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Abstract

Recursive algorithms are derived to compute the generalised frequency response function matrix of multi-input multi-output (MIMO) nonlinear systems as an analytical map from both nonlinear differential equation models and NARX (nonlinear autoregressive with eXogenous inputs) models of the system. The algorithm is computationally compact and exposes the explicit relationship between the model parameters and the elements of the generalised frequency response function matrix and can thus provide important insights into the behaviour of nonlinear systems

1 Introduction

In the past few years theories have been developed to model and analyse a large class of nonlinear systems through the introduction of the functional series representations of Volterra and Wiener. Volterra was the first to formalise the mathematical expressions for an integral series by means of higher degree kernels and Wiener showed how a related series could be orthogonalised. Early works on nonlinear system identification using the Volterra and Wiener kernels have been comprehensively reviewed by Hung and Stark(1977) and Billings(1980).

Following the work of Wiener, a great deal of research has been done on the efficient computation of the kernels of physical systems (Lee and Schetzen(1965); Sandberg and Stark(1968); Watnabe and Stark(1975); Marmarelis and Marmarelis(1978); Schetzen(1980); Rough(1981)). More recent results on kernel based identification have been published by Korenberg et al(1988a), Korenberg and Hunter(1990), Hung and Stark(1991). Most of these kernel based techniques have been used to model single input single output (SISO) systems. In contrast the Volterra modelling of multi input multi output(MIMO) systems has received little attention. Early works on multi input Volterra modelling were briefly introduced in Marmarelis and Marmarelis(1978) and recently Westwick and Kearney(1992) modelled a multi-input system using the Wiener theory.

The kernel based identification techniques can be considered as nonparametric methods. But analysis in the frequency domain has the advantage that the integral equations which relate the input-output in the time domain become algebraic in the frequency domain. The frequency domain representation of the system can be obtained by applying the multi dimensional Fourier transform to the Volterra kernels. This yields the so called nonlinear frequency response functions or generalised frequency response functions (GFRF), Volterra transfer functions or kernel transforms.

Identification of the Volterra kernels or GFRF directly from input-output data has been extensively studied by Kim and Powers (1988), Tseng and Linebarger(1991), Tseng and Powers (1993), Tseng and Powers(1995). An alternative approach suggested by Billings and co-workers (Billings and Tsang,1989; Peyton Jones and Billings,1989;Billings and Peyton Jones,1990; Zhang et al,1995) is to estimate a parametric NARMAX (Leontaritis and Billings,1985) model and subsequently to derive the frequency response functions from this model using harmonic probing.

Although all these techniques and approaches have been applied to map SISO systems into the frequency domain, the frequency response functions of MIMO systems has not received much attention apart from Chua and Ng(1979) and Marmarelis and Naka(1974) who introduced the idea of extending the Volterra series to multi input single output(MISO) systems. Recently Worden et al(1995) adopted the Volterra approach to suppress the nonlinear harmonics of a two degree freedom mechanical system.

The purpose of the present study is to derive expressions for the generalised frequency response function matrix (GFRFM) for general MIMO nonlinear systems as an analytical map from either nonlinear differential equation models or NARX (Nonlinear Auto Regressive with eXogenous inputs) models of the system. The effects of different types of nonlinear terms, pure input, pure output and input-output cross product nonlinear terms, on the GFRFM are shown independently and this provides a great deal of insight into the time and frequency domain relationships among different terms in the models.

The paper is organised as follows. Section-2 briefly reviews the Volterra modelling of

SISO systems. Section-3-4 introduces the concept of Volterra modelling for MIMO systems and defines the generalised kernel transform (GKERT) for MIMO systems. The procedure for estimating GKERT is discussed in section-5. The general form of nonlinear differential equation models for MIMO systems is presented in section-6 and a relationship which maps the parameters of these models directly into the generalised transfer function matrix is derived. Several examples are included to demonstrate the application of the algorithm. In sections-7 and 8 an algorithm to map the MIMO NARX model into the frequency domain is derived.

2 Volterra Modelling of SISO Systems

The output y(t) of a single input single output (SISO) analytic system may be expressed as a Volterra functional polynomial of the input u(t) (Volterra,1930) to give

$$y(t) = \sum_{n=1}^{N} y^{(n)}(t) \tag{1}$$

where the nth order output of the system $y^{(n)}(t)$ is given by

$$y^{(n)}(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots \tau_n) \prod_{i=1}^{n} u(t-\tau_i) d\tau_i, \quad n > 0; \tag{2}$$

and $h_n(\tau_1, \dots\dots\tau_n)$ is a real valued function of $\tau_1, \dots.\tau_n$, called the nth order impulse response or Volterra kernel of the system.

The multi-dimensional Fourier Transform of the nth order impulse response yields the nth order transfer function or generalised frequency response function (GFRF)

$$H_n(j\omega_1, \dots j\omega_n) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} h_n(\tau_1, \dots, \tau_n) e^{-j(\omega_1\tau_1 + \dots \omega_n\tau_n)} d\tau_1 \dots d\tau_n \tag{3}$$

When the output is expressed in terms of the GFRF's, eqn(2) becomes

$$y^{(n)}(t) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} H_n(j\omega_1, \dots j\omega_n) \prod_{i=1}^{n} U(j\omega_i) e^{j(\omega_1 + \dots + \omega_n)t} d\omega_i \tag{4}$$

where $U(j\omega_i)$ represents the input spectrum.

Inspection of eqn(3) and (4) shows that the n-th order kernel and the kernel transform are not necessarily unique because an interchange of arguments in $h_n(\tau_1, \dots \tau_n)$ may give different kernels without affecting the input-output relationship. To ensure that the GFRF's are unique these are symmetrised to give

$$H_n^{sym}(j\omega_1,\ldots j\omega_n) = \frac{1}{n!} \sum_{\substack{\text{all permutations of} \\ \omega_1,\ldots\omega_n}} H_n(j\omega_1,\ldots j\omega_n) \tag{5}$$

2.1 Computation of the Generalised Frequency Response Functions for Single Input Single Output Systems

Computation of the GFRF's directly from input-output data is computationally involved due to multi dimensional FFT,windowing and smoothing operations. Most of these problems can however be avoided by fitting a parametric model to the input-output data and then mapping this model into the frequency domain to estimate the Generalised Frequency Response Functions (Billings and Tsang,1989; Peyton-Jones and Billings,1989; Billings and Peyton-Jones,1990).

Consider a system where a parametric model is assumed to exist and is represented as

$$M(t;\theta,y,u) = 0 \tag{6}$$

where M(.) is a functional of the input u, output y and θ is a set of model parameters . When the y in eqn(6) is substituted by the Volterra functional representation from eqn(4), eqn(6) becomes

$$M(t;\theta,H,u) = 0 \tag{7}$$

So the 'y' is now replaced by the GFRF's $H \equiv \{H_1, H_2, \ldots\}$ in eqn(6). Computation of H(.) by manipulating eqn(7) for arbitrary inputs often results in solving complicated integral equations. However the harmonic probing technique (Bedrosian and Rice,1971 ; Billings and Tsang ,1989) can be used to compute 'H' from eqn(7). This involves applying a input consisting of R complex exponentials defined as

$$u(t) = \sum_{\sigma=1}^{R} e^{j\omega_\sigma t} \tag{8}$$

The spectrum of the input is

$$U(j\omega) = \sum_{\sigma=1}^{R} 2\pi\delta(j\omega - j\omega_\sigma) \tag{9}$$

The output of the system under the harmonic excitation of eqn(9) becomes

$$y(t) = \sum_{n=1}^{N_l} \sum_{\sigma_1,\ldots\sigma_n=1}^{R} H_n(j\omega_{\sigma_1},\ldots j\omega_{\sigma_n})e^{j(\omega_{\sigma_1}+\ldots+\omega_{\sigma_n})t}$$
$$= \sum_{n=1}^{N_l} \sum_{\text{[all perm. of R freq taken n at a time]}} \sum_{\text{[all perm. of } \omega_{\sigma_1},\ldots\omega_{\sigma_n}]} H_n(j\omega_{\sigma_1},\ldots j\omega_{\sigma_n})e^{j(\omega_{\sigma_1}+\ldots+\omega_{\sigma_n})t} \quad (10)$$

To find the n-th order GFRF $H_n(.)$, it is convenient to consider the special case $R = n$; so that there is only one non-repetitive combination of frequencies $\{\omega_1,\ldots\omega_n\}$ among all the possibilities.

Substituting eqn(10) and (8) into eqn(7) yields the following equation

$$M(t;\theta,H,\omega_\sigma) = 0 \quad (11)$$

where ω_σ includes the frequencies $\{\omega_1,\ldots\omega_R\}$. To compute $H_n(.)$, R is made equal to n. M (.) will contain many exponential terms but we are only interested in the term with non-repetitive frequencies $e^{j(\omega_1+\ldots+\omega_n)t}$.

The procedure of computing 'H' by solving eqn(11) was derived using an *extraction operator* $\epsilon_n[.]$ by Zhang et al(1995). For a given expression, the operator $\varepsilon_n[.]$ for SISO systems involves the execution of following steps.

- Substitute the harmonic input of eqn(8) and corresponding Volterra expansion of the output (eqn(10)) into the given expression
- Express the output y(t) as a function of H and ω_σ
- Extract the coefficient of $e^{j(\omega_1+\omega_2+\ldots\omega_n)t}$ from the resulting expression.

As an example applying ϵ_n to eqn(10)will give

$$\epsilon_n[y(t)] = \sum_{\text{all permutations of } \omega_1,\ldots\omega_n} H_n(j\omega_1,\ldots j\omega_n) \quad (12)$$

3 Volterra Modelling of Multi Input Multi Output (MIMO) Systems

Consider a MIMO nonlinear system having r-inputs and m-outputs. The output of the 'j_1-th' subsystem of a MIMO system possessing nonlinearity up to degree N_l may be expressed as

$$\begin{aligned} y_{j_1}(t) &= \sum_{n=1}^{N_l} y_{j_1}^{(n)}(t) \\ &= y_{j_1}^{(1)}(t) + y_{j_1}^{(2)}(t) + \dots y_{j_1}^{(N_l)}(t) \end{aligned} \tag{13}$$

where $y_{j_1}^{(n)}(t)$ is the n-th order component of the output $y_{j_1}(t)$. Each of these components are homogeneous of degree-n.

Let the r-inputs be denoted as $u_{\beta_1}(t), \dots u_{\beta_r}(t)$, Eqn.(13) can be expressed as

$$y_{j_1}(t) = \sum_{n=1}^{N_l} \sum_{\beta_1=1}^{r} \sum_{\beta_2=\beta_1}^{r} .. \sum_{\beta_n=\beta_{n-1}}^{r} \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} h_n^{(j_1:\beta_1,\dots\beta_n)}(\tau_1, \dots \tau_n) u_{\beta_1}(t-\tau_1) \dots . u_{\beta_n}(t-\tau_n) d\tau_1, ..d\tau_n \tag{14}$$

where $h_n^{(j_1:\beta_1,\dots\beta_n)}(\tau_1, \dots \tau_n)$ is the n-th order Volterra kernel of the j_1th subsystem. The superscripts $\beta_1, \dots\beta_n$ in the kernel correspond to the inputs $u_{\beta_1}(t), \dots u_{\beta_n}(t)$ that take part in the n-dimensional convolution with $h_n^{(j_1:\beta_1,\dots\beta_n)}(\tau_1, \dots \tau_n)$. Note that the output of any other subsystem (the j_i-th say), can be obtained from eqn(14) by replacing j_1 by j_i. Thus it will suffice to concentrate the subsequent analysis based on the j_1-th subsystem of a MIMO system. The analysis of other subsystems is straight forward.

