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Clifford Algebras, Dynamical Systems and Periodic Orbits

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Abstract

Certain differential systems can be lifted to algebras (of matrices) which greatly simplifies the use of global linearization for nonlinear dynamical systems. Here we shall use Clifford algebras to obtain an interesting collection of systems which exhibit a wide variety of behaviour. In a particular we shall use global linearization to show that Lyapunov stability theory for linear systems can be directly extended to this situation and that periodic orbits can be explicitly calculated.

Keywords :Global Stabilizability , Nonlinear Systems, Switching Manifolds .

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1 Introduction

The Carleman linearization technique has been developed in a number of papers [1],[2],[3]. For a nonlinear, one-dimensional equation of the form

$$\dot{x} = f(x) \ , \ x(0) = x_0 \ , \ x(t) \in \mathbf{R} \tag{1.1}$$

the method is very simple. We define the new variables

$$\phi_i = x^i .$$

Then

$$\begin{aligned} \dot{\phi}_i &= i x^{i-1} \dot{x} \\ &= i x^{i-1} f(x) \\ &= \sum_{j=0}^{\infty} a_{ij} x^j \end{aligned}$$

for some values a_{ij} depending on the derivatives of f. Then we can write the equation as an infinite-dimensional linear one:

$$\dot{\Phi} = A\Phi \tag{1.2}$$

where $\Phi = (\phi_0, \phi_1, \cdots)^T$ and $A = (a_{ij})_{0 \le i,j < \infty}$. The solution is

$$\Phi(t) = e^{At}\Phi_0 \ , \ 0 \le t \le T$$

where $\Phi_0 = (1, x_0, x_0^2, x_0^3, \cdots)^T$, provided the Taylor series of the solution of the original system (1.1) converges in $[0, T]$.

The main drawback with the method is that, for vector systems, we must consider Taylor monomials of the form

$$\phi_{i_1 \cdots i_n} = x_1^{i_1} \cdots x_n^{i_n}$$

and proceed as before. However, we then obtain a tensorial equation of the form

$$\dot{\Phi} = A\Phi$$

where $\Phi = (\phi_{i_1 \cdots i_n})_{(i_j \geq 0)}$ is a rank-n tensor and A is a tensor operator. This approach has been used in [4]. One can, of course, string Φ out into a long vector as in [6], but this destroys the essential structure of A.

In this paper we shall consider systems which can be 'lifted' to associative algebras, so that we have

$$\dot{x} = f(x) \ , \ x(0) = x_0 \ , \ x \in \mathcal{A} \tag{1.3}$$

where $x_0 \in \mathcal{A}$ for some associative algebra $\mathcal{A}$. Assuming that f has a convergent Taylor series in this algebra, then we can use the expansion functions x^i which are well-defined in the algebra. This will lead to a matrix linearization of (1.3) since these functions can be treated a essentially 'scalar' objects.

The most useful algebras for our purposes are the standard matrix algebras and the Clifford algebras, generalizing the complex numbers and quaternions. In the next section we shall give a brief introduction to the Clifford algebras, in section 3 we consider the lifting of dynamical systems to Clifford algebras, in section 4 their application to linearization will be discussed and finally in section 5 the existence of periodic orbits for Clifford systems will be studied.

2 Clifford Algebras

In this section we shall give a brief introduction to Clifford algebras and their properties which are needed for global linearization of systems. For more details see [7] . **A quadratic**

space (V,Q) is a vector space over a field $\mathbf{F}$ $(= \mathbf{R}, \mathbf{C}$ or $\mathbf{H})$ together with a nondegenerate quadratic form Q, i.e. a mapping $Q : V \longrightarrow \mathbf{F}$ such that

$$(i)\ \ Q(\lambda v) = \lambda^2 Q(v)\ ,\ \ \lambda \in \mathbf{F}\ ,\ \ v \in V$$

$$(ii)\ \ B(v,w) \triangleq \frac{1}{2}(Q(v) + Q(w) - Q(v-w))\ ,\ \ v,w \in V$$

is bilinear. The associated form $B(v,w)$ can be used to define an inner product on V and if $\{e_j\}$ is a basis for V, then

$$Q(v) = \sum_{j,k} B(e_j, e_k) v_j v_k\ \ ,$$

where $v = \sum_j v_j e_j$. We can always choose the basis to be B-orthogonal, so that, in this case

$$Q(v) = \sum_j Q(e_j) v_j^2.$$

2.1 Examples

(a) The quadratic form

$$Q_{p,q}(x) = -(x_1^2 + \cdots + x_p^2) + (x_{p+1}^2 + \cdots + x_{p+q}^2),$$

$x \in \mathbf{R}^{p+q}$ gives rise to a real quadratic Minkowski space denoted by $(\mathbf{R}^{p,q}, Q_{p,q})$. Note that $(\mathbf{R}^{n,0}, Q_{n,0})$ is just $\mathbf{R}^n$ with the quadratic form $-\|x\|^2$ while $(\mathbf{R}^{0,n}, Q_{0,n})$ is $\mathbf{R}^n$ with the quadratic form $\|x\|^2$ (i.e. usual Euclidean space).

