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UNFITTED FINITE ELEMENT METHODS USING BULK MESHES FOR SURFACE PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. In this paper, we define new unfitted finite element methods for numerically approximating the solution of surface partial differential equations using bulk finite elements. The key idea is that the n-dimensional hypersurface, $\Gamma \subset \mathbb{R}^{n+1}$, is embedded in a polyhedral domain in $\mathbb{R}^{n+1}$ consisting of a union, $\mathscr{T}_h$, of $(n+1)$-simplices. The unifying feature of the methodological approach is that the finite element approximating space is based on continuous piecewise linear finite element functions on the bulk triangulation $\mathscr{T}_h$ which is independent of Γ. Our first method is a sharp interface method (*SIF*) which uses the bulk finite element space in an approximating weak formulation obtained from integration on a polygonal approximation, Γ_h, of Γ. The full gradient is used rather than the projected tangential gradient and it is this which distinguishes *SIF* from the method of [M. A. Olshanskii, A. Reusken, and J. Grande, *SIAM J. Numer. Anal.*, 47 (2009), pp. 3339–3358]. The second method is a narrow band method (*NBM*) in which the region of integration is a narrow band of width $O(h)$. *NBM* is similar to the method of [K. Deckelnick et al., *IMA J. Numer. Anal.*, 30 (2010), pp. 351–376] but again the full gradient is used in the discrete weak formulation. The a priori error analysis in this paper shows that the methods are of optimal order in the surface L^2 and H^1 norms and have the advantage that the normal derivative of the discrete solution is small and converges to zero. Our third method combines bulk finite elements, discrete sharp interfaces, and narrow bands in order to give an unfitted finite element method for parabolic equations on evolving surfaces. We show that our method is conservative so that it preserves mass in the case of an advection-diffusion conservation law. Numerical results are given which illustrate the rates of convergence.

Key words. unfitted finite elements, cut cells, error analysis, narrow band, sharp interface, elliptic and parabolic surface equations

AMS subject classifications. 35R01, 65N30, 65N15, 65M60

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1. Introduction.

In this article we propose and analyze numerical methods based on bulk finite element meshes for the following model elliptic equation on a stationary surface.

Model elliptic equation on stationary surface. Let Γ be a smooth hypersurface in $\mathbb{R}^{n+1}$ and $f \in L^2(\Gamma)$. We seek solutions $u\colon \Gamma \to \mathbb{R}$ of

$$-\Delta_\Gamma u + u = f \quad \text{on } \Gamma. \tag{1.1}$$

The methods can be extended in natural ways to deal with variable coefficients and nonlinearities. The approach may be extended to the following advection-diffusion equation on a moving surface.

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Model parabolic equation on evolving surface. Let $\{\Gamma(t)\}$ be a family of smooth hypersurfaces in $\mathbb{R}^{n+1}$ for $t \in [0,T]$. Denoting by $\partial^\bullet u$ the material derivative of u and v the velocity of $\Gamma(t)$ (see section 5 for notation), we seek solutions $u\colon \bigcup_t \Gamma(t) \times \{t\}$ of the advection-diffusion equation

$$\partial^\bullet u + u \nabla_\Gamma \cdot v - \Delta_\Gamma u = f \quad \text{on} \bigcup_{t \in (0,T)} \Gamma(t) \times \{t\}, \tag{1.2a}$$

$$u(\cdot, 0) = u_0 \quad \text{on } \Gamma(0). \tag{1.2b}$$

Surface partial differential equations or partial differential equations (PDEs) on manifolds arise in a wide variety of applications in materials science, fluid dynamics, and biology, [5, 32, 24, 29, 25, 2, 28, 26]. Computational approaches include *surface finite elements on triangulated surfaces* [17, 19, 18, 15, 14, 23, 22], *bulk finite element or finite difference meshes for the approximation of implicit surface formulations* [9, 30, 10, 21, 11], *bulk finite element or finite difference meshes on narrow bands* [12, 38], and *bulk finite element meshes and sharp interface weak forms* [34, 33, 16].

An important feature of the methods cited above is the avoidance of *charts* both in the problem formulation and the numerical methods. For example, the surface finite element method is based simply on triangulated surfaces and requires the geometry solely through the knowledge of the vertices of the triangulation, whereas methods based on implicit surfaces require only the level set function Φ which encodes all the necessary geometry. Another feature of some of these methods is the use of unfitted bulk meshes. Here we use the terminology *unfitted finite element methods* (sometimes called cut cell methods) when the underlying meshes that form the computational domain are not fitted to the domain in which the PDE holds. The motivation for using finite element spaces on meshes not fitting to the domain came from the desire to solve free or moving boundary problems. Such methods were introduced in [3, 4] for elliptic equations in curved domains; see also [31, 8, 27]. In this setting we are concerned with bulk meshes independent of the surface.

The new methods. The new unfitted finite element methods for surface elliptic equations proposed in this paper are variants of the bulk finite element approaches using a sharp interface or a narrow band. The new scheme for advection diffusion on an evolving surface is a hybrid of these. In the following we sketch the main ideas of these methods describing the details in sections 3–5.

Sharp interface method (*SIF*). Given an interpolation Γ_h of Γ, we use a bulk finite element space V_h^I of the form

$$V_h^I = \{\phi_h \in C^0(U_h^I) \,|\, \phi_{h|T} \in P_1(T) \text{ for each } T \in \mathscr{T}_h^I\},$$

where $\mathscr{T}_h^I$ is a set of elements which intersect Γ_h and $U_h^I = \bigcup_{T \in \mathscr{T}_h^I} T$; see section 3. The discrete scheme approximating the model elliptic equation (1.1) is find $u_h \in V_h^I$ such that

$$\int_{\Gamma_h} \big(\nabla u_h \cdot \nabla \phi_h + u_h \phi_h\big) \, \mathrm{d}\sigma_h = \int_{\Gamma_h} f^e \phi_h \, \mathrm{d}\sigma_h \quad \text{for all } \phi_h \in V_h^I, \tag{1.3}$$

where f^e is an extension of f. The method is related to the following method of Olshanskii, Reusken, and Grande, introduced in [34]: find $u_h \in V_h^\Gamma$ such that

$$\int_{\Gamma_h} \big(\nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \phi_h + u_h \phi_h\big) \, \mathrm{d}\sigma_h = \int_{\Gamma_h} f^e \phi_h \, \mathrm{d}\sigma_h \quad \text{for all } \phi_h \in V_h^\Gamma. \tag{1.4}$$

There are two significant differences. Note the use of the full gradient in (1.3) as opposed to the tangential gradient. This gives control over the normal derivative of the finite element solution which is lacking in (1.4). Another difference relates to the use of the finite element space V_h^Γ, which essentially consists of the traces on Γ_h of elements in V_h^I. While V_h^I has a natural basis, this does not seem to be the case for V_h^Γ. The "standard basis" of finite element hat functions is only a spanning set for V_h^Γ.

Narrow band method (*NBM*). We use the bulk finite element space V_h^B on the triangulation $\mathscr{T}_h^B$,

$$V_h^B = \{\phi_h \in C^0(U_h^B) \,|\, \phi_{h|T} \in P_1(T) \text{ for each } T \in \mathscr{T}_h^B\}.$$

Here $\mathscr{T}_h^B$ consists of those triangles intersecting a narrow band domain D_h defined by the $\pm h$ level sets of an interpolated level set function $I_h\Phi$ and $U_h^B = \bigcup_{T\in\mathscr{T}_h^B} T$. The discrete scheme approximating the model elliptic equation (1.1) is find $u_h \in V_h^B$ such that

$$\int_{D_h} \big(\nabla u_h \cdot \nabla \phi_h + u_h\phi_h\big)\,|\nabla I_h\Phi|\,\mathrm{d}x = \int_{D_h} f^e\phi_h\,|\nabla I_h\Phi|\,\mathrm{d}x \quad \text{for all } \phi_h \in V_h^B. \tag{1.5}$$

This is similar to the method in [12] except that *NBM* uses the full instead of projected gradients thus avoiding the resulting degeneracy. As a result we are able to prove an optimal L^2-error bound which was not obtained for the method in [12]. It is also the case that the normal derivative of the discrete solution converges to zero.

Hybrid unfitted evolving surface method. The discrete problem approximating (1.2) is given $u_h^m \in V_h^m, m = 0,\dots,N-1$, find $u_h^{m+1} \in V_h^{m+1}$ such that

$$\begin{aligned} &\int_{\Gamma_h^{m+1}} u_h^{m+1}\phi_h\,\mathrm{d}\sigma_h - \int_{\Gamma_h^m} u_h^m\phi_h(\cdot + \tau_m v^{e,m+1})\,\mathrm{d}\sigma_h \\ &\quad + \frac{\tau_m}{2h}\int_{D_h^{m+1}} \nabla u_h^{m+1}\cdot\nabla\phi_h\,\left|\nabla I_h\Phi^{m+1}\right|\,\mathrm{d}x = \tau_m\int_{\Gamma_h^{m+1}} f^{e,m+1}\phi_h\,\mathrm{d}\sigma_h \end{aligned} \tag{1.6}$$

for all $\phi_h \in V_h^{m+1}$. Here $v^{e,m}$ denotes an extension of the surface velocity at time level m. We use time step labeled analogues of the notation for NBM; see section 5 for the details. Here, u_h^0 is appropriate initial data. Because of this combination of narrow band and sharp interface discretization, under some mild constraints on the discretization parameters (see section 5) our numerical scheme preserves the important property that solutions of (1.2) conserve mass in the case that $f \equiv 0$.

Outline. The paper is organized as follows: In section 2 we introduce our notation and collect some auxiliary results. In sections 3 and 4 we present and analyze unfitted methods for the model elliptic equation (1.1). In section 5 we describe how a combination of these two approaches can be used to calculate solutions of the advection-diffusion equation on evolving hypersurfaces, (1.2). Details of the implementation and several numerical examples illustrating the orders of convergence are presented in section 6.

2. Preliminaries.

2.1. Surface calculus. Let Γ be a connected compact smooth hypersurface embedded in $\mathbb{R}^{n+1}$ $(n = 1, 2)$. We assume that there exists a smooth function $\Phi : U \to \mathbb{R}$ such that

$$\Gamma = \{x \in U \mid \Phi(x) = 0\} \quad \text{and} \quad \nabla\Phi(x) \neq 0, x \in U,$$

where U is an open neighborhood of Γ. For a function $z : \Gamma \to \mathbb{R}$ we define its tangential gradient by

$$\nabla_\Gamma z(p) := \nabla\widetilde{z}(p) - \big(\nabla\widetilde{z}(p) \cdot \nu(p)\big)\nu(p), \quad p \in \Gamma, \tag{2.1}$$

where $\widetilde{z} : U \to \mathbb{R}$ is an arbitrary smooth extension of z to U and

$$\nu(x) = \frac{\nabla\Phi(x)}{|\nabla\Phi(x)|}$$

is a unit vector to the level sets of Φ. It can be shown that $\nabla_\Gamma z(p)$ is independent of the particular choice of $\widetilde{z}$. We denote by $\underline{D}_i z, 1 \leq i \leq n+1$, the components of $\nabla_\Gamma z$. Furthermore, we let

$$\Delta_\Gamma z = \nabla_\Gamma \cdot \nabla_\Gamma z = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i z$$

be the Laplace–Beltrami operator of z.

