THE GEOMETRY OF BRAUER GRAPH ALGEBRAS AND CLUSTER MUTATIONS

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Abstract. In this paper we establish a connection between ribbon graphs and Brauer graphs. As a result, we show that a compact oriented surface with marked points gives rise to a unique Brauer graph algebra up to derived equivalence. In the case of a disc with marked points we show that a dual construction in terms of dual graphs exists. The rotation of a diagonal in an m-angulation gives rise to a Whitehead move in the dual graph, and we explicitly construct a tilting complex on the related Brauer graph algebras reflecting this geometrical move.

Introduction

Recently there has been widespread renewed interest — spurred on by the many interesting developments in cluster algebras — in tilting phenomena. In particular, the geometrical interpretation of cluster-tilting objects in cluster categories as triangulations suggests the consideration of similar models in relation to other algebras. One such area of interest is the class of finite-dimensional self-injective algebras.

Finite dimensional self-injective algebras encompass a large variety of algebras, such as the group algebras of finite groups, Hecke algebras and special biserial algebras. The last family is particularly interesting, as large parts of the representation theory and (co)homology are well-studied and understood. The class of symmetric special biserial algebras coincides with the class of Brauer graph algebras [32]. Brauer graph algebras can be described in two important ways, firstly as a quotient of the path algebra of a quiver, and secondly via a graph, the Brauer graph, with a circular ordering on the edges around each vertex, together with some special (exceptional) vertices with associated multiplicities. This combinatorial description makes them particularly amenable to calculations and combinatorial studies.

We observe that the definition of a Brauer graph without exceptional vertices is essentially that of a ribbon graph (or fat graph). Such a graph has a natural filling embedding into an oriented surface (see [22]) giving rise to the cyclic ordering of the edges incident with each vertex. Conversely, such an embedding gives rise to a Brauer graph without exceptional vertices.

In particular, Kauer [21] shows that if a certain local move (which we call a *Kauer move*) is applied to the graph of a Brauer graph algebra, the resulting Brauer graph algebra is derived equivalent to the initial algebra. We show that if a Brauer graph arises from a triangulation of a compact oriented marked surface (regarded as the embedding of a graph), then the Kauer move coincides with the flip of the triangulation which corresponds to mutation in the cluster algebra associated to the surface [16]. It follows that we can associate a Brauer graph algebra to a compact oriented marked surface (by choosing a triangulation) which is unique up to derived equivalence. We also give a version of this for the case of a surface with boundary.

In the case of a disc with marked points on its boundary we consider not only triangulations but m-angulations. For such an m-angulation, regarded as a Brauer graph without exceptional vertex, the Kauer move again coincides with the mutation of the m-angulation [34] (see also [9, §11]) in the cluster

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sense. We may also consider the dual graph of the m-angulation, which is an $(m-1)$-ary tree. Mutation of the m-angulation induces a move which coincides with a *nearest neighbour interchange* (in the sense of [35, §2], which refers to [25]), or *Whitehead move*, i.e. the contraction and expansion of a given edge. Note that in the case of a triangulation the effect on the dual graph corresponds to the associativity rule as described, for example, in [15].

In the case of a triangulation of a disc, a mutation or flip of a diagonal corresponds to four different things simultaneously: a mutation of the associated cluster, a derived equivalence of the Brauer graph algebra whose graph is the triangulation, a derived equivalence of the Brauer tree algebra of the dual graph of the triangulation and, using [23, 28] (see also [5, 20]), a derived equivalence of endomorphism algebras in Frobenius categories in the cluster context.

We observe that in the case of Brauer graph algebras the Brauer graph algebra associated to an m-angulation is of tame representation type whereas the one associated to the dual graph is of finite representation type.

In Section 1 we recall the concept of a Brauer graph algebra and recall the pivotal result of Kauer on derived equivalences of Brauer graph algebras which lies at the heart of this paper. As we expect the audience of this paper to have a more algebraic background we begin Section 2 with a short recall of the theory of ribbon graphs and surfaces, and go on to show how up to derived equivalence there is a unique Brauer graph algebra associated to every compact oriented marked surface (with or without boundary). In Section 3 we explain how our results are connected with the theory of cluster algebras and cluster categories. In Section 4 we consider m-angulations of a disc with marked points on its boundary. We give an explicit two-term tilting complex, compatible with the geometry, realising the derived equivalence of Brauer tree algebras induced by the move on the dual graph induced by mutation of a diagonal in the m-angulation. Finally, in Section 5 we give counter-examples to show that the Brauer graph algebras of the dual graphs associated to two graphs related by the Kauer move needn't necessarily be derived equivalent in general, by considering the cases of a triangulation of a sphere and a punctured disk.

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1. Brauer graph algebras

Let K be an algebraically-closed field and $A = KQ/I$ be a finite-dimensional algebra given by the quiver Q and an admissible ideal I. All modules we consider are left modules and we denote the category of finite dimensional left A-modules by A-mod. If $\cdot \xrightarrow{\alpha} \cdot \xrightarrow{\beta} \cdot$ are two consecutive arrows in KQ, we write the corresponding path as $\beta\alpha$.

1.1. **Brauer Graph Algebras.** We will briefly recall the standard setup of Brauer graph algebras. Details can be found, for example, in [6, 21, 31].

Let Γ be a finite graph with at least one edge. Denote by Γ_0 the set of vertices of Γ and by Γ_1 the set of edges. Define a function $m : \Gamma_0 \to \mathbb{N} \setminus \{0\}$, called the *multiplicity function* of Γ. For any graph Γ, there is a local embedding of Γ into the plane with the property that an orientation of the plane induces a cyclic ordering around the vertices of Γ.

In general, we consider a fixed local embedding of Γ and, unless otherwise stated, the clockwise orientation of the plane and the induced cyclic order of the edges around every vertex of Γ. We call Γ, equipped with multiplicity function m and a fixed cyclic ordering a *Brauer graph.* The valency $\mathrm{val}(X)$ of a vertex $X \in \Gamma_0$ is the number of edges incident to X, where a loop counts twice. Given a Brauer graph Γ, the *Brauer graph algebra* A_Γ is the algebra kQ_Γ/I_Γ where the quiver Q_Γ is as descibed below. The ideal I_Γ is generated by relations ρ_Γ also described below.

Firstly, if Γ is the graph X —— Y with $m(X) = m(Y) = 1$ then $Q_\Gamma = \bullet \circlearrowright \alpha$ and $\rho_\Gamma = \{\alpha^2\}$ so that $A_\Gamma = K[\alpha]/(\alpha^2)$.

In all other cases, the edges a of Γ correspond to the vertices v_a of Q_Γ, and if b is the successor of a in the cyclic ordering around a vertex X then there is an arrow $v_a \xrightarrow{\alpha} v_b$ in Q_Γ.

There are three types of relations. Let $X \in \Gamma_0$, and let $a \in \Gamma_1$ be an edge incident with X. Let $a = a_1, a_2, \dots a_{\mathrm{val}(X)}$ be the edges around X in the cyclic ordering. Set $C_{a,X} = \alpha_{\mathrm{val}(X)} \dots \alpha_1$ to be the corresponding cycle of arrows in KQ_Γ. Then the relations ρ_Γ are defined as follows:
Relations of type I: If the edge $a \in \Gamma_1$ has endpoints X and Y so that a is not a leaf at X with $m(X) = 1$ or at Y with $m(Y) = 1$, then we have the relation $C_{a,X}^{m(X)} - C_{a,Y}^{m(Y)}$ in ρ_Γ.
Relations of type II: If the edge $a \in \Gamma_1$ with endpoints X and Y is a leaf at the vertex Y with $m(Y) = 1$ then $\alpha_1 C_{a,X}^{m(X)}$ is a relation in ρ_Γ.
Relations of type III: All paths $\alpha\beta$ of length 2 where $\alpha\beta$ is not a subpath of any cycle $C_{a,X}$ are relations in ρ_Γ.

Remark 1.1. (i) We do not allow truncated edges in the sense of [21] here.
(ii) In [18] it is shown that for any given Brauer graph there is a tower of coverings of Brauer graphs corresponding to a tower of coverings of Brauer graph algebras such that the topmost covering Brauer graph has no loops, no multiple edges and multiplicity function identical to 1.
(iii) We do not consider the quantized case of [18]. The Brauer graph algebras that we consider here are all symmetric algebras (Recall that a finite dimensional K-algebra A is symmetric if there exists an isomorphism of A-A-bimodules between A and $\mathrm{Hom}_K(A, K)$).

We recall part of a result of Kauer [21].