(b) $(\mathbf{C}^n, Q_n)$ is a complex quadratic space with

$$Q_n(z) = z_1^2 + \cdots + z_n^2.$$

The associated blinear form is $B_n(z,w) = \sum_{i=1}^n z_i w_i$ (rather than the usual form $\sum_{i=1}^n z_i \overline{w_i}$).

Let (V,Q) be a (finite-dimensional) quadratic space. A **Clifford algebra** (A,ν) for (V,Q) is an associative algebra A over $\mathbf{F}$ with identity 1 together with an $\mathbf{F}$-linear embedding

$\nu : V \longrightarrow A$ of V into A such that $(\nu(v))^2 = -Q(v)1$, $v \in V$ and A is generated by $\{\nu(v) : v \in V\}$ and $\{\lambda 1 : \lambda \in \mathbf{F}\}$. Identifying V as a subset of A we simply write the Clifford condition as

$$v^2 = -Q(v)1.$$

Given a quadratic space (V, Q) we can generate a Clifford algebra $\mathcal{A}$ as

$$\mathcal{A} = T(V)/I$$

where $T(V) = \sum_{k=0}^{\infty} V \otimes \cdots \otimes V$ is the tensor algebra over V and I is the two-sided ideal generated by the set $\{v \otimes v + Q(v)1 : v \in V\}$. The embedding $\nu : V \longrightarrow \mathcal{A}$ is just the projection $\pi : T(V) \longrightarrow T(V)/I = \mathcal{A}$ restricted to V.

For any nondegenerate quadratic space (V, Q) we can find an orthonormal basis e_i , $1 \leq i \leq n$ for which $Q(e_i) = \pm 1$. Expanding $(u+v)^2$ for any $u, v \in V$ we see that

$$B(u,v) = -\frac{1}{2}(uv + vu) \ , \ u, v \in V$$

so that

$$e_j e_k + e_k e_j = -2Q(e_j)\delta_{jk} \ , \ 1 \leq j,k \leq n. \tag{2.1}$$

Define

$$e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k} \ , \ e_\emptyset = 1$$

for each subset α of $\{1, \cdots, n\}$ such that $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$. From (2.2) we see that a Clifford algebra for V is generated by all the e_α , $\alpha \in 2^{\{1,\cdots,n\}}$, and so has dimension $\leq 2^n$. If the algebra has dimension 2^n it is called a **universal Clifford algebra**.

2.3 Examples We shall characterize the real Clifford algebra over $\mathbf{R}^{p,q}$ for $p + q \leq 2$.

These are denoted by $\mathcal{A}_{p,q}$. First let

$$1 = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

denote the usual Pauli matrices and define

$$E_0 = \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, E_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}; E_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

Then

$$\sigma_i^2 = I \;, \;, 0 \leq i \leq 3 \;, \; E_0^2 = I \;, \; E_i^2 = -I \;, 1 \leq i \leq 3.$$

Since $\mathbf{R}^{0,0}$ has dimension 0, its universal Clifford algebra has dimension $2^0 = 1$ and so

$$\mathcal{A}_{0,0} = \{\lambda 1 : \lambda \in \mathbf{R}\} \equiv \mathbf{R},$$

together with the embedding $\nu : \mathbf{R}^{0,0} \longrightarrow \mathcal{A}_{0,0}$ given by $\nu(0) = 0$. Since $\sigma_3^2 = I$ the embedding $\nu : \mathbf{R}^{1,0} \longrightarrow \mathcal{A}_{1,0}$ given by $\nu(y) = y\sigma_3$ gives rise to the realization

$$\mathcal{A}_{1,0} = \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix} : x, y \in \mathbf{R} \right\} \equiv \mathbf{R} \oplus \mathbf{R}$$

with basis $e_\emptyset = 1 = \sigma_0 \;, \; e_1 = \sigma_3$.