In what follows it will be convenient to use special coordinates which are adapted to Φ. Consider for $p \in \Gamma$ the system of ODEs

$$\gamma_p'(s) = \frac{\nabla\Phi(\gamma_p(s))}{|\nabla\Phi(\gamma_p(s))|^2}, \quad \gamma_p(0) = p. \tag{2.2}$$

It can be shown that there exists $\delta > 0$ so that the solution γ_p of (2.2) exists uniquely on $(-\delta, \delta)$ uniformly in $p \in \Gamma$, so that we can define the mapping $F : \Gamma \times (-\delta, \delta) \to \mathbb{R}^{n+1}$ by

$$F(p, s) := \gamma_p(s), \quad p \in \Gamma, |s| < \delta. \tag{2.3}$$

Since $\frac{d}{ds}\Phi(\gamma_p(s)) = 1$ and $\gamma_p(0) = p \in \Gamma$, we infer that $\Phi(\gamma_p(s)) = s, |s| < \delta$, and hence that $x = F(p, s)$ implies that $|\Phi(x)| < \delta$. As a result, we deduce that F is a diffeomorphism of $\Gamma \times (-\delta, \delta)$ onto $U_\delta := \{x \in U \mid |\Phi(x)| < \delta\}$ with inverse

$$F^{-1}(x) = (p(x), \Phi(x)), \quad x \in U_\delta, \tag{2.4}$$

where $p : U_\delta \to \mathbb{R}^{n+1}$ satisfies $p(x) \in \Gamma, x \in U_\delta$. For later purposes it is convenient to expand p and its derivatives in terms of Φ. Let us fix $x \in U_\delta$ and define the function

$$\eta(\tau) := F(p(x), (1-\tau)\Phi(x)), \tau \in [0, 1].$$

Since $\frac{\partial F}{\partial s}(p, s) = \gamma_p'(s)$ we have

$$\eta'(\tau) = -\Phi(x)\gamma_{p(x)}'((1-\tau)\Phi(x)) = -\Phi(x)\frac{\nabla\Phi(\gamma_{p(x)}((1-\tau)\Phi(x)))}{\big|\nabla\Phi(\gamma_{p(x)}((1-\tau)\Phi(x)))\big|^2}.$$

Observing that $\gamma_{p(x)}(\Phi(x)) = F(p(x),\Phi(x)) = x$ and using similar arguments to calculate $\eta''(\tau)$ we find that

$$\eta_k'(0) = -\Phi(x)\frac{\Phi_{x_k}(x)}{|\nabla\Phi(x)|^2},$$
$$\eta_k''(0) = \Phi(x)^2\sum_{l,r=1}^{n+1}\left(\delta_{kr} - \frac{2\Phi_{x_k}(x)\Phi_{x_r}(x)}{|\nabla\Phi(x)|^2}\right)\frac{\Phi_{x_l}(x)\Phi_{x_l x_r}(x)}{|\nabla\Phi(x)|^4},$$

$k = 1,\dots,n+1$. Since $\eta(1) = p(x), \eta(0) = x$ we deduce with the help of Taylor's theorem that

(2.5)

$$p_k(x) = x_k - \Phi(x)\frac{\Phi_{x_k}(x)}{|\nabla\Phi(x)|^2} + \frac{1}{2}\Phi(x)^2\sum_{l,r=1}^{n+1}\left(\delta_{kr} - \frac{2\Phi_{x_k}(x)\Phi_{x_r}(x)}{|\nabla\Phi(x)|^2}\right)\frac{\Phi_{x_l}(x)\Phi_{x_l x_r}(x)}{|\nabla\Phi(x)|^4}$$
$$+ \Phi(x)^3 r_k(x), \quad k = 1,\dots,n+1,$$

where r_k are smooth functions. In a similar way we may write

$$\nabla\Phi(x) = \nabla\Phi(p(x)) + \Phi(x)G(x), \tag{2.6}$$

where $G(x) = \int_0^1 D^2\Phi(F(p(x),\tau\Phi(x)))\frac{\partial F}{\partial s}(p(x),\tau\Phi(x))\,\mathrm{d}\tau$.

Let us next use the function p in order to define a particular extension of $z\colon\Gamma\to\mathbb{R}$:

$$z^e(x) := z(p(x)), \quad x\in U_\delta. \tag{2.7}$$

Since $p(F(p(x),s)) = p(x)$ we deduce that $s\mapsto z^e(F(p(x),s))$ is independent of s and thus

$$\nabla z^e(x)\cdot\nu(x) = 0, \quad x\in U_\delta. \tag{2.8}$$

In order to express the derivatives of z^e in terms of the tangential derivatives of z we first deduce from (2.5) that

$$p_{k,x_i}(x) = \delta_{ik} - \frac{\Phi_{x_k}(x)\Phi_{x_i}(x)}{|\nabla\Phi(x)|^2} - \frac{\Phi(x)\Phi_{x_k x_i}(x)}{|\nabla\Phi(x)|^2} + 2\Phi(x)\Phi_{x_k}(x)\sum_{l=1}^{n+1}\frac{\Phi_{x_l}(x)\Phi_{x_l x_i}(x)}{|\nabla\Phi(x)|^4}$$
$$+ \Phi(x)\Phi_{x_i}(x)\sum_{l,r=1}^{n+1}\left(\delta_{kr} - \frac{2\Phi_{x_k}(x)\Phi_{x_r}(x)}{|\nabla\Phi(x)|^2}\right)\frac{\Phi_{x_l}(x)\Phi_{x_l x_r}(x)}{|\nabla\Phi(x)|^4} + \Phi(x)^2\alpha_k^i(x).$$

Combining this relation with (2.6) we deduce that

(2.9)

$$p_{k,x_i}(x) = \delta_{ik} - \nu_i(p(x))\nu_k(p(x)) + a_{ik}(x)\Phi(x),$$

$$p_{k,x_i x_j}(x) = -\frac{\Phi_{x_i}(x)\Phi_{x_k x_j}(x)}{|\nabla\Phi(x)|^2} - \frac{\Phi_{x_j}(x)\Phi_{x_k x_i}(x)}{|\nabla\Phi(x)|^2}$$
$$+ \frac{\Phi_{x_i}(x)\Phi_{x_j}(x)}{|\nabla\Phi(x)|^2}\sum_{l=1}^{n+1}\frac{\Phi_{x_l}(x)\Phi_{x_k x_l}(x)}{|\nabla\Phi(x)|^2} + \beta_k^{ij}(x)\nu_k(p(x)) + \gamma_k^{ij}(x)\Phi(x), \tag{2.10}$$

where $a_{ik}, \beta_k^{ij}, \gamma_k^{ij}$ are smooth functions. Differentiating (2.7) and using (2.9), (2.10) as well as the fact that $\sum_{k=1}^{n+1} \underline{D}_k z(p(x))\nu_k(p(x)) = 0$ we obtain

(2.11)
$$\nabla z^e(x) = \big(I + \Phi(x)A(x)\big)\nabla_\Gamma z(p(x)),$$

(2.12)
$$\frac{1}{|\nabla\Phi(x)|}\nabla\cdot\big(|\nabla\Phi(x)|\nabla z^e(x)\big)$$
$$= (\Delta_\Gamma z)(p(x)) + \Phi(x)\left(\sum_{k,l=1}^{n+1} b_{lk}(x)\underline{D}_l\underline{D}_k z(p(x)) + \sum_{k=1}^{n+1} c_k(x)\underline{D}_k z(p(x))\right),$$

where $A = (a_{ik}), b_{lk}$, and c_k are again smooth.

2.2. Bulk finite element space and inequalities. In what follows we assume that U is polyhedral. Let $(\mathscr{T}_h)_{0<h\le h_0}$ be a family of triangulations of U consisting of closed simplices T with maximum mesh size $h := \max_{T\in\mathcal{T}_h} h(T)$, where $h(T) = \operatorname{diam}(T)$. We assume that $(\mathscr{T}_h)_{0<h\le h_0}$ is regular in the sense that there exists $\rho > 0$ such that

$$\operatorname{diam} B_T \ge \rho h(T) \quad \text{for all } T \in \mathscr{T}_h,\ 0 < h \le h_0, \tag{2.13}$$

where B_T is the largest ball contained in T. Let us denote by X_h the space of linear finite elements

$$X_h = \{\phi_h \in C^0(\bar{U}) \,|\, \phi_{h|T} \in P_1(T), T \in \mathcal{T}_h\},$$

and by $I_h : C^0(\bar{U}) \to X_h$ the usual Lagrange interpolation operator. We have

$$\|\eta - I_h\eta\|_{W^{k,p}(T)} \le Ch(T)^{2-k}\,\|\eta\|_{W^{2,p}(T)}, \quad T \in \mathcal{T}_h, \eta \in W^{k,p}(U), \tag{2.14}$$

for $k = 0, 1$ and $1 < p \le \infty$ with $2 - \frac{n+1}{p} > 0$. As a consequence,

$$\|\Phi - I_h\Phi\|_{L^\infty(U)} + h\,\|\nabla(\Phi - I_h\Phi)\|_{L^\infty(U)} \le Ch^2, \tag{2.15}$$

so that we may assume that there exist constants c_0, c_1 such that

$$c_0 \le |\nabla I_h\Phi(x)| \le c_1, \quad x \in U, 0 < h \le h_0. \tag{2.16}$$

Let us next define

$$\Gamma_h := \{x \in U \,|\, I_h\Phi(x) = 0\} \quad \text{and} \quad D_h := \{x \in U \,|\, |I_h\Phi(x)| < h\} \tag{2.17}$$

as approximations of the given hypersurface Γ and the neighborhood $D^h := \{x \in U \,|\, |\Phi(x)| < h\}$; see Figure 1, for example. Note that Γ_h is a polygon whose facets are line segments if $n = 1$ and a polyhedral surface whose facets consist of triangles or quadrilaterals if $n = 2$. The corresponding decomposition of Γ_h is in general not shape regular and can have arbitrary small elements.

Furthermore, we introduce $F_h : U \to \mathbb{R}^{n+1}$ by

$$F_h(x) := F(p(x), I_h\Phi(x)), \quad x \in U,$$

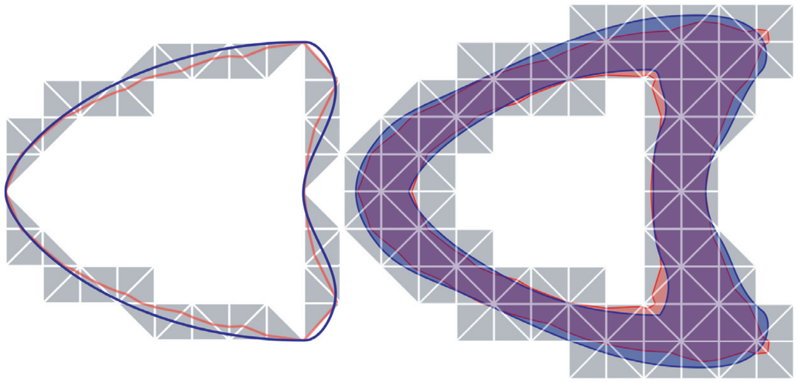

FIG. 1. *A cartoon of the domains of the sharp interface (left) and the narrow band (right) method. The surface Γ, resp., the set D^h is displayed in red, the approximations Γ_h, resp., D_h in blue, and the domains U_h^I, U_h^B in gray.*

where F was defined in (2.3). From the properties of F we infer that

$$p(F_h(x)) = p(x) \quad \text{and} \quad \Phi(F_h(x)) = I_h\Phi(x) \text{ if } F_h(x) \in U_\delta, \tag{2.18}$$

$$F_h(x) = p(x) \text{ if } x \in \Gamma_h. \tag{2.19}$$

LEMMA 2.1. *There exists $0 < h_1 \leq h_0$ such that for $0 < h \leq h_1$ the mapping $F_h : D_h \to D^h := \{x \in U \mid |\Phi(x)| < h\}$ is bi-Lipschitz with $F_h(\Gamma_h) = \Gamma$. Furthermore,*

$$\|F_h - \mathrm{Id}\|_{L^\infty(U)} + h\,\|DF_h - I\|_{L^\infty(U)} \leq ch^2, \tag{2.20}$$

$$\left\| |\det DF_h| - \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \right\|_{L^\infty(U)} \leq ch^2. \tag{2.21}$$

Proof. Since $F(p(x), \Phi(x)) = x$ we deduce with the help of (2.15)

$$|F_h(x) - x| = |F(p(x), I_h\Phi(x)) - F(p(x), \Phi(x))| \leq c|I_h\Phi(x) - \Phi(x)| \leq ch^2.$$