The interpretation of the notation can be better understood by representing a SISO system using the MIMO notations. Thus the output of a SISO system can be expressed as

$$\begin{aligned} y_1(t) &= \sum_{n=1}^{N_l} y_1^{(n)}(t) \\ &= y_1^{(1)}(t) + y_1^{(2)}(t) + \dots y_1^{(N_l)}(t) \end{aligned} \tag{15}$$

where

$$y_1^{(n)}(t) = \int_{-\infty}^{\infty} .. \int_{-\infty}^{\infty} h_n^{\overbrace{(1:1,\dots1)}^{}_{n-times}}(\tau_1, \dots \tau_n) u_1(t-\tau_1) \dots . u_1(t-\tau_n) d\tau_1, \dots d\tau_n \tag{16}$$

Thus the first order kernel for the SISO system $h_1(.)$ in eqn(1) corresponds to $h_1^{(1:1)}(.)$ in eqn(14) and the second order kernel $h_2(.)$ in eqn(1) corresponds to $h_2^{(1:11)}(\,.\,)$ in eqn(14) and so on.

The Volterra model of an N_l-th degree nonlinear SISO system is completely characterised by the N_l kernels $h_1(\tau_1), h_2(\tau_1,\tau_2), \ldots, h_{N_l}(\tau_1, \ldots \tau_n)$. The total number of kernels required to model a particular subsystem of an r-input nonlinear system possessing nonlinearity of degree N_l equals $Tk^{(r)}_{(N_l)}$ where

$$Tk^{(r)}_{(N_l)} = \sum_{i=1}^{N_l} Nk^{(r)}_{(i)} \tag{17}$$

$Nk^{(r)}_{(i)}$ represents the number of i-th order kernels of an r-input system and is calculated recursively using the relation

$$Nk^{(r)}_{(i)} = Nk^{(r)}_{(i-1)} + Nk^{(r-1)}_{(i-1)} + \ldots\ldots Nk^{(1)}_{(i-1)} \tag{18}$$

where $Nk^{(r)}_{(1)}$ = total number of first order kernels for an r-input system = r. As an example for a system with $r = 4$ and $N_l = 3$, the total number of kernels for a particular subsystem $Tk^{(4)}_{(3)}$ equals 34.

The kernels are called *self-kernels* when all the superscripts $\beta_1, \ldots\ldots \beta_n$ in $h_n^{(j_1:\beta_1,\ldots\beta_n)}(\tau_1, \ldots \tau_n)$ are equal ; otherwise they are called *cross-kernels.* Each self-kernel is convolved with one input while each cross-kernel is convolved with at least two different inputs. Clearly all the kernels of a SISO system are *self kernels.* The self-kernels of the MIMO system can be assumed to be symmetric but the cross-kernels may be either symmetric or asymmetric depending on the properties of the system (Chen,1995). For example the second order cross-kernel $h_2^{(j_1:\beta_1\beta_2)}(\tau_1,\tau_2)$ may not be equal to $h_2^{(j_1:\beta_2\beta_1)}(\tau_2,\tau_1)$. The nonlinear interaction amongst different input paths is reflected in the cross-kernels of the multi-input system.

When the effects of all the possible cross-kernels obtained from the interchange or permutation of the arguments are to be included in the output response, the output of the j_1th subsystem can be expressed as

$$y_{j_1}(t) = \sum_{n=1}^{N_l} \sum_{\beta_1=1}^{r} \sum_{\beta_2=1}^{r} \cdots \sum_{\beta_n=1}^{r} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n^{(j_1:\beta_1,\ldots\beta_n)}(\tau_1, ..\tau_n) u_{\beta_1}(t-\tau_1) \ldots u_{\beta_n}(t-\tau_n) d\tau_1, \ldots d\tau_n \tag{19}$$

4 Elements of the Generalised Frequency Response Function Matrix

The GFRF of a SISO system includes all the linear and higher order frequency response functions which are defined as the multi-dimensional Fourier transforms of the kernels. This will be referred to as the n-th order *self- kernel transform*. Similarly the multi-dimensional Fourier transform of the kernels of a MIMO system constitute the elements of the GFRFM. In section-2.1 the estimation of n-th order symmetric self-kernel transform of a SISO system was described using the operator $\epsilon_n[.]$. It will be shown below that the estimation of the *cross-kernel transform* of a MIMO system is not quite so straight forward.

Since the objective is to find an analytic map from the time domain system models into the frequency domain and to compute the elements of the GFRFM, it is worth beginning by considering a simple example and naively applying the probing techniques to indicate the problems involved.

The output of the j_1-th subsystem of a 2-input system which includes only kernels up to degree-2 from eqn(19) is

$$\begin{aligned} y_{j_1}(t) &= \int_{-\infty}^{\infty} h_1^{(j_1:1)}(\tau_1)u_1(t-\tau_1)d\tau_1 + \int_{-\infty}^{\infty} h_1^{(j_1:2)}(\tau_1)u_2(t-\tau_1)d\tau_1 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{(j_1:11)}(\tau_1,\tau_2)u_1(t-\tau_1)u_1(t-\tau_2)d\tau_1, d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{(j_1:12)}(\tau_1,\tau_2)u_1(t-\tau_1)u_2(t-\tau_2)d\tau_1, d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{(j_1:21)}(\tau_1,\tau_2)u_2(t-\tau_1)u_1(t-\tau_2)d\tau_1, d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{(j_1:22)}(\tau_1,\tau_2)u_2(t-\tau_1)u_2(t-\tau_2)d\tau_1, d\tau_2 \end{aligned} \tag{20}$$

However the dummy variables involved in the cross-kernel integrals may be rearranged and eqn(20) can be expressed as

$$\begin{aligned} y_{j_1}(t) &= \int_{-\infty}^{\infty} h_1^{(j_1:1)}(\tau_1)u_1(t-\tau_1)d\tau_1 + \int_{-\infty}^{\infty} h_1^{(j_1:2)}(\tau_1)u_2(t-\tau_1)d\tau_1 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{(j_1:11)}(\tau_1,\tau_2)u_1(t-\tau_1)u_1(t-\tau_2)d\tau_1, d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [h_2^{(j_1:12)}(\tau_1,\tau_2) + h_2^{(j_1:21)}(\tau_2,\tau_1)]u_1(t-\tau_1)u_2(t-\tau_2)d\tau_1, d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2^{(j_1:22)}(\tau_1,\tau_2)u_2(t-\tau_1)u_2(t-\tau_2)d\tau_1, d\tau_2 \end{aligned} \tag{21}$$

To compute the self kernel transform $H_2^{(j_1:11)}(j\omega_1, j\omega_2)$, set

$$u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \text{ and } u_2(t) = 0 \tag{22}$$

This input pattern is described by $e^{j\omega_1 t}$ and $e^{j\omega_2 t}$ which belong to input one. The output of the system with this input can from eqn(21) be expressed as

$$\begin{aligned} y_{j_1}(t) &= H_1^{(j_1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(j_1:1)}(j\omega_2)e^{j\omega_2 t} + [H_2^{(j_1:11)}(j\omega_1, j\omega_2) + H_2^{(j_1:11)}(j\omega_2, j\omega_1)]e^{j(\omega_1+\omega_2)t} \\ &\quad + \text{terms of repetitious combinations of frequencies} \end{aligned} \tag{23}$$

Since the self kernels are symmetric eqn(23) can be written as

$$\begin{aligned} y_{j_1}(t) &= H_1^{(j_1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(j_1:1)}(j\omega_2)e^{j\omega_2 t} + 2!H_{2sym}^{(j_1:11)}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} \\ &\quad + \text{terms of repetitious combinations of frequencies} \end{aligned} \tag{24}$$

To compute the cross-kernel transform $H_2^{(j_1:12)}(j\omega_1, j\omega_2)$, the two-tone input is split and applied at the two input points 1 and 2 according to $u_1(t) = e^{j\omega_1 t}$ and $u_2(t) = e^{j\omega_2 t}$. Thus $e^{j\omega_1 t}$ belongs to input point-1 and $e^{j\omega_2 t}$ belongs to input point-2. The output of the system under this excitation pattern from eqn(19) becomes

$$\begin{aligned} y_{j_1}(t) &= H_1^{(j_1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(j_1:2)}(j\omega_2)e^{j\omega_2 t} + [H_2^{(j_1:12)}(j\omega_1, j\omega_2) + H_2^{(j_1:21)}(j\omega_2, j\omega_1)]e^{j(\omega_1+\omega_2)t} \\ &\quad + \text{terms of repetitious combinations of frequencies} \end{aligned} \tag{25}$$

Comparing eqn(23) with eqn(25), the coefficients of $e^{j(\omega_1+\omega_2)t}$ in eqn(23) has been replaced by defining the symmetric direct kernel transform such that

$$\begin{aligned} 2!H_{2sym}^{(j_1:11)}(j\omega_1, j\omega_2) &= [H_2^{(j_1:11)}(j\omega_1, j\omega_2) + H_2^{(j_1:11)}(j\omega_2, j\omega_1)] \\ &= \sum_{\substack{\text{all permutations of} \\ \omega_1, \ldots \omega_n}} H_2^{(j_1:11)}(j\omega_1, j\omega_2) \end{aligned} \tag{26}$$

Note that the symmetric version of the self kernel transform is obtained by an averaging operation over all permutations of frequency arguments of the asymmetric kernel transforms. From this perspective the symmetric self kernel transform is essentially the *average* of the asymmetric self kernel transforms.

But for a multi-input system, the cross-kernel transforms do not enjoy the property of symmetry. Hence it is not possible to define the symmetric cross kernel transform. Instead an *average cross-kernel transform* will be defined which is the equivalent of the symmetric self kernel transform. In the present example, the coefficients of $e^{j(\omega_1+\omega_2)t}$ in eqn(25) is replaced by $2!H_{2_{avg}}^{(1:12)}(j\omega_1, j\omega_2)$ where

$$\begin{aligned} 2!H_{2_{avg}}^{(j_1:12)}(j\omega_1, j\omega_2) &= H_2^{(j_1:12)}(j\omega_1, j\omega_2) + H_2^{(j_1:21)}(j\omega_2, j\omega_1) \\ &= \sum_{\substack{\text{all permutations of} \\ [\omega,\beta]}} H_2^{(j_1:12)}(j\omega_1, j\omega_2) \end{aligned} \tag{27}$$

Note that when ω_1 is permutated to ω_2, the corresponding superscripts also change accordingly that is $H_2^{(j_1:12)}(j\omega_1, j\omega_2)$ changes to $H_2^{(j_1:21)}(j\omega_2, j\omega_1)$. Thus the permutation of the arguments of the cross kernels are not independent of the superscripts of the kernels, but follow a specific rule called $[\omega, \beta]$ permutation which will be explained in detail in section-5. Note that the average cross-kernel transform is analogous to the symmetric self kernel transform.

