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A Bound for the Magnitude Characteristics of Nonlinear Output Frequency Response Functions

Part II: Practical Computation of the Bound for Systems Described by the NARX Model

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A BOUND FOR THE MAGNITUDE CHARACTERISTICS OF NONLINEAR OUTPUT FREQUENCY RESPONSE FUNCTIONS

PART II: PRACTICAL COMPUTATION OF THE BOUND FOR SYSTEMS DESCRIBED BY THE NARX MODEL

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Abstract: In Part I of this paper the concept of a bound for the output frequency response magnitude characteristics of nonlinear systems was proposed and general calculation and analysis procedures were developed. In this, the second part of the paper, a new recursive algorithm for the computation of the gain bounds for the generalised frequency response functions of the polynomial NARX model is proposed and effective procedures for the practical computation of the new bound are developed. Simulated examples are included to verify the effectiveness of the proposed procedures.

1. INTRODUCTION

In Part I of this paper the concept of a bound for output frequency response magnitude characteristics of nonlinear systems was proposed. It was shown that the expression for the new bound is similar in form to the relationship between the input and output magnitude frequency characteristics of linear systems and that the practical computation of the bound can easily be performed for nonlinear systems of finite but arbitrary order nonlinearities.

In this the second part of the paper the practical computation of this new bound for nonlinear systems described by the polynomial NARX (Nonlinear AutoRegressive model with eXogenous input) model is investigated. The emphasis is placed upon developing a new recursive algorithm for the computation of the gain bounds for the generalised frequency response functions (GFRFs) of the polynomial NARX model and then using this together with the results from Part I of the paper to produce effective procedures for the practical computation of the new bound on this model. The results can be applied to analyse or synthesise a wide class of nonlinear systems to evaluate or formulate appropriate bounds on the output magnitude frequency domain responses. Simulation studies are included to illustrate the effectiveness of the proposed methods

2. PRELIMINARIES

2.1 Output Frequency Characteristics of Nonlinear Systems

The output frequency characteristics of nonlinear systems which are stable at the zero equilibrium point and can be described in the neighbourhood of the equilibrium point by the Volterra series can be expressed as (Lang and Billings 1994)

$$
\begin{cases}
\mathbf{Y}(j\omega)=\sum_{n=1}^{N}Y_n(j\omega) \qquad \text{for } \forall\omega \\
Y_n(j\omega)=\frac{1/\sqrt{n}}{(2\pi)^{n-1}}\int_{\omega_1+,\ldots,+\omega_n=\omega}Y_n(j\omega_1,\ldots,j\omega_n)d\sigma_\omega \\
Y_n(j\omega_1,\ldots,j\omega_n)=H_n(j\omega_1,\ldots,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)
\end{cases}
\tag{2.1}
$$

where $\mathbf{Y}(j\omega)$ and $U(j\omega)$ represent the Fourier transformations of the system output and input, $Y_n(j\omega)$ is the nth order output frequency characteristic, N is the maximum order of the dominant system nonlinearity, $H_n(j\omega_1,\ldots,j\omega_n)$ is the nth order GFRF and $\int_{\omega_1+,\ldots,+\omega_n=\omega}Y_n(j\omega_1,\ldots,j\omega_n)d\sigma_\omega$ denotes the integration of $Y_n(j\omega_1,\ldots,j\omega_n)$ over the n-dimensional hyperplane $\omega=\omega_1+,\ldots,+\omega_n$.

For an input frequency spectrum defined by

$$
U(j\omega)=\begin{cases} U(j\omega) & \omega\in[\mathrm{a},\mathrm{b}] \;\; \mathrm{b}>\mathrm{a}\geq 0 \\ 0 & \text{for other } \omega\geq 0 \end{cases}
\tag{2.2}
$$

R_n the nonnegative frequency range produced by the nth order nonlinear output can be calculated using the following algorithm (Lang and Billings 1994)

$$
\begin{cases}
R_n=\bigcup_{i=1}^{2^{n-1}}[x_n^i,y_n^i] \\
\begin{bmatrix} x_n^i \\ y_n^i \end{bmatrix}=\begin{bmatrix} f_1[\min\{B_nA_n'(i,:)\},\max\{B_nA_n'(i,:)\}] \\ f_2[\min\{B_nA_n'(i,:)\},\max\{B_nA_n'(i,:)\}] \end{bmatrix}, \quad i=1,2,\ldots,2^{n-1} \\
A_n=\begin{bmatrix} I_1A_{n-1}(1,:) & I_2 \\ \vdots & \vdots \\ I_1A_{n-1}(2^{n-2},:) & I_2 \end{bmatrix}, \quad B_n=\begin{bmatrix} I_1B_{n-1}(1,:) & B_1 \\ \vdots & \vdots \\ I_1B_{n-1}(2^{n-1},:) & B_1 \end{bmatrix}, \; n\geq 2 \\
A_1=[1], \; B_1=\begin{bmatrix} a \\ b \end{bmatrix}, \; I_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; and \; I_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{cases}
\tag{2.3}
$$

where the functions f_1 and f_2 are defined as

$$
f_1(x,y)=\begin{cases} x & \text{if } y>x\geq 0 \\ |y| & \text{if } 0\geq y>x \\ 0 & \text{if } y>0>x \end{cases}
\tag{2.4}
$$

and

$$
f_2(x,y)=\begin{cases} y & \text{if } y>x\geq 0 \\ |x| & \text{if } 0\geq y>x \\ \max[y,|x|] & \text{if } y>0>x \end{cases}
\tag{2.5}
$$

and $A_n(i,:)$ and $B_n(i,:)$ represent the ith row of A_n and B_n respectively. From R_n the frequency range produced by the nth order nonlinear output can easily be obtained as

$$R_n \cup R_{-n} = -R_n \cup R_n \setminus \{0\} \tag{2.6}$$

where $-R_n$ denotes the set which possesses elements of opposite signs to those in the set R_n.

The above results are natural extensions of the well known facts in linear systems that the output frequency characteristics are linearly related to that of the input by

$$\mathbf{Y}(j\omega) = H_1(j\omega)U(j\omega) \tag{2.7}$$

and that the frequency range of the system output is the same as that of the corresponding input.

2.2 General Expression and Calculation Procedures for the Bound on Output Frequency Response Magnitude Characteristics of Nonlinear Systems

In Part I of this paper it has been shown that for the nonlinear systems considered in Section 2.1 a bound for the output magnitude frequency domain characteristics can be expressed as

$$\mathbf{Y}^B(\omega) = \sum_{n=1}^{N} \frac{1}{(2\pi)^{n-1}} \left| H_n(j\omega_{n1}^{\#}, \ldots, j\omega_{nn}^{\#}) \right| \underbrace{\left| U \right| * \cdots * \left| U(j\omega) \right|}_{n} \tag{2.8}$$

where $\mathbf{Y}^B(\omega)$ denotes the bound, $\{\omega_{n1}^{\#}, \ldots, \omega_{nn}^{\#}\}$ represents the co-ordinates of a point on the n-dimensional hyperplane $\omega_1 +, \ldots, +\omega_n = \omega$ and

$$\underbrace{\left| U \right| * \cdots * \left| U(j\omega) \right|}_{n} = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \left| U(j\omega_1) \right| \cdots \left| U(j\omega_{n-1}) \right| \left| U[j(\omega - \omega_1 -, \cdots, -\omega_{n-1})] \right| d\omega_1 \cdots d\omega_{n-1}$$

represents the n-dimensional convolution integral of $\left| U(j\omega) \right|$ for discrete time nonlinear systems.

Part I of this paper also suggests the following general procedures for the practical computation of this bound.