Differentiating the relation $F_i(p(x), \Phi(x)) = x_i$, $i = 1, \ldots, n+1$, we obtain

$$\sum_{k=1}^{n+1} \underline{D}_k F_i(p(x), \Phi(x)) p_{k,x_j}(x) + \frac{\partial F_i}{\partial s}(p(x), \Phi(x)) \Phi_{x_j}(x) = \delta_{ij}, \quad i, j = 1, \ldots, n+1,$$

and hence

$$\begin{aligned} F_{hi,x_j}(x) &= \sum_{k=1}^{n+1} \underline{D}_k F_i(p(x), I_h\Phi(x)) p_{k,x_j}(x) + \frac{\partial F_i}{\partial s}(p(x), I_h\Phi(x))(I_h\Phi)_{x_j}(x) \\ &= \delta_{ij} + \frac{\partial F_i}{\partial s}(p(x), \Phi(x)) \big(I_h\Phi - \Phi\big)_{x_j}(x) \\ &\quad + \sum_{k=1}^{n+1} \Big(\underline{D}_k F_i(p(x), I_h\Phi(x)) - \underline{D}_k F_i(p(x), \Phi(x)) \Big) p_{k,x_j}(x) \\ &\quad + \left(\frac{\partial F_i}{\partial s}(p(x), I_h\Phi(x)) - \frac{\partial F_i}{\partial s}(p(x), \Phi(x)) \right) (I_h\Phi)_{x_j}(x) \\ &= \delta_{ij} + \frac{\Phi_{x_i}(x)}{|\nabla\Phi(x)|^2} \big(I_h\Phi - \Phi\big)_{x_j}(x) + r_{ij}(x), \end{aligned} \tag{2.22}$$

where $|r_{ij}(x)| \le ch^2$ in view of (2.15). This implies (2.20). In particular we deduce that F_h is bi-Lipschitz provided that h is sufficiently small, whereas the properties $F_h(D_h) = D^h$ and $F_h(\Gamma_h) = \Gamma$ follow from (2.18). Finally we deduce from (2.22) that

$$\begin{aligned}|\det DF_h| &= 1 + \frac{\nabla\Phi}{|\nabla\Phi|^2} \cdot \nabla(I_h\Phi - \Phi) + c_h = \frac{\nabla\Phi \cdot \nabla I_h\Phi}{|\nabla\Phi|^2} + c_h \\ &= \frac{|\nabla I_h\Phi|}{|\nabla\Phi|} - \frac{1}{2}\left|\frac{\nabla I_h\Phi}{|\nabla I_h\Phi|} - \frac{\nabla\Phi}{|\nabla\Phi|}\right|^2 \frac{|\nabla I_h\Phi|}{|\nabla\Phi|} + c_h = \frac{|\nabla I_h\Phi|}{|\nabla\Phi|} + d_h,\end{aligned}$$

where $|c_h|, |d_h| \le ch^2$ proving (2.21). ∎

We introduce $\mu_h : \Gamma_h \to \mathbb{R}$ via $\mathrm{d}\sigma(p(x)) = \mu_h(x)\,\mathrm{d}\sigma_h(x)$. It is well known (see Proposition 2.1 in [15], (3.37) in [34]) that

$$|1 - \mu_h| \le ch^2 \quad \text{on } \Gamma_h. \tag{2.23}$$

Using the properties of F_h together with the coarea formula and (2.9),(2.10), (2.11), (2.23) one can prove the following result on the equivalence of certain norms.

Lemma 2.2. *There exist constants $c_1, c_2 > 0$ which are independent of h, such that for all $z \in H^1(\Gamma)$*

$$\begin{aligned} c_1 \|z^e\|_{L^2(\Gamma_h)} &\le \|z\|_{L^2(\Gamma)} \le c_2 \|z^e\|_{L^2(\Gamma_h)}, \\ c_1 \frac{1}{\sqrt{h}} \|z^e\|_{L^2(D_h)} &\le \|z\|_{L^2(\Gamma)} \le c_2 \frac{1}{\sqrt{h}} \|z^e\|_{L^2(D_h)}, \\ c_1 \|\nabla z^e\|_{L^2(\Gamma_h)} &\le \|\nabla_\Gamma z\|_{L^2(\Gamma)} \le c_2 \|\nabla z^e\|_{L^2(\Gamma_h)}, \\ c_1 \frac{1}{\sqrt{h}} \|\nabla z^e\|_{L^2(D_h)} &\le \|\nabla_\Gamma z\|_{L^2(\Gamma)} \le c_2 \frac{1}{\sqrt{h}} \|\nabla z^e\|_{L^2(D_h)}. \end{aligned}$$

If in addition $z \in H^2(\Gamma)$ then

$$c_1 \frac{1}{\sqrt{h}} \left\|D^2 z^e\right\|_{L^2(D_h)} \le \|z\|_{H^2(\Gamma)}.$$

2.3. Variational form of elliptic equation and Strang's second lemma.

It is well known [1] that for every $f \in L^2(\Gamma)$ there exists a unique solution $u \in H^2(\Gamma)$ of (1.1) which satisfies

$$\|u\|_{H^2(\Gamma)} \le c\,\|f\|_{L^2(\Gamma)}. \tag{2.24}$$

Let us write (1.1) in weak form:

$$a(u, \varphi) = l(\varphi) \qquad \text{for all } \varphi \in H^1(\Gamma), \tag{2.25}$$

where

$$a(w, \varphi) = \int_\Gamma \left(\nabla_\Gamma w \cdot \nabla_\Gamma \varphi + w\varphi\right) \mathrm{d}\sigma, \quad l(\varphi) = \int_\Gamma f\varphi\,\mathrm{d}\sigma.$$

Next, suppose that V_h is a finite–dimensional space and $V^e := \{v^e \mid v \in H^1(\Gamma)\}$. Assume that $a_h : (V_h + V^e) \times (V_h + V^e) \to \mathbb{R}$ is a symmetric, positive semidefinite bilinear form which is, in addition, positive definite on $V_h \times V_h$. Furthermore, let $l_h : V_h \to \mathbb{R}$ be linear. Then the approximate problem

$$a_h(u_h, v_h) = l_h(v_h) \qquad \text{for all } v_h \in V_h \tag{2.26}$$

has a unique solution $u_h \in V_h$. Introducing

$$\|v\|_h := \sqrt{a_h(v,v)}, \qquad v \in V_h + V^e,$$

we have by Strang's second lemma

$$\|u^e - u_h\|_h \le 2 \inf_{v_h \in V_h} \|u^e - v_h\|_h + \sup_{\phi_h \in V_h} \frac{|a_h(u^e,\phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h}. \tag{2.27}$$

3. Sharp interface method (*SIF*).

3.1. Setting up the method. Let us begin by observing that if $T \in \mathscr{T}_h$ satisfies $\mathscr{H}^n(T \cap \Gamma_h) > 0$, then the following two cases can occur: (1) $\Gamma_h \cap \mathrm{int}(T) \neq \emptyset$, in which case $\mathscr{H}^n(\partial T \cap \Gamma_h) = 0$; (2) $T \cap \Gamma_h = \partial T \cap \Gamma_h$ in which case $T \cap \Gamma_h$ is the face between two elements. We may now define a unique subset $\mathscr{T}_h^I \subset \mathscr{T}_h$ by taking all elements satisfying case 1 and in case 2 taking just one of the two elements T. The numerical method does not depend on which element is chosen. We may therefore conclude that there exists $N \subset \Gamma_h$ with $\mathscr{H}^n(N) = 0$ and a subset $\mathscr{T}_h^I \subset \mathscr{T}_h$ such that every $x \in \Gamma_h \setminus N$ belongs to exactly one $T \in \mathscr{T}_h^I$. We then define

$$U_h^I = \bigcup_{T \in \mathscr{T}_h^I} T.$$

Clearly $U_h^I \subseteq U_\delta$ if h is small enough. We define the finite element space V_h^I by

$$V_h^I = \{\phi_h \in C^0(U_h^I) \,|\, \phi_{h|T} \in P_1(T) \text{ for each } T \in \mathscr{T}_h^I\}.$$

Note that $\nabla \phi_h$ is defined on $\Gamma_h \setminus N$ in view of the definition of $\mathscr{T}_h^I$. In particular the unit normal ν_h to Γ_h is given by

$$\nu_h = \frac{\nabla I_h \Phi}{|\nabla I_h \Phi|} \quad \text{on } \Gamma_h \setminus N, \tag{3.1}$$

and we use (3.1) in order to extend ν_h to U_h^I. Let us next turn to the approximation error for the space V_h^I. Note that for a function $z \in H^2(\Gamma)$ we have $z^e \in C^0(\bar{U}_\delta)$ so that $I_h z^e$ is well–defined.

LEMMA 3.1. *Let $z \in H^2(\Gamma)$. Then*

$$\|z^e - I_h z^e\|_{L^2(\Gamma_h)} + h \|\nabla(z^e - I_h z^e)\|_{L^2(\Gamma_h)} \le ch^2 \|z\|_{H^2(\Gamma)}. \tag{3.2}$$

Proof. We first observe that Theorem 3.7 in [34] yields

$$\|z^e - I_h z^e\|_{L^2(\Gamma_h)} + h \|\nabla_{\Gamma_h}(z^e - I_h z^e)\|_{L^2(\Gamma_h)} \le ch^2 \|z\|_{H^2(\Gamma)}. \tag{3.3}$$

Hence, it remains to bound $\|\nabla(z^e - I_h z^e)\cdot \nu_h\|_{L^2(\Gamma_h)}$. To do so, we start by considering an element $T \in \mathscr{T}_h^I$. Then we see that

$$\begin{aligned}
&\int_{T\cap\Gamma_h} |\nabla(z^e - I_h z^e)\cdot \nu_h|^2 \,\mathrm{d}\sigma_h \\
&\quad \le 2\int_{T\cap\Gamma_h} |\nabla z^e \cdot \nu_h|^2 \,\mathrm{d}\sigma_h + 2\int_{T\cap\Gamma_h} |\nabla(I_h z^e)\cdot \nu_h|^2 \,\mathrm{d}\sigma_h \\
&\quad \le 2\int_{T\cap\Gamma_h} |\nabla z^e \cdot (\nu_h - \nu)|^2 \,\mathrm{d}\sigma_h + ch(T)^{-1}\int_T |\nabla(I_h z^e)\cdot \nu_h|^2 \,\mathrm{d}x =: I_1 + I_2,
\end{aligned}$$

in view of (2.8) and the fact that $\mathscr{H}^n(T \cap \Gamma_h) \leq ch(T)^{-1}\mathscr{H}^{n+1}(T)$. Note that by (3.1) and (2.15)

$$\|\nu - \nu_h\|_{L^\infty(T)} = \left\| \frac{\nabla \Phi}{|\nabla \Phi|} - \frac{\nabla I_h \Phi}{|\nabla I_h \Phi|} \right\|_{L^\infty(T)} \leq ch(T) \tag{3.4}$$

so that

$$I_1 \leq ch^2 \int_{T \cap \Gamma_h} |\nabla z^e|^2 \, \mathrm{d}\sigma_h.$$

Furthermore, recalling (2.14) and using again (3.4)

$$I_2 \leq ch(T)^{-1} \int_T \left(|\nabla z^e \cdot (\nu_h - \nu)|^2 + |\nabla (z^e - I_h z^e)|^2 \right) \mathrm{d}x \leq ch \, \|z^e\|^2_{H^2(T)} \, .$$

We use the bounds for I_1, I_2 and sum over all elements $T \in \mathscr{T}_h^I$, then apply Lemma 2.2 to see

$$\int_{\Gamma_h} |\nabla (z^e - I_h z^e) \cdot \nu_h|^2 \, \mathrm{d}\sigma_h \leq ch^2 \|\nabla z^e\|^2_{L^2(\Gamma_h)} + ch \, \|z^e\|^2_{H^2(D_{c_1 h})} \leq ch^2 \, \|z\|^2_{H^2(\Gamma)} \, ,$$

since $T \subset D_{c_1 h}$ for all $T \in \mathcal{T}_h^I$ in view of (2.16). ▯

3.2. The method. Let us write (1.3) in the form find $u_h \in V_h^I$ such that

$$a_h(u_h, \phi_h) = l_h(\phi_h) \quad \text{for all } \phi_h \in V_h^I, \tag{3.5}$$

where

$$a_h(w_h, \phi_h) = \int_{\Gamma_h} \left(\nabla w_h \cdot \nabla \phi_h + w_h \phi_h \right) \mathrm{d}\sigma_h, \quad l_h(\phi_h) = \int_{\Gamma_h} f^e \phi_h \, \mathrm{d}\sigma_h.$$