Similarly, $\mathcal{A}_{0,1}$ has the realization

$$\mathcal{A}_{0,1} = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} : x, y \in \mathbf{R} \right\} \equiv \mathbf{C}$$

with basis $e_\emptyset = 1 = \sigma_0 \;, \; e_1 = E_2$.

For $\mathcal{A}_{2,0}$ note that

$$\begin{aligned}(x_1\sigma_1 + x_2\sigma_3)^2 = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}^2 &= \begin{bmatrix} x_1^2 + x_2^2 & 0 \\ 0 & x_1^2 + x_2^2 \end{bmatrix} \\ &= (x_1^2 + x_2^2)\sigma_0 \\ &= -Q_{2,0}(x_1, x_2)1\end{aligned}$$

and so $\nu : (x_1, x_2) \longrightarrow x_1\sigma_1 + x_2\sigma_3$ is the required embedding. Hence, $\mathcal{A}_{2,0}$ has basis $e_\emptyset = \sigma_0$, $e_1 = \sigma_1$, $e_2 = \sigma_3, e_{1,2} = \sigma_1\sigma_3$, giving the realization

$$\mathcal{A}_{2,0} = \left\{ \begin{bmatrix} x_0 + x_1 & x_2 + x_3 \\ x_2 - x_3 & x_0 - x_1 \end{bmatrix} : x_0, \cdots, x_3 \in \mathbf{R} \right\} \equiv \mathbf{R}^4$$

Similarly,

$$\mathcal{A}_{0,2} = \left\{ \begin{bmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{bmatrix} : x_0, \cdots, x_3 \in \mathbf{R} \right\} \equiv \mathbf{H}$$

Higher order Clifford algebras can be generated from the above special cases since it is easy to see that if $\mathcal{A}(V, Q)$ denotes the universal Clifford algebra for (V, Q) then

$$M(2, \mathcal{A}(V, Q)) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathcal{A}(V, Q) \right\}$$

is a universal Clifford algebra of dimension $4 \times \dim(\mathcal{A}(V, Q))$. For example, consider the quadratic space

$$(V \oplus \mathbf{R}^{0,2}, -Q(v) + x^2 + y^2).$$

It is realized by matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in \mathcal{A}(V, Q)$ under the embedding $\nu : (v, x, y) \longrightarrow \begin{bmatrix} ix & iy + v \\ iy - v & -ix \end{bmatrix}$. Similarly, $(V \oplus \mathbf{R}^{1,0}, -Q(v) - x^2)$ is realized by a subalgebra of $(V \oplus \mathbf{R}^{2,0}, -Q(v) - x^2 - y^2)$ under the embedding

$$\nu : (v, x) \longrightarrow \begin{bmatrix} 0 & v + x \\ -v + x & 0 \end{bmatrix}$$

while $(V \oplus \mathbf{R}^{0,1}, Q(v) + x^2)$ is realized by a subalgebra of $(V \oplus \mathbf{R}^{1,1}, Q(v) + x^2 - y^2)$ under the embedding

$$\nu : (v, x) \longrightarrow \begin{bmatrix} 0 & v+x \\ v-x & 0 \end{bmatrix}.$$

In fact, it can be shown that if $\mathcal{A}(V,Q)$ is the universal Clifford algebra for a non-degenerate real quadratic space (V,Q), then it is isomorphic to a real subalgebra of one of the matrix algebras $\mathbf{C}^{2^m \times 2^m}$ or $\mathbf{C}^{2^m \times 2^m} \oplus \mathbf{C}^{2^m \times 2^m}$ if dim V is even or odd, respectively.

Three important operators are defined on a Clifford algebra. These are the **principal automorphism** ('), given by

$$e'_\alpha = (-1)^{|\alpha|} e_\alpha \ , \ \alpha \in 2^{\{1,\cdots,n\}}$$

the **reversion** (*), given by

$$e^*_\alpha = (e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_k})^* = e_{\alpha_k} \cdots e_{\alpha_2} e_{\alpha_1}$$

and the **conjugation** operation ($^-$), given by

$$\overline{e}_\alpha = (e^*_\alpha)' = (e'_\alpha)^*.$$

All these operators are extended by linearity to the whole of the algebra. Using these operators we can define the **norm** function

$$\Delta : \mathcal{A} \longrightarrow \mathcal{A}$$

by

$$\Delta(x) = \overline{x}x.$$

Care should be taken, however, with definition, since it may not resemble the usual idea of norm in a vector space- it is more like the deteminant of a matrix, and indeed replaces the det function in the general theory of spin groups. This can be seen clearly in the cases $\mathcal{A}_{0,1}$ and $\mathcal{A}_{1,0}$. In the former case we have the expansion of any element $z = xe_0 + ye_2 \in \mathcal{A}_{0,1}$ in terms of the basis e_0, e_2. Then

