}^0$	0	0	$\alpha_{30}\omega_r^{-2}U^3 C_3/32$
	(odd i)	$(\gamma_{0.5}(3,0) = 0)$	$(\gamma_0(3,0) = 1)$
$x^3\dot{x}^1$	0	$-\alpha_{31}\omega_r^{-1}U^4 S_2/12$	0
	(odd i,j)	$(\gamma_1(3,1) = 2)$	$(\gamma_{0.5}(3,1) = 0)$
$x^3\dot{x}^2$	0	0	$-\alpha_{32}U^5 C_3/128$
	(odd i)	$(\gamma_{1.5}(3,2) = 0)$	$(\gamma_1(3,2) = 1)$
$x^3\dot{x}^3$	0	$\alpha_{33}\omega_r U^6 S_2/32$	0
	(odd i,j)	$(\gamma_2(3,3) = -3)$	$(\gamma_{1.5}(3,3) = 0)$

2.3. Complete response

Finally to calculate the full response, the resonant response amplitude U and phase ϕ are calculated using Eqs. (15) and (16) and then the harmonic responses can be calculated using Eqs. (18) and (19) respectively. Using Eq. (2), the full response to order ε is given by

$$x = U\cos(\omega_r t - \phi) + h_{(0)} + \sum_{k=2}^{I+J} h_{(k)} \tag{20}$$

where $u = u_p + u_m$ with $u_p = \frac{U}{2}\mathrm{e}^{\mathbf{i}(\omega_r t - \phi)}$ and $u_m = \frac{U}{2}\mathrm{e}^{-\mathbf{i}(\omega_r t - \phi)}$ has been used and $h_{(0)}$ and $h_{(k)}$ are given in Eqs. (18) and (19) respectively. From this response equation the nonlinear terms in N can be separated into 4 classes based on whether i and j are even or odd. This is summarised in Table 3.

2.4. Stability of the solution

The steady-state constant amplitude near-resonant response of a generalised nonlinear equation of motion, consisting of the RRF along with expressions for the harmonics, has now been found. We now consider the stability of the RRF solution. This is done by considering the amplitude of response U to be slowly varying, as is typically done in the multiple scales technique (see for example [3]). Recalling that $u = u_p + u_m = U\cos(\omega_r t - \phi)$, we now write

$$u = U_c\cos(\omega_r t) + U_s\sin(\omega_r t) \tag{21}$$

where $U_c = U\cos(\phi)$ and $U_s = U\sin(\phi)$ and to account for the slow amplitude variation with time both U_c and U_s are functions of εt (the ε is present to indicate the slow nature of the variation). The derivatives of u may be written

11

Table 3: Summary of the form of the response for combinations of odd and even i and j, note that higher harmonic terms where the γ_a term contains a non-integer a have been removed resulting in either even or odd values of k.

i	j	Contribution to RRF, Eq. (15)	Contribution to constant off-set, Eq. (18)	Higher harmonics response terms, Eq. (19)
even	odd	terms in $D(U)$	none	$\sin(k(\omega_r t - \phi)]$ terms, $k = 3, 5, \ldots, i+j$
odd	even	terms in $K(U)$	none	$\cos(k(\omega_r t - \phi)]$ terms, $k = 3, 5, \ldots, i+j$
odd	odd	none	none	$\sin(k(\omega_r t - \phi)]$ terms, $k = 2, 4, \ldots, i+j$
even	even	none	yes if $\gamma_{(i+j)/2} \neq 0$	$\cos(k(\omega_r t - \phi)]$ terms, $k = 2, 4, \ldots, i+j$

as

$$\dot{u} = \omega_r[-U_c\sin(\omega_r t) + U_s\cos(\omega_r t)] + \varepsilon[U_c'\cos(\omega_r t) + U_s'\sin(\omega_r t)]$$

$$\ddot{u} = -\omega_r^2 u + 2\varepsilon\omega_r[-U_c'\sin(\omega_r t) + U_s'\cos(\omega_r t)] + \mathcal{O}\{\varepsilon^2\} \quad (22)$$

where $\{\}'$ indicates the derivative with respect to εt.