Theorem 1.2. [21, 3.5] *Consider two Brauer graphs, locally embedded in the plane, which are the same except for a local move as shown in Figure 1.1. Then the corresponding Brauer graph algebras are derived equivalent.*

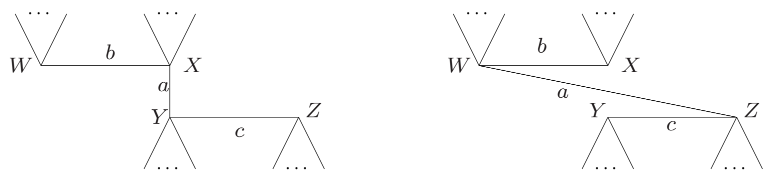


FIGURE 1. The Kauer move at the edge a.

We call the move appearing in Theorem 1.2 the *Kauer move* at the edge a.

We note that this derived equivalence has later also been considered in [1, 3, 13] where in some cases it is called a tilting mutation in analogy with the corresponding mutation in cluster theory (See Section 3).

2. Surfaces and derived equivalences of Brauer graph algebras

The aim of this section is to introduce ribbon graphs and to establish the fact that a Brauer graph is a ribbon graph with a vertex labelling corresponding to the values of the multiplicity function of the Brauer graph. The natural embedding of a ribbon graph into a compact oriented marked surface gives a natural way to associate a surface to a Brauer graph algebra. We will also see that, by considering triangulations, we can associate to every compact oriented marked surface a Brauer graph algebra unique up to derived equivalence (although we note that this is not the inverse of the above operation).

First, we recall part of the well-known theory of ribbon graphs and their embeddings into surfaces, following [22]. We will break this section into two parts. In the first part we consider marked surfaces without boundary and in the second we consider marked surfaces with boundary.

2.1. **Marked surfaces without boundary.**

We begin by giving the definition of a *ribbon graph* or *fat graph*. An *oriented graph* is a triple (V, E, φ), where V is the set of vertices, E is the set of edges, and $\varphi : E \to V \times V$ is a map sending $e \in E$ to the pair (e^-, e^+) of vertices. We regard e^- as the start of

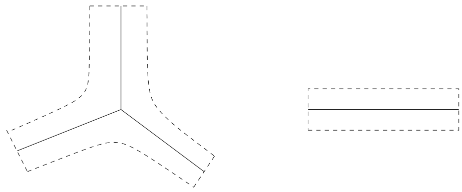


Figure 2. Surfaces corresponding to vertices and edges in Γ.

the arrow e and e^+ as the end. A *graph* is a pair (Γ, I), where $\Gamma = (V, E, \varphi)$ is an oriented graph and $I : E \to E$ is a fixed-point-free involution, $e \mapsto \overline{e}$, satisfying $(\overline{e})^+ = e^-$ and $(\overline{e})^- = e^+$ for all $e \in E$.

We call the pair $(e, \overline{e})$ an *undirected edge*. Thus we are regarding an undirected edge as a pair of directed edges in opposite directions between the same pair of vertices (which might be equal, in the case of a loop).

The *geometric realisation* $|\Gamma|_I$ of (Γ, I) is the topological space $|\Gamma|_I = (E \times [0,1])/\sim$, where $\sim$ is the equivalence relation defined by $(e,t) \sim (\overline{e}, 1-t)$ (for all $t \in [0,1]$), $(e,0) \sim (f,0)$ if $e^- = f^-$ and $(e,1) \sim (f,1)$ if $e^+ = f^+$, for all $e, f \in E$.

A *cyclic* ordering of a finite set S is a bijection $s : S \to S$ satisfying $\{s^n(x) : n \in \mathbb{Z}\} = S$, for all $x \in S$. The *predecessor* of $x \in S$ is $s^{-1}(x)$ and the *successor* of x is $s(x)$.

If $v \in V$, then the *star* of V is $E_v = \{e \in E : e^- = v\}$. A *ribbon graph* (or fat graph) is a graph together with a cyclic ordering on s_v on E_v for each $v \in V$. Morphisms and isomorphisms of ribbon graphs are then defined in the natural way. It is easy to see that:

Remark 2.1. A ribbon graph is a Brauer graph without exceptional vertices. In particular, the way loops work (see Section 1.1 above), ensures that the definitions of ribbon graphs and Brauer graphs coincide. Thus a Brauer graph can be regarded as a ribbon graph with a multiplicity attached to each of its vertices.

Examples of ribbon graphs include planar graphs and graphs locally embedded in the plane. An embedding of a graph into an oriented surface induces a ribbon graph structure on the graph, with cyclic ordering induced from the embedding around each vertex and the orientation of the surface.

Conversely, every Brauer graph with multiplicity function identically equal to 1 gives rise to an oriented surface with boundary:

Lemma 2.2. [22, 2.2.4] *Every ribbon graph Γ can be embedded in an oriented surface with boundary in such a way that the cyclic orderings around each of its vertices arise from the orientation of the surface.*

The *ribbon surface* S_Γ° of Γ is an example of such a surface. It can be constructed in the following way. Firstly, a surface is associated to each vertex and edge of Γ as in Figure 2. Then the structure of the graph determines a gluing of the corresponding surfaces giving an oriented surface together with an embedding of Γ.

Note that S_Γ° has a number of boundary components. A *face* of Γ is an equivalence class, up to cyclic permutation, of n-tuples $(e_1, e_2, \ldots, e_n)$ of edges satsifying $e_p^+ = e_{p+1}^-$ and $s_{e_p^+}(\overline{e_p}) = e_{p+1}$ for all p with $1 \leq p \leq n$ (taking subscripts modulo n). Then the boundary components of S_Γ° correspond to the faces of Γ.

An embedding $\iota : \Gamma \to S$ is said to be a *filling embedding* if $S \setminus |\Gamma| = \bigsqcup_{f \in F} D_f$, where $|\Gamma|$ denotes the image of ι, each D_f is a disc, and F is a finite set. That is, the complement of the embedding is a disjoint union of finitely many discs.

Proposition 2.3. *(see* [22, 2.2.7]*) Every ribbon graph has a filling embedding into a compact oriented surface such that the connected components of $S \setminus |\Gamma|$ are in bijection with the faces of Γ in the above sense.*

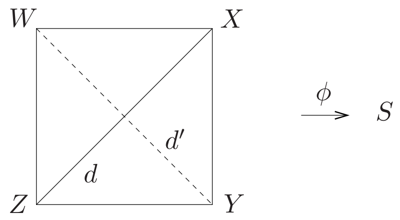


FIGURE 3. The flip of a triangulation.

This is proved by gluing discs onto the ribbon surface of the ribbon graph to fill in the boundary components. Such an embedding has the following uniqueness property:

Proposition 2.4. *(see* [22, 2.2.10]*) Let $\Gamma \to S$, $\Gamma' \to S'$ be filling embeddings of ribbon graphs of compact oriented surfaces S, S' and $\varphi : \Gamma \to \Gamma'$ an isomorphism of ribbon graphs. Then φ induces an orientation-preserving homeomorphism $\varphi : |\Gamma| \to |\Gamma'|$ extending to a homeomorphism from S to S'.*

Corollary 2.5. *If Γ is a ribbon graph, then there is a compact oriented surface S_Γ together with a filling embedding $\Gamma \to S_\Gamma$, unique up to homeomorphism.*

Thus we see that there is a filling embedding of an arbitrary Brauer graph (without, or ignoring, multiplicity) into an oriented surface in such a way that the cyclic ordering around each vertex arises from the orientation of the surface. In Section 3, we shall compare Kauer moves in this context with a certain kind of twist (or mutation) arising in cluster theory.

Remark 2.6. Note that [3, §3] has also associated a surface to a Brauer graph by associating a CW-complex, called the Brauer complex, to the quiver with relations of the corresponding graph algebra. Comparing the definitions, it can be seen that this gives rise to the same filling embedding as above (Corollary 2.5). In particular, the G-cycles in [4, §2] correspond to the faces of the Brauer graph as a ribbon graph.

Finally, we have the following:

Proposition 2.7. *(stated as* [22, 2.2.12]*) Every compact oriented surface admits a filling ribbon graph.*

One way to realise a filling ribbon graph as in Proposition 2.7 is through an (ideal) triangulation of the surface.

We now suppose that (S, M) is a compact oriented surface with marked points M in S. We assume in addition that (S, M) is not a sphere with 1 or 2 marked points and that each connected component of S contains at least one element of M to ensure that (S, M) has at least two triangulations (see [16, §2]: note that we allow a sphere with 3 marked points here).

Let $\mathcal{T}$ be a triangulation of (S, M) where the set of marked points M coincides with the set of vertices of the triangulation. Then we obtain a filling ribbon graph $\Gamma_\mathcal{T}$ whose vertices are the elements of M and whose edges are the arcs in $\mathcal{T}$. Denote by $A_\mathcal{T}$ the corresponding Brauer graph algebra.

We recall the definition of the *flip* of a triangulation from [10, §2.2]. Let a be an arc in $\mathcal{T}$ incident with two distinct triangles T_1 and T_2 in $\mathcal{T}$ (we call such an arc *flippable*). Then there is a map ψ from a square with diagonal d onto the union of T_1 and T_2 in S. The map ψ may fold the square, possibly identifying distinct points or edges. Then in the flip of $\mathcal{T}$ at a, a is replaced with $\psi(d') = a'$, where d' is the diagonal of the square distinct from d. See Figure 3.