$$\begin{aligned} \Delta(z) &= (xe_0 - ye_2)(xe_0 + ye_2) \\ &= (x^2 + y^2)I \end{aligned}$$

and this is indeed the standard Euclidean norm. In the latter case, however, we have $z = x\sigma_0 + y\sigma_3$ for any element in $\mathcal{A}_{1,0}$ and then

$$\begin{aligned} \Delta(z) &= (x\sigma_0 - y\sigma_3)(x\sigma_0 + y\sigma_3) \\ &= (x^2 - y^2)I \; . \end{aligned}$$

In fact, we have

$$\Delta(x) = ||x||^2 I \tag{2.2}$$

always in the Euclidean case, i.e. $\mathcal{A}_{0,n}$ for any n, where $||x||^2$ is the Euclidean norm, considering x in terms of a basis e_α, $\alpha \in 2^{\{1,\cdots,n\}}$ of $\mathcal{A}_{0,n}$: thus, if

$$x = \sum_{\alpha \in 2^{\{1,\cdots,n\}}} e_\alpha x_\alpha \tag{2.3}$$

then

$$||x||^2 = \sum_{\alpha \in 2^{\{1,\cdots,n\}}} x_\alpha^2 . \tag{2.4}$$

Moreover, this norm is independent of the basis. We shall use the norm given by (2.4) and (2.5) even in the non-Euclidean case, but it should be noted that (2.3) no longer holds and the norm is not independent of the basis $\{e_\alpha\}$ although all such norms are equivalent.

3 Lifting Nonlinear Dynamical Systems to Clifford Algebras

Consider a nonlinear differential equation of the form

$$\dot{x} = f(x) \ , \ x(0) = x_0 \in \mathbf{R}^n. \tag{3.1}$$

If $\mathcal{A}_{p,q}$ is a universal Clifford algebra, with $p+q \geq n$, we say that the equation (3.1) can be **lifted** to $\mathcal{A}_{p,q}$ if there is a linear embedding $L : \mathbf{R}^n \longrightarrow \mathcal{A}_{p,q}$ such that

$$L(x_1, \cdots, x_n) = x_1 e_{\alpha_1} + \cdots + x_n e_{\alpha_n}$$

for some basis elements e_{α_i} in $\mathcal{A}_{p,q}$ and a function

$$F : \mathcal{A}_{p,q} \longrightarrow \mathcal{A}_{p,q}$$

such that

$$\langle F, e_{\alpha_i} \rangle |_{\xi_j = 0} = f_i$$

where

$$X = \sum_{j \in J} \xi_j e_{\beta_j} + \sum_{i=1}^{n} x_i e_{\alpha_i}$$

and $\{e_{\beta_j}\}_{j \in J} \cup \{e_{\alpha_i}\}_{1 \leq i \leq n}$ is a complete basis of $\mathcal{A}_{p,q}$ such that $\{e_{\beta_j}\}_{j \in J}$ and $\{e_{\alpha_i}\}_{1 \leq i \leq n}$ are disjoint. We then consider the 'extended' differential equation

$$\dot{X} = F(X) \quad , \quad X = L(x_0) \in \mathcal{A}_{p,q}. \tag{3.2}$$

3.1 Lemma *If the equation (3.2) is a lifting of equation (3.1) to $\mathcal{A}_{p,q}$, then any solution $X(t)$ of (3.2) with $\xi_i = 0$, $j \in J$ gives rise to a solution $x(t)$ of (3.1):*

$$x_i(t) = \langle X, e_{\alpha_i} \rangle.$$

Proof We have

$$\langle \dot{X}, e_{\alpha_i} \rangle = \frac{d}{dt}\langle X, e_{\alpha_i} \rangle = \frac{d}{dt}x_i$$

and

$$\langle F(X), e_{\alpha_i} \rangle|_{\xi_j=0} = f_i(x)$$

and the result follows. □

3.2 Examples

(i) The equations

$$\begin{aligned} \dot{x}_1 &= x_1^2 + x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{aligned}$$

lifts to $\mathcal{A}_{1,0}$, since it can be written

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 & \dot{x}_2 \\ \dot{x}_2 & \dot{x}_1 \end{pmatrix} &= \begin{pmatrix} x_1^2 + x_2^2 & 2x_1x_2 \\ 2x_1x_2 & x_1^2 + x_2^2 \end{pmatrix} \\ &= \begin{pmatrix} x_1 & x_2 \\ x_2 & x_1 \end{pmatrix}^2 \end{aligned}$$

i.e.