When letting the amplitude of response vary slowly with time we firstly note that, to order ε^1, the derivation of the transformed equation of motion, Eq. (23), remains unchanged. This is because u is only present at order ε^1 in the relationship linking the original nonlinear term, the transformed nonlinear term and the transform (derived in the Appendix – Eq. (A2)). Hence including the modified to u, which is order ε^1, results in new terms at only order ε^2. Now taking the transformed equation of motion, Eq. (23), recognising it is accurate to order ε as $K(U)$ and $D(U)$ are order ε and making the substitution for u and $\ddot{u}$ gives

$$2\omega_r[-U_c'\sin(\omega_r t) + U_s'\cos(\omega_r t)] + D(U)\omega_r[-U_c\sin(\omega_r t) + U_s\cos(\omega_r t)]$$

$$+ \left(\omega_n^2 - \omega_r^2 + K(U)\right)[U_c\cos(\omega_r t) + U_s\sin(\omega_r t)] = P\cos(\omega_r t), \quad (23)$$

to order ε^1, once we have dropped the ε notation. Balancing sine and cosine terms results in the first-order differential equation

$$\begin{pmatrix} U_c \\ U_s \end{pmatrix}' = \frac{1}{2\omega_r}\begin{pmatrix} (K(U) + \omega_n^2 - \omega_r^2)U_s - \omega_r D(U)U_c \\ -(K(U) + \omega_n^2 - \omega_r^2)U_c - \omega_r D(U)U_s \end{pmatrix} + \begin{pmatrix} 0 \\ P \end{pmatrix} \quad (24)$$

The forcing frequency dynamics are now in the form $\mathbf{X}' = \mathbf{f}(\mathbf{X}, t) + \mathbf{P}$. The equilibrium of the constant amplitude solutions, $\bar{\mathbf{X}}$, can be found by considering a perturbation $\mathbf{X_p}$ away from equilibrium. Using a Taylor series expansion, the perturbation dynamics are $\mathbf{X_p}' = \mathbf{f_x}(\bar{\mathbf{X}}, t)\mathbf{X_p}$, where $\mathbf{f_x}$ is the

13

Jacobian of $\mathbf{f}$. Hence the equilibrium solution is stable when the real parts of both eigenvalues of $\mathbf{f_x}(\bar{\mathbf{X}}, t)$ are negative.

With some algebraic manipulation the eigenvalue equation may be written as

$$\lambda^2 + \lambda \left[D + U^2 D^* \right] + \frac{U^2(K(U) + \omega_n^2 - \omega_r^2)}{\omega_r} \frac{\mathrm{d}\omega_r}{\mathrm{d}U^2} = 0 \qquad (25)$$

where $D^* = \mathrm{d}D/\mathrm{d}U^2$ and λ is an eigenvalue of $\mathbf{f_x}(\bar{\mathbf{X}}, t)$. In deriving this equation we have used $U^2 = U_c^2 + U_s^2$, $\partial D/\partial U_s = 2U_s D^*$ etc and to simplify the λ^0 term have used the derivative of the equilibrium solution equation, Eq. (15), with respect to U^2 to give an expression for $\mathrm{d}\omega_r/\mathrm{d}U^2$. This eigenvalue equation is in the quadratic form $\lambda^2 + b\lambda + c = 0$, where b and c are real. For the real parts of the eigenvalues to be negative it is well known, and straightforward to show, that both b and c must be positive. Hence for the constant amplitude solutions, derived in section 2.1, to be stable we have the conditions

$$(K(U) + \omega_n^2 - \omega_r^2) \frac{\mathrm{d}\omega_r}{\mathrm{d}U^2} \geq 0 \qquad (26)$$

$$D + U^2 D^* \geq 0 \qquad (27)$$

Note that by inspection of Eq. (23) the backbone curve for the system, the curve defining the nonlinear natural frequency in the unforced, undamped system, ω_0, as a function of amplitude may be written as $\omega_0^2 = \omega_n^2 + K(U)$. As a result, considering the response curve defined by Eq. (23) in the usual $[U, \omega_r]$ plane, the first condition, Eq. (26), may be interpreted as requiring a positive gradient to the response curve when to the left of the backbone curve and a negative gradient to the right of the backbone curve for the RRF solution to be stable. This is consistent with a zero eigenvalue corresponding

14

to a fold. At the transition to instability via purely imaginary eigenvalues, i.e. when $D + U^2 D^* = 0$, a Neimark-Sacker bifurcation occurs resulting in quasi-periodic motion [22].

3. Example system

The relationships derived in the previous section are now applied to an example system. This system is the forced Van der Pol equation with an additional cubic stiffness nonlinearity giving

$$\ddot{x} + \omega_n^2 x - \mu(1 - x^2)\dot{x} + \alpha x^3 = P\cos(\Omega t) \tag{28}$$

where it is assumed that the nonlinear and damping terms are small compared to the linear ones and the forcing is near resonant such that Ω is close to the natural frequency, ω_n, and therefore $\omega_r = \Omega$.