We recall that, by [10, §2.1], the possible triangles that can appear in $\mathcal{T}$ are as in Figure 4.

Lemma 2.8. *With the notation in Figure 3, the following successor relations hold*

(a) The clockwise successor of $a = \psi(d)$ at $\psi(X)$ is $\psi(WX)$.
(b) The clockwise successor of $a = \psi(d)$ at $\psi(Z)$ is $\psi(ZY)$.

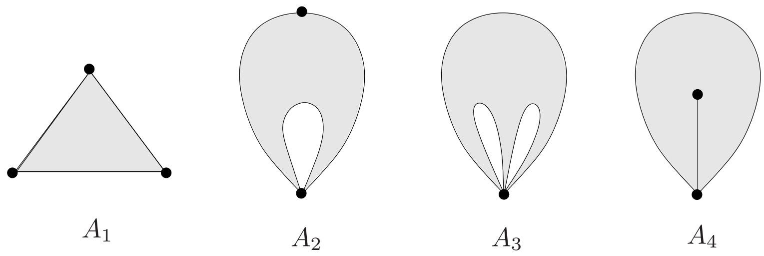


FIGURE 4. The possible triangles in $\mathcal{T}$.

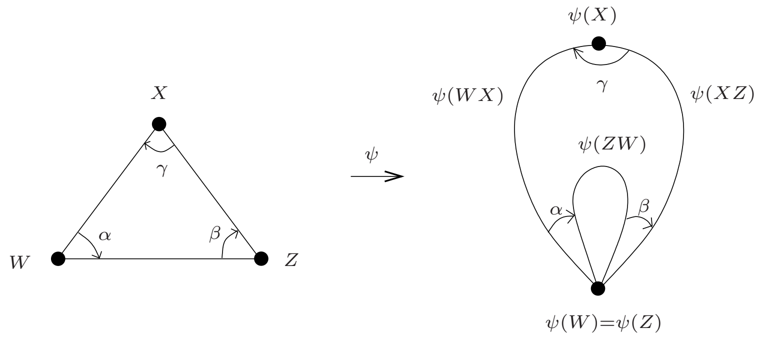


FIGURE 5. Proof of Lemma 2.8, case A_2.

Proof. This holds because ψ only acts by folding and identifying, so doesn't change successor relations. We describe this in detail.

Consider the triangle WXZ, which is folded by ψ according to A_1, A_2, A_3 or A_4 in Figure 4. The argument for the triangle XYZ is similar. We just need to check that ψ preserves the clockwise successors at each vertex.

Case A_1: This case is trivial.

Case A_2: This is illustrated in Figure 5. We indicate the successor relation by curved arrows, labelled α, β and γ to indicate the correspondence under ψ. Note that there are three possibilities (depending on which edge becomes the inside edge in the folding), but the argument in each case is similar.

Case A_3: This is indicated in Figure 6. We represent the folding in two steps, first folding as in case A_2 (labelled ψ_0), then a further folding, labelled ψ_1, to obtain $\psi = \psi_1 \circ \psi_0$.

Case A_4: This is indicated in Figure 7. Note that, in this case, ψ must map XZ to the outer edge of the triangle of type A_4, as $\psi(XZ)$ must be a flippable edge in $\mathcal{T}$.

□

Proposition 2.9. *Let $\mathcal{T}$ be a triangulation of a marked compact oriented surface (S, M). Regarding $\mathcal{T}$ as a Brauer graph, the flip of $\mathcal{T}$ at an arc a coincides with applying the Kauer move to $\mathcal{T}$ at a.*

Proof. This follows from Lemma 2.8 and the definition of a Kauer move (see Theorem 1.2). □

It is a consequence of Proposition 2.9 that given a marked surface (S, M), up to derived equivalence, there exists a unique Brauer graph algebra associated to (S, M):

Corollary 2.10. *Let (S, M) be a marked compact oriented surface. Then the Brauer graph algebra $A_{\mathcal{T}}$ associated to (S, M) via a triangulation $\mathcal{T}$ does not depend on the choice of $\mathcal{T}$, up to derived equivalence.*

Proof. This follows from Proposition 2.9 and Theorem 1.2, using the fact that any two triangulations of (S, M) are connected by a sequence of flips (see [10, Theorem 2], [16, Prop. 3.8]). □

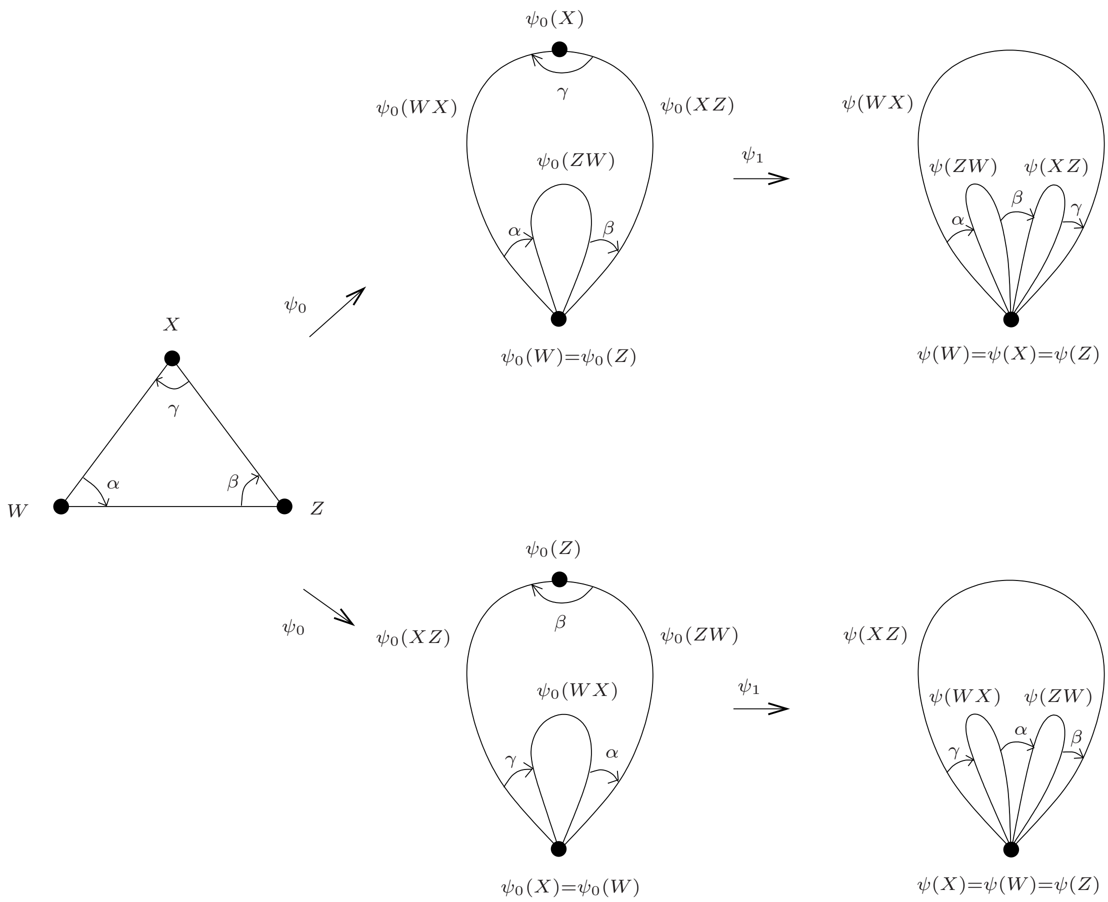


FIGURE 6. Proof of Lemma 2.8, case A_3.

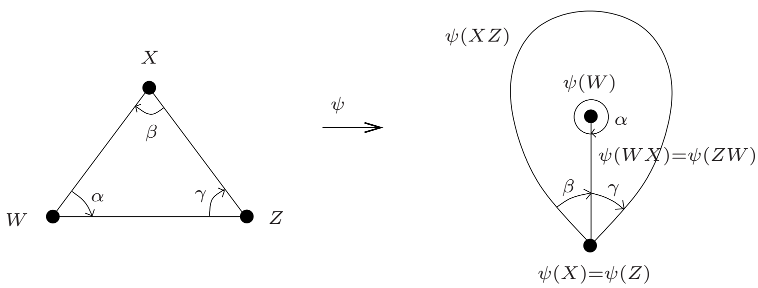


FIGURE 7. Proof of Lemma 2.8, case A_4.

2.2. **Marked surfaces with boundary.** We first note that every oriented surface with boundary is homeomorphic to a surface with boundary obtained from a surface without boundary by removing a disjoint collection of discs.

Let (S, M) be a marked surface with (non-empty) boundary. The marked points M can be anywhere on the surface, on or off the boundary. We assume that there is at least one marked point on each connected component of S and on each connected component of the boundary of S.