In order to verify that the symmetric bilinear form a_h is positive definite on $V_h^I \times V_h^I$ we note that $a_h(\phi_h, \phi_h) = 0$ implies that

$$\int_{\Gamma_h \cap T} \left(|\nabla \phi_h|^2 + \phi_h^2 \right) \mathrm{d}\sigma_h = 0 \quad \text{for all } T \in \mathscr{T}_h^I.$$

Since $\mathscr{H}^n(T \cap \Gamma_h) > 0$ for $T \in \mathscr{T}_h^I$ we infer that $\nabla \phi_h = 0$ and hence ϕ_h is constant on these elements. Using again that $\mathscr{H}^n(T \cap \Gamma_h) > 0$ we deduce that $\phi_h = 0$ on each $T \in \mathscr{T}_h^I$ so that $\phi_h \equiv 0$ in V_h^I. Hence (3.5) has a unique solution $u_h \in V_h^I$ and

$$\|u_h\|_h = \left(\|\nabla u_h\|^2_{L^2(\Gamma_h)} + \|u_h\|^2_{L^2(\Gamma_h)} \right)^{\frac{1}{2}} \leq c \, \|f^e\|_{L^2(\Gamma_h)} \leq c \, \|f\|_{L^2(\Gamma)} \, . \tag{3.6}$$

3.3. Error analysis. The following error bounds hold.

THEOREM 3.2. *Let u be the solution of* (1.1) *and u_h the solution of the finite element scheme* (3.5). *Then*

$$\|u^e - u_h\|_{L^2(\Gamma_h)} + h \, \|\nabla (u^e - u_h)\|_{L^2(\Gamma_h)} \leq ch^2 \, \|f\|_{L^2(\Gamma)} \, . \tag{3.7}$$

Proof. In view of the definition of $\|\cdot\|_h$, (2.27), and Lemma 3.1 we have for $e_h := u^e - u_h$

$$
\begin{aligned}
(3.8) \qquad & \big(\|e_h\|^2_{L^2(\Gamma_h)} + \|\nabla e_h\|^2_{L^2(\Gamma_h)}\big)^{\frac{1}{2}} \\
&\le 2\big(\|u^e - I_h u^e\|^2_{L^2(\Gamma_h)} + \|\nabla(u^e - I_h u^e)\|^2_{L^2(\Gamma_h)}\big)^{\frac{1}{2}} \\
&\quad + \sup_{\phi_h \in V_h^I} \frac{|a_h(u^e,\phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h} \\
&\le ch\|u\|_{H^2(\Gamma)} + \sup_{\phi_h \in V_h^I} \frac{|a_h(u^e,\phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h}.
\end{aligned}
$$

In order to estimate the second term we choose $\phi_h \in V_h^I$, then for $\varphi_h := \phi_h \circ F_h^{-1}$,

$$
a_h(u^e,\phi_h) - l_h(\phi_h) = \big(a_h(u^e,\phi_h) - a(u,\varphi_h)\big) + \big(l(\varphi_h) - l_h(\phi_h)\big) \equiv I + II.
$$

Using the transformation rule and (2.11) we obtain

$$
\begin{aligned}
\int_\Gamma \big(\nabla_\Gamma u \cdot \nabla_\Gamma \varphi_h + u\varphi_h\big)\,\mathrm{d}\sigma &= \int_{\Gamma_h} \big((\nabla_\Gamma u)\circ p \cdot (\nabla_\Gamma \varphi_h)\circ p + (u\circ p)\,(\varphi_h \circ p)\big)\mu_h\,\mathrm{d}\sigma_h \\
(3.9) \qquad &= \int_{\Gamma_h} \big((I+\Phi A)^{-1}\nabla u^e \cdot (I+\Phi A)^{-1}\nabla \varphi_h^e + u^e\,\varphi_h^e\big)\mu_h\,\mathrm{d}\sigma_h.
\end{aligned}
$$

Since $\varphi_h^e(x) = \varphi_h(p(x)) = \phi_h(F_h^{-1}(p(x)))$ we derive

$$
\begin{aligned}
\nabla\varphi_h^e(x) &= [Dp(x)]^T[DF_h^{-1}(p(x))]^T\nabla\phi_h(F_h^{-1}(p(x))) \\
&= [Dp(x)]^T[DF_h(F_h^{-1}(p(x)))]^{-T}\nabla\phi_h(F_h^{-1}(p(x))).
\end{aligned}
$$

We infer from (2.22) that

$$
(3.10) \qquad (DF_h)^{-T} = I - \frac{1}{|\nabla\Phi|}\nabla\eta_h \otimes \nu + B_h \quad \text{with } |B_h| \le ch^2,
$$

where $\eta_h = I_h\Phi - \Phi$. It follows from (2.19) that $F_h^{-1}(p(x)) = x, x \in \Gamma_h$, which together with (2.9) implies

$$
\nabla\varphi_h^e = (I - \nu\otimes\nu)\left(I - \frac{1}{|\nabla\Phi|}\nabla\eta_h\otimes\nu\right)\nabla\phi_h + q_h \quad \text{on } \Gamma_h, \quad |q_h| \le ch^2|\nabla\phi_h|.
$$

Taking into account that $\nabla u^e \cdot \nu = 0$ we therefore have

$$
\nabla u^e \cdot \nabla\varphi_h^e = \nabla u^e \cdot \nabla\phi_h - \frac{1}{|\nabla\Phi|}(\nabla u^e\cdot\nabla\eta_h)(\nabla\phi_h\cdot\nu) + \nabla u^e\cdot q_h \quad \text{on } \Gamma_h.
$$

Inserting this relation into (3.9) and recalling the definition of a_h we find that

$$
\begin{aligned}
|I| &\le \int_{\Gamma_h} \big(\big|\nabla u^e\cdot\nabla\phi_h - \mu_h(I+\Phi A)^{-T}(I+\Phi A)^{-1}\nabla u^e\cdot\nabla\varphi_h^e\big| + |(\mu_h - 1)u^e\varphi_h^e|\big)\,\mathrm{d}\sigma_h \\
&\le ch^2\,\|u^e\|_h\,\|\phi_h\|_h + ch\|u^e\|_h\int_{\Gamma_h}|(\nabla\phi_h\cdot\nu)|\,\mathrm{d}\sigma_h,
\end{aligned}
$$

where we used (2.15), (2.23), and the fact that $\varphi_h^e = \phi_h$ on Γ_h. Similarly,

$$|II| = \left| \int_\Gamma f \varphi_h \, \mathrm{d}\sigma - \int_{\Gamma_h} f^e \phi_h \, \mathrm{d}\sigma_h \right| \le \int_{\Gamma_h} |1 - \mu_h| \, |f^e| \, |\phi_h| \, \mathrm{d}\sigma_h$$
$$\le ch^2 \, \|f^e\|_{L^2(\Gamma_h)} \, \|\phi_h\|_h \le ch^2 \, \|f\|_{L^2(\Gamma)} \, \|\phi_h\|_h \, .$$

Combining these estimates with (2.24) we have

$$|a_h(u^e, \phi_h) - l_h(\phi_h)| \le ch^2 \|f\|_{L^2(\Gamma)} \, \|\phi_h\|_h + ch \|f\|_{L^2(\Gamma)} \int_{\Gamma_h} |\nabla \phi_h \cdot \nu| \, \mathrm{d}\sigma_h \tag{3.11}$$

for all $\phi_h \in V_h^I$, which inserted into (3.8) yields

$$\|e_h\|_{L^2(\Gamma_h)} + \|\nabla e_h\|_{L^2(\Gamma_h)} \le ch \, \|f\|_{L^2(\Gamma)} \, . \tag{3.12}$$

In order to improve the L^2-error bound we employ the usual Aubin–Nitsche argument. Denote by $w \in H^2(\Gamma)$ the solution of the dual problem

$$a(\varphi, w) = \int_\Gamma \widetilde{e}_h \varphi \, \mathrm{d}\sigma \qquad \text{for all } \varphi \in H^1(\Gamma) \quad \text{with } \widetilde{e}_h = e_h \circ F_h^{-1},$$

which satisfies

$$\|w\|_{H^2(\Gamma)} \le c \, \|\widetilde{e}_h\|_{L^2(\Gamma)} \, . \tag{3.13}$$

We have in view of (1.3)

$$\begin{aligned} \|\widetilde{e}_h\|_{L^2(\Gamma)}^2 &= a(\widetilde{e}_h, w) = \big(a(\widetilde{e}_h, w) - a_h(e_h, w^e) \big) \\ &\quad + a_h(e_h, w^e - I_h w^e) + \big(a_h(u^e, I_h w^e) - l_h(I_h w^e) \big) \\ &\equiv I + II + III. \end{aligned} \tag{3.14}$$

Similarly as above we deduce with the help of (3.12) and Lemma 2.2

$$|I| \le ch \, \|e_h\|_h \, \|w^e\|_h \le ch^2 \, \|f\|_{L^2(\Gamma)} \, \|w\|_{H^1(\Gamma)} \, .$$

Next, Lemma 3.1 and (3.12) imply

$$|II| \le \|e_h\|_h \, \|w^e - I_h w^e\|_h \le ch^2 \, \|f\|_{L^2(\Gamma)} \, \|w\|_{H^2(\Gamma)} \, .$$

Finally, (3.11), the fact that $\nabla w^e \cdot \nu = 0$, and Lemma 3.1 yield

$$|III| \le ch^2 \, \|f\|_{L^2(\Gamma)} \, \|I_h w^e\|_h + ch \, \|f\|_{L^2(\Gamma)} \int_{\Gamma_h} |\nabla (I_h w^e - w^e) \cdot \nu| \, \mathrm{d}\sigma_h$$
$$\le ch^2 \, \|f\|_{L^2(\Gamma)} \, \|w\|_{H^2(\Gamma)} \, .$$

Inserting the above estimates into (3.14) and recalling (3.13) we obtain

$$\|\widetilde{e}_h\|_{L^2(\Gamma)} \le ch^2 \, \|f\|_{L^2(\Gamma)} \, ,$$

which together with Lemma 2.2 completes the proof since $\widetilde{e}_h^e = e_h$ on Γ_h. $\square$

4. Narrow band method (NBM).

4.1. Setting up the method. Recalling the definition of D_h (2.17), we consider

$$\mathscr{T}_h^B = \{T \in \mathscr{T}_h \,|\, \mathscr{H}^{n+1}(T \cap D_h) > 0\} \quad \text{and} \quad U_h^B = \bigcup_{T \in \mathscr{T}_h^B} T.$$

We define the finite element space V_h^B on the triangulation $\mathscr{T}_h^B$ by

$$V_h^B = \{\phi_h \in C^0(U_h^B) \,|\, \phi_{h|T} \in P_1(T) \text{ for each } T \in \mathscr{T}_h^B\}.$$

Let us first examine the approximation error for the space V_h^B.