First compute $\left| H_n(j\omega_1, \ldots, j\omega_n) \right|_{\omega}^{B}$, a bound for $\left| H_n(j\omega_1, \ldots, j\omega_n) \right|$ with $\omega_1, \ldots, \omega_n$ satisfying the constraint $\omega_1 +, \ldots, +\omega_n = \omega$; then calculate $\underbrace{\left| U \right| * \cdots * \left| U(j\omega) \right|}_{n}$ using the developed algorithm

$$\begin{cases} \underbrace{\left| U \right| * \cdots * \left| U(j\frac{2\pi}{MT} i) \right|}_{n} = T\tilde{\tilde{U}}[i + (\frac{M}{2} - 1)n](\frac{2\pi}{M})^{n-1} \quad i = -n(\frac{M}{2} - 1), \ldots, 0, \ldots, n\frac{M}{2} \\ \{\tilde{\tilde{U}}[0] \ldots, \tilde{\tilde{U}}[n(M-1)]\} = Conv\{\underbrace{[\tilde{U}(0), \ldots, \tilde{U}(M-1)], \ldots, [\tilde{U}(0), \ldots, \tilde{U}(M-1)]}_{n}\} \\ \tilde{U}(i) = \left| U_d[j\frac{2\pi}{M}(i - \frac{M}{2} + 1)] \right|, \quad i = 0, 1, \ldots, M-1 \end{cases} \tag{2.9}$$

where $U_d(.) = U(.)$ and T=1 when only the cases of discrete time nonlinear systems are considered, M is a sufficiently large even number to be given a priori and $Conv[\underbrace{x,\ldots,x}_{n}]$ denotes the n-dimensional convolution of the vector x. Finally evaluate

$$\overline{\mathbf{Y}}^{B}(\omega) = \sum_{n=1}^{N} \frac{1}{(2\pi)^{n-1}} \left|H_n(j\omega_1,\ldots,j\omega_n)\right|_{\omega}^{B} \underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{2.10}$$

to yield an approximate result for $\mathbf{Y}^{B}(\omega)$ which possesses the same properties as $\mathbf{Y}^{B}(\omega)$. The difference between the exact result $\mathbf{Y}^{B}(\omega)$ and the approximation $\overline{\mathbf{Y}}^{B}(\omega)$ is that the maximum value of the term $\left|H_n(j\omega_{n1}^{\#},\ldots,j\omega_{nn}^{\#})\right|$ is used in the later.

2.3 Descriptions of the Polynomial NARX Model in Both the Time and Frequency Domains

A large class of deterministic discrete time nonlinear systems can be described by the NARX model (Chen and Billings 1989). The time domain description of this model is

$$y(k) = \sum_{m=1}^{\overline{M}} y_m(t) \tag{2.11}$$

where $y_m(t)$, the 'NARX mth-order output' of the system, is given by

$$y_m(t) = \sum_{p=0}^{m} \sum_{k_1,k_{p+q}=1}^{K} c_{pq}(k_1,\ldots,k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \tag{2.12}$$

with

$$p+q=m, \quad k_i = 1,\ldots,K, \; i=1,\ldots,p+q, \text{ and } \sum_{k_1,k_{p+q}=1}^{K} \equiv \sum_{k_1=1}^{K}\cdots\sum_{k_{p+q}=1}^{K} \tag{2.13}$$

Peyton Jones and Billings (1989) have proved that the GFRFs of this nonlinear model can be computed recursively from the time domain parameters in (2.12) using the following algorithm

$$\{1-\sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j(\omega_1+,\cdots,+\omega_n)k_1]\} H_n(j\omega_1,\ldots,j\omega_n)$$

$$= \sum_{k_1,k_n=1}^{K} c_{0n}(k_1,\ldots,k_n)\exp[-j(\omega_1 k_1+,\cdots,+\omega_n k_n)]$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K} c_{pq}(k_1,\ldots,k_{p+q})\exp[-j(\omega_{n-q+1}k_{p+1}+\cdots+\omega_n k_{p+q})] H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} c_{p0}(k_1,\ldots,k_p) H_{n,p}(j\omega_1,\ldots,j\omega_n) \tag{2.14}$$

where

$$H_{np}(j\omega_1,\ldots,j\omega_n) = \sum_{i=1}^{n-p+1} H_i(j\omega_1,\ldots,j\omega_i) H_{n-i,p-1}(j\omega_{i+1},\ldots,j\omega_n)\exp[-j(\omega_1+\cdots+\omega_i)k_p] \tag{2.15}$$

3. RECURSIVE CALCULATION OF THE GAIN BOUNDS FOR THE GFRFS OF THE POLYNOMIAL NARX MODEL

The gain bounds for the GFRFs of nonlinear systems are an important concept which can be applied to measure the significant orders of nonlinearities associated with system properties. In Chua and Liao (1991), this problem was addressed and an experimental algorithm was developed to find the highest significant order of the GFRFs. In Zhang and Billings (1994), an expression for the gain bound for each order of the GFRFs was derived for nonlinear systems which can be described by the polynomial NARX model . In this section, an explicit recursive algorithm for the computation of the gain bounds on the polynomial NARX model is developed to give a calculation procedure which can easily be applied in practice. The new algorithm will be taken as an important procedure in the next section to calculate the bounds for the output frequency response magnitude characteristics of nonlinear systems described by this model.

3.1 Derivation of the New Recursive Algorithm

From (2.14), $H_n(j\omega_1,\ldots,j\omega_n)$ can be written as

$$H_n(j\omega_1,\ldots,j\omega_n)=\frac{1}{(1-\sum_{k_1=1}^{K}c_{10}(k_1)\exp[-j(\omega_1+,\cdots,+\omega_n)k_1])}\times$$

$$\{\sum_{k_1,k_n=1}^{K}c_{0n}(k_1,\ldots,k_n)\exp[-j(\omega_1k_1+,\cdots,+\omega_nk_n)]$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K}c_{pq}(k_1,\ldots,k_{p+q})\exp[-j(\omega_{n-q+1}k_{p+1}+,\cdots,+\omega_nk_{p+q})]H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K}c_{p0}(k_1,\ldots,k_p)H_{n,p}(j\omega_1,\ldots,j\omega_n)\} \tag{3.1}$$

From (3.1) it is obvious that

$$\left|H_n(j\omega_1,\ldots,j\omega_n)\right|\le\frac{1}{\left|1-\sum_{k_1=1}^{K}c_{10}(k_1)\exp[-j(\omega_1+,\cdots,+\omega_n)k_1]\right|}\times$$

$$\{\sum_{k_1,k_n=1}^{K}\left|c_{0n}(k_1,\ldots,k_n)\right|+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K}\left|c_{pq}(k_1,\ldots,k_{p+q})\right|\left|H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})\right|$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K}\left|c_{p0}(k_1,\ldots,k_p)\right|\left|H_{n,p}(j\omega_1,\ldots,j\omega_n)\right|\}$$

$$\le\frac{1}{L_n}\{\sum_{k_1,k_n=1}^{K}\left|c_{0n}(k_1,\ldots,k_n)\right|+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K}\left|c_{pq}(k_1,\ldots,k_{p+q})\right|\left|H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})\right|^{B}$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K}|c_{p0}(k_1,\ldots,k_p)||H_{n,p}(j\omega_1,\ldots,j\omega_n)|^B\} \tag{3.2}$$

where

$$L_n = \min_{\omega\in R_n\cup R_{-n}}\left|1-\sum_{k_1=1}^{K}c_{10}(k_1)\exp(-j\omega k_1)\right| \tag{3.3}$$

with $R_n \cup R_{-n}$ given by (2.6), and $|H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})|^B$ and $|H_{n,p}(j\omega_1,\ldots,j\omega_n)|^B$ represent the bounds of $|H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})|$ and $|H_{n,p}(j\omega_1,\ldots,j\omega_n)|$ respectively.