Since we would like to consider flips of triangulations of surfaces, we assume, that (S, M) is not one of the following degenerate cases, in order to ensure that (S, M) has at least two triangulations (see [16]).

(a) a sphere with 1 or 2 marked points;

(b) a monogon with 0 or 1 marked points;
(c) a digon with no marked points;
(d) a triangle with no marked points.

In particular,

Proposition 2.11. *Every marked oriented surface with boundary admits a filling ribbon graph.*

Proof. As in Proposition 2.7, take a triangulation of (S, M) with vertex set M, and include boundary arcs (i.e. parts of the boundary of S between adjacent, possibly equal, marked points on the boundary) in the graph. □

Some of the faces of the corresponding ribbon graph will correspond to boundary components. So we have a distinguished subset, F_B of the set of faces.

We define a *ribbon graph with boundary* to be a tuple $\widetilde{\Gamma} = (\Gamma, I, \{s_v\}_{v\in V}, F_B)$, where $(\Gamma, I, \{s_v\}_{v\in V})$ is a ribbon graph and F_B is a subset of its set of faces with the property that, for every undirected edge $(e, \overline{e})$ of Γ, at most one of $e, \overline{e}$ lies in a face in F_B.

Let $S_{\Gamma,I,\{s_v\}} = S_\Gamma$ be the surface constructed as in the ribbon graph case, in which there is a filling embedding of Γ (see Proposition 2.3). Then $S_\Gamma \setminus |\Gamma| = \sqcup_{f\in F} D_f$ where F is a finite set and each D_f is an open disc. We define $S_{\widetilde{\Gamma}}$ to be $S_\Gamma \setminus \cup_{f\in F_B} D_f$, a surface with boundary obtained by removing the discs from S_Γ corresponding to the faces in F_B. Then $\widetilde{\Gamma} \to S_{\widetilde{\Gamma}}$ is a filling embedding of $\widetilde{\Gamma}$ into $S_{\widetilde{\Gamma}}$.

We extend the notion of morphism of ribbon graphs to ribbon graphs with boundary in the natural way (i.e. such maps should preserve the marked set, F_B, of faces). A morphism of ribbon graphs with boundary is an isomorphism if it has an inverse which is also a morphism of ribbon graphs with boundary.

Lemma 2.12. *Let $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$ be ribbon graphs with boundary together with filling embeddings $\widetilde{\Gamma} \to \widetilde{S}$ and $\widetilde{\Gamma}' \to \widetilde{S}'$ mapping the boundary faces to the boundaries of the boundary components in each case. If $\varphi : \widetilde{\Gamma} \to \widetilde{\Gamma}'$ is an isomorphism of ribbon graphs with boundary then it induces a homeomorphism $\varphi : |\widetilde{\Gamma}| \to |\widetilde{\Gamma}'|$ which extends to a homeomorphism from $\widetilde{S}$ to $\widetilde{S}'$.*

Proof. Let S, S' be the surfaces without boundary obtained from $\widetilde{S}$ and $\widetilde{S}'$ by gluing discs into their boundary components. The filling embeddings $\Gamma \to \widetilde{S}$ and $\Gamma' \to \widetilde{S}'$ induce filling embeddings $\Gamma \to \widetilde{S}$ and $\Gamma \to S'$ of the underlying ribbon graphs. By Proposition 2.4, there is a homeomorphism $\varphi : S \to S'$ restricting to a homeomorphism from $|\Gamma|$ to $|\Gamma'|$. By the construction of this homeomorphism, it restricts to a homeomorphism from $\widetilde{S}$ to $\widetilde{S}'$ with the required properties. □

Proposition 2.13. *Let $\widetilde{\Gamma}$ be a ribbon graph with boundary. Then there is an oriented surface with boundary S and filling embedding $\widetilde{\Gamma} \to S$, mapping the boundary faces to the boundaries of the boundary components, which is unique up to homeomorphism.*

Proof. Existence is guaranteed by the argument preceding Lemma 2.12, and uniqueness follows from Lemma 2.12. □

We say that an arc in a triangulation of a marked surface with boundary is *flippable* if it is incident with two triangles in $\mathcal{T}$ and if it is not a boundary arc.

Corollary 2.14. *Let $\mathcal{T}$ be a triangulation of a marked oriented surface with boundary (S, M). Regarding $\mathcal{T}$ as a Brauer graph, the flip of $\mathcal{T}$ at a flippable arc a in $\mathcal{T}$ coincides with applying the Kauer move to $\mathcal{T}$ at a.*

Proof. The proof is the same as the proof of Proposition 2.9. □

Similarly to the case of a marked surface without boundary, for a marked surface with boundary, up to derived equivalence, there exists a unique Brauer graph algebra associated to that surface:

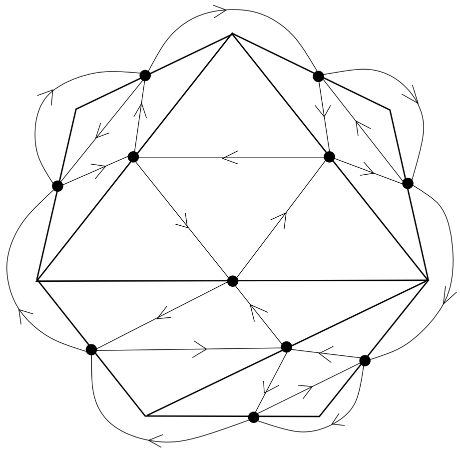


FIGURE 8. Example of the quiver $Q_{\mathcal{T}}$ of a triangulation $\mathcal{T}$.

Corollary 2.15. *Let (S, M) be a marked oriented surface with boundary. Then the derived equivalence class of the Brauer graph algebra $A_{\mathcal{T}}$ associated to a triangulation $\mathcal{T}$ of (S, M) does not depend on the choice of triangulation $\mathcal{T}$.*

Proof. As for the proof of Corollary 2.10. □

3. Cluster Algebras

In this section we discuss the relationship between cluster theory and the results above.

Let P_n be a polygon with n marked points on its boundary, and $\mathcal{T}$ a triangulation of P_n. Then regarding $\mathcal{T}$ as a Brauer graph, there is a corresponding quiver $Q_{\mathcal{T}}$. For an example in the case $n = 7$, see Figure 8. We call arrows outside P_n *boundary arrows*.

Let R denote the polynomial ring $K[x]$, and let Λ denote the Gorenstein tiled R-order considered in [23], defined as a matrix algebra by the $n \times n$ matrix:

$$\begin{pmatrix} R & R & R & \cdots & R & (x^{-1}) \\ (x) & R & R & \cdots & R & R \\ (x^2) & (x) & R & \cdots & R & R \\ \vdots & & & \ddots & & \vdots \\ (x^2) & (x^2) & (x^2) & \cdots & R & R \\ (x^2) & (x^2) & (x^2) & \cdots & (x) & R \end{pmatrix}.$$

Let $\mathcal{F}$ be the category of (maximal) Cohen-Macaulay modules over Λ. We recall some of the results of [23] (see also [5, 20]).

Theorem 3.1. [23, Thm. 1]

(a) *The category $\mathcal{F}$ is Frobenius and its stable category $\mathcal{C}$ is triangle equivalent to the cluster category of type A_{n-3}.*
(b) *The indecomposable objects in $\mathcal{F}$ are in bijection with the arcs joining vertices in P_n (including boundary arcs).*
(c) *The bijection in (b) induces a bijection $\mathcal{T} \mapsto M_{\mathcal{T}}$ between the triangulations in P_n and the cluster-tilting objects in $\mathcal{F}$.*
(d) *For each triangulation $\mathcal{T}$ of P_n, the algebra $\mathrm{End}_{\mathcal{F}}(M_{\mathcal{T}})^{\mathrm{op}}$ is a frozen Jacobian algebra associated to a frozen quiver with potential.*

Remark 3.2. The quiver of the frozen Jacobian algebra in Theorem 3.1 can be obtained from the quiver $Q_{\mathcal{T}}$ by deleting boundary arrows whose end points correspond to two sides of the same triangle of $\mathcal{T}$. The boundary vertices are frozen. Note also that $Q_{\mathcal{T}}$ coincides exactly with the quiver of the Postnikov diagram associated to $\mathcal{T}$ in [5, §12] using a modified version of the construction in [33, Cor. 2]. The associated dimer algebra in [5] is isomorphic to the frozen Jacobian algebra referred to above by [5, Lemma 11.1].

The following is a special case of [28, Prop. 4], using $\operatorname{Mod}\Lambda$ to denote the category of all modules over Λ.