LEMMA 4.1. *We have for each function $z \in H^2(\Gamma)$ that $I_h z^e \in V_h^B$ satisfies*

$$\frac{1}{\sqrt{h}} \|z^e - I_h z^e\|_{L^2(D_h)} + \sqrt{h} \|\nabla(z^e - I_h z^e)\|_{L^2(D_h)} \le ch^2 \|z\|_{H^2(\Gamma)} . \tag{4.1}$$

Proof. We infer from (2.14) and Lemma 2.2 that

$$\begin{aligned}
&\frac{1}{h}\|z^e - I_h z^e\|^2_{L^2(D_h)} + h\|\nabla(z^e - I_h z^e)\|^2_{L^2(D_h)} \\
&\quad \le \sum_{T \cap D_h \neq \emptyset} \left(\frac{1}{h}\|z^e - I_h z^e\|^2_{L^2(T)} + h\|\nabla(z^e - I_h z^e)\|^2_{L^2(T)}\right) \\
&\quad \le ch^3 \sum_{T \cap D_h \neq \emptyset} \|z^e\|^2_{H^2(T)} \le ch^3 \|z^e\|^2_{H^2(D_{(1+c_1)h})} \le ch^4 \|z\|^2_{H^2(\Gamma)},
\end{aligned}$$

since $T \subset D_{(1+c_1)h}$ for all $T \cap D_h \neq \emptyset$ in view of (2.16). $\square$

4.2. The method. Let us write (1.5) in the form find $u_h \in V_h^B$ such that

$$a_h(u_h, \phi_h) = l_h(\phi_h) \quad \text{for all } \phi_h \in V_h^B, \tag{4.2}$$

where

$$\begin{aligned}
a_h(w_h, \phi_h) &= \frac{1}{2h} \int_{D_h} \big(\nabla w_h \cdot \nabla \phi_h + w_h \phi_h\big) \, |\nabla I_h \Phi| \, \mathrm{d}x, \\
l_h(\phi_h) &= \frac{1}{2h} \int_{D_h} f^e \phi_h \, |\nabla I_h \Phi| \, \mathrm{d}x.
\end{aligned}$$

Note that the factors $\frac{1}{h}$ in each of the above terms are there to aid the notation for the error analysis. In a similar way as for SIF one can verify that a_h is positive definite on $V_h^B \times V_h^B$. Hence, the finite element scheme (4.2) has a unique solution $u_h \in V_h^B$ which satisfies

$$\|u_h\|_h = \left(\frac{1}{2h} \int_{D_h} \big(|\nabla u_h|^2 + u_h^2\big) |\nabla I_h \Phi| \, \mathrm{d}x\right)^{\frac{1}{2}} \le c \, \|f\|_{L^2(\Gamma)} . \tag{4.3}$$

4.3. Error analysis. Before we prove our main error bound we formulate a technical lemma which will be helpful in the error analysis.

LEMMA 4.2. *Suppose that $u \in H^2(\Gamma)$ is a solution of* (1.1). *Then,*

$$a_h(u^e, \phi) = \frac{1}{2h} \int_{D^h} f^e \phi \circ F_h^{-1} \, |\nabla \Phi| \, \mathrm{d}x + \frac{1}{2h} \int_{D_h} (\nabla u^e \cdot \nabla \eta_h)(\nabla \phi \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \, \mathrm{d}x + \langle S, \phi \rangle$$

for all $\phi \in H^1(D_h)$, *where* $\eta_h = I_h\Phi - \Phi$ *and*

$$|\langle S, \phi\rangle| \le Ch^2 \|u\|_{H^2(\Gamma)} \|\phi\|_h .$$

Proof. To begin, we derive from (2.12) and (1.1) that

$$-\frac{1}{|\nabla\Phi|}\nabla\cdot\big(|\nabla\Phi|\,\nabla u^e\big) + u^e = f^e + R \quad \text{ in } U_\delta, \tag{4.4}$$

where

$$R(x) = -\Phi(x)\left(\sum_{k,l=1}^{n+1} b_{lk}(x)\underline{D}_l\underline{D}_k u(p(x)) + \sum_{k=1}^{n+1} c_k(x)\underline{D}_k u(p(x))\right). \tag{4.5}$$

We multiply (4.4) by $\phi \circ F_h^{-1}|\nabla\Phi|, \phi \in H^1(D_h)$, and integrate over D^h. Since $\frac{\partial u^e}{\partial\nu} = 0$ on ∂D^h we obtain after integration by parts

$$\begin{aligned}&\int_{D^h} \nabla u^e\cdot\nabla(\phi\circ F_h^{-1})|\nabla\Phi|\;\mathrm{d}x + \int_{D^h} u^e\,\phi\circ F_h^{-1}|\nabla\Phi|\;\mathrm{d}x\\ &\quad= \int_{D^h} f^e\,\phi\circ F_h^{-1}|\nabla\Phi|\,\mathrm{d}x + \int_{D^h} R\,\phi\circ F_h^{-1}|\nabla\Phi|\;\mathrm{d}x.\end{aligned} \tag{4.6}$$

Observing that $\nabla(\phi\circ F_h^{-1}) = [(DF_h)^{-T}\circ F_h^{-1}]\nabla\phi\circ F_h^{-1}$, the transformation rule and Lemma 2.1 imply that

$$I := \int_{D^h} \nabla u^e\cdot\nabla(\phi\circ F_h^{-1})|\nabla\Phi|\;\mathrm{d}x = \int_{D_h} \nabla u^e\circ F_h\cdot(DF_h)^{-T}\nabla\phi\,|\nabla\Phi\circ F_h|\;|\det DF_h|\;\mathrm{d}x.$$

Recalling (2.7) and (2.18) we have

$$z^e(x) = z(p(x)) = z(p(F_h(x)) = z^e(F_h(x)), \tag{4.7}$$

from which we deduce by differentiation

$$\nabla z^e\circ F_h = (DF_h)^{-T}\nabla z^e, \tag{4.8}$$

so that

$$I = \int_{D_h} (DF_h)^{-T}\nabla u^e\cdot(DF_h)^{-T}\nabla\phi\;|\nabla\Phi\circ F_h|\,|\det DF_h|\;\mathrm{d}x.$$

Recalling (3.10), we find with the help of $\nabla u^e\cdot\nu = 0$ that

$$(DF_h)^{-T}\nabla u^e = \nabla u^e + B_h\nabla u^e, \quad (DF_h)^{-T}\nabla\phi = \nabla\phi - \frac{1}{|\nabla\Phi|}(\nabla\phi\cdot\nu)\nabla\eta_h + B_h\nabla\phi,$$

where $|B_h| \le ch^2$. Furthermore, Lemma 2.1 implies that

$$|\nabla\Phi\circ F_h|\;|\det DF_h| = |\nabla I_h\Phi| + \gamma_h, \quad \text{ where } |\gamma_h| \le ch^2,$$

so that in conclusion

$$I = \int_{D_h} \nabla u^e\cdot\nabla\phi\;|\nabla I_h\Phi|\;\mathrm{d}x - \int_{D_h} (\nabla u^e\cdot\nabla\eta_h)(\nabla\phi\cdot\nu)\frac{|\nabla I_h\Phi|}{|\nabla\Phi|}\;\mathrm{d}x + \langle R_h^1, \phi\rangle,$$

where

$$\left|\langle R_h^1, \phi\rangle\right| \le ch^2 \left\|\nabla u^e\right\|_{L^2(D_h)} \left\|\nabla \phi\right\|_{L^2(D_h)} \le ch^3 \left\|u\right\|_{H^2(\Gamma)} \left\|\phi\right\|_h,$$

in view of Lemma 2.2 and the definition of $\|\cdot\|_h$. Similarly, (4.7) and (2.21) yield

$$\int_{D^h} u^e\, \phi \circ F_h^{-1} |\nabla \Phi| \, \mathrm{d}x = \int_{D_h} u^e \phi \, |\nabla I_h \Phi| \, \mathrm{d}x + \langle R_h^2, \phi\rangle$$

with $\left|\langle R_h^2, \phi\rangle\right| \le ch^3 \left\|u\right\|_{H^2(\Gamma)} \left\|\phi\right\|_h$. Inserting the above identities into (4.6) and dividing by $2h$ we derive

(4.9)

$$\begin{aligned} a_h(u^e, \phi) = {} & \frac{1}{2h} \int_{D^h} f^e\, \phi \circ F_h^{-1} |\nabla \Phi| \, \mathrm{d}x + \frac{1}{2h} \int_{D^h} R\, \phi \circ F_h^{-1} |\nabla \Phi| \, \mathrm{d}x \\ & + \frac{1}{2h} \int_{D_h} (\nabla u^e \cdot \nabla \eta_h)(\nabla \phi \cdot \nu) \frac{|\nabla I_h \Phi|}{|\nabla \Phi|} \, \mathrm{d}x - \frac{1}{2h} \langle R_h^1, \phi\rangle - \frac{1}{2h} \langle R_h^2, \phi\rangle. \end{aligned}$$

In order to rewrite the integral over D^h we note that $F(\cdot, s)$ maps Γ onto $\Gamma_s = \{\Phi = s\}$ and that $\mathrm{d}\sigma_s = (1 + O(s))\, \mathrm{d}\sigma_p$, where $\mathrm{d}\sigma_s, \mathrm{d}\sigma_p$ are the surface elements of Γ_s, Γ respectively. The coarea formula then yields for integrable $g : D^h \to \mathbb{R}$

$$\int_{D^h} g(x) \, \mathrm{d}x = \int_{-h}^{h} \int_{\Gamma_s} g(x) \frac{1}{|\nabla \Phi(x)|} \, \mathrm{d}\sigma_s \, \mathrm{d}s = \int_{-h}^{h} \int_{\Gamma} g(F(p,s)) \mu(p,s) \, \mathrm{d}\sigma_p \, \mathrm{d}s$$

$$\text{where} \quad \left|\mu(p,s) - \frac{1}{|\nabla \Phi(F(p,s))|}\right| \le C\,|s|\,, \ |s| < h, p \in \Gamma. \tag{4.10}$$

Hence,

$$\begin{aligned} & \int_{D^h} R\, \phi \circ F_h^{-1} |\nabla \Phi| \, \mathrm{d}x \\ & = \int_{-h}^{h} \int_{\Gamma} R \circ F\, \phi \circ F_h^{-1} \circ F \; |\nabla \Phi \circ F| \, \mu \, \mathrm{d}\sigma_p \, \mathrm{d}s \\ & = \int_{-h}^{h} \int_{\Gamma} R \circ F\, \phi \circ F_h^{-1} \circ F \, \mathrm{d}\sigma_p \, \mathrm{d}s + \int_{-h}^{h} \int_{\Gamma} \widetilde{r}\, \phi \circ F_h^{-1} \circ F \, \mathrm{d}\sigma_p \, \mathrm{d}s \\ & \equiv T_1 + T_2, \end{aligned}$$

where $\widetilde{r}(p,s) = R(F(p,s))\big(\mu(p,s)\, |\nabla \Phi(F(p,s))| - 1\big)$. In order to treat T_1 we deduce from (4.5) and the fact that $\Phi(F(p,s)) = s$ that

$$R(F(p,s)) = -s \left(\sum_{k,l=1}^{n+1} b_{lk}(F(p,s)) \underline{D}_l \underline{D}_k u(p) + \sum_{k=1}^{n+1} c_k(F(p,s)) \underline{D}_k u(p) \right).$$

Since $\int_{-h}^{h} s \, \mathrm{d}s = 0$, the first term in T_1 can be written as

$$-\int_{-h}^{h} \int_{\Gamma} \sum_{k,l=1}^{n+1} s \underline{D}_l \underline{D}_k u(p) \Big\{ b_{lk}(F(p,s))\, \phi \circ F_h^{-1}(F(p,s)) - b_{lk}(p)\, \phi \circ F_h^{-1}(p) \Big\} \, \mathrm{d}\sigma_p \, \mathrm{d}s.$$

Treating the second term in T_1 in the same way and observing that $p = F(p, 0)$ we deduce with the help of the fundamental theorem of calculus that

$$|T_1| \le ch^{\frac{5}{2}} \left\|u\right\|_{H^2(\Gamma)} \left(\int_{D^h} \left(\left|\nabla \phi \circ F_h^{-1}\right|^2 + \left|\phi \circ F_h^{-1}\right|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}} \le ch^3 \left\|u\right\|_{H^2(\Gamma)} \left\|\phi\right\|_h.$$

Next, we infer from (4.5) and (4.10) that

$$|\widetilde{r}(p,s)| \le cs^2\big(|\nabla_\Gamma u(p)| + \big|D^2_\Gamma u(p)\big|\big),$$

so that

$$|T_2| \le Ch^{\frac{5}{2}} \|u\|_{H^2(\Gamma)} \left(\int_{D^h} \left|\phi \circ F_h^{-1}\right|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \le Ch^3 \|u\|_{H^2(\Gamma)} \|\phi\|_h .$$