$$\dot{X} = X^2$$

in $\mathcal{A}_{1,0}$. The system

$$\begin{aligned} \dot{x}_1 &= x_1^2 - x_2^2 \\ \dot{x}_2 &= 2x_1x_2 \end{aligned}$$

can be lifted to $\mathcal{A}_{0,1}$, since

$$\begin{pmatrix} \dot{x}_1 & \dot{x}_2 \\ -\dot{x}_2 & \dot{x}_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}^2$$

(ii) The system

$$\begin{aligned} \dot{x}_1 &= x_1^3 - 3x_2^2x_1 - 3x_3^2x_1 - 3x_4^2x_1 \\ \dot{x}_2 &= 3x_1^2x_2 - x_2^3 - x_3^2x_2 - x_4^2x_1 \\ \dot{x}_3 &= 3x_1^2x_3 - x_3^3 - x_2^2x_3 - x_4^2x_3 \\ \dot{x}_4 &= 3x_1^2x_4 - x_4^3 - x_2^2x_4 - x_3^2x_4 \end{aligned} \tag{3.3}$$

can be lifted to the system

$$\dot{X} = X^3$$

on $\mathcal{A}_{0,2}$, where

$$X = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} .$$

□

Now let

$$\dot{X} = F(X) \quad , \quad X_0 \in \mathcal{A}_{p,q} \tag{3.4}$$

be a differential equation defined on a universal Clifford algebra $\mathcal{A}_{p,q}$ and suppose that F has a convergent Taylor series (in the Hilbert space norm introduced in section 2). Thus,

$$F(X) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \underbrace{X \cdot X \cdots X}_{i} \tag{3.5}$$

where $a_i \in \mathbf{F}$, $i \geq 0$. The solution of (3.4) is given directly by the Lie series:

$$X(t) = \exp\left(tF\frac{d}{dX}\right) X\bigg|_{X=X_0} . \tag{3.6}$$

3.3 Example Consider the system

$$\dot{X} = X^3$$

on $\mathcal{A}_{0,2}$ given in example 3.2, with $X_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The solution is given by (3.6), i.e.

$$\begin{aligned} X(t) &= \exp\left(tX^3 \frac{d}{dX}\right) X\Big|_{X=X_0} \\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!} \left(X^3 \frac{d}{dX}\right)^i X\Big|_{X=X_0} \\ &= \sum_{i=0}^{\infty} \frac{t^i}{i!}(2i-1)!!X_0^{2i+1} \end{aligned}$$

where $n!!$ is the double factorial given by

$$n!! = n(n-2)(n-4)\cdots \quad , \quad 1!! = 1 \ , \ 0!! = 1 \ , \ n!! = 1 \ , n < 0.$$

Thus, if

$$X_0 = \begin{bmatrix} x_{10} + ix_{20} & x_{30} + ix_{40} \\ -x_{30} + ix_{40} & x_{10} - ix_{20} \end{bmatrix}$$

we have

$$\begin{bmatrix} \dot{x}_1 + i\dot{x}_2 & \dot{x}_3 + i\dot{x}_4 \\ -\dot{x}_3 + i\dot{x}_4 & \dot{x}_1 - i\dot{x}_2 \end{bmatrix} = \sum_{i=0}^{\infty} \frac{t^i}{i!}(2i-1)!! \begin{bmatrix} x_{10} + ix_{20} & x_{30} + ix_{40} \\ -x_{30} + ix_{40} & x_{10} - ix_{20} \end{bmatrix}^{2i+1}$$

which gives the solution to (3.3) in the form

$$\begin{aligned} x_1(t) &= x_{10} + t(x_{10}^3 - 3x_{20}^2 x_{10} - 3x_{30}^2 x_{10} - 3x_{40}^2 x_{10} + \cdots \\ x_2(t) &= x_{20} + t(-x_{20}^3 + 3x_{10}^2 x_{20} - x_{30}^2 x_{20} - x_{40}^2 x_{20} + \cdots \\ x_3(t) &= x_{30} + t(-x_{30}^3 + 3x_{10}^2 x_{30} - x_{20}^2 x_{30} - x_{40}^2 x_{30} + \cdots \qquad (3.7) \\ x_4(t) &= x_{40} + t(-x_{40}^3 + 3x_{10}^2 x_{40} - x_{20}^2 x_{40} - x_{30}^2 x_{40} + \cdots \end{aligned}$$