Through comparison with Eq. (1), the nonlinear stiffness and damping term may be written as

$$N_x(x, \dot{x}, r) = \mu(x^2 - 1)\dot{x} + \alpha x^3 \tag{29}$$

such that just the $[i, j] = [0, 1]$, $[2, 1]$ and $[3, 0]$ terms in the summation, Eq. (6), exist with coefficients $\alpha_{01} = -\mu$ $\alpha_{21} = \mu$ and $\alpha_{30} = \alpha$ respectively. Immediately from Eqs. (14) and (15) the resonant response may be written as

$$\frac{U}{P} = \frac{1}{\sqrt{\left(\omega_n^2 - \omega_r^2 + \frac{3\alpha U^2}{4}\right)^2 + \mu^2 \omega_r^2 \left(-1 + \frac{U^2}{4}\right)^2}} \tag{30}$$

with $\omega_r = \Omega$. This equation can be rewritten as a quadratic in ω_r^2 and solved for a range of U to give the resonant response curve.

Figure 1 shows an example resonant response function for the system with parameters $\omega_n = 1$, $\mu = 0.08$ and $\alpha = 0.04$ and forcing amplitude $P = 0.4$. Time-stepping simulation results, using a variable-step Runge-Kutta solver (matlab function ode45), are shown for comparison – dots and circles are the simulation results for steps of increasing and decreasing frequency respectively. The amplitudes plotted are based on an FFT applied to the response of the system after the initial transients have decayed away.

Considering the stability conditions, Eq. (26) and Eq. (27), the dashed line indicates unstable solutions, where the first condition, Eq. (26), isn't met — in this case the response is to the right of the backbone curve and the gradient negative. The second condition, Eq. (27), simply requires $U \geq \sqrt{2}$, and solutions outside this are indicated by the dotted lines. Stable solutions which satisfy both Eq. (26) and Eq. (27) are indicated by solid lines, and it can be seen that in the stable solution regions there is very good agreement between the prediction using the general RRF equation, Eq. (23), and the time-stepping simulations. The time stepping solutions for the case where the frequency is just above or below that corresponding to $U = \sqrt{2}$ have amplitudes that are oscillatory in nature. This is indicated by the lack of agreement between the circles, dots in the regions of the dotted lines.

The constant offset can be predicted using Eq. (18) and is $h_{(0)} = 0$ since all three terms in the nonlinear expression have odd values of j. The higher harmonics are predicted from Eq. (19). For the $[i, j] = [0, 1]$ term it can be seen that $\gamma_{(i+j-k)/2} = \gamma_{(1-k)/2}$ is zero for all valid $k = 2, 3, 4, \ldots$, since γ_a is only non-zero for positive integer values of a. For both the $[i, j] = [2, 1]$ and the $[i, j] = [3, 0]$ terms, $\gamma_{(i+j-k)/2} = \gamma_{(3-k)/2}$ and so, for valid k, only the case
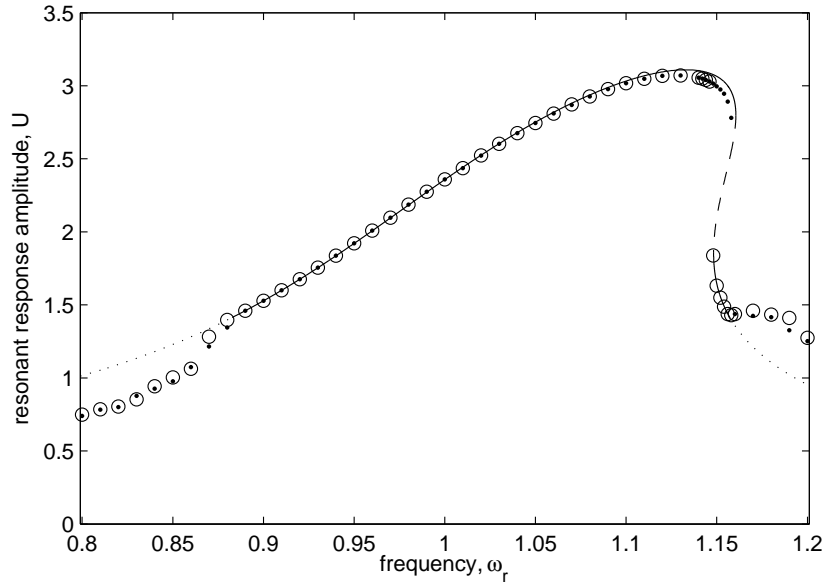
16