From (2.15) it is known that

$$|H_{np}(j\omega_1,\ldots,j\omega_n)| \le \sum_{i=1}^{n-p+1}|H_i(j\omega_1,\ldots,j\omega_i)||H_{n-i,p-1}(j\omega_{i+1},\ldots,j\omega_n)|$$

$$\le \sum_{i=1}^{n-p+1}|H_i(j\omega_1,\ldots,j\omega_i)|^B|H_{n-i,p-1}(j\omega_{i+1},\ldots,j\omega_n)|^B = |H_{np}(j\omega_1,\ldots,j\omega_n)|^B \tag{3.4}$$

where $|H_i(j\omega_1,\ldots,j\omega_i)|^B$ denotes the bound for $|H_i(j\omega_1,\ldots,j\omega_i)|$ and $|H_{np}(j\omega_1,\ldots,j\omega_n)|^B$ is defined as

$$|H_{np}(j\omega_1,\ldots,j\omega_n)|^B = \sum_{i=1}^{n-p+1}|H_i(j\omega_1,\ldots,j\omega_i)|^B|H_{n-i,p-1}(j\omega_{i+1},\ldots,j\omega_n)|^B \tag{3.5}$$

Denoting

$$\begin{cases}\sum_{k_1,k_n=1}^{K}|c_{0n}(k_1,\ldots,k_n)| = \sum c_{0n}\\ \sum_{k_1,k_{p+q}=1}^{K}|c_{pq}(k_1,\ldots,k_{p+q})| = \sum c_{pq}\\ \sum_{k_1,k_p=1}^{K}|c_{p0}(k_1,\ldots,k_p)| = \sum c_{p0}\end{cases} \tag{3.6}$$

and

$$\begin{cases}|H_{n,p}(j\omega_1,\ldots,j\omega_n)|^B = H_{n,p}^B\\ |H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})|^B = H_{n-q,p}^B\\ |H_i(j\omega_1,\ldots,j\omega_i)|^B = H_i^B\end{cases} \tag{3.7}$$

in (3.2) and (3.5) and unfolding the summations for p and q in (3.2) yields

$$|H_n(j\omega_1,\ldots,j\omega_n)| \le \frac{1}{L_n}\{\sum c_{0n} + \sum c_{20}H_{n2}^B +,\cdots,+\sum c_{n0}H_{nn}^B$$

$$+\sum c_{11}H_{n-11}^B +,\cdots,\sum c_{1n-2}H_{21}^B + \sum c_{1n-1}H_{11}^B$$

$$+\sum c_{21}H_{n-12}^B +,\cdots,+\sum c_{2n-2}H_{22}^B$$

$$\vdots \qquad\qquad \vdots$$

$$+\sum c_{n-21}H^B_{n-1n-2}+\sum c_{n-22}H^B_{n-2n-2}$$
$$+\sum c_{n-11}H^B_{n-1n-1}\}$$

$$=\frac{1}{L_n}\{\sum c_{0n}+\sum c_{20}H^B_{n2}+,\cdots,+\sum c_{n0}H^B_{nn}$$
$$+\sum c^n_1(H^{B_n}_1)^T+\sum c^n_2(H^{B_n}_2)^T+,\cdots,+\sum c^n_{n-1}(H^{B_n}_{n-1})^T\} \tag{3.8}$$

where

$$\begin{cases}\sum c^n_1=[\sum c_{11},\sum c_{12},\ldots,\sum c_{1n-1}]\\ \vdots\\ \sum c^n_k=[\sum c_{k1},\sum c_{k2},\ldots,\sum c_{kn-k}]\\ \vdots\\ \sum c^n_{n-1}=[\sum c_{n-11}]\end{cases} \tag{3.9}$$

$$\begin{cases}H^{B_n}_1=[H^B_{n-11},H^B_{n-21},\ldots,H^B_{11}]\\ \vdots\\ H^{B_n}_k=[H^B_{n-1k},H^B_{n-2k},\ldots,H^B_{(n-(n-k))k}]\\ \vdots\\ H^{B_n}_{n-1}=[H^B_{n-1n-1}]\end{cases} \tag{3.10}$$

In view of (3.5), it can be shown that

$$\begin{cases}H^B_{n2}=H^B_1H^B_{n-11}+H^B_2H^B_{n-21}+,\cdots,+H^B_{n-1}H^B_{11}=\overline{H}_{n-1}(H^{B_n}_1)^T\\ H^B_{n3}=H^B_1H^B_{n-12}+H^B_2H^B_{n-22}+,\cdots,+H^B_{n-2}H^B_{22}=\overline{H}_{n-2}(H^{B_n}_2)^T\\ \vdots\\ H^B_{nk}=H^B_1H^B_{n-1k-1}+H^B_2H^B_{n-2k-1}+,\cdots,+H^B_{n-k+1}H^B_{k-1k-1}=\overline{H}_{n-(k-1)}(H^{B_n}_{k-1})^T\\ \vdots\\ H^B_{nn}=H^B_1H^B_{n-1n-1}=\overline{H}_1(H^{B_n}_{n-1})^T\end{cases} \tag{3.11}$$

where

$$\overline{H}_{n-i}=[H^B_1,\ldots,H^B_{n-i}],\quad i=1,2,\ldots,n-1 \tag{3.12}$$

which yields

$$\sum c_{20}H^B_{n2}+,\cdots,+\sum c_{n0}H^B_{nn}=\sum c_{20}\overline{H}_{n-1}(H^{B_n}_1)^T+\sum c_{30}\overline{H}_{n-2}(H^{B_n}_2)^T+,\cdots,+\sum c_{n0}\overline{H}_1(H^{B_n}_{n-1})^T \tag{3.13}$$

Substituting (3.13) into (3.8) gives

$$|H_n(j\omega_1,\ldots,j\omega_n)|\le\frac{1}{L_n}\{\sum c_{0n}+\sum c_{20}\overline{H}_{n-1}(H^{B_n}_1)^T+\sum c_{30}\overline{H}_{n-2}(H^{B_n}_2)^T+,\cdots,+\sum c_{n0}\overline{H}_1(H^{B_n}_{n-1})^T$$
$$+\sum c^n_1(H^{B_n}_1)^T+\sum c^n_2(H^{B_n}_2)^T+,\cdots,+\sum c^n_{n-1}(H^{B_n}_{n-1})^T\}$$
$$=\frac{1}{L_n}\{\sum c_{0n}+[\sum c^n_1+\sum c_{20}\overline{H}_{n-1}](H^{B_n}_1)^T+,\cdots,+[\sum c^n_{n-1}+\sum c_{n0}\overline{H}_1](H^{B_n}_{n-1})^T\} \tag{3.14}$$
$$=|H_n(j\omega_1,\ldots,j\omega_n)|^B$$

where $|H_n(j\omega_1,\ldots,j\omega_n)|^B$ is defined as

$$|H_n(j\omega_1,\ldots,j\omega_n)|^B = \frac{1}{L_n}\{\sum c_{0n} + [\sum c_1^n + \sum c_{20}\overline{H}_{n-1}](H_1^{B_n})^T +,\cdots,+ [\sum c_{n-1}^n + \sum c_{n0}\overline{H}_1](H_{n-1}^{B_n})^T\}$$

$$= \frac{1}{L_n}[\sum c_{0n}, \sum c_1^n + \sum c_{20}\overline{H}_{n-1},\ldots,\sum c_{n-1}^n + \sum c_{n0}\overline{H}_1][1, H_1^{B_n},\ldots,H_{n-1}^{B_n}]^T$$