(3.8)

4 Global Linearization of Clifford Differential Systems

Consider now a Clifford differential equation of the form

$$\dot{X} = F(X) \quad , \quad X(0) = X_0 \in \mathcal{A}_{p,q} \tag{4.1}$$

where

$$F(X) = \sum_{i=0}^{\infty} \frac{a_i}{i!} \underbrace{X \cdot X \cdots X}_{i}$$

Define the Taylor monomials by

$$\phi_i = \underbrace{X \cdot X \cdots X}_{i} = X^i \quad , \quad i \geq 0.$$

Then

$$\begin{aligned}
\dot{\phi}_i &= \sum_{j=1}^{i} X \cdots \dot{X} \cdots X \\
&= \sum_{j=1}^{i} \sum_{k=0}^{\infty} X \cdots \left(\frac{a_k}{k!} \underbrace{X \cdot X \cdots X}_{k} \right) \cdots X \\
&= \sum_{k=0}^{\infty} i \frac{a_k}{k!} X^{k+i-1} \\
&= \sum_{k=0}^{\infty} i \frac{a_k}{k!} \phi_{k+i-1} \\
&= \sum_{\ell=0}^{\infty} \alpha_{i\ell} \phi_\ell
\end{aligned}$$

where

$$\alpha_{i\ell} = \begin{cases} i \frac{a_{\ell-i+1}}{(\ell-i+1)!} & \text{if } \ell + 1 \geq i \\ 0 & \text{otherwise} \end{cases}$$

Hence, writing $\Phi = (\phi_0, \phi_1, \phi_2, \cdots)^T$, we have

$$\dot{\Phi} = A\Phi \tag{4.2}$$

where $A = (\alpha_{ij})_{0 \le i,j < \infty}$. Note that $\Phi \in \mathcal{A}_{p,q}^{\infty} \triangleq \oplus_{k=0}^{\infty} \mathcal{A}_{p,q}$ and the latter can be made into a Banach space by defining

$$||\Phi||_{\mathcal{A}_{p,q}^{\infty}} = \sum_{i=0}^{\infty} \frac{||\phi_i||_{\mathcal{A}_{p,q}}}{i!}. \tag{4.3}$$

Then (as has been shown in) e^{At} exists as a bounded operator for $t \in [0, \tau)$ on $\left(\mathcal{A}_{p,q}^{\infty,T}, ||\cdot||_{\mathcal{A}_{p,q}^{\infty}}\right)$ and the solution of (4.2) is given by

$$\Phi(t) = e^{At}\Phi(0)$$

where

$$\Phi(0) = (1, X(0), X^2(0), \cdots)^T \quad ,$$

τ is the maximal time of existence of the solution and $\mathcal{A}_{p,q}^{\infty,T}$ is the subset of $\mathcal{A}_{p,q}^{\infty}$ consisting of all elements of the form $(1, X, X^2, X^3, \cdots)^T$.

4.1 Remark If $F(0) = 0$, i.e. $a_0 = 0$, we can consider the vector $\Phi = (\phi_1, \phi_2, \cdots)^T$, i.e. we only need to take into account Taylor monomials for $i > 0$.

4.2 Example Consider the system in example 3.2(ii), i.e.

$$\dot{X} = X^3.$$

Then

$$\phi_i = X^i \ , \ i > 0$$

and so

$$\begin{aligned} \dot{\phi}_i &= \sum_{k=0}^{i-1} X^k \dot{X} X^{i-k-1} \\ &= iX^{i+2} \\ &= i\phi_{i+2} \ . \end{aligned}$$

Hence the system is equivalent to the system

$$\dot{\Phi} = A\Phi$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & \cdot & \cdot & \cdot & \cdots \\ 0 & 0 & 0 & 2 & \cdots & \cdot & \cdots \\ 0 & 0 & 0 & 0 & 3 & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} .$$

This is easily checked to lead to the same solution as in (3.7). □

The linearized structure (4.2) is particularly useful in generalizing a variety of linear results to nonlinear systems which lift to Clifford algebras. First note that since the Clifford norm $\|\cdot\|_{\mathcal{A}_{p,q}}$ induces a norm on $\mathbf{R}^{p+q}$ equivalent to the Euclidean norm, the stability of a Clifford system (4.1) is equivalent to that of the underlying system in $\mathbf{R}^{p+q}$. The next result is proved in formally the same way as Lyapunov's classical theorem:

4.1 Theorem *If $F(0) = 0$, the system (4.1) is stable if and only if, given any positive definite (infinite) matrix Q, there exists a positive definite matrix P such that*

$$A^T P + PA = -Q ,$$

where by **positive definite** *we mean that*

$$\overline{\Phi}^T P\Phi > 0$$

for all $\Phi \in \mathcal{A}_{p,q}^{\infty}$ of the form $\Phi = (X, X^2, X^3, \cdots)$. □

(Here, $\overline{\Phi} = (\overline{X}, \overline{X}^2, \overline{X}^3, \cdots)$ where $\overline{X}$ denotes the conjugate of X.) A Lyapunov function is then given directly by

$$V = \overline{\Phi}^T P\Phi .$$

4.2 Example Consider the system

$$\dot{X} = -X^3$$

on $\mathcal{A}_{0,2}$, where

$$X = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}.$$

Let

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 1 & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot \\ 1 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Then,

$$A = -\begin{pmatrix} 0 & 0 & 1 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 2 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and so

$$A^T P + PA = -Q.$$

Also,

$$V = \overline{\Phi}^T P \Phi = \overline{X} X = \Delta(X) = \|x\|^2$$

and

$$\dot{V} = -\overline{\Phi}^T Q \Phi = -\overline{X}^3 X^3 = -\|x\|^6.$$

Remark 'Positive definite' now means something rather different than in the usual sense.

5 Periodic Orbits

The proof of the existence of periodic orbits of nonlinear dynamical systems has a long history, including justifications of the harmonic linearization techniques [8], Lyapunov theory [5] and the use of index theory [9]. The method developed here will allow us to prove directly the existence of periodic orbits for systems which can be lifted to a Clifford algebra. Thus, consider again the system (4.1) on $\mathcal{A}_{p,q}$ and its linearization (4.2) on $\mathcal{A}_{p,q}^{\infty}$. Suppose that $X(0)$ lies on a periodic orbit of period τ. Then

$$e^{A\tau}\Phi(0) = \Phi(0) \tag{5.1}$$

where $\Phi(0) = (X(0), X^2(0), X^3(0), \cdots)^T$. Hence, $e^{A\tau}$ has an eigenvalue 1 with eigenvector $\Phi(0)$. Thus, although A is not a bounded operator on $\mathcal{A}_{p,q}^{\infty,T}$, an obvious extension of the spectral mapping theorem implies that A has an eigenvalue $2\pi i/\tau$ (although note that A will **not** generally have an eigenvector of the form $\Phi(0)$). Hence we have

5.1 Theorem *A necessary condition for (4.1) to have a periodic orbit of period τ is that the associated operator $A \subseteq \mathcal{L}(\mathcal{A}_{p,q}^{\infty})$ has an eigenvalue of $2\pi i/\tau$.* □

A sufficient condition is that A satisfies the necessary condition of theorem 5.1 and that $e^{A\tau}$ has an eigenvector corresponding to the eigenvalue $2\pi i/\tau$ of the form $\Phi(0)$ above. Moreover, each element of the vector $e^{At}\Phi(0)$ is a formal power series which must converge for all $t \in [0, \tau]$. In many cases we can evaluate e^{At} directly and determine conditions for the existence of periodic orbits.

5.2 Example Consider the system

$$\dot{X} = iX + X^3 \tag{5.2}$$

for X in a complex Clifford algebra A. The linearization (4.2) is given by the operator

$$\begin{pmatrix} i & 0 & 1 & & & & & \\ & 2i & 0 & 2 & & & & \\ & & 3i & 0 & 3 & & & \\ & & & 4i & 0 & 4 & & \\ & & & & & \ddots & & \\ & & & & & & & \ddots \end{pmatrix}.$$

Clearly A (as an operator on $\mathcal{A}^\infty$) has i as an eigenvalue. To determine e^{At} consider the systems of equations

$$\begin{aligned} \dot{x}_1 &= ix_1 + x_3 \\ \dot{x}_2 &= 2ix_2 + 2x_4 \\ &\cdots \\ \dot{x}_k &= kix_k + kx_{k+2} \\ &\cdots \end{aligned} \tag{5.3}$$

Put $y_k = e^{-ikt}x_k$. Then we have

$$\begin{aligned} \dot{y}_k &= -ike^{-ikt}x_k + e^{-ikt}\dot{x}_k \\ &= ke^{2it}y_{k+1} \quad , \quad 1 \leq k < \infty. \end{aligned}$$