Figure 1: The resonant response for Eq. (28) for the case where $\omega_n = 1$, $\mu = 0.08$ and $\alpha = 0.04$ and forcing amplitude is $P = 0.4$. The line shows the normal form prediction, it is dashed and dotted in regions where the solution is unstable based on Eqs. (26) and (27) respectively. The dots and circles show the time-stepping simulation result for increasing and decreasing frequency steps respectively.
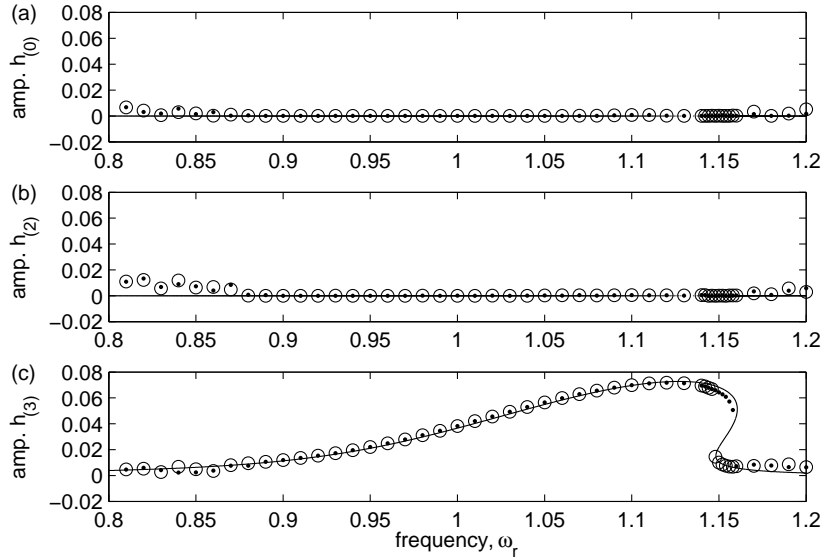
17

Figure 2: The system response at (a) zero frequency, (b) $2\omega_r$ and (c) $3\omega_r$ for $\omega_n = 1$, $\mu = 0.08$ and $\alpha = 0.04$ and $P = 0.4$. The lines shown the normal form prediction and the dots and circles time-stepping simulation results (with steps of increasing and decreasing in frequency respectively). The stability is defined by the curves in Fig. 1.

where $k = 3$ is $\gamma \neq 0$ giving

$$h_{(3)} = \frac{U^3}{32\omega_r^2} \left[-\mu\omega_r \sin(3\omega_r t - 3\phi) + \alpha \cos(3\omega_r t - 3\phi) \right] \tag{31}$$

Figure 2 shows the constant offset and higher harmonic resonance amplitudes for the system. It can be seen that the normal form prediction shows good agreement with the time-stepping simulation results in the regions where the constant amplitude solution is stable.

4. Conclusion

In this paper we have developed expressions for the resonance response functions (RRFs) of a class of nonlinear oscillators with polynomial nonlin-

18

earities based on a constant amplitude steady-state response using the second order normal form technique of [4]. Using this framework, the contributions of the polynomial type nonlinear terms to the resonance response function and the system response have been categorised. Sample results for different types of polynomial nonlinear terms are presented in Tables 1 and 2. Table 3 summarises the effect of odd and even i j values in the polynomial nonlinearity, and the subsequent contributions to both the resonance response function and the harmonic response at other frequencies. The stability of the constant amplitude steady-state response has also been considered and results in two simple conditions that must be met for solution stability.

The method has been applied to an example system, a forced Van der Pol oscillator with an additional cubic stiffness nonlinearity. The resonance response function results show excellent agreement with time stepping simulations for the stable solution regions. In addition the harmonic contributions away from the primary resonance have been computed, and are also in close agreement with time stepping simulations.

In summary, this analysis has defined a clear relationship between the polynomial nonlinearities and their subsequent contributions to the resonance response function (RRF). The complete resonance response function is given by Eq. (15). The harmonics term is given by Eq. (A8), the constant shift as Eq. (18) and the higher harmonic terms are given by Eq. (19). These are summations of individual contributions, and the complete response is given by Eq. (20), which is also a summation of the different contributions. The summation can be interpreted as a form of indirect superposition which is a useful feature of the normal form method in terms of helping us to

19

characterise the response of nonlinear systems. This superposition feature associated with normal form methods has previously been noted by Jezequel and Lamarque [11] using a first order normal form approach.