(3.15)

In (3.15) $H_k^{B_n}$, $k = 1,2,\ldots,n-1$ can be expressed as

$$\begin{cases} H_1^{B_n} = [H_{n-11}^B, H_{n-21}^B,\ldots,H_{11}^B] = [H_{n-11}^B H_1^{B_{n-1}}] \\ \vdots \\ H_k^{B_n} = [H_{n-1k}^B, H_{n-2k}^B,\ldots,H_{(n-(n-k))k}^B] = [H_{n-1k}^B, H_k^{B_{n-1}}] \\ \vdots \\ H_{n-2}^{B_n} = [H_{n-1n-2}^B, H_{n-2n-2}^B] = [H_{n-1n-2}^B, H_{n-2}^{B_{n-1}}] \\ H_{n-1}^{B_n} = [H_{n-1n-1}^B] \end{cases} \quad (3.16)$$

From (3.11) it is known that the first elements of $H_k^{B_n}$, for $k = 2,\ldots,n-1$ can be written as

$$\begin{cases} H_{n-12}^B = \overline{H}_{n-2}(H_1^{B_{n-1}})^T \\ \vdots \\ H_{n-1k}^B = \overline{H}_{n-1-(k-1)}(H_{k-1}^{B_{n-1}})^T \\ \vdots \\ H_{n-1n-1}^B = \overline{H}_1(H_{n-2}^{B_{n-1}})^T \end{cases}$$

(3.17)

and H_{n-11}^B, the first element of $H_1^{B_n}$, can also be expressed similarly as

$$H_{n-11}^B = \overline{H}_{n-1-(1-1)}(H_0^{B_{n-1}})^T = \overline{H}_{n-1}(H_0^{B_{n-1}})^T$$

(3.18)

if $H_0^{B_{n-1}}$ is defined as below

$$H_o^{B_{n-1}} = [H_{n-20}^B, H_{n-30}^B,\ldots,H_{00}^B] = [\underbrace{0,\ldots,0}_{n-2},1]$$

(3.19)

Thus $H_k^{B_n}$, for $k = 1,2,\ldots,n-1$, can be expressed uniformly as

$$\begin{cases} H_k^{B_n} = [H_{n-1k}^B, H_k^{B_{n-1}}] = [\overline{H}_{n-k}(H_{k-1}^{B_{n-1}})^T, H_k^{B_{n-1}}],\ k = 1,2,\ldots,n-2 \\ H_{n-1}^{B_n} = [\overline{H}_1(H_{n-2}^{B_{n-1}})^T] \end{cases} \quad (3.20)$$

and (3.15) can therefore be written as

$$|H_n(j\omega_1,\ldots,j\omega_n)|^B = H_n^B = \frac{1}{L_n}[\overbrace{0,\ldots,0}^{n-1}, \sum c_{0n}, \sum c_1^n + \sum c_{20}\overline{H}_{n-1},\ldots,\sum c_{n-1}^n + \sum c_{n0}\overline{H}_1][H_0^{B_n}, H_1^{B_n},\ldots,H_{n-1}^{B_n}]^T$$

(3.21)

Moreover combining (3.21) , (3.19), (3.20) as well as (3.9) and (3.12) yields the new recursive algorithm for computing the bounds on the GFRFs of the polynomial NARX model as follows

$$\begin{cases} H_n^B = \frac{1}{L_n}[0_{1\times(n-1)}, \sum c_{0n}, \sum c_1^n + \sum c_{20}\overline{H}_{n-1}, \ldots, \sum c_{n-1}^n + \sum c_{n0}\overline{H}_1][H_0^{B_n}, H_1^{B_n}, \ldots, H_{n-1}^{B_n}]^T, \\ H_0^{B_n} = [0_{1\times(n-1)}, 1],\ H_k^{B_n} = [\overline{H}_{n-k}(H_{k-1}^{B_{n-1}})^T,\ H_k^{B_{n-1}}],\ k = 1,2,\ldots,n-2,\ H_{n-1}^{B_n} = [\overline{H}_1(H_{n-2}^{B_{n-1}})^T], \\ \sum c_k^n = [\sum c_{k1}, \ldots, \sum c_{kn-k}],\ k = 1,\ldots,n-1,\ \sum c_{pq} = \sum_{k_1,k_{p+q}=1}^{K} \left| c_{pq}(k_1,\ldots,k_{p+q}) \right|, \\ \overline{H}_{n-k} = [H_1^B, \ldots, H_{n-k}^B],\ k = 1,\ldots,n-1. \end{cases} \tag{3.22}$$

Equation (3.22) shows that H_n^B, the nth step result of the recursive algorithm, is obtained by the multiplication of $1/L_n$ and two vectors. The first vector is composed of parameters of the NARX model and the gain bounds associated with the model GFRFs from order one to n-1 which are determined in previous recursive calculation steps. The second vector is composed of $H_0^{B_n}, H_1^{B_n}, \ldots, H_{n-1}^{B_n}$ which are obtained from $H_0^{B_{n-1}}, H_1^{B_{n-1}}, \ldots, H_{n-2}^{B_{n-1}}$ by the recursive computation in (3.20) and $H_0^{B_n} = [0, H_0^{B_{n-1}}] = [0_{1\times(n-1)}, 1]$ where the initial value for this recursive process is $H_0^{B_1} = [1]$. Therefore using the algorithm (3.22) H_n^B can eventually be described as a function of L_n, the model parameters as well as the gain bounds for the GFRFs from the first to n-1th order. In the next section, the computation for the first three gain bounds for the GFRFs, H_1^B, H_2^B, and H_3^B, are performed to illustrate the recursive calculation processes and corresponding results.

3.2 Illustration of the Recursive Calculation Process

For n=1 in (3.22)

$$H_1^B = \frac{1}{L_1}[\sum c_{01}][H_0^{B_1}]^T = \frac{1}{L_1}[\sum c_{01}][1] = \frac{\sum c_{01}}{L_1} \tag{3.23}$$

For n=2,

$$\begin{aligned} H_2^B &= \frac{1}{L_2}[0_{1\times1}, \sum c_{02}, \sum c_1^2 + c_{20}\overline{H}_1][H_0^{B_2}, H_1^{B_2}]^T \\ &= \frac{1}{L_2}[0, \sum c_{02}, \sum c_1^2 + c_{20}\overline{H}_1][0, 1, H_1^{B_2}]^T \end{aligned} \tag{3.24}$$

Substituting

$$H_1^{B_2} = [\overline{H}_1(H_0^{B_1})^T] = [\overline{H}_1 \times 1] = H_1^B \tag{3.25}$$

into (3.24) yields

$$H_2^B = \frac{1}{L_2}[0_{1\times 1}, \sum c_{02}, \sum c_1^2 + c_{20}\overline{H}_1][H_0^{B_2}, H_1^{B_2}]^T = \frac{1}{L_2}[0, \sum c_{02}, \sum c_{11} + c_{20}H_1^B][0,1,H_1^B]^T$$

$$= \frac{1}{L_2}[\sum c_{02} + \sum c_{11}H_1^B + \sum c_{20}(H_1^B)^2]$$

(3.26)

For n=3

$$H_3^B = \frac{1}{L_3}[0_{1\times 2}, \sum c_{03}, \sum c_1^3 + \sum c_{20}\overline{H}_2, \sum c_2^3 + \sum c_{30}\overline{H}_1][H_0^{B_3}, H_1^{B_3}, H_2^{B_3}]^T$$

$$= \frac{1}{L_3}[0,0, \sum c_{03}, [\sum c_{11}, \sum c_{12}] + \sum c_{20}[H_1^B, H_2^B], \sum c_{21} + \sum c_{30}H_1^B][0,0,1,H_1^{B_3}, H_2^{B_3}]^T$$