The result now follows from (4.9) together with the bounds on R_h^1 and R_h^2. ∎

THEOREM 4.3. *Let u be the solution of* (1.1) *and u_h the solution of the finite element scheme* (4.2). *Then*

$$\|u^e - u_h\|_{L^2(\Gamma_h)} + h\left(\frac{1}{2h}\int_{D_h} |\nabla(u^e - u_h)|^2 |\nabla I_h \Phi| \,\mathrm{d}x\right)^{\frac{1}{2}} \le ch^2 \|f\|_{L^2(\Gamma)} . \tag{4.11}$$

Proof. Let us write $e_h := u^e - u_h$. We infer from (2.27) and Lemma 4.1 that

$$\|e_h\|_h \le ch\|u\|_{H^2(\Gamma)} + \sup_{\phi_h \in V_h^B} \frac{|a_h(u^e,\phi_h) - l_h(\phi_h)|}{\|\phi_h\|_h}. \tag{4.12}$$

The second term on the right-hand side can be estimated with the help of Lemma 4.2. The transformation rule together with (4.7) yields

$$\frac{1}{2h}\int_{D^h} f^e \phi_h \circ F_h^{-1} |\nabla\Phi| \,\mathrm{d}x = \frac{1}{2h}\int_{D_h} f^e \circ F_h\, \phi_h\, |\nabla\Phi \circ F_h|\, |\det DF_h| \,\mathrm{d}x,$$

so that we deduce from Lemma 4.2

$$\begin{aligned} a_h(u^e,\phi_h) - l_h(\phi_h) =& \frac{1}{2h}\int_{D_h} (\nabla u^e \cdot \nabla \eta_h)\,(\nabla\phi_h \cdot \nu)\frac{|\nabla I_h \Phi|}{|\nabla\Phi|} \\ &+ \frac{1}{2h}\int_{D_h} f^e \phi_h \Big(|\nabla\Phi \circ F_h|\, |\det DF_h| - |\nabla I_h\Phi|\Big) \,\mathrm{d}x + \langle S, \phi_h\rangle. \end{aligned}$$

Using (2.21), (2.15), (2.24), and Lemmas 2.1 and 2.2 we infer that for $\phi_h \in V_h^B$

$$\begin{aligned} |a_h(u^e,\phi_h) - l_h(\phi_h)| \le& \, c\|\nabla u^e\|_{L^2(D_h)} \left(\int_{D_h} |\nabla\phi_h \cdot \nu|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \\ &+ ch\|f^e\|_{L^2(D_h)} \|\phi_h\|_{L^2(D_h)} + ch^2\|u\|_{H^2(\Gamma)} \|\phi_h\|_h \\ \le& \, ch\|f\|_{L^2(\Gamma)} \left(\frac{1}{2h}\int_{D_h} |\nabla\phi_h \cdot \nu|^2 \,dx\right)^{\frac{1}{2}} + ch^2 \|f\|_{L^2(\Gamma)} \|\phi_h\|_h , \end{aligned} \tag{4.13}$$

so that (4.12) implies the following intermediate result:

$$\|e_h\|_h = \left(\frac{1}{2h}\int_{D_h} \big(|\nabla e_h|^2 + e_h^2\big)|\nabla I_h\Phi| \,\mathrm{d}x\right)^{\frac{1}{2}} \le ch\|f\|_{L^2(\Gamma)} . \tag{4.14}$$

In order to improve the L^2-error bound we define $\widetilde{e}_h := e_h \circ F_h^{-1}$ as well as

$$\widetilde{E}_h(p) := \frac{1}{2h}\int_{-h}^{h} \widetilde{e}_h(F(p,s)) \,\mathrm{d}s, \quad p \in \Gamma,$$

with F as above. We denote by $w \in H^2(\Gamma)$ the unique solution of

$$-\Delta_\Gamma w + w = \widetilde{E}_h \quad \text{ on } \Gamma,$$

which satisfies

$$\|w\|_{H^2(\Gamma)} \le c \left\| \widetilde{E}_h \right\|_{L^2(\Gamma)}. \tag{4.15}$$

Similarly to (4.4) the extension w^e solves

$$-\frac{1}{|\nabla\Phi|}\nabla\cdot\left(|\nabla\Phi|\,\nabla w^e\right) + w^e = \widetilde{E}_h^e + \widetilde{R} \quad \text{ in } U_\delta,$$

where $\widetilde{R}$ is obtained from (4.5) by replacing u by w. Using the transformation rule together with (4.10) we obtain

$$\begin{aligned}
\left\|\widetilde{E}_h\right\|^2_{L^2(\Gamma)} &= \frac{1}{2h}\int_{-h}^{h}\int_\Gamma \widetilde{E}_h\widetilde{e}_h\circ F\,\mathrm{d}\sigma_p\,\mathrm{d}s = \frac{1}{2h}\int_{-h}^{h}\int_\Gamma \widetilde{E}_h^e\circ F\,\widetilde{e}_h\circ F\,|\nabla\Phi\circ F|\,\mu\,\mathrm{d}\sigma_p\,\mathrm{d}s\\
&\quad + \frac{1}{2h}\int_{-h}^{h}\int_\Gamma \widetilde{E}_h\,\widetilde{e}_h\circ F\,(1-|\nabla\Phi\circ F|\,\mu)\,\mathrm{d}\sigma_p\,\mathrm{d}s\\
&= \frac{1}{2h}\int_{D^h} \widetilde{E}_h^e\,e_h\circ F_h^{-1}|\nabla\Phi|\,\mathrm{d}x\\
&\quad + \frac{1}{2h}\int_{-h}^{h}\int_\Gamma \widetilde{E}_h\,\widetilde{e}_h\circ F\,(1-|\nabla\Phi\circ F|\,\mu)\,\mathrm{d}\sigma_p\,\mathrm{d}s.
\end{aligned}$$

The first term can be rewritten with the help of Lemma 4.2 (applied to w instead of u) to give

$$\begin{aligned}
\left\|\widetilde{E}_h\right\|^2_{L^2(\Gamma)} &= a_h(w^e,e_h) - \langle\widetilde{S},e_h\rangle - \frac{1}{2h}\int_{D_h}(\nabla w^e\cdot\nabla\eta_h)(\nabla e_h\cdot\nu)\frac{|\nabla I_h\Phi|}{|\nabla\Phi|}\,\mathrm{d}x\\
&\quad + \frac{1}{2h}\int_{-h}^{h}\int_\Gamma \widetilde{E}_h\,\widetilde{e}_h\circ F\,(1-|\nabla\Phi\circ F|\,\mu)\,\mathrm{d}\sigma_p\,\mathrm{d}s \equiv \sum_{k=1}^{4} I_k.
\end{aligned}$$

In view of Lemma 4.1, (4.13), the fact that $\nabla w^e\cdot\nu = 0$, and (4.14) we have

$$\begin{aligned}
|I_1|+|I_2| &\le |a_h(w^e - I_h w^e, e_h)| + |a_h(u^e, I_h w^e) - l_h(I_h w^e)| + |\langle\widetilde{S},e_h\rangle|\\
&\le ch\,\|w\|_{H^2(\Gamma)}\,\|e_h\|_h + ch^2\,\|f\|_{L^2(\Gamma)}\,\|I_h w^e\|_h\\
&\quad + ch\,\|f\|_{L^2(\Gamma)}\left(\frac{1}{2h}\int_{D_h}|\nabla(I_h w^e - w^e)\cdot\nu|^2 dx\right)^{\frac{1}{2}} + ch^2\,\|w\|_{H^2(\Gamma)}\,\|e_h\|_h\\
&\le ch^2\,\|f\|_{L^2(\Gamma)}\,\|w\|_{H^2(\Gamma)}.
\end{aligned}$$

Furthermore, (2.15), (4.10), and (4.14) imply

$$|I_3|+|I_4| \le ch\left(\|w^e\|_h + \left\|\widetilde{E}_h\right\|_{L^2(\Gamma)}\right)\|e_h\|_h \le ch^2\,\|f\|_{L^2(\Gamma)}\left(\|w\|_{H^2(\Gamma)} + \left\|\widetilde{E}_h\right\|_{L^2(\Gamma)}\right),$$

so that we obtain together with (4.15)

$$\left\|\widetilde{E}_h\right\|_{L^2(\Gamma)} \le ch^2\,\|f\|_{L^2(\Gamma)}.$$

Next, since $F(p,0)=p$ we may write for $p\in\Gamma$

$$\widetilde{E}_h(p)-\widetilde{e}_h(p)=\frac{1}{2h}\int_{-h}^{h}\int_0^s \nabla\widetilde{e}_h(F(p,\tau))\cdot\frac{\partial F}{\partial s}(p,\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

and hence we obtain with the help of (4.14)

$$\left\|\widetilde{E}_h-\widetilde{e}_h\right\|_{L^2(\Gamma)}\le c\sqrt{h}\left(\int_{D^h}|\nabla\widetilde{e}_h|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\le ch\left\|\nabla e_h\right\|_h\le ch^2\left\|f\right\|_{L^2(\Gamma)}.$$

In conclusion we deduce that

$$\|e_h\|_{L^2(\Gamma_h)}\le c\,\|\widetilde{e}_h\|_{L^2(\Gamma)}\le\left\|\widetilde{E}_h-\widetilde{e}_h\right\|_{L^2(\Gamma)}+\left\|\widetilde{E}_h\right\|_{L^2(\Gamma)}\le ch^2\left\|f\right\|_{L^2(\Gamma)}$$

and the theorem is proved. ∎

5. A hybrid method for equations on evolving surfaces.

5.1. The setting. The aim of this section is to combine ideas employed in sections 3 and 4 for the stationary problem in order to develop a finite element method for an advection–diffusion equation on an evolving hypersurface in which there is an underlying conservation law with a diffusive flux, [18]. More precisely, let $(\Gamma(t))_{t\in[0,T]}$ be a family of compact, connected smooth hypersurfaces embedded in $\mathbb{R}^{n+1}$ for $n=1,2$. We suppose that

$$\Gamma(t)=\{x\in\mathcal{N}(t)\,|\,\Phi(x,t)=0\},\qquad\text{where }\nabla\Phi(x,t)\neq 0, x\in\mathcal{N}(t),$$

and $\mathcal{N}(t)$ is an open neighborhood of $\Gamma(t)$. We assume that $\mathcal{N}(t)$ is chosen so small that we can construct the function $p(\cdot,t)$ as in section 2.1. Given a velocity field $v(\cdot,t):\Gamma(t)\to\mathbb{R}^{n+1}$, not necessarily in the normal direction, we then consider the following initial value problem

$$\partial^\bullet u+u\nabla_\Gamma\cdot v-\Delta_\Gamma u=f\quad\text{on}\bigcup_{t\in(0,T)}\Gamma(t)\times\{t\},\tag{5.1a}$$

$$u(\cdot,0)=u_0\quad\text{on }\Gamma(0).\tag{5.1b}$$

Here, $\partial^\bullet\eta$ denotes the material derivative of a function $\eta:\bigcup_{t\in(0,T)}\Gamma(t)\times\{t\}\to\mathbb{R}$ which is given by

$$\partial^\bullet\eta=\partial_t\eta+v\cdot\nabla\eta$$

if η is extended into a neighborhood of $\bigcup_{t\in(0,T)}\Gamma(t)\times\{t\}$.