4.1 Generalised Kernel Transform of Multi Input Multi Output Nonlinear Systems

From the example of section-4 it is straight forward to infer that the procedure of computing each kernel transform of a subsystem differs from another and is calculated independently of the other. Instead of deriving the expression for each kernel transform it will be logical to define a generalised kernel and derive the expression for the generalised kernel transform (GKERT). All the elements of the GFRFM can be shown to be the special case of GKERT.

The generalised kernel of the j_1-th subsystem of the MIMO system is denoted as

$$\text{GKER} = h_n^{(j_1:\underbrace{\beta_1,\ldots,}_{\gamma_1 \text{times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(\tau_1, \ldots \tau_n)$$

where

n_d = the number of distinct inputs present in the kernel

γ_1 = number of times β_1 occurs in the superscript and

γ_{n_d} = number of times β_{n_d} appears in the kernel

As an example, for the kernel $h_3^{(1:112)}(.)$, $n_d = 2$, $\gamma_1 = 2$, $\gamma_2 = 1$. $\beta_1 = 1$ and $\beta_2 = 2$.

The Generalised Kernel Transform is defined as the multidimensional Fourier transform of the generalised kernel and may be denoted as

$$\mathrm{GKERT} = \mathrm{H}_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n)$$

It is appropriate to mention here that for a SISO system the symmetric version of the GFRF is computed since this is unique and takes into account the effects of permutating the arguments of the Volterra kernels on the output. Analogously for a MIMO system it is necessary to compute the *average* of the kernel transforms.

5 ϵ_n-Operator for MIMO System

The extraction operator $\epsilon_n[.]$ (Zhang et al,1995) for computing the GKERT differs from the SISO case as discussed below.

Remark-1 : (Splitting n-tone Inputs)

While estimating the generalised cross kernel transform where the number of inputs is more than one, the n-tone input will no longer be applied at a single input point but is instead split and applied at various input points. The splitting pattern depends on the cross-kernel to be estimated. For example to compute $\mathrm{H}_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n)$ the n-tone inputs is split as

$$\begin{array}{lcl}
u_{\beta_1}(t) & = & e^{j\omega_1 t} + \ldots\ldots + e^{j\omega_{\gamma_1} t} \\
u_{\beta_2}(t) & = & e^{j\omega_{1+\gamma_1} t} + \ldots\ldots + e^{j\omega_{\gamma_1+\gamma_2} t} \\
\vdots & \vdots & \vdots \\
u_{\beta_{n_d}}(t) & = & e^{j\omega_{1+\gamma_1+\ldots+\gamma_{n_d-1}} t} + \ldots\ldots + e^{j\omega_n t}
\end{array} \tag{28}$$

Thus for a given expression the operator $\epsilon_n[.]$ for a MIMO system involves the execution of the following steps.

- Split the n-tone input according to eqn(28)
- Express the outputs of the system due to the above input excitation using eqn(19).
- Substitute the inputs and the outputs into the given expression

- Extract the coefficients of $e^{j(\omega_1+\dots+\omega_n)t}$.

Before illustrating the effects of applying the $\epsilon_n[.]$ operator to specific expressions,the concept of $[\omega,\beta]$ permutation and average kernel transform is introduced.

Remark-1.1($[\omega,\beta]$ permutation)

Since the cross kernels are not symmetric, permutation of the arguments of the cross-kernel transforms must be followed by a corresponding change or permutation of the superscripts. Corresponding in this context has a special meaning which will be explained through an example because neither the frequency ω nor the input points 'β' of the cross kernel transform can be permutated independently.

Example

To clarify the above remarks consider the estimation of the cross-kernel transform $H_3^{(j_1:\beta_1,\beta_2,\beta_3)}(\omega_1,\omega_2,\omega_3)$. This kernel can be obtained from the generalised kernel transform with $n_d = 3$, $\gamma_1 = 1$, $\gamma_2 = 1$ and $\gamma_3 = 1$. In order to compute this transform the 3-tone input is split among the three inputs such that

$$u_{\beta_1}(t) = e^{j\omega_1 t};\ u_{\beta_2}(t) = e^{j\omega_2 t} \text{ and } u_{\beta_3}(t) = e^{j\omega_3 t} \tag{29}$$

This implies that $e^{j\omega_1 t}$ belongs to the input point β_1, $e^{j\omega_2 t}$ belongs to the input point β_2 and $e^{j\omega_3 t}$ belongs to the input point β_3.

Applying $[\omega,\beta]$ permutation to $H_3^{(j_1:\beta_1,\beta_2,\beta_3)}(\omega_1,\omega_2,\omega_3)$ yields

$$\begin{aligned}\sum_{\substack{\text{all permutations}\\ [\omega,\beta]}} H_3^{(j_1:\beta_1,\beta_2,\beta_3)}(\omega_1,\omega_2,\omega_3) &= H_3^{(j_1:\beta_1,\beta_2,\beta_3)}(\omega_1,\omega_2,\omega_3)\\ &+H_3^{(j_1:\beta_1,\beta_3,\beta_2)}(\omega_1,\omega_3,\omega_2)+H_3^{(j_1:\beta_2,\beta_3,\beta_1)}(\omega_2,\omega_3,\omega_1)\\ +H_3^{(j_1:\beta_2,\beta_1,\beta_3)}(\omega_2,\omega_1,\omega_3)+H_3^{(j_1:\beta_3,\beta_1,\beta_2)}(\omega_3,\omega_1,\omega_2) &+ H_3^{(j_1:\beta_3,\beta_2,\beta_1)}(\omega_3,\omega_2,\omega_1)\end{aligned} \tag{30}$$

It is worth emphasising that $[\omega,\ \beta]$ permutation is crucial in the derivation of the expressions for the GKERT. Since the n-tone inputs are split among different inputs; before applying $[\omega,\ \beta]$ permutation it is essential to identify the different frequency components of the n-tone signal and the corresponding inputs where these belong or are applied. Then $[\omega,\ \beta]$ permutation would imply that when any argument of a kernel transform $H_n^{(.)}(.)$, ω_i (say) which belongs to input point β_i is permutated to ω_k (say), β_i must change to β_k such that $e^{j\omega_k t}$ belongs to the input point β_k.

Average Generalised Kernel Transform

The average generalised kernel transform is computed by taking the $[\omega, \beta]$ permutation of the asymmetric kernels and is defined as

$$H_{n_{avg}}^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}},\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n)$$

$$= \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ [\omega,\beta]}} H_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}},\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n) \tag{31}$$

Note that the left hand side of eqn(30) can be denoted as $3!H_{3_{avg}}^{(j_1:\beta_1,\beta_2,\beta_3)}(\omega_1,\omega_2,\omega_3)$.

Applying $\epsilon_n[.]$ to y_{j_1} gives

$$\epsilon_n[y_{j_1}] = n!H_{n_{avg}}^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}},\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n) \tag{32}$$

Example-1

In order to further illustrate the use of the extraction operator, the first second and third order GFRFM of a MISO system denoted as $GFRFM^{(1)}$, $GFRFM^{(2)}$ and $GFRFM^{(3)}$ is considered below.

Consider a MISO system described by the equation

$$a_1\dot{y}_1 + a_2y_1 = b_1u_1 + b_2u_2 + c_1y_1^2 + c_2y_1^3 + c_3y_1u_2^2 \tag{33}$$

For this system

$$\begin{aligned} GFRFM^{(1)} &= [H_1^{(1:1)}(j\omega_1), H_1^{(1:2)}(j\omega_1)] \\ GFRFM^{(2)} &= [H_2^{(1:11)}(j\omega_1, j\omega_2), H_2^{(1:12)}(j\omega_1, j\omega_2), H_2^{(1:22)}(j\omega_1, j\omega_2)] \\ GFRFM^{(3)} &= [H_3^{(1:111)}(j\omega_1, j\omega_2, j\omega_3), H_3^{(1:112)}(j\omega_1, j\omega_2, j\omega_3), \\ &\quad H_3^{(1:122)}(j\omega_1, j\omega_2, j\omega_3), H_3^{(1:222)}(j\omega_1, j\omega_2, j\omega_3)] \end{aligned} \tag{34}$$

The estimation procedure of a few of the kernel transforms will be illustrated before proceeding to analyse a MIMO system.

Computation of $H_1^{(1:1)}(j\omega_1)$

To compute $H_1^{(1:1)}(j\omega_1)$, the 1-tone input $e^{j\omega_1 t}$ is applied at input point-1, and all the other inputs are made equal to zero i.e.

$$u_1(t) = e^{j\omega_1 t} \text{ and } u_2(t) = 0 \tag{35}$$

The output of the system will therefore be expressed as

$$y_1(t) = H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} \tag{36}$$

Substituting the value of $y_1(t)$ and $u_1(t)$ in eqn(33) and equating the coefficients of $e^{j\omega_1 t}$ gives

$$H_1^{(1:1)}(j\omega_1) = \frac{b_1}{a_1(j\omega_1) + a_2} \tag{37}$$

Similarly the computation of $H_1^{(1:2)}(j\omega_1)$ will be carried out by applying the probing signal at input point-2 to give

$$u_2(t) = e^{j\omega_1 t}, \text{ and } u_1(t) = 0 \tag{38}$$

Substituting these in eqn(33) and solving the resulting equation gives

$$H_1^{(1:2)}(j\omega_1) = \frac{b_2}{a_1(j\omega_1) + a_2} \tag{39}$$

Computation of $H_2^{(1:12)}(j\omega_1, j\omega_2)$

Note that the number of inputs or input points to this kernel transform are two. So the 2-tone input $e^{j\omega_1 t} + e^{j\omega_2 t}$ is split such that

$$u_1(t) = e^{j\omega_1 t} \text{ and } u_2(t) = e^{j\omega_2 t} \tag{40}$$

The output of the system under the present input excitation pattern becomes

$$\begin{aligned} y_1(t) &= H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(1:2)}(j\omega_2)e^{j\omega_2 t} + [H_2^{(1:12)}(j\omega_1, j\omega_2) + H_2^{(1:21)}(j\omega_2, j\omega_1)]e^{j(\omega_1+\omega_2)t} \\ &\quad + \text{terms of repetitious combinations of frequencies} \end{aligned}$$

$$= H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(1:2)}(j\omega_2)e^{j\omega_2 t} + 2!H_{2_{avg}}^{(1:12)}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t}$$
$$+\text{terms of repetitious combinations of frequencies} \quad (41)$$

Substituting $y_1(t)$, $u_1(t)$ and $u_2(t)$ in eqn(33) and extracting the coefficients of $e^{j(\omega_1+\omega_2)t}$ yields

$$2!H_{2_{avg}}^{(1:12)}(.) = \frac{c_1[H_1^{(1:1)}(j\omega_1)H_1^{(1:2)}(j\omega_2) + H_1^{(1:2)}(j\omega_2)H_1^{(1:1)}(j\omega_1)]}{a_1(j\omega_1 + j\omega_2) + a_2} \quad (42)$$

Computation of $H_3^{(1:112)}(j\omega_1, j\omega_2, j\omega_3)$

The three tone input is split so that

$$u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \text{ and } u_2(t) = e^{j\omega_3 t} \quad (43)$$