Theorem 3.3. [28] *Let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau triangulated category which is the stable category of a Frobenius category $\mathcal{F}$. Let T, T' be cluster-tilting objects in $\mathcal{C}$ with preimages M, M' in $\mathcal{F}$. Then $D(\operatorname{Mod}\operatorname{End}(M)^{\mathrm{op}}) \simeq D(\operatorname{Mod}\operatorname{End}(M')^{\mathrm{op}})$.*

Theorem 3.3 applies in this case, and we have:

Proposition 3.4. *Let $\mathcal{T}, \mathcal{T}'$ be triangulations of P_n, and $M_{\mathcal{T}}$, $M_{\mathcal{T}'}$ the corresponding cluster-tilting objects in $\mathcal{F}$. Then the categories $D(\operatorname{Mod}\operatorname{End}(M_{\mathcal{T}})^{\mathrm{op}})$ and $D(\operatorname{Mod}\operatorname{End}(M_{\mathcal{T}'})^{\mathrm{op}})$ are equivalent.*

Remark 3.5. It is interesting to compare the above with Corollary 2.15 (in the disc case). We see that, in terms of derived equivalences, the algebras $\operatorname{End}(M_{\mathcal{T}})$, for $\mathcal{T}$ a triangulation of a disc, behave in a similar way to the corresponding Brauer graph algebras $A_{\mathcal{T}}$.

Using [11, Lemma 3.1] and Remark 3.2, we have:

Proposition 3.6. *Let $\mathcal{T}$ be a triangulation of (S, M) and $\mathcal{T}'$ the triangulation obtained from $\mathcal{T}$ by flipping an internal arc, γ. Then the quiver of the Brauer graph algebra associated to $\mathcal{T}'$, i.e., $Q_{\mathcal{T}'}$, can be obtained from $Q_{\mathcal{T}}$ by applying Fomin-Zelevinsky quiver mutation* [17] *to $Q_{\mathcal{T}}$ at the vertex corresponding to γ, leaving the boundary arrows unchanged.*

A version of Proposition 3.6 holds in a more general context. Suppose that (S, M) is a marked oriented surface in which every element of M lies on the boundary of S (i.e. the *unpunctured* case). Let $\iota : \Gamma \to S$ be a filling embedding of a ribbon graph Γ into S, such that $M = \iota(\Gamma_0)$ and boundary faces are mapped to the boundaries of the boundary components.

Proposition 3.7. *Let e be an edge of Γ with image $\gamma = \iota(e)$ not on the boundary. Then, applying a Kauer move to Γ at e corresponds to twisting γ in the sense of* [24, §3] *with respect to the set of remaining edges of $|\Gamma|$ which are not on the boundary. If Γ is a triangulation, then the change in the quiver Q_Γ of the Brauer graph algebra of Γ under this twist coincides with Fomin-Zelevinsky quiver mutation* [17].

Proof. The first statement follows from a comparison of the definition of the Kauer move and the definition of twisting in [24, §3]. The second statement can be checked directly as in [16, Prop. 4.8]. □

Note that the twist of arcs considered in [24] is shown to correspond to a categorical mutation in the cluster category [8] associated to the surface.

We note that A. Dugas [13] has remarked on similarities between quiver mutation (in fact mutation of quivers with potential in the sense of [12]) and tilting mutations considered by Aihara [1]. Note that these tilting mutations can be regarded, in the case of the complement of a single vertex, as a special case of the Kauer derived equivalences [21] discussed above.

4. The Brauer tree of an m-angulation of a disc

In this section we restrict the surface we consider to the disc with n marked points on the boundary. Here we allow m-angulations (or even angulations, in general) and not only triangulations of the disc. Given an m-angulation of the disc we define a dual graph of the m-angulation. We then give an explicit tilting complex of the Brauer graph algebra of this dual graph realizing the change of graph induced by a Kauer move or mutation of one of the diagonals of the m-angulation of the disc.

Let P_n be a polygon with n vertices. For $n = k(m-2)+2$ for some $k, m \in \mathbb{Z}^+$, an *m-angulation* $\mathcal{M}$ of P_n is a collection of arcs joining the vertices of P_n and dividing it up into m-gons. The set of edges of P_n constitutes the set of *boundary edges* of $\mathcal{M}$. We call an *internal arc* of $\mathcal{M}$ any arc that is not a boundary edge.

Regarding $\mathcal{M}$ as a (locally) embedded graph in the plane, we have seen that it can be regarded as a Brauer graph by choosing an orientation of the plane which then induces an ordering of the edges around each vertex. As discussed in the previous sections, unless otherwise stated we choose the clockwise orientation.

With this set-up there is a corresponding Brauer graph algebra which we denote by $A_{\mathcal{M}}$. Note that $A_{\mathcal{M}}$ has multiplicity function $m \equiv 1$.

Given an m-angulation $\mathcal{M}$ and an internal arc a, there is a new m-angulation $\mu_a(\mathcal{M}) = \mathcal{M}'$ obtained in the following way. Removing a leaves a $2m-2$-gon H which had a as one of its diagonals. Then $\mathcal{M}'$ is obtained by rotating a one-step clockwise within H. This move is a generalization of the the flip in previous section and it is generally known in cluster theory as a *mutation* at a (in the $(m-2)$-cluster sense; [34] (see also [9, §11], [14]).

The following is easy to check.

Lemma 4.1. *Let $\mathcal{M}$ be an m-angulation of P_n and a an internal arc of $\mathcal{M}$. Then mutating $\mathcal{M}$ at a (in the way described above) coincides with applying the Kauer move to $\mathcal{M}$ at a.*

We also observe the following:

Lemma 4.2. *Let $\mathcal{M}$ be an m-angulation of P_n and a an internal arc of $\mathcal{M}$. Then, regarding $\mathcal{M}$ as a Brauer graph (with an appropriate orientation of P_n), the full subquiver of $A_{\mathcal{M}}$ on the vertices corresponding to non-boundary arcs coincides with the quiver of the corresponding $(m-2)$-cluster-tilted algebra as given by* [26, Prop. 4.1.8].

Corollary 4.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two m-angulations of P_n. Then $A_{\mathcal{M}_1}$ and $A_{\mathcal{M}_2}$ are derived equivalent.*

Proof. This follows from Lemma 4.1 and the fact that any two m-angulations are connected by a sequence of mutations (this can be seen using [9, Prop. 7.1] or [36, Prop. 4.5], combined with [9, §11]). □

We also consider the completed m-ary tree, $\overline{\mathcal{M}}^*$, corresponding to $\mathcal{M}$. Its interior vertices correspond to the m-gons in $\mathcal{M}$, with a leaf vertex corresponding to each boundary edge. Two interior vertices are connected by an edge if the corresponding m-gons share a common edge in $\mathcal{M}$, and an interior vertex and a leaf vertex are connected by an edge if the boundary edge corresponding to the leaf vertex forms part of the boundary of the m-gon corresponding to the interior vertex.

Remark 4.4. If we identify all of the leaf vertices in $\overline{\mathcal{M}}^*$, we obtain the dual graph of $\mathcal{M}$.

As for $\mathcal{M}$, the graph $\overline{\mathcal{M}}^*$, via its embedding in the plane, can be considered as a Brauer graph. So we again have a corresponding Brauer graph algebra, $A_{\overline{\mathcal{M}}^*}$. Note that $\overline{\mathcal{M}}^*$ is in fact a tree and thus $A_{\overline{\mathcal{M}}^*}$ is a Brauer tree algebra.

In a similar way, we may also consider $\overline{\mathcal{M}}^*$ without its leaf vertices (deleting all incident edges also). We call this non-boundary version $\mathcal{M}^*$.

Remark 4.5. If we adopt the rule for $\mathcal{M}$ that the successor of an edge is anticlockwise from the edge, while in $\overline{\mathcal{M}}^*$ we use the usual (clockwise) rule then there is a close correspondence between the quivers of $A_{\mathcal{M}}$ and $A_{\overline{\mathcal{M}}^*}$: the arrows are the same except that $Q_{\mathcal{M}}$ has an extra cycle of arrows around its boundary. If, however, we were to consider instead the dual graph of $\mathcal{M}$, i.e. if all of the boundary vertices are identified, then the arrows in the quiver corresponding to the dual graph would be the same as in $Q_{\mathcal{M}}$. It will become clear in Section 5 why we choose to work with $\overline{\mathcal{M}}^*$ instead of the dual graph of $\mathcal{M}$.

Given an internal edge a of $\mathcal{M}$, denote by a^* the unique edge of $\overline{\mathcal{M}}^*$ intersecting a.

Definition 4.6. Mutating $\mathcal{M}$ at an internal edge a, induces a corresponding move on $\overline{\mathcal{M}}^*$: the first edge clockwise of a^* at each end-point of a^* is moved along a^* (together with the subtrees attached to the edges we are moving) so that it becomes incident with the other end-point of a^* instead. We call this a *dual Kauer move* at a^* and we denote the resulting tree by $\mu^*_{a^*}(\overline{\mathcal{M}}^*)$.