(3.27)

From (3.20)

$$\begin{cases} H_1^{B_3} = [\overline{H}_2(H_0^{B_2})^T, H_1^{B_2}] = [(H_1^B, H_2^B)\begin{pmatrix}0\\1\end{pmatrix}, H_1^{B_2}] = [H_2^B, H_1^{B_2}] \\ H_2^{B_3} = [\overline{H}_1(H_1^{B_2})^T] = H_1^B H_1^{B_2} \end{cases} \quad (3.28)$$

Substituting (3.28) into (3.27) gives

$$H_3^B = \frac{1}{L_3}[0,0, \sum c_{03}, \sum c_{11} + \sum c_{20}H_1^B, \sum c_{12} + \sum c_{20}H_2^B, \sum c_{21} + \sum c_{30}H_1^B][0,0,1,H_2^B, H_1^{B_2}, H_1^B H_1^{B_2}]^T$$

$$= \frac{1}{L_3}[\sum c_{03} + H_2^B(\sum c_{11} + \sum c_{20}H_1^B) + H_1^{B_2}(\sum c_{12} + \sum c_{20}H_2^B) + H_1^B H_1^{B_2}(\sum c_{21} + \sum c_{30}H_1^B)]$$

(3.29)

Finally using (3.25), (3,29) can be written as

$$H_3^B = \frac{1}{L_3}[\sum c_{03} + \sum c_{11}H_2^B + \sum c_{20}H_1^B H_2^B + \sum c_{12}H_1^B + \sum c_{20}H_2^B H_1^B + \sum c_{21}(H_1^B)^2 + \sum c_{30}(H_1^B)^3]$$

(3.30)

These recursive calculations demonstrate that the new algorithm can easily be implemented by hand if the orders concerned are not very high. In addition the results from (3.23), (3.26) and (3.30) also reveal which model terms make contributions to the gain bound for each GFRF and how these contributions are made. For example, it can be seen from the subscripts of the coefficients in (3.26) that in total four kinds of NARX model polynomial terms make contributions to the gain bound H_2^B. These are

$$y(k-k_1),\ u(k-k_1)u(k-k_2),\ u(k-k_1)y(k-k_2) \text{ and } y(k-k_1)y(k-k_2)$$

and the effects of these terms on H_2^B are reflected by

$$1/L_2,\ \sum c_{02},\ \sum c_{11}H_1^B \text{ and } \sum c_{20}(H_1^B)^2$$

respectively.

By extending the above recursive calculations to an arbitrary order n a general expression for H_n^B in terms of the model parameters and $H_1^B, \ldots, H_{n-1}^B$ can be obtained. This provides a basis for formulating the gain bounds on the GFRFs for the polynomial NARX model. This is not directly related to the topic considered here and will be discussed in detail in another paper.

4. PRACTICAL COMPUTATION OF THE BOUND FOR SYSTEMS DESCRIBED BY THE POLYNOMIAL NARX MODEL

Consider systems described by the polynomial NARX model (2.11) and satisfying the conditions mentioned in Section 2.1. Then the bound for the output magnitude frequency domain characteristics can generally be written in the form of Equation (2.8) and the practical computation of this bound can be carried out by calculating an approximate value of the theoretical result $\mathbf{Y}^B(\omega)$ as shown in (2.10).

From (3.1) it is known that

$$|H_n(j\omega_1,\ldots,j\omega_n)|_{\omega_1+,\cdots,+\omega_n=\omega} = \frac{1}{\left|(1-\sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j\omega k_1])\right|}\times$$

$$|\{\sum_{k_1,k_n=1}^{K} c_{0n}(k_1,\ldots,k_n)\exp[-j(\omega_1 k_1+,\cdots,+\omega_n k_n)]$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{k_1,k_{p+q}=1}^{K} c_{pq}(k_1,\ldots,k_{p+q})\exp[-j(\omega_{n-q+1}k_{p+1}+,\cdots,+\omega_n k_{p+q})]H_{n-q,p}(j\omega_1,\ldots,j\omega_{n-q})$$

$$+\sum_{p=2}^{n}\sum_{k_1,k_p=1}^{K} c_{p0}(k_1,\ldots,k_p)H_{n,p}(j\omega_1,\ldots,j\omega_n)\}|_{\omega_1+,\cdots,+\omega_n=\omega}$$

$$\leq \frac{1}{\left|(1-\sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j\omega k_1])\right|} L_n H_n^B \tag{4.1}$$

where L_n and H_n^B are given by (3.3) and (3.15) respectively. Defining

$$|H_n(j\omega_1,\ldots,j\omega_n)|_\omega^B = \frac{L_n H_n^B}{\left|(1-\sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j\omega k_1])\right|} \tag{4.2}$$

and substituting (4.2) into (2.10) yields

$$\overline{\mathbf{Y}}^B(\omega) = \sum_{n=1}^{N}\frac{1}{(2\pi)^{n-1}}\frac{L_n H_n^B}{\left|(1-\sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j\omega k_1])\right|}\underbrace{|U|*\cdots*|U(j\omega)|}_{n}$$

$$= C(\omega)\sum_{n=1}^{N}\frac{1}{(2\pi)^{n-1}} L_n H_n^B \underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{4.3}$$

where

$$C(\omega) = \frac{1}{\left|(1 - \sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j\omega k_1])\right|} \tag{4.4}$$

In (4.3) N denotes the maximum order of the dominant system nonlinearities. Generally N can not be determined directly from parameters of the polynomial NARX model because of the well known fact that in practical cases the amount of nonlinearity exhibited in the output of a system depends on the system model and on the magnitude of the input. Consequently an appropriate N should be determined based on both the GFRFs and $|U(j\omega)|$.

Notice that in (4.3)

$$\frac{1}{(2\pi)^{n-1}} C(\omega) L_n H_n^B \underbrace{|U|*\cdots*|U(j\omega)|}_{n}$$

actually reflects a bound for $|Y_n(j\omega)|$, the magnitude frequency domain characteristic of the nth order nonlinear output. Defining

$$G_n = \frac{1}{(2\pi)^{n-1}} L_n H_n^B \max_{\omega \in R_n \cup R_{-n}} C(\omega) \underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{4.5}$$

and substituting (3.3) and (4.4) into (4.5) yields

$$G_n = \frac{1}{(2\pi)^{n-1}} H_n^B L_n \max_{\omega \in R_n \cup R_{-n}} C(\omega) \max_{\omega \in R_n \cup R_{-n}} \underbrace{|U|*\cdots*|U(j\omega)|}_{n}$$

$$= \frac{1}{(2\pi)^{n-1}} H_n^B L_n \frac{1}{L_n} \max_{\omega \in R_n \cup R_{-n}} \underbrace{|U|*\cdots*|U(j\omega)|}_{n} = \frac{1}{(2\pi)^{n-1}} H_n^B \max_{\omega \in R_n \cup R_{-n}} \underbrace{|U|*\cdots*|U(j\omega)|}_{n} \tag{4.6}$$

Clearly this can be taken as a measure for the effect of the nth order system nonlinearity on the system output under the input excitation with a frequency characteristic $U(j\omega)$.

Based on the above results and assuming that parameters of the NARX model, the frequency characteristic of the input $U(j\omega)$, and an integer $N_M > N$ are known a priori, procedures for the practical computation of the bound for systems described by the polynomial NARX model are proposed as below.

(i) Compute R_n for n=1,2,...,N_M using the algorithm (2.3)

(ii) From (3.3), obtain L_n, for n=1,2,...,N_M.