The output of the system under the present input excitation pattern becomes

$$\begin{aligned} y_1(t) &= H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(1:1)}(j\omega_2)e^{j\omega_2 t} + H_1^{(1:2)}(j\omega_3)e^{j\omega_3 t} \\ &+2!H_{2_{avg}}^{(1:11)}(j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + 2!H_{2_{avg}}^{(1:12)}(j\omega_2, j\omega_3)e^{j(\omega_2+\omega_3)t} \\ &+2!H_{2_{avg}}^{(1:12)}(j\omega_1, j\omega_3)e^{j(\omega_1+\omega_3)t} + 3!H_{3_{avg}}^{(1:112)}(j\omega_1, j\omega_2, j\omega_3)e^{j(\omega_1+\omega_2+\omega_3)t} \\ &+\text{terms of repetitious combinations of frequencies} \end{aligned} \quad (44)$$

Substituting the values of $y_1(t)$, $u_1(t)$ and $u_2(t)$ in eqn(33) gives

$$\begin{aligned} 3!H_{3_{avg}}^{(1:112)}(j\omega_1, j\omega_2, j\omega_3) &= c_1 \sum_{\substack{\text{all permutation} \\ [\omega,\beta]}} H_1^{(1:1)}(j\omega_1)H_2^{(1:12)}(j\omega_2, j\omega_3) \\ &+c_2 \sum_{\substack{\text{all permutation} \\ [\omega,\beta]}} H_1^{(1:1)}(j\omega_1)H_1^{(1:1)}(j\omega_2)H_1^{(1:1)}(j\omega_3) \end{aligned} \quad (45)$$

6 Nonlinear Differential Equation Models for Multi Input Multi Output Systems

A wide class of nonlinear systems can be described by nonlinear differential equations. The dynamics of 'j_1-th' subsystem of an r-input m-output MIMO nonlinear system can be represented by

$$\sum_{n=1}^{N_l}\sum_{p=0}^{n}\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{\beta_1=1}^{r}\sum_{\beta_2=\beta_1}^{r}\cdots\sum_{\beta_q=\beta_{q-1}}^{r}$$
$$\sum_{l_1,..l_{p+q}=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\beta_1,..\beta_q}(j_1 : l_1,..l_{p+q})\prod_{i=1}^{p} D^{l_i} y_{\alpha_i}(t)\prod_{i=p+1}^{p+q} D^{l_i} u_{\beta_{i-p}} = 0 \quad (46)$$

where $p+q=n$ and the operator D^{l_i} is defined as

$$D^{l_i}x(t) = \frac{d^{l_i}x(t)}{dt} \quad (47)$$

'L' is the order of maximum differential. The parameter $c_{pq}^{\alpha_1,..\alpha_p,\beta_1,..\beta_q}(j_1 : l_1,..l_{p+q})$ is associated with the term $\prod_{i=1}^{p} D^{l_i} y_{\alpha_i}(t)\prod_{i=p+1}^{p+q} D^{l_i} u_{\beta_{i-p}}$ in the j_1-th subsystem.

For example the equation

$$\begin{aligned} a_1\dot{y}_1 + y_1 + b_1\dot{y}_2 + b_2y_2 + c_1u_1 + d_1y_1^2 + \\ d_2y_1y_2 + d_3y_2^2 + d_4\dot{y}_1\dot{u}_2 + d_5y_2\dot{u}_1 &= 0 \end{aligned} \quad (48)$$

which may describe the first subsystem of a 2-input 2-output system would be represented in the above form as $c_{10}^{1}(1:1) = a_1$ $c_{10}^{1}(1:0) = 1.0$ $c_{10}^{2}(1:1) = b_1$ $c_{10}^{2}(1:0) = b_2$, $c_{01}^{1}(1:0) = c_1$ $c_{20}^{11}(1:00) = d_1$, $c_{20}^{12}(1:00) = d_2$, $c_{20}^{22}(1:00) = d_3$, $c_{11}^{12}(1:11) = d_4$, $c_{11}^{21}(1:01) = d_5$.

Example-2

Prior to deriving a general expression for the frequency domain mapping of differential equation models, consider a simple quadratically nonlinear system and map this into the frequency domain to calculate the linear and second order GFRFM. The equations of the system are

$$\begin{array}{l} \frac{d^2y_1}{dt^2} + 0.20\frac{dy_1}{dt} + 1.1y_1 + 1.4y_2 = u_1 - 0.1y_1^2 - 0.2y_1y_2 - 0.1y_2^2 \\ \frac{d^2y_2}{dt^2} + 0.2\frac{dy_2}{dt} + 4.06y_2 + 0.72y_1 = u_2 - 0.1y_1^2 - 0.2y_1y_2 - 0.1y_2^2 \end{array} \quad (49)$$

The $GFRM^{(1)}$ and $GFRM^{(2)}$ which denote the first and second order GFRFM are

$$GFRFM^{(1)} = \begin{bmatrix} H_1^{(1:1)}(j\omega_1) & H_1^{(1:2)}(j\omega_1) \\ H_1^{(2:1)}(j\omega_1) & H_1^{(2:2)}(j\omega_1) \end{bmatrix} \quad (50)$$

$$\text{GFRFM}^{(2)} = \begin{bmatrix} H_2^{(1:11)}(j\omega_1, j\omega_2) & H_2^{(1:12)}(j\omega_1, j\omega_2) & H_2^{(1:22)}(j\omega_1, j\omega_2) \\ H_2^{(2:11)}(j\omega_1, j\omega_2) & H_2^{(2:12)}(j\omega_1, j\omega_2) & H_2^{(2:22)}(j\omega_1, j\omega_2) \end{bmatrix} \quad (51)$$

Each column of the GFRFM corresponds to a particular configuration of the inputs. To calculate the first column of the $\text{GFRFM}^{(1)}$, apply $u_1(t) = e^{j\omega_1 t}$ and $u_2(t) = 0$ to get the probing expression

$$y_1(t) = H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} \text{ and } y_2(t) = H_1^{(2:1)}(j\omega_1)e^{j\omega_1 t} \quad (52)$$

Substituting these expressions in eqn(49) and equating the coefficients of $e^{j\omega_1 t}$ gives

$$\begin{bmatrix} H_1^{(1:1)}(j\omega_1) \\ H_1^{(2:1)}(j\omega_1) \end{bmatrix} = \begin{bmatrix} (j\omega_1)^2 + 0.2(j\omega_1) + 1.1 & 1.4 \\ 0.72 & (j\omega_1)^2 + 0.2(j\omega_1) + 4.06 \end{bmatrix}^{-1} \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix} \quad (53)$$

Similarly to compute the second column of the transfer function matrix apply $u_2(t) = e^{j\omega_1 t}$ and $u_1(t) = 0$ to get the probing expression

$$y_1(t) = H_1^{(1:2)}(j\omega_1)e^{j\omega_1 t} \text{ and} y_2(t) = H_1^{(2:2)}(j\omega_1)e^{j\omega_1 t} \quad (54)$$

Substituting these expressions in eqn(49) and equating the coefficients of $e^{j\omega_1 t}$ gives

$$\begin{bmatrix} H_1^{(1:1)}(j\omega_1) \\ H_1^{(2:1)}(j\omega_1) \end{bmatrix} = \begin{bmatrix} (j\omega_1)^2 + 0.2(j\omega_1) + 1.1 & 1.4 \\ 0.72 & (j\omega_1)^2 + 0.2(j\omega_1) + 4.06 \end{bmatrix}^{-1} \begin{bmatrix} 0.0 \\ 1.0 \end{bmatrix} \quad (55)$$

Computation of Second Order $\text{GFRFM}^{(2)}$

From the $\text{GFRFM}^{(2)}$, the first and third column are the direct kernel transforms and their calculation is similar to the calculation of the GFRF of a SISO system. To illustrate the procedure, consider the estimation of second column of $\text{GFRFM}^{(2)}$ which consists of $H_2^{(1:12)}(j\omega_1, j\omega_2)$ and $H_2^{(2:12)}(j\omega_1, j\omega_2)$. The two-tone input $e^{j\omega_1 t} + e^{j\omega_2 t}$ is split such that

$$u_1(t) = e^{j\omega_1 t} \text{ and } u_2(t) = e^{j\omega_2 t} \quad (56)$$

The output of the first and second subsystem under the present input excitation pattern becomes

$$
\begin{aligned}
y_1(t) &= H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(1:2)}(j\omega_2)e^{j\omega_2 t} + \underbrace{[H_2^{(1:12)}(j\omega_1, j\omega_2) + H_2^{(1:21)}(j\omega_2, j\omega_1)]}_{2!H_{2avg}^{(1:12)}(j\omega_1, j\omega_2)} e^{j(\omega_1+\omega_2)t} \\
&\quad + \text{terms of repetitious combinations of frequencies} \\
y_2(t) &= H_1^{(2:1)}(j\omega_1)e^{j\omega_1 t} + H_1^{(2:2)}(j\omega_2)e^{j\omega_2 t} + \underbrace{[H_2^{(2:12)}(j\omega_1, j\omega_2) + H_2^{(2:21)}(j\omega_2, j\omega_1)]}_{2!H_{2avg}^{(2:12)}(j\omega_1, j\omega_2)} e^{j(\omega_1+\omega_2)t} \\
&\quad + \text{terms of repetitious combinations of frequencies} \qquad (57)
\end{aligned}
$$

Substituting the values of $y_1(t)$, $y_2(t)$, $u_1(t)$ and $u_2(t)$ in the system equation and extracting the coefficients of $e^{j(\omega_1+\omega_2)t}$ yields

$$
\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} 2!H_{2avg}^{(1:12)}(j\omega_1, j\omega_2) \\ 2!H_{2avg}^{(2:12)}(j\omega_1, j\omega_2) \end{bmatrix} = \begin{bmatrix} n_{11} \\ n_{22} \end{bmatrix} \qquad (58)
$$

where

$$
d_{11} = (j\omega_1 + j\omega_2)^2 + 0.2(j\omega_1 + j\omega_2) + 1.1, \quad d_{12} = 1.4
$$

$$
d_{21} = 0.72 \text{ and } d_{22} = (j\omega_1 + j\omega_2)^2 + 0.2(j\omega_1 + j\omega_2) + 4.06
$$

$$
\begin{aligned}
n_{11} &= -0.1 \sum_{\substack{\text{all permutation} \\ [\omega, \beta]}} H_1^{(1:1)}(j\omega_1)H_1^{(1:2)}(j\omega_2) \\
&\quad -0.2 \sum_{\substack{\text{all permutation} \\ [\omega, \beta]}} H_1^{(2:1)}(j\omega_1)H_1^{(1:2)}(j\omega_2) \\
&\quad -0.1 \sum_{\substack{\text{all permutation} \\ [\omega, \beta]}} H_1^{(2:1)}(j\omega_1)H_1^{(2:2)}(j\omega_2)
\end{aligned} \qquad (59)
$$

$$
\begin{aligned}
n_{22} &= -0.1 \sum_{\substack{\text{all permutation} \\ [\omega, \beta]}} H_1^{(1:1)}(j\omega_1)H_1^{(1:2)}(j\omega_2) \\
&\quad -0.2 \sum_{\substack{\text{all permutation} \\ [\omega, \beta]}} H_1^{(2:1)}(j\omega_1)H_1^{(1:2)}(j\omega_2)
\end{aligned}
$$

$$-0.1 \sum_{\substack{\text{all permutation} \\ [\omega,\beta]}} H_1^{(2:1)}(j\omega_1)H_1^{(2:2)}(j\omega_2) \tag{60}$$

6.1 Frequency Domain Mapping of Nonlinear Differential Equations

Example-2 demonstrates the computational procedure of the Generalised Frequency Response Function Matrix of a simple quadratic nonlinear system. When the order of nonlinearity and dimensions of the system increases, extraction of the elements of the GFRFM using a step by step procedure becomes tedious. It is therefore necessary to find the general expression for the elements of n-th order GFRFM denoted as $GFRFM^{(n)}$. This will be derived in the present section.