Acknowledgment

Appendix

Appendix A. Summary of the second-order normal form method for single-degree-of-freedom systems

Normal form analysis is a method of transforming the equation of motion for a weakly nonlinear system into a form in which the response is at just one frequency – the dominant response frequency. The response at other frequencies being captured by the transform equation.

Consider oscillators of the form of Equation 1. The following discussion gives the near-identify transformation between transformed coordinate u and x. Initially it is necessary to introduce ε as a bookkeeping device to establish the different magnitudes of each orders when using Poincaré asymptotic expansions

$$
\begin{aligned}
N_x(x, \dot{x}, \mathbf{r}) &= \varepsilon n_{x1}(x, \dot{x}, \mathbf{r}) + \varepsilon^2 n_{x2}(x, \dot{x}, \mathbf{r}) + \cdots \\
N_u(u, \dot{u}, \mathbf{r}) &= \varepsilon n_{u1}(u, \dot{u}, \mathbf{r}) + \varepsilon^2 n_{u2}(u, \dot{u}, \mathbf{r}) + \cdots \\
h(u, \dot{u}, \mathbf{r}) &= \varepsilon h_1(u, \dot{u}, \mathbf{r}) + \varepsilon^2 h_2(u, \dot{u}, \mathbf{r}) + \cdots
\end{aligned} \tag{A1}
$$

20

Substituting Eqs. (3) and (A1) into Eq. (1), yields

$$\mathbf{P_u}\mathbf{r} - \varepsilon n_{u1} + \varepsilon \omega_n^2 h_1 + \varepsilon n_{x1} + \varepsilon \frac{\mathrm{d}^2}{\mathrm{d}t^2} h_1 + \cdots = \mathbf{P_x}\mathbf{r} \tag{A2}$$

where terms of order ε^2 or larger have been ignored and where n_{u1} and h_1 are the function vectors whose variable parameters are u, $\dot{u}$, $\mathbf{r}$ and n_{x1} is a function of x, $\dot{x}$, $\mathbf{r}$. In order to make the functions have the same variables we substitute $x = u + h(u, \dot{u}, \mathbf{r})$ into n_{x1}, and apply a Taylor expansion results in $n_{x1}(x, \dot{x}, \mathbf{r}) = n_{x1}(u, \dot{u}, \mathbf{r}) + \mathcal{O}(\varepsilon^1)$. Substituting this into Eq. (A2), and equating powers of ε, gives

$$\varepsilon^0 : \mathbf{P_u} = \mathbf{P_x} \tag{A3}$$

$$\varepsilon^1 : n_{u1} = \omega_n^2 h_1 + n_{x1} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} h_1 \tag{A4}$$

where n_{u1}, h_1 and n_{x1} are all written as functions of u, $\dot{u}$ and $\mathbf{r}$.

The next step is to select $n_{u1}(u, \dot{u}, \mathbf{r})$ and $h_1(u, \dot{u}, \mathbf{r})$ for the given nonlinearity $n_{x1}(u, \dot{u}, \mathbf{r})$ to satisfy Eq. (A4). To do this we set $u = u_p + u_m$, where $u_p = \frac{U}{2}e^{\mathbf{i}(\omega_r t - \phi)}$ and $u_m = \frac{U}{2}e^{-\mathbf{i}(\omega_r t - \phi)}$. Substituting this into $n_{x1}(u, \dot{u}, \mathbf{r})$ results in terms of the type u_p, u_m, u_p^2, u_m^2, ...etc. The expansion of the non-linear terms in u_p and u_m is defined by a vector $\mathbf{u}^*$ which consists of all the combinations of u_p and u_m for that particular nonlinearity and the forcing terms r_p and r_m. This is an important step because then all the function vectors can be rewritten as a coefficient matrix multiplied by vector $\mathbf{u}^*$ [26] such that

$$n_{x1}(u, \dot{u}, \mathbf{r}) = \mathbf{n_x}\mathbf{u}^*, \quad n_{u1}(u, \dot{u}, \mathbf{r}) = \mathbf{n_u}\mathbf{u}^*, \quad h(u, \dot{u}, \mathbf{r}) = \mathbf{h}\mathbf{u}^* \tag{A5}$$

Define the form of any element in the vector $\mathbf{u}^*$ by writing the lth element as

$$u_\ell^* = r_p^{v_{\ell p}} r_m^{v_{\ell m}} u_p^{s_{\ell p}} u_m^{s_{\ell m}}, \tag{A6}$$