(iii) From the algorithm (3.22) and using the L_n obtained in (ii), calculate $H_1^B, \ldots, H_{N_M}^B$.

(iv) For an a priori given M, compute $\underbrace{|U|*\cdots*\left|U(j\frac{2\pi}{M}i)\right|}_{n}$ for $i = -(M/2 - 1), \ldots, 0, \ldots, M/2$

and n=1,2,...,N_M using the algorithm (2.9) with $U_d(.) = U(.)$ and T=1.

(v) Evaluate G_n as

$$G_n = \frac{1}{(2\pi)^{n-1}} H_n^B \max_{i \in \{0,1,\ldots,M/2\}} \underbrace{|U|*\cdots*\left|U(j\frac{2\pi}{M}i)\right|}_{n} \quad \text{for n=1,...,}N_M \tag{4.7}$$

to find an n^* such that

$$G_n < \varepsilon \quad \text{for } n > n^*$$

where ε is a small number given a priori. Then take $N = n^*$.

(vi) Evaluate the bound at the frequencies of interest within the range $\omega = \frac{2\pi}{M} i,\ i = -(\frac{M}{2} - 1), \ldots, 0, \ldots, \frac{M}{2}$ using (4.3).

In view of the simplicity and practicability of the algorithms used in (i), (iii) and (iv), the proposed procedures can easily be applied in practice.

It is worth emphasising that when using the above procedures N the maximum order of the dominant system nonlinearities is determined by evaluating the values of G_n given by (4.6) for n=1,...,N_M which reflects the effects of both the GFRFs and $U(j\omega)$. This implies that for different $U(j\omega)$ the maximum order of dominant system nonlinearities may be different and even if the gain bounds for the GFRFs do not converge as n increases it is still possible to describe the system output frequency characteristics using a model of finite order nonlinearities. These conclusions are in fact an extension of the well-known engineering concept that a practical system can usually be approximated quite well by a linear model when the system input varies within a small neighbourhood of the working point but have not previously been seen in the frequency domain analysis of nonlinear systems.

5. SIMULATION STUDIES

Example 1.

Consider a discrete time nonlinear system described by the NARX model

$$y(k) = c_{01}(1)u(k-1) + c_{10}(1)y(k-1) + c_{10}(2)y(k-2) + c_{02}(1,3)u(k-1)u(k-3) + c_{03}(1,1,2)u^2(k-1)u(k-2) \tag{5.1}$$

where $c_{01}(1) = 0.5$, $c_{10}(1) = 0.8$, $c_{10}(2) = -0.64$, $c_{02}(1,3) = -0.4$, $c_{03}(1,1,2) = 10$ and the input is defined by the sequence

$$u(k) = \frac{1}{2\pi}\left[\frac{\sin\alpha kT_s - \sin\beta kT_s}{kT_s}\right] \quad \text{for } k = 0, \pm 1, \pm 2, \ldots \tag{5.2}$$

with $\alpha = 5$, $\beta = 1$, and $T_s = 0.2$.

For this simple example there is no need to follow the procedures in Section 4 step by step to evaluate the bound $\overline{\mathbf{Y}}^B(\omega)$. From (3.23) (3.26) and (3.30) it is known that for this particular case

$$H_1^B = \frac{\sum c_{01}}{L_1} = \frac{|c_{01}(1)|}{L_1} = \frac{0.5}{L_1} \qquad (\text{i.e. } H_1^B L_1 = 0.5) \tag{5.3}$$

$$H_2^B = \frac{1}{L_2}[\sum c_{02} + \sum c_{11} H_1^B + \sum c_{20}(H_1^B)^2] = \frac{1}{L_2}[|c_{02}(1,3)|] = \frac{0.4}{L_2}, \quad (\text{i.e. } H_2^B L_2 = 0.4) \tag{5.4}$$

and

$$H_3^B = \frac{1}{L_3}[\sum c_{03} + \sum c_{11} H_2^B + \sum c_{20} H_1^B H_2^B + \sum c_{12} H_1^B + \sum c_{20} H_2^B H_1^B + \sum c_{21} (H_1^B)^2 + \sum c_{30} (H_1^B)^3]$$

$$= \frac{1}{L_3} |c_{03}(1,1,2)| = \frac{10}{L_3} \qquad \text{(i.e.} \quad H_3^B L_3 = 10)$$

(5.5)

and the maximum order of dominant system nonlinearities is N=3.

Thus substituting N=3 , equations (5.3)-(5.5) and

$$C(\omega) = \frac{1}{\left|(1-\sum_{k_1=1}^{K} c_{10}(k_1)\exp[-j\omega k_1])\right|} = \frac{1}{|1-0.8\exp(-j\omega)+0.64\exp(-j2\omega)|}$$

into (4.3) yields

$$\overline{\mathbf{Y}}^B(\omega) = \frac{1}{|1-0.8\exp(-j\omega)+0.64\exp(-j2\omega)|}[L_1 H_1^B |U(j\omega)| + \frac{1}{2\pi} L_2 H_2^B |U| * |U(j\omega)| + \frac{1}{(2\pi)^2} L_3 H_3^B |U| * |U| * |U(j\omega)|]$$

$$= \frac{1}{|1-0.8\exp(-j\omega)+0.64\exp(-j2\omega)|}[0.5|U(j\omega)| + \frac{0.4}{2\pi}|U| * |U(j\omega)| + \frac{10}{(2\pi)^2}|U| * |U| * |U(j\omega)|]$$

(5.6)

Using the algorithm (2.9) with $U_d(.) = U(.)$, T=1 and M=2000, $\underbrace{|U| * \cdots * |U(j\omega)|}_{n}$ for n=1,2,3 can be obtained at the frequencies $\omega = \frac{\pi i}{1000}$, $i = -999, \ldots, 0, \ldots, 1000$. Combining $\underbrace{|U| * \cdots * |U(j\omega)|}_{n}$ for n=1,2,3 and $\frac{1}{|1-0.8\exp(-j\omega)+0.64\exp(-j2\omega)|}$ as indicated in (5.6) then yields $\overline{\mathbf{Y}}^B(\omega)$ as shown in Fig.1.

To enable a comparison between $\overline{\mathbf{Y}}^B(\omega)$ and $|\mathbf{Y}(j\omega)|$, $y(k)$, $k = 0, \pm 1, \ldots$ can be computed using (5.1) and (5.2). $\mathbf{Y}(j\omega)$, at frequencies $\omega = \frac{\pi i}{1000}$, $i = -999, \ldots, 0, \ldots, 1000$, is then obtained by applying an FFT routine to $y(k)$ $k = -999, \ldots, 0, \ldots, 1000$ in order to compute $|\mathbf{Y}(j\omega)|$. The results in Fig.1 indicate that the trend in the variation of $|\mathbf{Y}(j\omega)|$ is well reflected by the computed bound $\overline{\mathbf{Y}}^B(\omega)$.

Example 2.

Consider a discrete time nonlinear system described by the NARX model

$$y(k) = c_{01}(1)u(k-1) + c_{10}(1)y(k-1) + c_{10}(2)y(k-2) + c_{02}(1,3)u(k-1)u(k-3) + c_{11}(1,1)y(k-1)u(k-1)$$

(5.7)

where $c_{01}(1) = 0.5$, $c_{10}(1) = 0.8$, $c_{10}(2) = -0.64$, $c_{02}(1,3) = -0.4$, $c_{11}(1,1) = 0.2$ and the input is defined by the sequence

$$u(k) = 2 \times \frac{1}{2\pi}\left[\frac{\sin\alpha k T_s - \sin\beta k T_s}{k T_s}\right] \quad \text{for } k = 0, \pm 1, \pm 2, \ldots \tag{5.8}$$

where $\alpha = 2$, $\beta = 1$, and $T_s = 0.2$.