Note that the model of eqn(46) consists of various terms which can be divided into three types : pure inputs, pure outputs and input-output cross-product terms. Although applying the ϵ_n operator (Zhang et al,1995) to the system equation will result in an algebraic matrix equation whose solution gives the GFRFM, it will be physically more appealing if the contribution of each type of nonlinearity to the GFRFM can be explicitly expressed independent of the other nonlinear terms. Effects of individual nonlinear terms on the GFRF of a SISO system have been studied by Peyton Jones and Billings(1989), Billings and Peyton Jones(1990), Zhang et al (1995). A similar type of study will be carried out to determine the contribution of different types of nonlinearity to the GFRFM. These will be described in the following remarks.

Remark-2 : Pure Input Nonlinear Terms

While computing the generalised kernel transform $H_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n)$, the effect of applying ϵ_n operator to a pure input nonlinear term denoted as $[U^{N_l}]$ is given by

$$\begin{aligned} \epsilon_n\left[U^{N_l}\right] = & \sum_{\substack{\text{all permutations of} \\ \omega_1,\ldots\omega_{\gamma_1}}} (j\omega_1)^{l_1}\ldots(j\omega_{\gamma_1})^{l_{\gamma_1}} \sum_{\substack{\text{all permutations of} \\ \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2}}} (j\omega_{1+\gamma_1})^{l_{1+\gamma_1}}\ldots(j\omega_{\gamma_1+\gamma_2})^{l_{\gamma_1+\gamma_2}} \\ & \ldots \sum_{\substack{\text{all permutations of} \\ \omega_{1+\gamma_1+\ldots\gamma_{n_d-1}},\ldots\ldots\omega_n}} (j\omega_{1+\gamma_1+\ldots+\gamma_{n_d-1}})^{l_{1+\gamma_1+\ldots+\gamma_{n_d-1}}}\ldots(j\omega_n)^{l_n} \end{aligned} \tag{61}$$

where U^{N_l}; the N_l-th order nonlinear terms of the input consists of

$$n_1\text{-th order factor of } u_{\beta_1},$$
$$n_2\text{-th order factor of } u_{\beta_2},$$
$$\ldots\ldots\ \ldots\ldots\ \ldots\ldots\ \ldots\ldots$$
$$n_{n_d}\text{-th order factor of } u_{\beta_{n_d}}, \text{ such that}$$
$$n_1 + n_2 + \ldots\ldots + n_{n_d} = N_1 \text{ and}$$
$$\max(n_1) \leq \gamma_1$$
$$\max(n_2) \leq \gamma_2$$
$$\max(n_{n_d}) \leq \gamma_{n_d}$$

The terms that constitute the subclass of pure input nonlinear terms $[U^{N_1}]$ are $\prod_{i_1=1}^{\gamma_1} D^{l_{i_1}} u_{\beta_1}(t) \prod_{i_2=1+\gamma_1}^{\gamma_1+\gamma_2} D^{l_{i_2}} u_{\beta_2}(t) \ldots \prod_{i_{n_d}=1+\gamma_1+\ldots+\gamma_{n_d-1}}^{n} D^{l_{i_{n_d}}} u_{\beta_{n_d}}(t)$

The coefficients of these terms are of the form

$$c_{0q}^{\underbrace{\beta_1,\ldots,}_{\gamma_1 \text{ times}}\underbrace{\beta_2,\ldots,}_{\gamma_2 \text{ times}}\ldots\underbrace{\beta_{n_d}\ldots}_{\gamma_{n_d} \text{ times}}}(j_1 : l_1, \ldots l_n)$$

As an example consider the calculation of the kernel transforms $H_3^{(j_1:\beta_1,\beta_1,\beta_2)}(.)$. When this kernel transform is represented in terms of GKERT this gives $\gamma_1 = 2$, $\gamma_2 = 1$ and $n_d = 2$. The terms that will contribute to this kernel transform are the subset of all the nonlinear input terms which are of the form $D^{l_1} u_{\beta_1}(t) D^{l_2} u_{\beta_1}(t) D^{l_3} u_{\beta_2}(t)$ having a coefficient $c_{0,3}^{\beta_1,\beta_1,\beta_2}(j_1 : l_1, l_2, l_3)$

Remark-3: Pure Output Nonlinear Terms

For a SISO system, a pure output nonlinear term of degree p will contribute to the n-th order GFRF when $p \leq n$. Since the number of subsystems of the SISO system is only one, the p-th order output nonlinear term consists of output terms of the form $\prod_{i=1}^{p} D^{l_i} y_1(t)$.

The p-th order factor of pure output nonlinear terms of a MIMO system denoted as Y^p, will contain the terms of the form $\prod_{i=1}^{p} D^{l_i} y_{\alpha_i}(t)$. When all $\alpha_i, i = 1, \ldots p$ are equal Y^p is called a pure inter subsystem nonlinear output term; otherwise it is called an intra-subsystem nonlinear output term. The coefficients of these terms are of the form $c_{p,q}^{\alpha_1,\ldots\alpha_p}(j_1 : l_1, \ldots, l_p)$. Applying the ϵ_n operator to Y^P gives

$$\begin{aligned} \epsilon[Y^p] &= \sum_{\substack{\text{all permutations of} \\ [\omega,\beta]}} H_{n,p}^{\alpha_p,\ldots\ldots\alpha_1}(j\omega_1, \ldots j\omega_n) \quad \text{for } p \leq n \\ &= 0 \qquad \text{for } p > n \end{aligned} \tag{62}$$

where $H_{n,p}^{\alpha_p,\ldots\ldots\alpha_1}(\,.\,)$ denotes the contribution of the p-th order nonlinear output terms of the form $\prod_{i=1}^{p} D^{l_i} y_{\alpha_i}(t)$ to the n-th order nonlinearity. This is estimated recursively as

$$H_{n,p}^{\alpha_p,\alpha_{p-1},\ldots\alpha_1}(.) = \sum_{i=1}^{n-p+1} H_i^{(\alpha_p:\beta_{\sigma_1},\ldots\beta_{\sigma_i})}(j\omega_1,\ldots j\omega_i) H_{n-i,p-1}^{\alpha_{p-1}\ldots\alpha_1}(j\omega_{i+1},..j\omega_n)(j\omega_1+\ldots+j\omega_i)^{l_p}. \quad (63)$$

where β_{σ_1} corresponds to the input point where $e^{j\omega_1 t}$ belongs and β_{σ_i} corresponds to the input point to which $e^{j\omega_i t}$ belongs.

The recursion finishes with $p = 1$ and $H_{n,1}^{\alpha_1}(j\omega_1,\ldots.j\omega_n)$ has the property

$$H_{n,1}^{\alpha_1}(.) = H_n^{(\alpha_1:\beta_{\sigma_1},\ldots\beta_{\sigma_n})}(j\omega_1,\ldots j\omega_n)(j\omega_1+\ldots.+j\omega_n)^{l_1} \quad (64)$$

Remark-4: Input-Output Cross Product Terms

For a SISO system the p-th order factor of the output in conjunction with the q-th order factor of the input contribute to the n-th order GFRF provided $p + q \leq n$. For a MIMO system the p-th order factor of the output belongs to the class Y^p and the q-th order factor of the input belongs to the class U^q which consists of

n_1-th order factor of u_{β_1},

n_2-th order factor of u_{β_2},

....

n_{n_d}-th order factor of $u_{\beta_{n_d}}$, subject to the following constraints

$$n_1 + n_2 + \ldots\ldots + n_{n_d} = q \text{ and}$$
$$\max(n_1) \leq \gamma_1$$
$$\max(n_2) \leq \gamma_2$$
$$\max(n_{n_d}) \leq \gamma_{n_d}$$

The general form of the subclass of the cross-product terms that qualify to contribute to the GKERT are expressed as

$$\begin{aligned} [Y^p U^q] &= \prod_{i=1}^{p} D^{l_i} y_{\alpha_i}(t) \prod_{i=p+1}^{p+n_1} D^{l_i} u_{\beta_1}(t) \prod_{i=p+n_1+1}^{p+n_1+n_2} D^{l_i} u_{\beta_2}(t) \\ &\ldots \prod_{i=p+n_1+n_2+\ldots+n_{n_{d-1}}}^{p+q} D^{l_i} u_{\beta_{n_d}}(t) \end{aligned} \quad (65)$$

The coefficients of these terms are of the form

$$c_{pq}^{\alpha_1,..\alpha_p,\, \underbrace{\beta_1,\ldots}_{n_1 \text{ times}} \underbrace{\beta_2,\ldots}_{n_2 \text{ times}} \underbrace{\beta_{n_d},\ldots}_{n_{n_d} \text{ times}}}(j_1 : l_1, ..l_{p+q})$$

The contribution of each possible $[U^q]$ differs from another. To find the contribution of the general term $[U^q]$ the following notations are introduced.

Let $\Omega_1^{\gamma_1}, \ldots \Omega_{n_1}^{\gamma_1}$ represent the variable of each combination of γ_1 frequencies belonging to the input $u_{\beta_1}(t)$ taken n_1 at a time.

Similarly let $\Omega_1^{\gamma_i}, \ldots \Omega_{n_i}^{\gamma_i}$ represent the variables of each combination of γ_i frequencies belonging to the input $u_{\beta_i}(t)$ taken n_i at a time.