21

where the $v_{\ell p}$, $v_{\ell m}$, $s_{\ell p}$ and $s_{\ell m}$, constants indicate the power of the r_p, r_m, u_p and u_m terms respectively. When considering Eq. (A4), the second derivatives of the u_ℓ^* are also needed, which are given as

$$\frac{\mathrm{d}^2 u_l^*}{\mathrm{d}t^2} = \{\mathbf{i}[(v_{\ell p} - v_{\ell m})\Omega + (s_{\ell p} - s_{\ell m})\omega_r]\}^2 \, u_\ell^* = -\hat{\omega}_\ell^2 u_\ell^* \qquad \text{(A7)}$$

from which we infer that $\ddot{\mathbf{u}}^* = -\hat{\omega}^2\mathbf{u}^*$ where $\hat{\omega}^2$ is a diagonal matrix containing the $\hat{\omega}_\ell^2$ elements. Using this and substituting Eq. (A5) into Eq. (A4), yields

$$(-\mathbf{h}\hat{\omega}^2 + \omega_n^2\mathbf{h} - \mathbf{n_u} + \mathbf{n_x})\mathbf{u}^* = 0 \qquad \text{(A8)}$$

and for non-zero $\mathbf{u}^*$ solutions we can write

$$\mathbf{n_x} - \mathbf{n_u} = \mathbf{h}\hat{\omega}^2 - \omega_n^2\mathbf{h} = \tilde{\mathbf{h}}. \qquad \text{(A9)}$$

It is convenient here to apply a detuning approximation [4] in which we note that $\omega_r \approx \omega_n$ such that $\omega_n^2 = \omega_r^2 + \varepsilon\delta$. Substituting this into Eq. (A9) and recalling that this equation represents the ε^1 terms in Eq. (A2) gives

$$\mathbf{n_x} - \mathbf{n_u} = \mathbf{h}\hat{\omega}^2 - \omega_r^2\mathbf{h} = \tilde{\mathbf{h}}. \qquad \text{(A10)}$$

Note that the $\varepsilon\delta$ term modifies the equation relating the ε^2 terms in Eq. (A2).

The ℓth row of this vector equation may be written as

$$n_{x\ell} - n_{u\ell} = \tilde{h}_\ell. \qquad \text{(A11)}$$

where the ℓth element of $\mathbf{n_x}$ is $n_{x\ell}$ etc. Using Eqs. (A7) and (A8), the relationship between $\tilde{h}_\ell$ and h_ℓ may be written as

$$\tilde{h}_\ell = h_\ell\{[(v_{\ell p} - v_{\ell m})\Omega + (s_{\ell m} - s_{\ell m})\omega_r]^2 - \omega_r^2\} = \beta_\ell h_\ell. \qquad \text{(A12)}$$

22

The coefficients in $\mathbf{n_x}$ are known, they are defined from the nonlinear terms in the original equation of motion. Now using these equations the coefficients of $\mathbf{n_u}$ and $\mathbf{h}$ may be selected. Ideally we want as many of the $\mathbf{n_u}$ coefficients as possible to be zero, which will eliminate nonlinear terms in the transformed dynamic equation. Therefore the default is to write, for each element ℓ in turn, $n_{u\ell} = 0$ and $h_\ell = n_{x\ell}/\beta_\ell$. However for the cases where $\beta_\ell \approx 0$, which corresponds to a resonant term, this would result in large terms in the near-identity transform and hence invalidate the assumption that the transform terms are of order ε^1. Therefore for elements where $\beta_\ell \approx 0$ we write $n_{u\ell} = n_{x\ell}$ and $h_\ell = 0$ and retain the nonlinear term in the transformed equations. For more details about this transformation see [4].

Appendix B. Application of normal forms to polynomial nonlinearity

Here we identify the resonant and harmonic terms present in the generalised polynomial nonlinearity defined by N_x in Eq. (1)

Appendix B.1. The nonlinear resonant terms

Considering the process to select the near-identity transform, the key coefficient for the ℓth term is β_ℓ, see Eq. (A12), which is given by

$$\beta_\ell = [(m_{\ell p} - m_{\ell m})\Omega + (s_{\ell p} - s_{\ell m})\omega_r]^2 - \omega_r^2 = 0. \qquad \text{(A1)}$$