The frequency characteristics of $u(k)$, which are shown in Fig.2, indicates that the nonnegative frequency range of the input is [a,b] with a=0.2 and b=0.4. N_M in this example is taken as 8.

Following the procedures in Section 4, R_n for n=1,...,8 can be obtained by using the algorithm (2.3) to give the results in Table 1. Then from (3.3), L_n, for n=1,...,8 , is computed to give the results in Table 2.

Table 1

n	R_n
1	[0.2, 0.4]
2	[0, 0.2]U[0.4, 0.8]
3	[0, 1.2]
4	[0, 1.6]
5	[0, 2]
6	[0, 2.4]
7	[0, 2.8]
8	[0, 3.2]

Table 2

n	L_n
1	0.7243
2	0.4291
3	0.3118
4	0.3118
5	0.3118
6	0.3118
7	0.3118
8	0.3118

The new algorithm for calculating the gain bounds $H_1^B,\ldots,H_8^B$ for the GFRFs can then be employed to yield the results shown in Fig.3.

The algorithm in (2.9) can now be applied with $U_d(.) = U(.)$, T=1 and M=2000 to compute

$$\underbrace{|U|*\cdots*\left|U(j\frac{2\pi}{2000}i)\right|}_{n}, \text{ for } i=-999,\ldots,0,\ldots,1000 \text{ and } n=2,\ldots,8$$

and the results for n=2,4,6,8 are shown in Fig.4.

At the fifth step, G_n, for $n=1,2,\ldots,8$ are evaluated using (4.7) and shown in Fig.5 which indicates that n^* can be taken as 4 if ε is chosen to be 0.1. Therefore take N=4.

Finally evaluate the bound $\overline{\mathbf{Y}}^B(\omega)$ using (4.3) with N=4. The results for $\omega = -\frac{999}{1000}\pi,\ldots,0,\ldots,\frac{1000}{1000}\pi$ are shown in Fig.6 together with $|\mathbf{Y}(j\omega)|$ at these frequencies. The results indicate that the trend in the variation of $|\mathbf{Y}(j\omega)|$ is reflected quite well by the bound $\overline{\mathbf{Y}}^B(\omega)$ and illustrate the effectiveness of the proposed procedures .

In this example N is finally taken as 4 because the values of G_n are almost zero when $n \geq 5$. But from Fig.3 it can be seen that although H_n^B will converge as n increases, the values of H_n^B are still considerable when $n \geq 5$. This indicates that because of the properties of the input signal the influence of the system nonlinearities at orders higher than 5 on the system output are negligible. In fact this phenomenon can be seen even in the cases where H_n^B does not converge as n increases.

Example 3

Consider a nonlinear system the description of which is exactly the same as that of the system considered in Example 2 except that $c_{11}(1,1) = 0.9$ rather than 0.2 where the input sequence is also given by (5.8).

Take $N_M = 8$ and apply the proposed procedures to compute the output frequency domain magnitude bound. The results for steps (i) (ii) and (iv) are obviously the same as in Example 2.

For step (iii) the result is shown in Fig.7 indicating that as n the order of nonlinearity increases H_n^B the gain bound for the GFRFs does not converge. But the result of step (v) shown in Fig.8 indicates that G_n will eventually converge to zero so that the output response of the system only depends on finite order nonlinearities under the given input excitation.

The result of evaluating the bound $\overline{\mathbf{Y}}^B(\omega)$ using (4.3) with $N = N_M = 8$ is shown in Fig.9 together with $|\mathbf{Y}(j\omega)|$ for this particular case.

This example demonstrates the analysis given in the last section regarding the influence of the convergence of the GFRFs gain bounds on the output characteristic descriptions of nonlinear systems and reflects the importance of considering the effects of the input in the frequency domain analysis of nonlinear systems.

6. CONCLUSIONS

A new recursive algorithm for the computation of the gain bounds for generalised frequency response functions of polynomial NARX models has been proposed. By combining the new algorithm with techniques developed previously effective procedures for the practical computation of the bound for the output frequency response magnitude characteristics of nonlinear systems described by the NARX model have been developed. The effectiveness of the proposed procedures has been verified by simulation studies and it has been shown that based on the new procedures the maximum order of dominant system nonlinearities is determined by the influence of both the GFRFs and the system input. This implies conclusions which are natural extensions of the well-known engineering concept that a practical system can usually be approximated quite well by a linear model when the system input varies within a small neighbourhood of the working point. These results provide a basis for the analysis and synthesis of a wide class of nonlinear systems to evaluate or formulate appropriate bounds on the output magnitude frequency domain responses.

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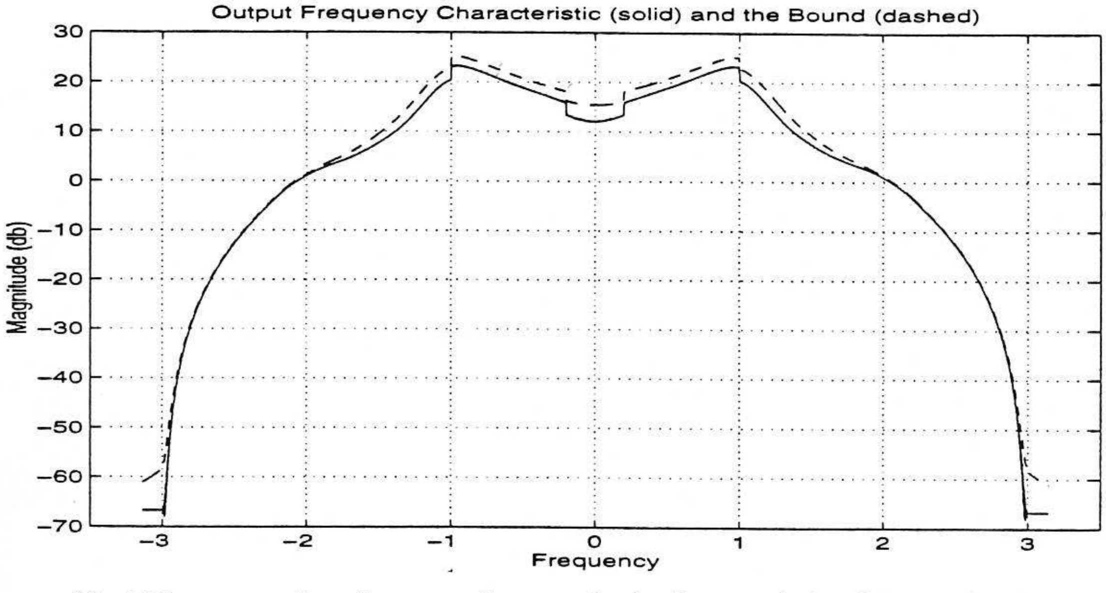

Fig.1 The comparison between the magnitude characteristic of the output frequency response and the bound in Example 1.