Applying the ϵ_q operator to the $[U^q]$ part of the cross-product term which will extract the coefficients of $e^{j(\Omega_1^{\gamma_1}+\ldots+\Omega_{n_{n_d}}^{\gamma_{n_d}})t}$ gives

$$\begin{aligned}
\epsilon_q[U^q] &= \sum_{\substack{\text{all combinations of } \omega_1,\omega_{\gamma_1} \\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_1},\ldots\Omega_{n_1}^{\gamma_1}}} (j\Omega_1^{\gamma_1})^{l_{p+1}} \ldots (j\Omega_{n_1}^{\gamma_1})^{l_{p+n_1}} \ldots \\
&\quad \ldots \sum_{\substack{\text{all } \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2} \\ \text{taken } n_2 \text{ at a time}}} \sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_2},\ldots\Omega_{n_2}^{\gamma_2}}} (j\Omega_1^{\gamma_2})^{l_{p+n_1+1}} \ldots (j\Omega_{n_2}^{\gamma_2})^{l_{p+n_1+n_2}} \ldots \\
&\quad \ldots \sum_{\substack{\text{all } \omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}},\ldots\omega_n \\ \text{taken } n_{n_d} \text{ at a time}}} \sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_{n_d}},\ldots\Omega_{n_{n_d}}^{\gamma_{n_d}}}} (j\Omega_1^{\gamma_{n_d}})^{l_{p+n_1+\ldots+n_{d-1}+1}} \ldots (j\Omega_{n_d}^{\gamma_{n_d}})^{l_q} \\
&= \sum_{\substack{\text{all } \omega_1,\omega_{\gamma_1} \\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all } \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2} \\ \text{taken } n_2 \text{ at a time}}} \cdots \sum_{\substack{\text{all } \omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}},\ldots\omega_n \\ \text{taken } n_{n_d} \text{ at a time}}} \\
&\quad \cdot \left\{ \left(\sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_1},\ldots\Omega_{n_1}^{\gamma_1}}} (j\Omega_1^{\gamma_1})^{l_{p+1}} \ldots (j\Omega_{n_1}^{\gamma_1})^{l_{p+n_1}} \right) \right. \\
&\quad \left(\sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_2},\ldots\Omega_{n_2}^{\gamma_2}}} (j\Omega_1^{\gamma_2})^{l_{p+n_1+1}} \ldots (j\Omega_{n_2}^{\gamma_2})^{l_{p+n_1+n_2}} \right) \ldots \\
&\quad \left. \left(\sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_{n_d}},\ldots\Omega_{n_{n_d}}^{\gamma_{n_d}}}} (j\Omega_1^{\gamma_{n_d}})^{l_{p+n_1+\ldots+n_{d-1}+1}} \ldots (j\Omega_{n_d}^{\gamma_{n_d}})^{l_q} \right) \right\} \\
&= \sum_{\substack{\text{all } \omega_1,\omega_{\gamma_1} \\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all } \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2} \\ \text{taken } n_2 \text{ at a time}}} \cdots \sum_{\substack{\text{all } \omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}},\ldots\omega_n \\ \text{taken } n_{n_d} \text{ at a time}}} H_{n,qc}^{(U)}(\Omega_q) \qquad (66)
\end{aligned}$$

where $H_{n,qc}^{(U)}(\Omega_q)$ represents the expression inside $\{.\}$ of the above equation(66) and

$$\Omega_q = [\Omega_1^{\gamma_1}, \ldots \Omega_{n_1}^{\gamma_1}, \ldots \Omega_1^{\gamma_{n_d}}, \ldots \Omega_{n_d}^{\gamma_{n_d}}] \qquad (67)$$

$D^{l_1}y_{\alpha_1}(t)D^{l_2}u_{\beta_1}(t)$	$(j\omega_1)^{l_2}(j\omega_2+j\omega_3)^{l_1}\sum_{\text{all perm. }[\omega,\beta]}H_2^{(\alpha_1:\beta_1\beta_2)}(j\omega_2,j\omega_3)$ $+(j\omega_2)^{l_2}(j\omega_1+j\omega_3)^{l_1}\sum_{\text{all perm. }[\omega,\beta]}H_2^{(\alpha_1:\beta:\beta_2)}(j\omega_1,j\omega_3)$
$D^{l_1}y_{\alpha_1}(t)D^{l_2}u_{\beta_2}(t)$	$(j\omega_3)^{l_2}(j\omega_1+j\omega_2)^{l_1}\sum_{\text{all perm. }[\omega,\beta]}H_2^{(\alpha_1:\beta_1\beta_1)}(j\omega_1,j\omega_2)$
$D^{l_1}y_{\alpha_1}(t)D^{l_2}u_{\beta_1}(t)D^{l_3}u_{\beta_1}(t)$	$(\sum_{\substack{\text{all perm.}\\ \omega_1,\omega_2}}(j\omega_1)^{l_2}(j\omega_2)^{l_3}).(j\omega_3)^{l_1}H_1^{(\alpha_1:\beta_2)}(j\omega_3)$
$D^{l_1}y_{\alpha_1}(t)D^{l_2}u_{\beta_1}(t)D^{l_3}u_{\beta_2}(t)$	$(j\omega_1)^{l_2}(j\omega_3)^{l_3}).(j\omega_2)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_2)$ $+(j\omega_2)^{l_2}(j\omega_3)^{l_3}).(j\omega_1)^{l_1}H_1^{(\alpha_1:\beta_1)}(j\omega_1)$
$D^{l_1}y_{\alpha_1}(t)D^{l_2}u_{\beta_2}(t)D^{l_3}u_{\beta_2}(t)$	zero ; since β_2 occurs once in the kernel transform a second order factor of $[u_{\beta_2}]$ will not contribute to the kernel transform
$D^{l_1}y_{\alpha_1}(t)D^{l_2}y_{\alpha_2}(t)D^{l_3}u_{\beta_1}(t)$	$(j\omega_1)^{l_3}\sum_{\substack{\text{all perm.}\\ [\omega,\beta]}}H_{22}^{\alpha_2,\alpha_1}(j\omega_2,j\omega_3)$ $+(j\omega_2)^{l_3}\sum_{\substack{\text{all perm.}\\ [\omega,\beta]}}H_{22}^{\alpha_2,\alpha_1}(j\omega_1,j\omega_3)$
$D^{l_1}y_{\alpha_1}(t)D^{l_2}y_{\alpha_2}(t)D^{l_3}u_{\beta_2}(t)$	$(j\omega_3)^{l_3}\sum_{\substack{\text{all perm.}\\ [\omega,\beta]}}H_{22}^{\alpha_2,\alpha_1}(j\omega_1,j\omega_2)$

Table 1: Contribution of input-output cross product terms to $H_3^{(j_1:\beta_1,\beta_1,\beta_2)}(j\omega_1,j\omega_2,j\omega_3)$

Now applying the extraction operator to $[Y^pU^q]$ gives

$$\epsilon_n[Y^pU^q] = \sum_{\substack{\text{all }\omega_1,\omega_{\gamma_1}\\ \text{taken } n_1 \text{ at a time}}}\ \sum_{\substack{\text{all }\omega_{1+\gamma_1},\dots\omega_{\gamma_1+\gamma_2}\\ \text{taken } n_2 \text{ at a time}}} \cdots \sum_{\substack{\text{all }\omega_{1+\gamma_1+\dots\gamma n_{d-1}},\dots\omega_n\\ \text{taken } n_{n_d} \text{ at a time}}} H_{n,qc}^{(U)}(\Omega_q)\sum_{\text{perm }[\omega,\beta]}H_{n-q,p}^{\alpha_p,\dots\alpha_1}(\Omega_{n-q}) \tag{68}$$

where

$$\begin{aligned}\Omega_{n-q} &= [\omega_1,\omega_2,\dots\omega_n]\bigcap[\Omega_1^{\gamma_1},..\Omega_{n_1}^{\gamma_1},\Omega_1^{\gamma_{n_d}},\dots\Omega_{n_d}^{\gamma_{n_d}}]\\ &= [\omega_1,\omega_2,\dots\omega_n]\bigcap[\Omega_q]\end{aligned} \tag{69}$$

that is Ω_{n-q} is disjoint from the frequencies $[\omega_1,\omega_2,\dots\omega_n]$ and is obtained from the intersection of the two sets given above.

To clarify the notational complexities of the above expressions, the effect of applying the extraction operator to a certain class of input-output cross product terms while computing the kernel transform $H_3^{(j_1:\beta_1,\beta_1,\beta_2)}(j\omega_1,j\omega_2,j\omega_3)$ are illustrated in the Table-1

6.2 Mapping Nonlinear Differential Equation Models of Multi Input Multi Output Systems into the Frequency Domain

The final phase of the present study is to map the full differential equation of the system into the frequency domain. From remarks 2,3 and 4 it is obvious that among all the terms present in eqn(46), only the linear output terms will produce a term $e^{j(\omega_1+\dots+\omega_n)t}$ with $H_n^{(\cdot)}(.)$ appearing as a coefficient. All the other terms will only produce terms with lower order $H_i^{(.)}(.)$, $i < n$, as the coefficients. Also for a valid input-output map to exist it is essential that there must be at least one linear output term present in the system model. That linear output term can belong to any subsystem. Since applying the ϵ_n operator to linear output terms produces a term with coefficient $H_n^{(.)}(.)$, the linear output terms from eqn(46) are brought to the left hand side and all other terms are taken to the right hand side. Thus eqn(46) becomes

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1 : l_1)D^{l_1}y_{\alpha_1}(t)\right]$$
$$=\sum_{n=1}^{N_l}\sum_{p=0}^{n}\sum_{\alpha_1=1}^{m}\sum_{\alpha_2=\alpha_1}^{m}..\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{\beta_1=1}^{r}\sum_{\beta_2=\beta_1}^{r}..\sum_{\beta_q=\beta_{q-1}}^{r}$$
$$\sum_{l_1,.l_{p+q}=0}^{L} C_{pq}^{\alpha_1,..\alpha_p,\beta_1,..\beta_q}(j_1 : l_1, ..l_{p+q})\prod_{i=1}^{p} D^{l_i}y_{\alpha_i}(t)\prod_{i=p+1}^{p+q} D^{l_i}u_{\beta_{i-p}} \tag{70}$$

Applying the ϵ_n operator to eqn(70) gives

$$-\sum_{\alpha_1=1}^{m}\left[\sum_{l_1=0}^{L} c_{10}^{\alpha_1}(j_1 : l_1)(j\omega_1 + \ldots j\omega_n)^{l_1}\right] n!H_{n_{avg}}^{(\alpha_1:\underbrace{\beta_1,\ldots}_{\gamma_1\,times},\underbrace{\beta_2,\ldots}_{\gamma_2\,times}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d}\,times})}(j\omega_1,\ldots\ldots j\omega_n)$$
$$=\sum_{l_1,.l_n=0}^{L} c_{0,q}^{\underbrace{\beta_1,\ldots}_{\gamma_1\,times}\underbrace{\beta_2,\ldots}_{\gamma_2\,times}\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d}\,times}}(j_1 : l_1, ..l_q)\epsilon_n\left[U^n\right]$$
$$+\sum_{p=2}^{n}\sum_{\alpha_1=1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{l_1,.l_p=0}^{L} c_{p,0}^{\alpha_1,..\alpha_p}(j_1 : l_1, ..l_p)\epsilon_n\left[Y^p\right]$$
$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{l_1,.l_{p+q}=0}^{L} c_{pq}^{\alpha_1,..\alpha_p,\underbrace{\beta_1,\ldots}_{n_1\,times}\underbrace{\beta_2,\ldots}_{n_2\,times}\underbrace{\beta_{n_d},\ldots}_{n_{n_d}\,times}}(j_1 : l_1, ..l_{p+q})\epsilon_n\left[Y^pU^q\right] \tag{71}$$

Note that eqn(71) will have m-number of variables $H_{n_{avg}}^{(\alpha_1:\ldots)}(.)$, $\alpha_1 = 1, \ldots m$. Applying the extraction operator to the remaining $m - 1$ subsystems will finally give 'm' number of equations which can be solved to get all the GKERT corresponding to the particular column.

After deriving the general expression for the kernel transform it is simple to calculate the GFRFM of any system by using eqn(71). This is illustrated in the next section.

Computation of $H_2^{(1:11)}(j\omega_1, j\omega_2)$ of Example-1

When the parameters of eqn(33) are represented following the general notation of eqn(46) this gives $c^1_{1,0}(1:0) = a_2$, $c^1_{1,0}(1:1) = a_1$ $c^1_{0,1}(1:0) = -b_1$, $c^2_{0,1}(1:0) = -b_2$, $c^{11}_{2,0}(1:00) = -c_1$, $c^{111}_{3,0}(1:000) = -c_2$, $c^{122}_{1,2}(1:000) = -c_3$.