When β_ℓ is equal to (or close to) zero, the corresponding nonlinear term is resonant and so the ℓth element in $\mathbf{n_u}$ matrix, $n_{u\ell}$, is set equal to the term in the $\mathbf{n_x}$ matrix, $n_{x\ell}$. With no parametric forcing terms present we can write,

for the lth term, that if

$$|s_{\ell p} - s_{\ell m}| = 1 \qquad\qquad (A2)$$

then the term is resonant, and hence $n_{u\ell} = n_{x\ell}$ and $h_\ell = 0$ such that the term is kept in the equation of motion. Taking the nonlinearity in the form of the summation, Eq. (8), i.e. the terms present in vector $\mathbf{u}^*$ are $u_p^{i+j-k}u_m^k$ for $k = 0$ to $k = i + j$, and with reference to Eq. (A6), Eq. (A2) becomes

$$|i + j - 2k| = 1 \qquad\qquad (A3)$$

for the kth element.

Inspecting Eq. (A3) it can be seen that for the case where $i + j$ is even no resonance terms exist and hence N_{ij} is removed from the dynamic equations, its effects being represented in the transform equation. When $i + j$ is odd two resonance terms exist. These cases can be summarised as

$$\text{even } i + j: \qquad\qquad N_{ij,res} = 0 \qquad\qquad (A4)$$

$$\text{odd } i + j: \quad N_{ij,res} = \alpha_{ij}(\mathbf{i}\omega_r)^j \left(\gamma_k u_p^{k+1}u_m^k + \gamma_{k+1}u_p^k u_m^{k+1}\right), \quad k = \tfrac{i+j-1}{2}$$

where the subscript res indicates that only the resonant terms are included.

Appendix B.2. The harmonics in the system response

As with N, Eq. (6), taking h to be a summation over i and j where terms consist of $u^i \dot{u}^j$, the (i, j)th term in h may be written as

$$h_{(ij)} = \alpha_{ij}(\mathbf{i}\omega_r)^j \sum_{k=0}^{i+j} \delta_k \gamma_k u_p^{i+j-k}u_m^k, \qquad\qquad (A5)$$

i.e. in a similar way to Eq. (8). Here δ_k is determined by whether $u_p^{i+j-k}u_m^k$ is resonant or non-resonant and, using Eq. (A12) and the subsequent discussion,

24

is given by

$$\text{non-resonant:} \quad \delta_k = \frac{1}{[(i+j-2k)^2 - 1]\omega_r^2}, \quad k \neq \frac{i+j\pm 1}{2} \qquad \text{(A6)}$$

$$\text{resonant:} \qquad \delta_k = 0, \quad k = \frac{i+j\pm 1}{2}. \qquad \text{(A7)}$$

To proceed the terms in the summation are collected in pairs, $u_p^{i+j-k}u_m^k$ and $u_p^k u_m^{i+j-k}$, by writing

$$h_{(ij)} = \alpha_{ij}(\mathbf{i}\omega_r)^j \left[\delta_k \gamma_k u_p^k u_m^k \big|_{k=(i+j)/2} + \sum_{k=0}^{(i+j-2)/2} \delta_k \gamma_k (u_p^{i+j-k} u_m^k + (-1)^j u_p^k u_m^{i+j-k}) \right],$$

$$\qquad \text{(A8)}$$

using Eq. (A5) and noting that $\delta_k = \delta_{i+j-k}$. Note that when $i+j$ is odd the first term in the square brackets is set to zero.

Appendix B.2.1. The constant time-invariant response

Considering the first term in the square bracket of Eq. (A8), making the substitutions $u_p = \frac{U}{2}e^{\mathbf{i}(\omega_r t - \phi)}$ and $u_m = \frac{U}{2}e^{-\mathbf{i}(\omega_r t - \phi)}$ and using Eqs. (A6) and (A7) gives

$$\delta_k \gamma_k u_p^k u_m^k \big|_{k=(i+j)/2} = \begin{cases} -\dfrac{\gamma_k}{\omega_r^2}\left(\dfrac{U}{2}\right)^{2k}\bigg|_{k=(i+j)/2} & \text{even } i+j \\[4mm] 0 & \text{odd } i+j \end{cases} \qquad \text{(A9)}$$

This term represents the constant off-set due to the nonlinearity in the system. Note that this expression can be further simplified by realising that $\gamma_k = 0$ for $k = (i+j)/2$ for odd values of j, using Eq. (A5). Recalling that the near-identity transform h is made up of a summation of $h_{(ij)}$ terms over i and j, the full constant off-set response may be written as

$$h_{(0)} = \sum_{i=0,e}^{I} \sum_{j=0,e}^{J} \alpha_{ij}(\mathbf{i}\omega_r)^{j-2}\gamma_{(i+j)/2}\left(\frac{U}{2}\right)^{i+j} \qquad \text{(A10)}$$

25

where the subscript (0) for h indicates the zero frequency response and e in the summation indicates only even terms are used (elsewhere o is used for odd terms).