5.2. The method. In order to discretize the above problem we choose a partition $0=t_0<t_1<\cdots<t_N=T$ of $[0,T]$ with $\tau_m:=t_{m+1}-t_m, m=0,\ldots,N-1$, and $\tau:=\max_{m=0,\ldots,N-1}\tau_m$. Also, let $\mathscr{T}_h$ be an unfitted regular triangulation with mesh size h of a region containing $\mathcal{N}(t), t\in[0,T]$. For $m=0,1,\ldots,N$ we set

$$\Gamma_h^m=\{x\in\mathcal{N}(t_m)\,|\,I_h\Phi(x,t_m)=0\}\quad\text{and}\quad D_h^m=\{x\in\mathcal{N}(t_m)\,|\,|I_h\Phi(x,t_m)|<h\},$$

as well as

$$\mathscr{T}_h^m:=\{T\in\mathscr{T}_h\,|\,\mathscr{H}^{n+1}(T\cap D_h^m)>0\}\quad\text{and}\quad U_h^m:=\bigcup_{T\in\mathscr{T}_h^m}T.$$

Here we assume that $0 < h \le h_0$, where h_0 is chosen so small that there exists $c_0, c_1 > 0$ such that

$$c_0 \le |\nabla I_h \Phi(x,t)| \le c_1, \quad (x,t) \in \bigcup_{t\in(0,T)} \mathcal{N}(t)\times\{t\}.$$

Finally, we introduce

$$V_h^m = \{\phi_h \in C^0(U_h^m)\,|\,\phi_h|_T \in P_1(T) \text{ for each } T \in \mathscr{T}_h^m\}.$$

In what follows we shall frequently use the abbreviation $z^m(x) := z(x,t_m)$.

In order to motivate our method we fix $m \in \{0,1,\dots,N-1\}$ and let Ψ be the solution of

$$\Psi_t(x,t) + D\Psi(x,t)v^e(x,t) = 0, \quad \Psi(x,t_{m+1}) = x,$$

where $v^e(x,t) := v(p(x,t),t)$. For a sufficiently smooth function $\varphi : \mathcal{N}(t_{m+1}) \to \mathbb{R}$ we define $\eta(x,t) := \varphi(\Psi(x,t))$. Clearly, $\eta(\cdot,t_{m+1}) = \varphi$ and a short calculation shows that $\partial^\bullet\eta = 0$. Assuming that u is a solution of (5.1a) we obtain with the help of the Leibniz formula and integration by parts

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma(t)} u\eta\,\mathrm{d}\sigma_{|t=t_{m+1}} &= \int_{\Gamma(t_{m+1})} \big(\partial^\bullet(u\eta) + u\eta\nabla_\Gamma\cdot v\big)\,\mathrm{d}\sigma\\
&= \int_{\Gamma(t_{m+1})} \varphi\big(\partial^\bullet u + u\nabla_\Gamma\cdot v\big)\,\mathrm{d}\sigma\\
&= \int_{\Gamma(t_{m+1})} \big(\varphi\Delta_\Gamma u + \varphi f\big)\,\mathrm{d}\sigma\\
&= -\int_{\Gamma(t_{m+1})} \nabla_\Gamma u\cdot\nabla_\Gamma\varphi\,\mathrm{d}\sigma + \int_{\Gamma(t_{m+1})} f\varphi\,\mathrm{d}\sigma.
\end{aligned}$$

Since $\Psi(\cdot,t_{m+1}) \equiv \mathrm{id}$, a Taylor expansion shows that

$$\begin{aligned}
\Psi(x,t_m) &= \Psi(x,t_{m+1}-\tau_m) \approx x - \tau_m\Psi_t(x,t_{m+1})\\
&= x + \tau_m D\Psi(x,t_{m+1})v^{e,m+1}(x) = x + \tau_m v^{e,m+1}(x).
\end{aligned}$$

Thus we may approximate the left-hand side of the above relation by

$$\frac{d}{dt}\int_{\Gamma(t)} u\eta\,\mathrm{d}\sigma_{|t=t_{m+1}} \approx \frac{1}{\tau_m}\left\{\int_{\Gamma(t_{m+1})} u^{m+1}\varphi\,\mathrm{d}\sigma - \int_{\Gamma(t_m)} u^m\varphi(\cdot+\tau_m v^{e,m+1})\,\mathrm{d}\sigma\right\}.$$

The above calculations motivate the following scheme, in which we use the narrow band approach in order to discretize the elliptic part. Given $u_h^m \in V_h^m$, $m = 1,\dots,N-1$, find $u_h^{m+1} \in V_h^{m+1}$ such that

$$\begin{aligned}
&\int_{\Gamma_h^{m+1}} u_h^{m+1}\phi_h\,\mathrm{d}\sigma_h - \int_{\Gamma_h^m} u_h^m\phi_h(\cdot+\tau_m v^{e,m+1})\,\mathrm{d}\sigma_h\\
&\quad + \frac{\tau_m}{2h}\int_{D_h^{m+1}} \nabla u_h^{m+1}\cdot\nabla\phi_h\,|\nabla I_h\Phi^{m+1}|\,\mathrm{d}x = \tau_m\int_{\Gamma_h^{m+1}} f^{e,m+1}\phi_h\,\mathrm{d}\sigma_h
\end{aligned} \tag{5.2}$$

for all $\phi_h \in V_h^{m+1}$. Here, $u_h^0 = I_h u^0$. Existence and uniqueness of u_h^{m+1} follows in a similar way as for NBM in the elliptic case.

5.3. Mass conservation. An important property of solutions of (5.1a) is conservation of mass in the case that $\int_{\Gamma(t)} f(\cdot,t)\,\mathrm{d}\sigma = 0$. The following lemma shows that our numerical scheme preserves this property under some mild constraints on the discretization parameters. In fact this discrete conservation law is a remarkable property of the scheme and relies on the use of both a sharp interface and a narrow band approach.

LEMMA 5.1. *Let $u_h^m \in V_h^m, m = 1,\dots,N$, be the solutions of* (5.2) *in the case $\int_{\Gamma_h^{m+1}} f^{e,m+1}\,\mathrm{d}\sigma_h = 0$, $m = 0,\dots,N-1$. Then provided that $0 < h \le h_1$ and $\tau \le \gamma\sqrt{h}$*

$$\int_{\Gamma_h^m} u_h^m\,\mathrm{d}\sigma_h = \int_{\Gamma_h^0} u_h^0\,\mathrm{d}\sigma_h. \tag{5.3}$$

Proof. Let us first observe that

$$\{x + \tau_m v^{e,m+1}(x) \,|\, x \in \Gamma_h^m\} \subset U_h^{m+1}, \quad m = 0,\dots,N-1, \tag{5.4}$$

provided that h, τ are sufficiently small. To see this, let $x \in \Gamma_h^m$ and choose an element $T \in \mathscr{T}_h$ such that $x \in T$. Then,

$$\Phi^{m+1}(x + \tau_m v^{e,m+1}(x)) = \Phi^m(x) + \tau_m \nabla\Phi^m(x)\cdot v^{e,m+1}(x) + \tau_m \Phi_t(x,t_m) + R_m(x),$$

where $|R_m(x)| \le c\tau_m^2$. Observing that $\Phi_t + \nabla\Phi\cdot v = 0$ on $\bigcup_{t\in(0,T)} \Gamma(t)\times\{t\}$ we write

$$\begin{aligned}
&\nabla\Phi^m(x)\cdot v^{e,m+1}(x) + \Phi_t(x,t_m)\\
&\quad = \nabla\Phi^m(x)\cdot v^{m+1}(p^{m+1}(x)) + \Phi_t(x,t_m)\\
&\quad = \big(\nabla\Phi^m(x) - \nabla\Phi^{m+1}(p^{m+1}(x))\big)\cdot v^{m+1}(p^{m+1}(x))\\
&\qquad + \Phi_t(x,t_m) - \Phi_t(p^{m+1}(x),t_{m+1}),
\end{aligned}$$

so that

$$\begin{aligned}
\left|\Phi^{m+1}(x + \tau_m v^{e,m+1}(x))\right| &\le |\Phi^m(x)| + c\tau_m \left|x - p^{m+1}(x)\right| + c\tau_m^2\\
&\le |\Phi^m(x)| + c\tau_m \left|\Phi^{m+1}(x)\right| + c\tau_m^2 \le c(h^2 + \tau_m^2),
\end{aligned}$$

in view of (2.5) and since $|\Phi^m(x)| = |\Phi^m(x) - I_h\Phi^m(x)| \le ch^2$. As a result,

$$\left|(I_h\Phi^{m+1})(x + \tau_m v^{e,m+1}(x))\right| \le \left\|I_h\Phi^{m+1} - \Phi^{m+1}\right\|_{L^\infty} + c(h^2+\tau_m^2) \le c(h^2+\tau_m^2) < h$$

provided that $0 < h \le h_1$ and $\tau \le \gamma\sqrt{h}$. Hence, $x + \tau_m v^{e,m+1}(x) \in D_h^{m+1} \subset U_h^{m+1}$ proving (5.4). The result of the lemma follows from inserting $\phi_h \equiv 1 \in V_h^{m+1}$ into (5.2) and using (5.4) together with our assumption that $\int_{\Gamma_h^{m+1}} f^{e,m+1}\,\mathrm{d}\sigma_h = 0$. $\square$

6. Numerical experiments.

6.1. Notes on implementation. The methods were implemented using the Distributed and Unified Numerics Environment (DUNE) [6, 7, 13]. Assembly of the matrices is nonstandard in that the method requires integration over partial elements.

To do so we subdivide the integration areas into simplices using the Triangle [39, 40] and Tetgen [41] packages. In each case, the linear system is solved with the conjugate gradient method until the residual is reduced by a factor of 10^{-8} in comparison to its initial value in the ℓ^2 norm. Due to the lack of shape regularity of Γ_h and D_h, the matrix systems are ill conditioned [37] and so we used a Jacobi preconditioner in order to speed up the convergence of our iterative solver. In practice, we will take U_h to be a subset of a cube shaped domain. The triangulation $\mathscr{T}_h$ will be computed adaptively refining only those elements which intersect the computational domain, either Γ_h or D_h. Given errors E_i and E_{i-1} at two different mesh sizes h_i and h_{i-1}, we calculate the experimental order of convergence by $(\mathrm{eoc})_i = \frac{\log(E_i/E_{i-1})}{\log(h_i/h_{i-1})}$.

6.2. Poisson equation. To test our methods, we present two numerical examples. The first is on a torus and is taken from [34] and the second is on a potato-like surface from [17]. We define the torus through the signed distance function

$$\Gamma = \{x \in \mathbb{R}^3 \,|\, d(x) = 0\}, \quad d(x) = \sqrt{\left(\sqrt{x_1^2 + x_2^2} - R\right)^2 + x_3^2} - r$$

for $R = 1, r = 0.6$. For this example, (2.4) can be calculated analytically. To compute our exact solution, we parameterize the torus by

$$x_1 = (R + r\cos\theta)\cos\varphi, \quad x_2 = (R + r\cos\theta)\sin\varphi, \quad x_3 = r\sin\theta \quad \text{for } \theta, \varphi \in (-\pi, \pi)$$

and take the exact solution

$$u(\theta, \varphi) = \cos(3\varphi)\sin(3\theta + \varphi).$$

For our second example, we set $\Gamma = \{x \in \mathbb{R}^3 \,|\, \Phi(x) = 0\}$, where Φ is given by

$$\Phi(x) = (x_1 - x_3^2)^2 + x_2^2 + x_3^2 - 1.$$

From Φ, we calculate the normal $\nu = \frac{\nabla\Phi}{|\nabla\Phi|}$ and the mean curvature by

$$H = \nabla \cdot \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{1}{|\nabla\Phi|} \sum_{j,k=1}^{3} \left(\delta_{jk} - \frac{\Phi_{x_j}\Phi_{x_k}}{|\nabla\Phi|^2}\right) \Phi_{x_j x_k}.$$

As an exact solution, we take $u(x) = x_1 x_2$ and calculate the right-hand side $f = -\Delta_\Gamma u + u$ as

$$f(x) = 2\nu_1(x)\nu_2(x) + H(x)(x_2\nu_1(x) + x_1\nu_2(x)), \quad x \in \Gamma.$$

For this example, (2.4) cannot be calculated exactly so we approximate using a gradient-descent-like iteration from [36] originally for the closest point operator. The errors $\|u^e - u_h\|_{L^2(\Gamma_h)}$ and the Dirichlet seminorm used in the inner product for *SIF* and *NBM* are shown in Tables 1 and 2. The numerical results confirm the theoretical