For this kernel transform $\beta_1 = 1$, $\gamma_1 = 2$ and $m = 1$. Putting these values in eqn(71) gives

$$\begin{aligned} -[a_2(j\omega_1 + j\omega_2) + a_1]\, 2! H^{(1:11)}_{2_{avg}}(j\omega_1, j\omega_2) &= c_1 \sum_{\substack{\text{permutation}\\ [\omega,\beta]}} H^{11}_{22}(j\omega_1, j\omega_2) \\ &= c_1 \sum_{\substack{\text{permutation}\\ [\omega,\beta]}} H^{(1:1)}_1(j\omega_1) H^{(1:1)}_1(j\omega_2) \\ &= c_1[H^{(1:1)}_1(j\omega_1) H^{(1:1)}_1(j\omega_2) + H^{(1:1)}_1(j\omega_2) H^{(1:1)}_1(j\omega_1)] \end{aligned} \quad (72)$$

It is simple to show that for this example $H_2^{(1:22)}(j\omega_1, j\omega_2)$ equals $H_2^{(1:11)}(j\omega_1, j\omega_2)$

Computation of $H_3^{(1:122)}(j\omega_1, j\omega_2, j\omega_3)$ of Example-1

For this kernel transform $\beta_1 = 1$, $\beta_2 = 2$, $\gamma_1 = 1$ and $\gamma_2 = 2$. Putting these values in the general expression of eqn(71) gives

$$\begin{aligned} -[a_2(j\omega_1 + j\omega_2 + j\omega_3) + a_1]\, 3! H^{(1:122)}_{3_{avg}}(j\omega_1, j\omega_2, j\omega_3) &= c_1 \sum_{\substack{\text{permutation}\\ [\omega,\beta]}} H^{11}_{32}(j\omega_1, j\omega_2, j\omega_3) \\ +c_2 \sum_{\substack{\text{permutation}\\ [\omega,\beta]}} H^{111}_{33}(j\omega_1, j\omega_2, j\omega_3) &+ c_3[\sum_{\substack{\text{all permutation}\\ \omega_2,\omega_3}} (j\omega_2)^0 (j\omega_3)^0] H^{11}_{11}(j\omega_1) \end{aligned} \quad (73)$$

Dividing the coefficient of $H^{(1:122)}_{3_{avg}}(j\omega_1, j\omega_2, j\omega_3)$ on the right hand side with the left hand side of eqn(73) gives required result.

7 Representation of Multi Input Multi Output Systems in the Discrete Domain : The MIMO NARX Model

The output of the j_1-th subsystem of an r-input m-output system when represented by a discrete Nonlinear Autoregressive with eXogenous input (NARX) model may be expressed as

$$y_{j_1}(k) = \sum_{n=1}^{N_l} \sum_{p=0}^{n} \sum_{\alpha_1=1}^{m} \sum_{\alpha_2=\alpha_1}^{m} .. \sum_{\alpha_p=\alpha_{p-1}}^{m} \sum_{\beta_1=1}^{r} \sum_{\beta_2=\beta_1}^{r} .. \sum_{\beta_q=\beta_{q-1}}^{r}$$

$$\sum_{k_1,.k_{p+q}=1}^{K} C_{pq}^{\alpha_1,..\alpha_p,\beta_1,..\beta_q}(j_1 : k_1, ..k_{p+q}) \prod_{i=1}^{p} y_{\alpha_i}(k-k_i) \prod_{i=p+1}^{p+q} u_{\beta_{i-p}}(k-k_i) \qquad (74)$$

Note that the NARX model is the subset of the NARMAX (Nonlinear Autoregressive Moving Average with eXogenous inputs) model without the noise or moving average terms. As an example consider a 2-input 2-output system

$$\begin{aligned} y_1(k) &= 0.5y_1(k-1) + u_1(k-2) + 0.7y_2(k-1) + 0.1y_2(k-1)u_1(k-1) \\ y_2(k) &= 0.9y_2(k-2) + u_2(k-1) + 0.3y_1(k-1) + 0.2y_2(k-1)u_2(k-2) \end{aligned} \qquad (75)$$

This would be represented in the general form with the following parameters :
$c^1_{10}(1:1) = 0.5$, $c^2_{10}(1:1) = 0.7$, $c^1_{01}(1:2) = 1.0$, $c^{21}_{11}(1:11) = 0.1$, $c^2_{10}(2:2) = 0.9$, $c^1_{10}(2:1) = 0.3$, $c^2_{01}(2:1) = 1.0$, $c^{22}_{11}(2:12) = 0.2$.

Computation of the GFRFM of the system from the NARX model can be done following a similar procedure adopted for differential equation models. The contribution of pure input,output and cross-product terms to the generalised kernel transforms can be derived analogously. Before deriving the expressions it will be convenient to begin with the mapping of eqn(75) into the frequency domain.

Computation of $H_1^{(1:1)}(j\omega_1)$

The elements of the first and the second order GFRFM of the system denoted as $GFRFM^{(1)}$ and $GFRFM^{(2)}$ are the same as shown in eqns(50) and (51).

To calculate the first column of $GFRFM^{(1)}$, apply $u_1(k) = e^{j\omega_1 t}$ and $u_2(k) = 0$ to get the probing expression

$$y_1(k) = H_1^{(1:1)}(j\omega_1)e^{j\omega_1 t} \text{ and } y_2(k) = H_1^{(2:1)}(j\omega_1)e^{j\omega_1 t} \tag{76}$$

Substitute these expressions in eqn(75) and equate coefficients of $e^{j\omega_1 t}$ to give

$$\begin{bmatrix} H_1^{(1:1)}(j\omega_1) \\ H_1^{(2:1)}(j\omega_1) \end{bmatrix} = \begin{bmatrix} 1-0.5e^{-j\omega_1} & -0.7e^{-j\omega_1} \\ -0.3e^{-j\omega_1} & 1-0.9e^{-j2\omega_1} \end{bmatrix}^{-1} \begin{bmatrix} e^{-j2\omega_1} \\ 0.0 \end{bmatrix} \tag{77}$$

Similarly putting $u_2(k) = e^{j\omega_1 t}$ and $u_1(k) = 0$, the second column of the linear GFRM can then be expressed as

$$\begin{bmatrix} H_1^{(1:2)}(j\omega_1) \\ H_1^{(2:2)}(j\omega_1) \end{bmatrix} = \begin{bmatrix} 1-0.5e^{-j\omega_1} & -0.7e^{-j\omega_1} \\ -0.3e^{-j\omega_1} & 1-0.9e^{-j2\omega_1} \end{bmatrix}^{-1} \begin{bmatrix} 0.0 \\ e^{-j\omega_1} \end{bmatrix} \tag{78}$$

Computation of $H_2^{(1:12)}(j\omega_1, j\omega_2)$

Note that the number of inputs or input points of this kernel transform are two. So the 2-tone input $e^{j\omega_1 t} + e^{j\omega_2 t}$ is split such that

$$u_1(k) = e^{j\omega_1 t} \text{ and } u_2(k) = e^{j\omega_2 t} \tag{79}$$

Following a similar procedure as in Example-2, the expression for the cross-kernel transform becomes

$$\begin{bmatrix} 2!H_{2_{avg}}^{(1:12)}(j\omega_1, j\omega_2) \\ 2!H_{2_{avg}}^{(2:12)}(j\omega_1, j\omega_2) \end{bmatrix} = \begin{bmatrix} 1-0.5e^{-j(\omega_1+\omega_2)} & -0.7e^{-j(\omega_1+\omega_2)} \\ -0.3e^{-j(\omega_1+\omega_2)} & 1-0.9e^{-j2(\omega_1+\omega_2)} \end{bmatrix}^{-1} \begin{bmatrix} 0.1e^{-j\omega_2}H_1^{(2:2)}(j\omega_2)e^{-j\omega_1} \\ 0.2e^{-j\omega_1}H_1^{(2:1)}(j\omega_1)e^{-j2\omega_2} \end{bmatrix} \tag{80}$$

8 Frequency Domain Mapping of Nonlinear Discrete Time Systems

The model of eqn(74) can be divided into pure input,pure output and input-output cross product terms. The contribution of each of these terms can be found by applying the ϵ_n operator. These are given in the following remarks.

Remark-5 : Pure Input Nonlinear Terms

While computing the generalised kernel transform $H_n^{(j_1:\underbrace{\beta_1,\ldots}_{\gamma_1 \text{times}},\underbrace{\beta_2,\ldots}_{\gamma_2 \text{times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{times}})}(j\omega_1,\ldots\ldots j\omega_n)$, the effect of applying the ϵ_n operator to a pure input nonlinear term denoted as $[U^{N_l}]$ is given by

$$\epsilon_n\left[U^{N_l}\right] = \sum_{\substack{\text{all permutations of} \\ \omega_1,\ldots\omega_{\gamma_1}}} e^{-j(\omega_1 k_1+\ldots+\omega_{\gamma_1}k_{\gamma_1})} \sum_{\substack{\text{all permutations of} \\ \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2}}} e^{-j(\omega_{1+\gamma_1}k_{1+\gamma_1}+\ldots+\omega_{\gamma_1+\gamma_2}k_{\gamma_1+\gamma_2})}$$
$$\ldots \sum_{\substack{\text{all permutations of} \\ \omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}},\ldots\omega_n}} e^{-j(\omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}}k_{1+\gamma_1+\ldots\ldots+\gamma_{n_{d-1}}}+\ldots+\omega_n k_n)} \quad (81)$$

The above expression is true for $N_l = n$ else $\epsilon_n\left[U^{N_l}\right] = 0$ for $N_l \neq n$.

Compare eqn(81) with eqn(61); roughly the difference is that the terms of the form $(j\omega_i)^{l_i}$ have been replaced by terms of the form $e^{-j\omega_i k_i}$. Eqn(81) may be called the discrete equivalent of eqn(61).

Remark-6: Pure Output Nonlinear Terms

The p-th order factor of pure output nonlinear terms will be of the form $\prod_{i=1}^{p} y_{\alpha_i}(k-k_i)$ and will be denoted as Y^p. Applying the ϵ_n operator to Y^P gives

$$\begin{aligned}\epsilon[Y^p] &= \sum_{\text{all permutations of}\,\omega_1,\ldots\omega_n} H_{n,p}^{\alpha_p,\ldots\ldots\alpha_1}(j\omega_1,\ldots j\omega_n) \quad \text{for} p \leq n \\ &= 0 \qquad \text{for } p > n \end{aligned} \quad (82)$$

where $H_{n,p}^{\alpha_p,\ldots\ldots\alpha_1}(\,.\,)$ denotes the contribution of p-th order nonlinear output terms of the form $\prod_{i=1}^{p} y_{\alpha_i}(k-k_i)$ to the n-th order nonlinearity. This is estimated recursively as

$$H_{n,p}^{\alpha_p,\alpha_{p-1},\ldots\alpha_1}(.) = \sum_{i=1}^{n-p+1} H_i^{(\alpha_p:\beta_{\sigma_1},\ldots\beta_{\sigma_i})}(j\omega_1,\ldots j\omega_i) H_{n-i,p-1}^{\alpha_{p-1}\ldots\alpha_1}(j\omega_{i+1},\ldots j\omega_n)e^{-j(\omega_1+\ldots+\omega_i)k_p}. \quad (83)$$

where β_{σ_1} corresponds to the input point where $e^{j\omega_1 t}$ belongs and so on.