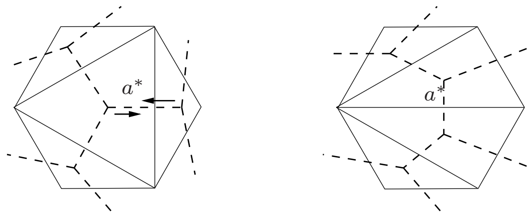


FIGURE 9. Example of a dual Kauer move at the edge a^*. The dotted lines in the left figure denote $\overline{\mathcal{M}}^*$ and the dotted lines in the right figure denote $\mu^*_{a^*}(\overline{\mathcal{M}}^*)$.

Remark 4.7.
(1) The dual Kauer move is known in graph theory under the name of nearest neighbour interchange (NNI) [35, §2] or also as a Whitehead move.
(2) Note that this rule does not apply to $\mathcal{M}^*$. For an example of this, see Figure 10 where going from the left hand figure to the right hand figure, the change of in $\mathcal{M}^*$ induced by a Kauer move in the heptagon at the edge corresponding to a^* is not a dual Kauer move. But notice that going the other way, that is from the right hand figure to the left hand one, if we mutate the edge corresponding to a^* in $\mathcal{M}$ and then take the dual graph this does correspond to a dual Kauer move.

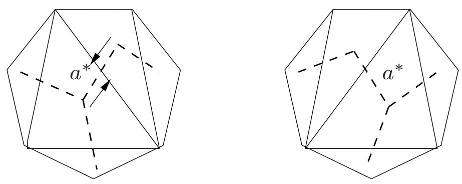


FIGURE 10. Mutating at a in $\mathcal{M}$ does not induce a dual Kauer move on $\mathcal{M}^*$ (the latter indicated by the arrows in the left hand picture).

In the following we see that when working in the dual context of $\overline{\mathcal{M}}^*$, a dual Kauer move also induces a derived equivalence. Note that by [30] it is already known that the two corresponding Brauer tree algebras are derived equivalent, since they both have the same number of vertices and the same multiplicity function. However, our interest here is in constructing an explicit derived equivalence compatible with the geometry.

In general, for a finite dimensional algebra A, let $\mathcal{P}(A)$ denote the category of finitely generated projective A-modules and let $\mathcal{K}^\flat(\mathcal{P}(A))$ denote the bounded homotopy category of complexes of projective A-modules. Recall that an object T in $\mathcal{K}^\flat(\mathcal{P}(A))$ is said to be a *tilting complex* provided:

(i) $\mathrm{Hom}_{\mathcal{K}^\flat(\mathcal{P}(A))}(T, T[n]) = 0$ for all $n \neq 0$, and
(ii) T generates $\mathcal{K}^\flat(\mathcal{P}(A))$ as a triangulated category.

Following Rickard [29], in order to show that two two finite dimensional algebras A and A' are derived equivalent it is enough to show that there is a tilting complex T in $\mathcal{K}^\flat(\mathcal{P}(A))$ such that A' is isomorphic to $\mathrm{End}_{K^\flat(\mathcal{P}(A))}(T)^{\mathrm{opp}}$.

Consider a dual Kauer move at an edge a in $\overline{\mathcal{M}}^*$. Let X and Y denote the two vertices in $\overline{\mathcal{M}}^*$ incident with a. We label the edges near a as follows. In clockwise orientation around X we have the following

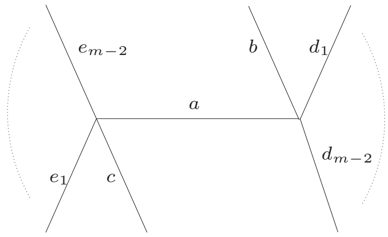


FIGURE 11. Configuration I: Local configuration around the edge a.

edges: $a, b, d_1, \dots, d_{m-2}$ after which we return to a. Similarly, around Y, we have $a, c, e_1, \dots, e_{m-2}$, after which we return to a. We call this *Configuration I*, see figure 11.

After a dual Kauer move at a (recall that this corresponds to mutating the only edge of the m-angulation of the polygon that the edge a intersects and then taking our version of the dual graph), we have the following configuration, which we call *Configuration II*: around X we have $a, d_1, \dots, d_{m-2}, c$ and then we return to a and around Y we have $a, e_1, \dots, e_{m-2}, b$ and then we return to a. See also figure 12.

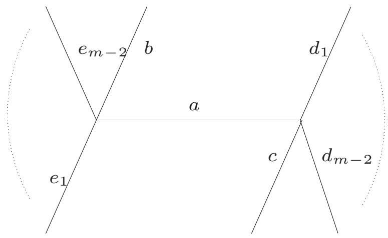


FIGURE 12. Configuration II: Local configuration around the edge a after application of dual Kauer move at edge a.

Let I be the set of edges of $\overline{\mathcal{M}}^*$. For $i = b, d_1, \dots, d_{m-2}, c, e_1, \dots, e_{m-2}$, let G_i be the subtree of $\overline{\mathcal{M}}^*$ at the vertex of the edge i which is not incident with a, together with i itself.

Define an object $T = \bigoplus_{i \in I} T_i$ in $\mathcal{K}^\flat(\mathcal{P}(A_{\overline{\mathcal{M}}^*}))$ as follows:

$$\begin{array}{lcl} T_a & := & 0 \to P_a; \\ T_b & := & P_b \to P_a; \\ T_c & := & P_c \to P_a; \\ T_i & := & \begin{cases} P_i \to 0, & \text{if } i \in G_b \setminus \{b\} \text{ or } i \in G_c \setminus \{c\}; \\ 0 \to P_i, & \text{if } i \in G_{d_j} \text{ or } i \in G_{e_j} \text{ for some } j, \end{cases} \end{array}$$

where P_s denotes the projective indecomposable module at vertex v_s in the quiver of $A_{\overline{\mathcal{M}}^*}$ (which corresponds to the edge s in $\overline{\mathcal{M}}^*$). For an object X in an additive category, we denote by $\mathrm{add}(X)$ the full subcategory consisting of direct summands of finite direct sums of copies of X.

Remark 4.8. (a) Let $A' = End_{K^\flat(\mathcal{P}(A_{\overline{\mathcal{M}}^*}))}(T)$. For $i \in I$, let $P'_i = \mathrm{Hom}_{K^\flat(\mathcal{P}(A_{\overline{\mathcal{M}}^*}))}(T, T_i)$ be an indecomposable projective A'-module and let $P' = \bigoplus_{i \in I} P'_i$. Then the functor $\mathrm{Hom}_{K^\flat(\mathcal{P}(A_{\overline{\mathcal{M}}^*}))}(T, -)$ induces an equivalence of categories $\mathrm{add}(T) \to \mathrm{add}(P')$.

(b) We note that according to our conventions the arrows in the quiver of $A_{\overline{\mathcal{M}}^*}$ correspond to the clockwise orientation in $\overline{\mathcal{M}}^*$, so that homomorphisms between the projective indecomposables go anticlockwise to the orientation in $\overline{\mathcal{M}}^*$.

The fact that $T = \bigoplus_{i \in I} T_i$ in $K^\flat((P(A_{\overline{\mathcal{M}}^*}))$ is a tilting complex is easily verified and also follows directly from the fact that T is an Okuyama-Rickard complex [27]. It then follows from [29] that $A_{\overline{\mathcal{M}}^*}$ and $A' = \mathrm{End}_{K^\flat((P(A_{\overline{\mathcal{M}}^*}))}(T)^{\mathrm{opp}}$ are derived equivalent.

We now show that A' is the Brauer tree algebra obtained from A through a dual Kauer move at the edge a.

Lemma 4.9. *A' is a Brauer tree algebra with no exceptional vertex.*

Proof. Firstly, note that A is symmetric, as it is a Brauer tree algebra. Rickard [30] states that if a symmetric finite dimensional algebra Λ is derived equivalent to another finite dimensional algebra Λ', then Λ' is also symmetric. Hence, in our case, A' is also symmetric. Next, [30, 4.2] states that a Brauer tree algebra is determined up to derived equivalence by its exceptional vertex multiplicity and its number of edges. Hence A is derived equivalent to the Brauer tree algebra A'' corresponding to a star with multiplicity 1. Thus A' is derived equivalent to A''. By [30, 2.2] if two finite dimensional symmetric algebras A' and B' are derived equivalent then they are stably equivalent. Hence A' and A'' are stably equivalent. According to [2, X, 3.14] if two finite dimensional algebras Λ and Λ' are stably equivalent and if Λ is a symmetric Nakayama Brauer tree algebra (*e.g.* a star) then Λ' is a Brauer tree algebra. Hence A' is a Brauer tree algebra and by [30, 4.2] it has multiplicity function equal to 1. □

Theorem 4.10. *There is an isomorphism of algebras from $A' = \mathrm{End}_{K^b(P(A_{\overline{\mathcal{M}}^*}))}(T)^{\mathrm{opp}}$ to $A_{\overline{\mathcal{N}}^*}$, where $\overline{\mathcal{N}}^* = \mu_a^*(\overline{\mathcal{M}}^*)$, that is $\overline{\mathcal{N}}^*$ is obtained from $\overline{\mathcal{M}}^*$ by a dual Kauer move on the edge a. Hence, $A_{\overline{\mathcal{M}}^*}$ and $A_{\overline{\mathcal{N}}^*}$ are derived equivalent.*

Lemma 4.11. *With the notation and the set-up of Theorem 4.10, suppose for a summand T_i of T we find two non-zero loops of morphisms*
(a) $T_i \to T_{i_1} \to T_{i_2} \to \cdots \to T_{i_r} \to T_i$;
(b) $T_i \to T_{j_1} \to T_{j_2} \to \cdots \to T_{j_s} \to T_i$
and show that
(c) $\mathrm{Hom}_{K^b(P(A_{\overline{\mathcal{M}}^*}))}(T_i, T_k) = 0$ for all k such that T_k is not one of the components in the loops (a) and (b) above.