Table 1

Error tables of SIF *and* NBM *for the first test problem. These calculations took successively* $38, 69, 128, 240, 359, 641$ *conjugate gradient iterations for* SIF *and* $33, 54, 97, 182, 392, 634$ *conjugate gradient iterations for* NBM*.*

h	$\left\lVert u^e - u_h^{SIF} \right\rVert_{L^2(\Gamma_h)}$	(eoc)	$\left\lVert u^e - u_h^{NBM} \right\rVert_{L^2(\Gamma_h)}$	(eoc)
$2^{-1}\sqrt{3}$	1.67739	—	1.61943	—
$2^{-2}\sqrt{3}$	$7.10825 \cdot 10^{-1}$	1.238650	$7.07220 \cdot 10^{-1}$	1.195260
$2^{-3}\sqrt{3}$	$1.90004 \cdot 10^{-1}$	1.903470	$2.32053 \cdot 10^{-1}$	1.607700
$2^{-4}\sqrt{3}$	$4.73865 \cdot 10^{-2}$	2.003480	$7.17605 \cdot 10^{-2}$	1.693190
$2^{-5}\sqrt{3}$	$1.19721 \cdot 10^{-2}$	1.984800	$1.97350 \cdot 10^{-2}$	1.862430
$2^{-6}\sqrt{3}$	$3.01376 \cdot 10^{-3}$	1.990040	$5.08158 \cdot 10^{-3}$	1.957410

h	$\left\lVert \nabla \left(u^e - u_h^{SIF} \right) \right\rVert_{L^2(\Gamma_h)}$	(eoc)	$\left(\frac{1}{2h} \left\lVert \nabla \left(u^e - u_h^{NBM} \right) \right\rVert_{L^2(D_h)}^2 \right)^{\frac{1}{2}}$	(eoc)
$2^{-1}\sqrt{3}$	$1.13951 \cdot 10^{+1}$	—	$1.14682 \cdot 10^{+1}$	—
$2^{-2}\sqrt{3}$	7.95596	0.518299	8.83655	0.376084
$2^{-3}\sqrt{3}$	4.07793	0.964199	5.13567	0.782931
$2^{-4}\sqrt{3}$	2.04879	0.993065	2.59979	0.982155
$2^{-5}\sqrt{3}$	1.03454	0.985780	1.30732	0.991779
$2^{-6}\sqrt{3}$	$5.17692 \cdot 10^{-1}$	0.998826	$6.54845 \cdot 10^{-1}$	0.997393

Table 2

Error tables of SIF *and* NBM *for the second test problem. These calculations took successively* $25, 53, 103, 196, 298, 585, 1151$ *conjugate gradient iterations for* SIF *and* $37, 68, 116, 153, 297, 580, 1137$ *conjugate gradient iterations for* NBM*.*

h	$\left\lVert u^e - u_h^{SIF} \right\rVert_{L^2(\Gamma_h)}$	(eoc)	$\left\lVert u^e - u_h^{NBM} \right\rVert_{L^2(\Gamma_h)}$	(eoc)
$2^{-1}\sqrt{3}$	$3.31237 \cdot 10^{-1}$	—	$2.31128 \cdot 10^{-1}$	—
$2^{-2}\sqrt{3}$	$9.97842 \cdot 10^{-2}$	1.730980	$9.06054 \cdot 10^{-2}$	1.351020
$2^{-3}\sqrt{3}$	$2.57329 \cdot 10^{-2}$	1.955200	$2.57213 \cdot 10^{-2}$	1.816630
$2^{-4}\sqrt{3}$	$6.59538 \cdot 10^{-3}$	1.964090	$7.43214 \cdot 10^{-3}$	1.791110
$2^{-5}\sqrt{3}$	$1.64586 \cdot 10^{-3}$	2.002610	$1.94710 \cdot 10^{-3}$	1.932450
$2^{-6}\sqrt{3}$	$4.10269 \cdot 10^{-4}$	2.004200	$4.99422 \cdot 10^{-4}$	1.962990
$2^{-7}\sqrt{3}$	$1.02735 \cdot 10^{-4}$	1.997640	$1.26086 \cdot 10^{-4}$	1.985850

h	$\left\lVert \nabla \left(u^e - u_h^{SIF} \right) \right\rVert_{L^2(\Gamma_h)}$	(eoc)	$\left(\frac{1}{2h} \left\lVert \nabla \left(u^e - u_h^{NBM} \right) \right\rVert_{L^2(D_h)}^2 \right)^{\frac{1}{2}}$	(eoc)
$2^{-1}\sqrt{3}$	1.15310	—	1.22526	—
$2^{-2}\sqrt{3}$	$6.46853 \cdot 10^{-1}$	0.834010	$7.78185 \cdot 10^{-1}$	0.654906
$2^{-3}\sqrt{3}$	$3.41718 \cdot 10^{-1}$	0.920634	$4.31069 \cdot 10^{-1}$	0.852196
$2^{-4}\sqrt{3}$	$1.71480 \cdot 10^{-1}$	0.994768	$2.33518 \cdot 10^{-1}$	0.884384
$2^{-5}\sqrt{3}$	$8.55564 \cdot 10^{-2}$	1.003090	$1.21377 \cdot 10^{-1}$	0.944044
$2^{-6}\sqrt{3}$	$4.28811 \cdot 10^{-2}$	0.996532	$6.17478 \cdot 10^{-2}$	0.975030
$2^{-7}\sqrt{3}$	$2.14321 \cdot 10^{-2}$	1.000570	$3.10965 \cdot 10^{-2}$	0.989635

bounds from Theorems 3.2 and 4.3. To compare with other methods, we also include a plot of the error in the L^2-norm (Figure 2) and number of conjugate gradient iterations (Figure 3) against number of degrees of freedom for *SIF* and *NBM* along with the unfitted finite element methods of [34, 12], as well as the surface finite element method (sfem) [17]. The plot shows that the error on Γ_h is almost the same for each of the four methods considered. The errors from the method using full gradients are slightly higher than using projected gradients. However, the computations take fewer conjugate gradient iterations which indicates that the matrices are better conditioned.

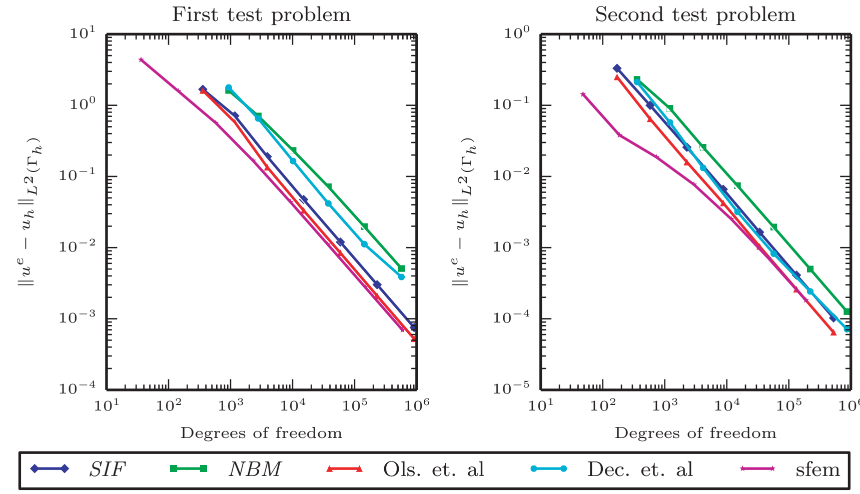

FIG. 2. *Plot of the L^2-error of various methods.*

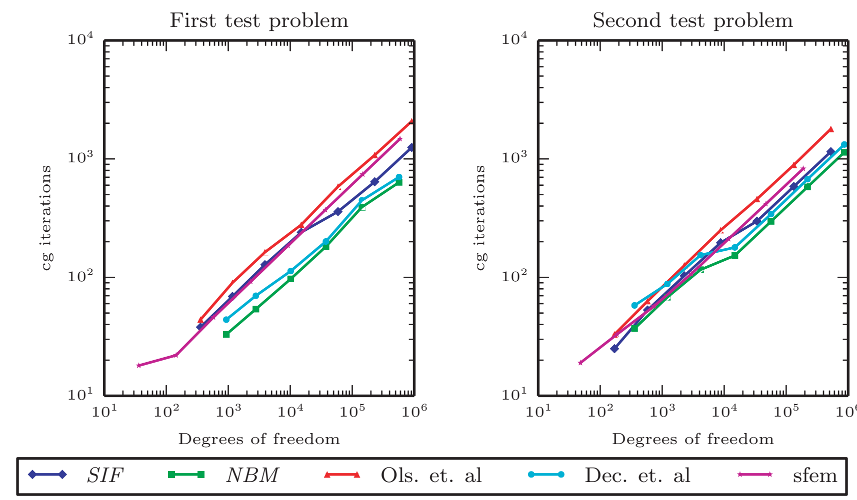

FIG. 3. *Plot of the number of conjugate gradient iterations required to solve various methods.*

Finally, we give a plot of the computation domain and solution for the second problem in Figure 4. Further numerical examples are available in [35].

6.3. Parabolic equation on an evolving curve. For an example of an evolving curve we take $\Gamma(t) = \{x \in \mathbb{R}^2 \,|\, \Phi(x,t) = 0\}$ for

$$\Phi(x,t) = \frac{x^2}{1 + \frac{1}{4}\sin(2\pi t)} + y^2 - 1$$

for $t \in [0, \frac{1}{2}]$. We calculate a right-hand side f so that the exact solution is $u(x,t) = \exp(-4t)x_1x_2$. Here we take $\tau = 2h^2$, so that the expected order of temporal and spatial errors coincide. Indeed the scheme demonstrates second order convergence in the $L^2(\Gamma_h^m)$-norm; see Table 3. Since the chosen time step satisfies the sufficient

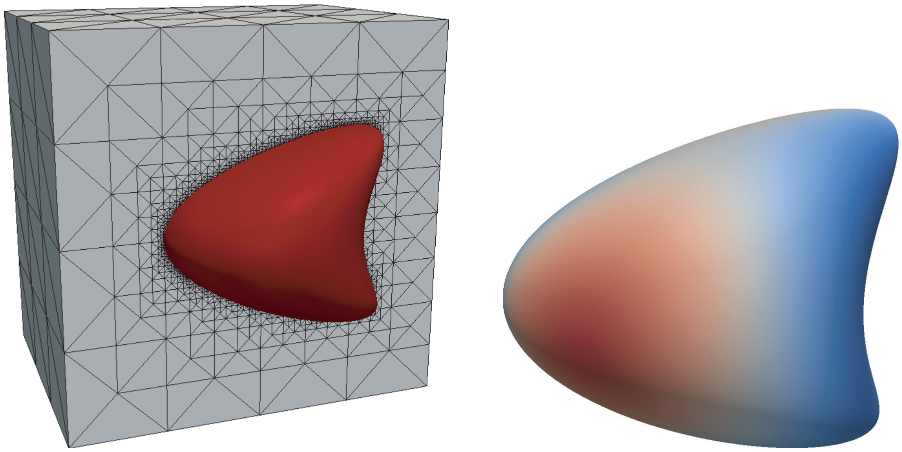

Fig. 4. *Plots of the computation domain (left) and solution (right) with $h = 2^{-6}\sqrt{3}$ for the second problem.*

Table 3
Results of the hybrid scheme for a parabolic equation on an evolving curve.

h	$\max_m \|u(t^m) - u_h^m\|_{L^2(\Gamma_h^m)}$	(eoc)
$2^{-1}\sqrt{2}$	$1.15457 \cdot 10^{-1}$	—
$2^{-2}\sqrt{2}$	$3.25344 \cdot 10^{-2}$	1.82732
$2^{-3}\sqrt{2}$	$8.64172 \cdot 10^{-3}$	1.91258
$2^{-4}\sqrt{2}$	$2.13241 \cdot 10^{-3}$	2.01883
$2^{-5}\sqrt{2}$	$5.42960 \cdot 10^{-4}$	1.97357

constraint on the time step of Lemma 5.1, mass is conserved to the tolerance of the linear solver, as proven for the exact solution of the scheme.

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