We must solve these equations with the initial condition $(0, 0, \cdots, \underbrace{1}_{2p+1}, 0, \cdots)$. Thus,

$$\begin{aligned} \dot{y}_{2p+1} &= 0 \Longrightarrow y_{2p+1} = 1 \\ \dot{y}_{2p-1} &= (2p-1)e^{2it} \cdot 1 \end{aligned}$$

so

$$y_{2p-1} = (2p-1)\frac{1}{2i}(e^{2it} - 1).$$

Next,

$$\begin{aligned} \dot{y}_{2p-3} &= (2p-3)e^{2it}y_{2p-1} \\ &= (2p-3)(2p-1)\frac{1}{2i}e^{2it}(e^{2it}-1) \end{aligned}$$

and so

$$y_{2p-3} = (2p-3)(2p-1)\frac{1}{2i}\frac{1}{4i}(e^{2it}-1)]^2 \ .$$

Continuing in this way we obtain

$$\begin{aligned} y_1 &= (2p-1)!!\frac{1}{2i}\frac{1}{4i}\cdots\frac{1}{p\cdot 2i}(e^{2it}-1)^p \\ &= \frac{(2p-1)!!}{(2i)^p p!}(e^{2it}-1)^p. \end{aligned}$$

Thus,

$$x_1 = \frac{(2p-1)!!}{(2i)^p p!}e^{2it}(e^{2it}-1)^p.$$

It follows that the solution of (5.2) through X_0 is given by

$$X(t) = e^{it}X_0\sum_{p=0}^{\infty}\frac{(2p-1)!!}{p!}\left(\frac{(e^{2it}-1)X_0^2}{2i}\right)^p \ .$$

This power series has radius of convergence

$$\begin{aligned} r &= 1/\limsup_p \sqrt[p]{\frac{(2p-1)!!}{p!}} \\ &= 1/limsup_p \sqrt[p]{\left(2-\frac{1}{p}\right)\left(2-\frac{1}{p-1}\right)\cdots\left(2-\frac{1}{1}\right)} \\ &\geq \frac{1}{2} \ . \end{aligned}$$

Hence, if

$$\frac{|(e^{2it}-1)X_0^2|}{2} \leq \frac{1}{2}$$

then the solution is given by the above power series. Since A has i as an eigenvalue, all solutions with

$$|X_0|^2 \leq \frac{1}{sup_{0 \leq t \leq 2\pi} |e^{2it} - 1|} = \frac{1}{2}$$

lie on periodic orbits of period 2π. In particular, if $\mathcal{A} = \mathcal{A}_{0,1} \cong \mathbf{C}$, then the equation becomes

$$\frac{d}{dt}(x_1 + ix_2) = i(x_1 + ix_2) + (x_1 + ix_2)^3$$

i.e.

$$\begin{aligned} \dot{x}_1 &= -x_2 + x_1^3 - 3x_1x_2^2 \\ \dot{x}_2 &= x_1 - x_2^3 + 3x_1^2x_2 \end{aligned}$$

The phase plane portrait of this system is shown in fig. 1.

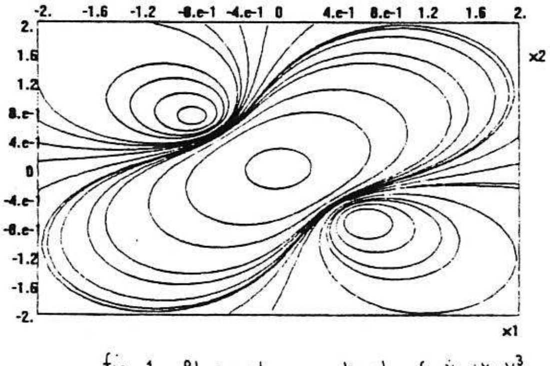

fig. 1. Phase-plane portrait of $\dot{x} = ix + x^3$

6 Conclusions

In this paper we have studied nonlinear dynamical systems which can be lifted to Clifford algebras. Such systems have a considerable degree of symmetry and exhibit interesting behaviour. In particular, we have shown that Lyapunov's equation for linear system stability directly generalizes to Clifford systems and that periodic orbits may often be computed explicitly for these types of system. We have presented only some very simple examples to illustrate the theory. Clearly, many other kinds of systems are expressible in this form and a thorough study of some higher-dimensional examples should provide some very interesting behaviour. For example, are any of these systems chaotic? If so, the global linearization technique will provide a method for obtaining explicit characterization of their properties, as in the case of periodic orbits. These questions will be examined in a future paper.

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