Then, the edges incident with the end-points of i in the Brauer graph of A' must be, in the clockwise orientation, $i, i_1, \ldots, i_r, i_1, i$ on one vertex of i and $i, j_1, \ldots, j_s, i$ on the other vertex of i.

Proof. By Remark 4.8, the edges in the Brauer graph of A' can be identified with those in the Brauer graph of A. By (a),(b) and (c), the edges incident with i must be $\{i_1, \ldots, i_r, j_1, \ldots, j_s\}$ (since $\mathrm{Hom}_{A_{\overline{\mathcal{M}}^*}}(T_i, T_k) = 0$ implies that $\mathrm{Hom}_{A'}(P'_i, P'_k) = 0$; see Remark 4.8(a)).

Since the Brauer graph of A' is a tree with multiplicity function equal to 1 there are no loops and the edges around a vertex appear only once, i.e. there are no repeats. Since $\mathrm{Hom}(T_{i_1}, T_{i_2}) \neq 0$, i_2 must be incident with the same end of i as i_1 by Lemma 4.9. If the edge i_2 was clockwise of i and anticlockwise of i_1 (about the common vertex), then the map $T_i \to T_{i_1}$ would factor through T_{i_2}. Therefore the map in (a) would factor $T_i \to T_{i_2} \to T_{i_1} \to T_{i_2} \to T_i$. However, this composition is zero by Remark 4.8 and Lemma 4.9 as any composition that is longer than a cycle is zero in a Brauer tree algebra with multiplicity function 1. Thus we get a contradiction, and hence i_2 lies clockwise of i_1 and anticlockwise of i. If we repeat this argument for all the edges incident with i, the claim follows. In the case of a leaf, there is only one non-zero loop, so (a) and (c) imply the clockwise ordering of the edges around the vertex of i which is not of valency 1. □

Proof of Theorem 4.10. By Lemma 4.9, A' is a Brauer tree algebra with multiplicity function equal to 1. We now apply the technique of Lemma 4.11 to each indecomposable summand T_i of T. We consider the following compositions in $K^b(A_{\overline{\mathcal{M}}^*})$

$$T_a \longrightarrow T_b \longrightarrow T_{e_{m-2}} \longrightarrow T_{e_{m-3}} \longrightarrow \cdots \longrightarrow T_{e_1} \longrightarrow T_a$$

and

$$T_a \longrightarrow T_c \longrightarrow T_{d_{m-2}} \longrightarrow T_{d_{m-3}} \longrightarrow \cdots \longrightarrow T_{d_1} \longrightarrow T_a$$

It is easy to see that we get a non-zero composition in both cases. Now we check that condition (c) of the claim above holds. If $j \in G_b \setminus \{b\} \cup G_c \setminus \{c\}$ then any map of complexes from $T_a = (0 \to P_a)$ to $T_j = (P_j \to 0)$ must be zero. If $j \in G_{d_k} \setminus \{d_k\} \cup G_{e_k} \setminus \{e_k\}$ for some k, then any map of complexes from $T_a = (0 \to P_a)$ to $T_j = (0 \to P_j)$ must be zero since, by the defining relations of $A_{\overline{\mathcal{M}}^*}$, there is no non-zero map $P_a \to P_j$. Thus, in both of these cases we have $\mathrm{Hom}_{K^\flat((P(A_{\overline{\mathcal{M}}^*}))}(T_a, T_j) = 0$. It follows that the edges around a are as in Configuration II above.

Next we consider non-zero loops at T_b. The edge b has two vertices X and B. Recall that the edges around vertex X are given in clockwise order by $b, d_1, d_2, \ldots, d_{m-2}, a$ and back to b. Let the edges around vertex B be given by $b, x_1, x_2, \ldots, x_{m-1}$ and back to b. After the dual Kauer move at the edge a, we have the following configuration of edges around the vertices of b. At vertex X we have in clockwise order $b, a, e_1, e_2, \ldots, e_{m-1}$ and back to b and the order around B remains unchanged. It is then easy to see that we have the following two non-zero loops of morphisms starting and ending at T_b

$$T_b \longrightarrow T_{e_{m-2}} \longrightarrow T_{e_{m-3}} \longrightarrow \cdots \longrightarrow T_{e_1} \longrightarrow T_a \longrightarrow T_b$$

and

$$T_b \longrightarrow T_{x_{m-1}} \longrightarrow T_{x_{m-2}} \longrightarrow \cdots \longrightarrow T_{x_1} \longrightarrow T_b.$$

If $j \in G_b \backslash \{b, x_1, \ldots, x_{m-1}\} \cup G_c$, then there is no non-zero map $T_b \to T_j$ since $\mathrm{Hom}_{A_{\overline{\mathcal{M}}^*}}(P_b, P_j) = 0$. Similarly, if $j \in G_{d_k} \backslash \{d_k\}$ or G_{e_k} for some k, then there is no non-zero map $T_b \to T_j$ in $K^\flat(\mathcal{P}(A_{\overline{\mathcal{M}}^*{}_*}))$ since $\mathrm{Hom}_{A_{\overline{\mathcal{M}}^*}}(P_a, P_j) = 0$. If $j = d_k$, then there is no non-zero map $T_b \to T_j$ in $K^\flat(\mathcal{P}(A_{\overline{\mathcal{M}}^*}))$ since the corresponding diagram

$$\begin{tikzcd} P_b \arrow[r] \arrow[d] & P_a \arrow[d] \\ 0 \arrow[r] & P_j \end{tikzcd}$$

cannot commute. Therefore the edges incident with b in $\mathcal{N}$ are as claimed. Note that if b is a leaf then the edges x_i are missing and we can just omit the corresponding part of this proof.

Similar arguments show that for $k = 1, 2, \ldots, m-2$, the edges in $\overline{\mathcal{M}}^*$ around edge d_k with vertices X and D_k are given clockwise around vertex X by $d_k, d_{k+1}, \ldots, d_{m-2}, a, b, d_1, d_2, \ldots, d_{k-1}$ and back to d_k and around vertex D_k by $d_k, y_1, y_2, \ldots, y_{m-1}$ and back to d_k. After the dual Kauer move at a the edges around X are given by $d_k, d_{k+1}, \ldots, d_{m-2}, c, a, d_1, d_2, \ldots, d_{k-1}$ and back to d_k and around vertex D_k they remain the same. By arguments similar to those used above there are non-zero loops of morphisms in the T_i around both vertices, leading to non-zero maps in the quiver of A'. Furthermore, $\mathrm{Hom}_{K^\flat(P(A_{\overline{\mathcal{M}}^*}))}(T_{d_k}, T_j) = 0$ if j is an edge in any of the subtrees or unions of subtrees G_b, $G_c \setminus \{c\}$, $\bigcup_{l \neq k} G_{d_l} \backslash \{d-l\}$, $G_{d_k} \backslash \{d-k, y_1, \ldots, y_{m-1}\}$, or $\bigcup_l G_{e_l}$. Thus the edges around d_k in the tree of A' are as claimed.

Finally, consider an edge x_k, for $k = 1, 2, \ldots, m-1$ with vertices B and X_k. Then in $\overline{\mathcal{M}}^*$ we have the configuration $x_k, x_{k+1}, \ldots, x_{m-1}, b, x_1, x_2, \ldots, x_{k-1}$ around the vertex B in the clockwise direction and around X_k we have the configuration $x_k, z_1, z_2, \ldots, z_{m-1}$. The dual Kauer move on the edge a does not change the configuration around the vertices of x_k. There are non-zero loops of morhpisms

$$T_{x_k} \to T_{x_{k-1}} \to \cdots \to T_{x_1} \to T_b \to T_{x_{m-1}} \to T_{x_{m-2}} \to T_{x_k}$$

and

$$T_{x_k} \to T_{z_{m-1}} \to T_{z_{m-2}} \to \cdots T_{z_1} \to T_{x_k}.$$

Moreover, it is easy to check that $\mathrm{Hom}_{K^\flat(P(A_{\overline{\mathcal{M}}^*}))}(T_{d_k}, T_j) = 0$ for all j in $\{c\}$, $G_b \setminus \{b, x_1, x_2, \ldots, x_{m-1}\}$, $\bigcup_k G_{d_k} \setminus \{d_k\}$, $G_c \setminus \{c, w_1, w_2, \ldots, w_{m-1}\}$ (where the edges $w_1, w_2, \ldots, w_{m-1}$ are those incident to c) or G_{e_k} for some k. Thus the arrows in the quiver of A' are the same as in $A_{\overline{\mathcal{N}}^*}$.