The general form of this constant off-set value is contributed by different nonlinearities suggests that a form of superposition can be used here.

Appendix B.2.2. Response at higher harmonic

Now, considering the second term in the square bracket in Eq. (A8) and making the substitutions $u_p = \frac{U}{2}e^{\mathbf{i}(\omega_r t - \phi)}$ and $u_m = \frac{U}{2}e^{-\mathbf{i}(\omega_r t - \phi)}$ gives

$$
h_{(ij,\emptyset)} = \begin{cases} 2\alpha_{ij}(\mathbf{i}\omega_r)^j \left(\dfrac{U}{2}\right)^{i+j} \displaystyle\sum_{k'=2}^{i+j} \dfrac{1}{(k'^2-1)\omega_r^2}\gamma_{(i+j-k')/2} \cos[k'(\omega_r t - \phi)] & \text{even } j \\[4ex] 2\alpha_{ij}\mathbf{i}(\mathbf{i}\omega_r)^j \left(\dfrac{U}{2}\right)^{i+j} \displaystyle\sum_{k'=2}^{i+j} \dfrac{1}{(k'^2-1)\omega_r^2}\gamma_{(i+j-k')/2} \sin[k'(\omega_r t - \phi)] & \text{odd } j \end{cases}
$$

(A11)

where subscript $\emptyset$ indicates that the zero frequency response is excluded, the substitution $k' = i + j - 2k$ has been made and Eq. (A6) has been used. Note also that for non-integer values and negative values of a; $\gamma_a = 0$ is defined. Again, recalling that the near-identity transform h is made up of a summation of $h_{(ij)}$ terms over i and j, the response at the kth harmonic may be written as

$$
h_{(k)} = \sum_{i=0}^{I} \sum_{j=0,e}^{J} 2\alpha_{ij}(\mathbf{i}\omega_r)^j \left(\frac{U}{2}\right)^{i+j} \frac{1}{(k^2-1)\omega_r^2}\gamma_{(i+j-k)/2} \cos[k(\omega_r t - \phi)] +
$$
$$
\sum_{i=0}^{I} \sum_{j=0,o}^{J} 2\alpha_{ij}\mathbf{i}(\mathbf{i}\omega_r)^j \left(\frac{U}{2}\right)^{i+j} \frac{1}{(k^2-1)\omega_r^2}\gamma_{(i+j-k)/2} \sin[k(\omega_r t - \phi)]
$$

(A12)

Note that the harmonic response is potentially non-zero for either $k = 2, 4, \ldots i + j$ for the case where $i + j$ is even or $k = 3, 5, \ldots i + j$ when

26

$i + j$ is odd (since at other k values the corresponding γ_a has a non-integer a).

Appendix C. The coefficient γ

The definition of coefficient γ_k is given in Eq. (9)

$$\gamma_k = \sum_v C_i^v C_j^{k-v} (-1)^{k-v} \tag{A1}$$

where C is the binomial coefficient, taking $C_0^0 = 1$ and $C_a^{a+b} = C_a^{-b} = 0$ for positive values of b. Here a simple relationship between γ_k and γ_{i+j-k} is derived. To do this consider γ_{i+j-k}, given by

$$\gamma_{i+j-k} = \sum_v C_i^v C_j^{i+j-k-v} (-1)^{i+j-k-v} \tag{A2}$$

and make the substitution $v = i - v'$ to give

$$\gamma_{i+j-k} = \sum_{v'} C_i^{i-v'} C_j^{j-k+v'} (-1)^{j-k+v'}. \tag{A3}$$

Using the relationship $C_a^b = C_a^{a-b}$ and noting that $(-1)^a = (-1)^{-a}$ results in

$$\gamma_{i+j-k} = \sum_{v'} C_i^{v'} C_j^{k-v'} (-1)^{-j+k-v'}. \tag{A4}$$

Inspecting Eqs. (A1) and (A4) leads to the relationship

$$\gamma_k = (-1)^j \gamma_{i+j-k} \tag{A5}$$

The resonant terms for the i, j term of N, Eq. (8), will now be considered, followed by the harmonic terms.

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List of Figures