The recursion finishes with $p = 1$ and $H_{n,1}^{\alpha_1}(j\omega_1,\ldots j\omega_n)$ has the property

$$H_{n,1}^{\alpha_1} = H_n^{(\alpha_1:\beta_{\sigma_1},\ldots\beta_{\sigma_i})}(j\omega_1,\ldots j\omega_n)e^{-j(\omega_1+\ldots+\omega_n)k_1}. \quad (84)$$

Remark-7 : Input-Output Cross Product Terms

For a MIMO system the p-th order factor of the output belongs to the class Y^p and the q-th order factor of the input belongs to class U^q which consists of

n_1-th order factor of u_{β_1},

n_2-th order factor of u_{β_2},

....

n_{n_d}-th order factor of $u_{\beta_{n_d}}$, subject to the following constraints

$$n_1 + n_2 + \ldots\ldots + n_{n_d} = q \text{ and}$$
$$\max(n_1) \leq \gamma_1$$
$$\max(n_2) \leq \gamma_2$$
$$\max(n_{n_d}) \leq \gamma_{n_d}$$

The general form of the subclass of the cross-product terms that qualify to contribute to the GKERT is expressed as

$$\begin{aligned}[][Y^pU^q] &= \prod_{i=1}^{p} y_{\alpha_i}(k-k_i) \prod_{i=p+1}^{p+n_1} u_{\beta_1}(k-k_i) \prod_{i=p+n_1+1}^{p+n_1+n_2} u_{\beta_2}(k-k_i) \\ &\quad \ldots \prod_{i=p+n_1+n_2+\ldots+n_{n_{d-1}}}^{p+q} u_{\beta_{n_d}}(k-k_i) \end{aligned} \tag{85}$$

The coefficients of these terms are of the form

$$c_{pq}^{\alpha_1,..\alpha_p,\underbrace{\beta_1,\ldots}_{n_1 \text{ times}}\underbrace{\beta_2,\ldots}_{n_2 \text{ times}}\underbrace{\beta_{n_d},\ldots}_{n_{n_d} \text{ times}}}(j_1 : k_1, ..k_{p+q})$$

Applying the ϵ_q operator to the $[U^q]$ part of the cross product term, gives

$$\begin{aligned} \epsilon_q[U^q] &= \sum_{\substack{\text{all combinations of } \omega_1,\omega_{\gamma_1} \\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_1},..\Omega_{n_1}^{\gamma_1}}} e^{-j(\Omega_1^{\gamma_1}k_{p+1}+\ldots+\Omega_{n_1}^{\gamma_1}k_{p+n_1})} \ldots \\ &\quad \ldots \sum_{\substack{\text{all } \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2} \\ \text{taken } n_2 \text{ at a time}}} \sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_2},..\Omega_{n_2}^{\gamma_2}}} e^{-j(\Omega_1^{\gamma_2}k_{p+n_1+1}+\ldots+\Omega_{n_2}^{\gamma_2}k_{p+n_1+n_2})} \ldots \\ &\quad \ldots \sum_{\substack{\text{all } \omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}},\ldots\omega_n \\ \text{taken } n_{n_d} \text{ at a time}}} \sum_{\substack{\text{all perm. of} \\ \Omega_1^{\gamma_{n_d}},..\Omega_{n_{n_d}}^{\gamma_{n_d}}}} e^{-j(\Omega_1^{\gamma_{n_d}}k_{p+n_1+\ldots+n_{d-1}+1}+\ldots+\Omega_{n_d}^{\gamma_{n_d}}k_q)} \\ &= \sum_{\substack{\text{all } \omega_1,\omega_{\gamma_1} \\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all } \omega_{1+\gamma_1},\ldots\omega_{\gamma_1+\gamma_2} \\ \text{taken } n_2 \text{ at a time}}} \cdots \sum_{\substack{\text{all } \omega_{1+\gamma_1+\ldots\gamma_{n_{d-1}}},\ldots\omega_n \\ \text{taken } n_{n_d} \text{ at a time}}} \end{aligned}$$

$$\left\{\left(\sum_{\substack{\text{all perm. of}\\ \Omega_1^{\gamma_1},..\Omega_{n_1}^{\gamma_1}}} e^{-j(\Omega_1^{\gamma_1}k_{p+1}+...+\Omega_{n_1}^{\gamma_1}k_{p+n_1})}\right)\right.$$

$$\left(\sum_{\substack{\text{all perm. of}\\ \Omega_1^{\gamma_2},..\Omega_{n_2}^{\gamma_2}}} e^{-j(\Omega_1^{\gamma_2}k_{p+n_1+1}+...+\Omega_{n_2}^{\gamma_2}k_{p+n_1+n_2})}\right)\dots$$

$$\left.\left(\sum_{\substack{\text{all perm. of}\\ \Omega_1^{\gamma_{n_d}},..\Omega_{n_{n_d}}^{\gamma_{n_d}}}} e^{-j(\Omega_1^{\gamma_{n_d}}k_{p+n_1+...+n_{d-1}+1}+...+\Omega_{n_d}^{\gamma_{n_d}}k_q)}\right)\right\}$$

$$= \sum_{\substack{\text{all } \omega_1,..\omega_{\gamma_1}\\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all } \omega_{1+\gamma_1},..\omega_{\gamma_1+\gamma_2}\\ \text{taken } n_2 \text{ at a time}}} \dots \sum_{\substack{\text{all } \omega_{1+\gamma_1+...+\gamma_{n_{d-1}}},..\omega_n\\ \text{taken } n_{n_d} \text{ at a time}}} H_{n,qd}^{(U)}(\Omega_q) \tag{86}$$

where $H_{n,qd}^{(U)}(\Omega_q)$ represents the expression inside $\{.\}$ of the above equation(86) and

$$\Omega_q = [\Omega_1^{\gamma_1},\dots\Omega_{n_1}^{\gamma_1},\dots\Omega_1^{\gamma_{n_d}},\dots\Omega_{n_d}^{\gamma_{n_d}}] \tag{87}$$

Now applying the extraction operator to $[Y^pU^q]$ gives

$$\epsilon_n[Y^pU^q] = \sum_{\substack{\text{all } \omega_1,..\omega_{\gamma_1}\\ \text{taken } n_1 \text{ at a time}}} \sum_{\substack{\text{all } \omega_{1+\gamma_1},..\omega_{\gamma_1+\gamma_2}\\ \text{taken } n_2 \text{ at a time}}} \dots \sum_{\substack{\text{all } \omega_{1+\gamma_1+...+\gamma_{n_{d-1}}},..\omega_n\\ \text{taken } n_{n_d} \text{ at a time}}} H_{n,qd}^{(U)}(\Omega_q) \sum_{\text{perm }[\omega,\beta]} H_{n-q,p}^{\alpha_p,..\alpha_1}(\Omega_{n-q}) \tag{88}$$

where

$$\begin{aligned}\Omega_{n-q} &= [\omega_1,\omega_2,\dots\omega_n]\bigcap[\Omega_1^{\gamma_1},..\Omega_{n_1}^{\gamma_1},\Omega_1^{\gamma_{n_d}},\dots\Omega_{n_d}^{\gamma_{n_d}}]\\ &= [\omega_1,\omega_2,\dots\omega_n]\bigcap[\Omega_q]\end{aligned} \tag{89}$$

that is Ω_{n-q} is disjoint from the frequencies $[\omega_1,\omega_2,\dots\omega_n]$ and is obtained from the intersection of the two sets given above.

8.1 Mapping the NARX Model into the Frequency Domain

Following identical arguments to the differential equation mapping, the expression for the frequency domain equivalent of eqn(74) will be given as

$$\left[1-\sum_{k_1=1}^{K} c_{10}^{j_1}(j_1:k_1)e^{-j(\omega_1+\ldots+\omega_n)k_1}\right] n!H_{n_{avg}}^{(j_1:\underbrace{\beta_1,\ldots,}_{\gamma_1 \text{ times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{ times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{ times}})}(j\omega_1,\ldots\ldots j\omega_n)$$

$$-\sum_{\substack{\alpha_1=1\\ \alpha_1\neq j_1}}^{m}\left[\sum_{k_1=1}^{K} c_{10}^{j_1}(j_1:k_1)e^{-j(\omega_1+\ldots+\omega_n)k_1}\right] n!H_{n_{avg}}^{(j_1:\underbrace{\beta_1,\ldots,}_{\gamma_1 \text{ times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{ times}}\ldots\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{ times}})}(j\omega_1,\ldots\ldots j\omega_n)$$

$$=\sum_{k_1,..k_n=1}^{K} c_{0,n}^{\underbrace{\beta_1,\ldots}_{\gamma_1 \text{ times}}\underbrace{\beta_2,\ldots}_{\gamma_2 \text{ times}}\underbrace{\beta_{n_d},\ldots}_{\gamma_{n_d} \text{ times}}}(j_1:k_1,..k_n)\epsilon_n\left[U^n\right]$$

$$+\sum_{p=2}^{n}\sum_{\alpha_1=1}^{m}\cdots\sum_{\alpha_p=\alpha_{p-1}}^{m}\sum_{k_1,k_p=1}^{K} c_{p,0}^{\alpha_1,..\alpha_p}(j_1:k_1,..k_p)\epsilon_n\left[Y^p\right]$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K} c_{pq}^{\alpha_1,..\alpha_p,\underbrace{\beta_1,\ldots}_{n_1 \text{ times}}\underbrace{\beta_2,\ldots}_{n_2 \text{ times}}\underbrace{\beta_{n_d},\ldots}_{n_{n_d} \text{ times}}}(j_1:k_1,..k_{p+q})\epsilon_n\left[Y^pU^q\right] \tag{90}$$

Example

After getting the general expression rederive the expression for $H_2^{(1:12)}(j\omega_1, j\omega_2)$ and $H_2^{(2:12)}(j\omega_1, j\omega_2)$. For these kernel transforms $\beta_1 = 1$, $\beta_2 = 1$, $\gamma_1 = 1$,$\gamma_2 = 1$,$n_d = 2$, and $n = 2$. Substituting these values in eqn(90) gives

$$(1-0.5e^{-j(\omega_1+\omega_2)})2!H_{2_{avg}}^{(1:12)}(j\omega_1,j\omega_2)-(0.7e^{-j(\omega_1+\omega_2)})2!H_{2_{avg}}^{(2:12)}(j\omega_1,j\omega_2)$$
$$=0.1e^{-j\omega_2}H_1^{(2:2)}(j\omega_2)e^{-j\omega_1}$$
$$(1-0.9e^{-j2(\omega_1+\omega_2)})2!H_{2_{avg}}^{(2:12)}(j\omega_1,j\omega_2)-(0.3e^{-j(\omega_1+\omega_2)})2!H_{2_{avg}}^{(1:12)}(j\omega_1,j\omega_2)$$
$$=0.2e^{-j\omega_1}H_1^{(2:1)}(j\omega_1)e^{-j2\omega_2} \tag{91}$$

Solution of this equation gives an identical result to eqn(80).

9 Conclusions

Analytical expressions for the generalised frequency response function matrix for multi-input multi-output nonlinear systems have been derived for both continuous time nonlinear differential equation models and polynomial NARX models. The expression provides a great deal of insight into the relationship between the time and frequency domain representations of nonlinear MIMO systems and can be used to to study the sensitivity of the frequency domain effects due to parameter variations in the models.

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