We now have considered all cases and have shown that the Brauer tree of A' is as claimed. $\square$

Remark 4.12. (a) The same arguments (with easy minor modifications) apply if $\overline{\mathcal{M}}^*$ is replaced with $\mathcal{M}^*$, that is if we remove the leaf vertices (and the adjacent edges).

(b) The same arguments also apply if we consider a general 'angulation' of P_n, that is, with no restrictions on the sizes of the subpolygons. This is the case since the sizes of the polygons do not play a role in the proofs.

(c) Theorem 4.10 gives a third derived equivalence in the disk case, to add to the other two discussed in Remark 3.5.

5. Choice of graphs and counter-examples

In this section we describe a number of situations where there is no derived equivalence corresponding to the dual Kauer move defined in the previous section. These counter-examples have motivated our choice of dual graph in the disc case. The proofs all rely on the same principle, based on Sylvester's law of inertia, which for the convenience of the reader we will recall here (see e.g. [19]). This principle provides us with a useful criterion for distinguishing when two finite-dimensional algebras are not derived equivalent.

Theorem 5.1 (Sylvester's law of inertia)**.** *Let A and B be symmetric real square matrices. Assume A is congruent to B, that is there exists a matrix $P \in \mathrm{GL_n}(\mathbb{R})$ such that $PAP^T = B$. Then A and B have the same number of strictly positive eigenvalues, strictly negative eigenvalues, and zero eigenvalues.*

We recall the following result from [7].

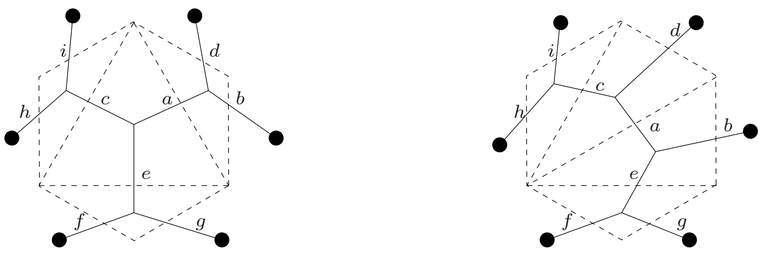
Proposition 5.2. *Let A and B be two finite-dimensional, derived equivalent algebras with Cartan matrices C_A and C_B respectively. Let n denote the number of simple modules of A and B (up to isomorphism). Then there exists a matrix $P \in \mathrm{GL_n}(\mathbb{Z})$ such that $PC_AP^T = C_B$.*

Combining Sylvester's law of inertia with Proposition 5.2 gives a citerion which can be used to show that two finite-dimensional algebras are not derived equivalent.

Corollary 5.3. *Let A and B be two finite-dimensional, derived equivalent algebras with Cartan matrices C_A and C_B respectively. Then C_A and C_B have the same number of strictly positive, strictly negative and zero eigenvalues.*

5.1. **Dual graph of a triangulation of a polygon.** Given a triangulation $\mathcal{T}$ of a polygon, instead of considering the graph $\overline{\mathcal{T}}^*$ as in Section 4 we could have considered the dual graph of $\mathcal{T}$, which can be obtained from $\overline{\mathcal{T}}^*$ by identifying all of its boundary vertices. We call this graph $\mathcal{T}^\vee$. Note that this graph is not necessarily a Brauer tree anymore, but it is a Brauer graph (where, we think of the graph embedded in a sphere and as usual we use a local embedding in the plane to get the cyclic ordering). Let a^* be one of the internal edges of $\mathcal{T}^\vee$. We denote by $\mu^*_{a^*}(\mathcal{T}^\vee)$ the Brauer graph obtained by applying a dual Kauer move to $\mathcal{T}^\vee$ at a^*. We shall see that the Brauer graph algebras $A_{\mathcal{T}^\vee}$ and $A_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ are not always derived equivalent.

We consider first the example in Figure 13. We will use the criterion based on Sylvester's law of inertia to show that there is no derived equivalence in this case.



$\mathcal{T}^\vee$ $\qquad\qquad$ $\mu^*_{a^*}(\mathcal{T}^\vee)$

Figure 13. In the above figures the black vertices are all identified, that is in each figure they represent one single vertex, namely a vertex corresponding to a point external to the disc.

The Cartan matrices $\mathcal{C}_{\mathcal{T}^\vee}$ and $\mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ of $A_{\mathcal{T}^\vee}$ and $A_{\mu^*_{a^*}(\mathcal{T}^\vee)}$, respectively, are given by

$$\mathcal{C}_{\mathcal{T}^\vee} = \begin{array}{c|c} & \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i \end{array} & \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1\\ 1 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 1\\ 1 & 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1\\ 1 & 0 & 1 & 0 & 2 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2\\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \end{pmatrix} \end{array} \text{ and } \mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)} = \begin{array}{c|c} & \begin{array}{ccccccccc} a & b & c & d & e & f & g & h & i \end{array} \\ \begin{array}{c} a\\ b\\ c\\ d\\ e\\ f\\ g\\ h\\ i \end{array} & \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 1\\ 1 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 1\\ 1 & 1 & 0 & 0 & 2 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 1 & 0 & 1 & 1 & 2 & 2 & 1 & 1\\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2\\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 & 2 \end{pmatrix} \end{array}$$

The characteristic polynomial of $\mathcal{C}_{\mathcal{T}^\vee}$ is

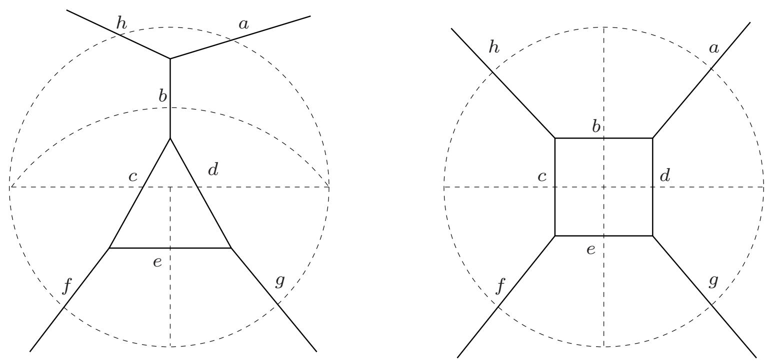
$$P_{\mathcal{T}^\vee}(x) = x^9 - 18x^8 + 111x^7 - 288x^6 + 270x^5,$$

and $\mathcal{C}_{\mathcal{T}^\vee}$ has 5 eigenvalues equal to zero. The characteristic polynomial of $\mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ is

$$P_{\mu^*_{a^*}(\mathcal{T}^\vee)}(x) = x^9 - 18x^8 + 113x^7 - 316x^6 + 395x^5 - 180x^4$$

and $\mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ has 4 eigenvalues equal to zero. It follows from Corollary 5.3 that the algebras $A_{\mathcal{T}^\vee}$ and $A_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ are not derived equivalent.

5.2. **Punctured disc case.** Even without making the boundary identification in the dual graph as in the previous example, in the punctured disc case we have an example of a triangulation $\mathcal{T}$ and an edge a, where the Brauer graph algebras corresponding to the graphs $\overline{\mathcal{T}}^*$ and $\mu^*_{a^*}(\overline{\mathcal{T}}^*)$ (as defined in Section 4) are not derived equivalent.



$\overline{\mathcal{T}}^*$ $\mu^*_{a^*}(\overline{\mathcal{T}}^*)$

FIGURE 14. Triangulation of a punctured disc and dual graph

In Figure 14 we consider the dual graph of a triangulation of a disc with 4 boundary vertices and one puncture. Note that the boundary vertices are not identified in this set-up.

The Cartan matrices $\mathcal{C}_{\mathcal{T}^\vee}$ and $\mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ of $A_{\mathcal{T}^\vee}$ and $A_{\mu^*_{a^*}(\mathcal{T}^\vee)}$, respectively, are given by

$$\mathcal{C}_{\mathcal{T}^\vee} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array} \text{ and } \mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{array}$$

Then the determinant of $\mathcal{C}_{\mathcal{T}^\vee}$ is 4 and the determinant of $\mathcal{C}_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ is 0 and thus by proposition 5.2 the algebras $A_{\mathcal{T}^\vee}$ and $A_{\mu^*_{a^*}(\mathcal{T}^\vee)}$ are not derived equivalent.

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