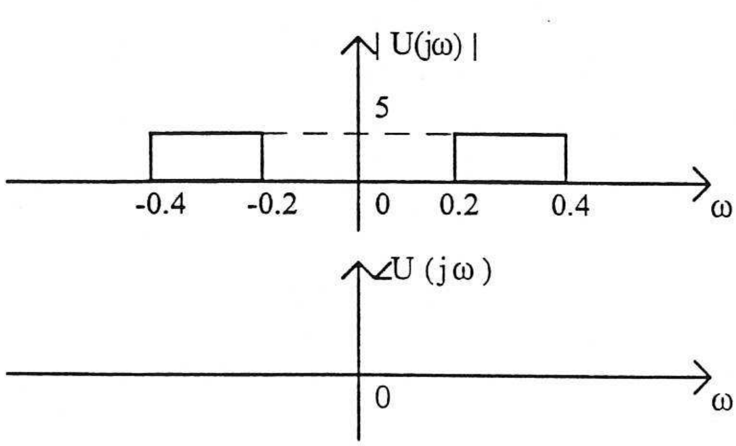

Fig.2 The frequency characteristic of $u(k)$ in Example 2

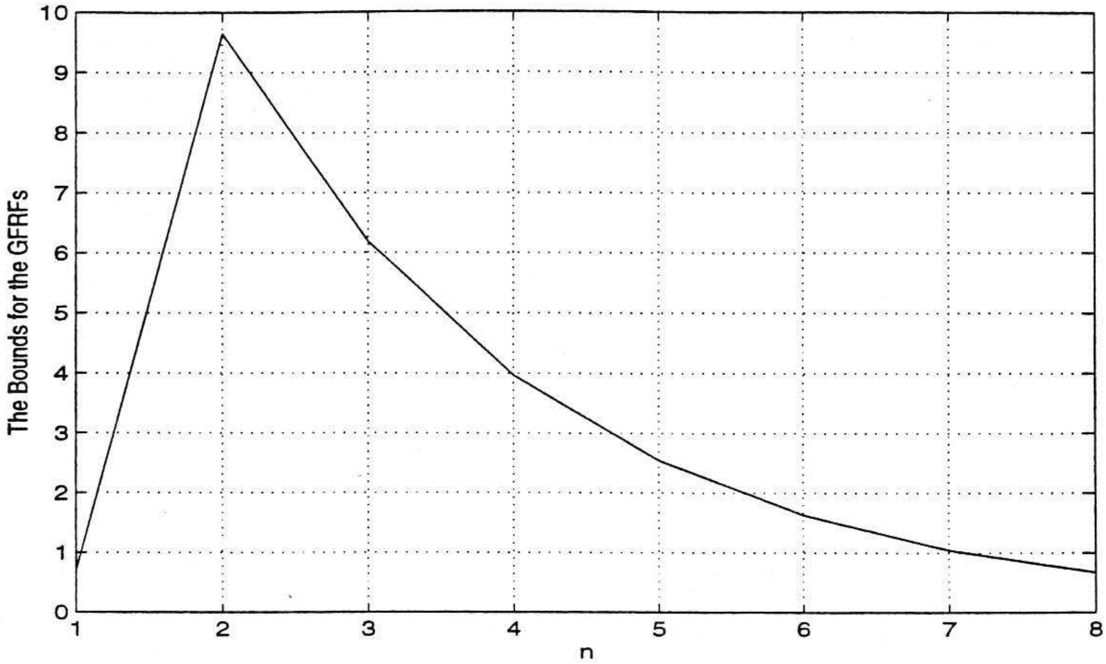

Fig.3 The bounds for the GFRFs of the system in Example 2

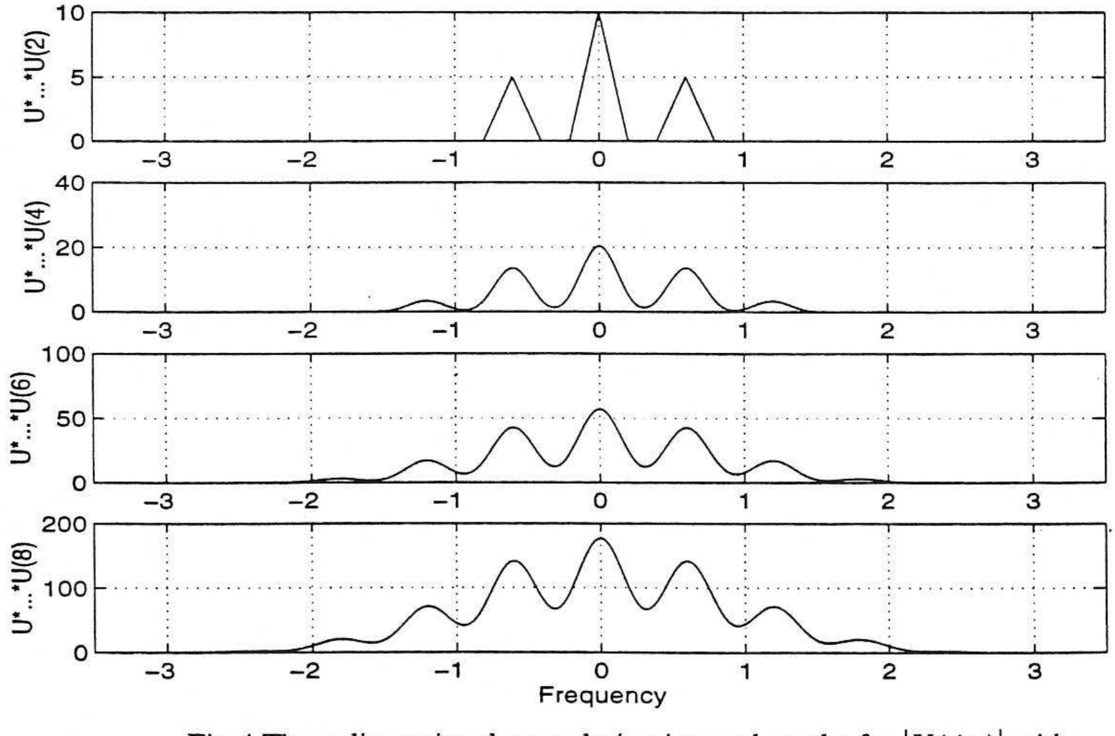

Fig.4 The n dimensional convolution integral results for $|U(j\omega)|$ with n=2,4,6,8 in Example 2

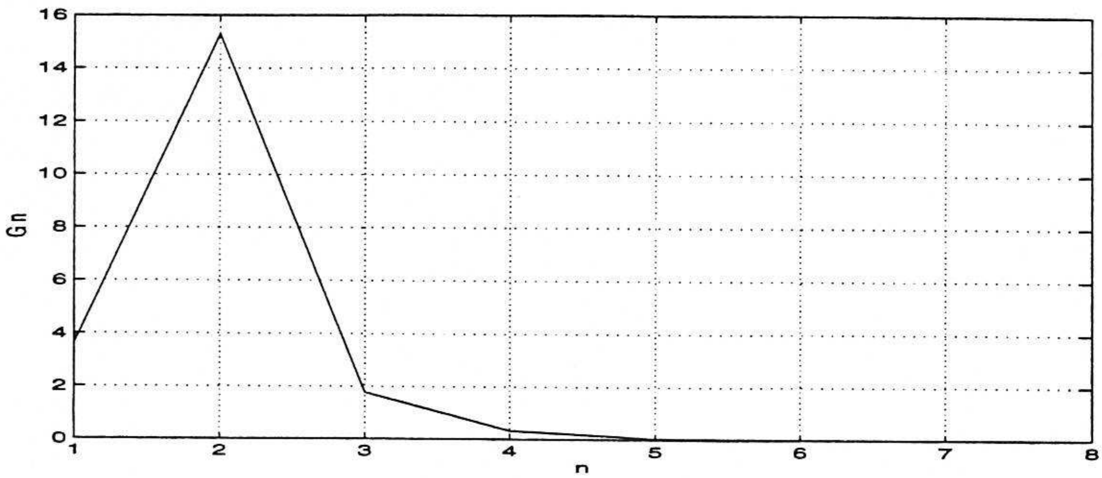

Fig.5. Illustration of the measurement for the effect of nth order nonlinearity on the system output in Example 2

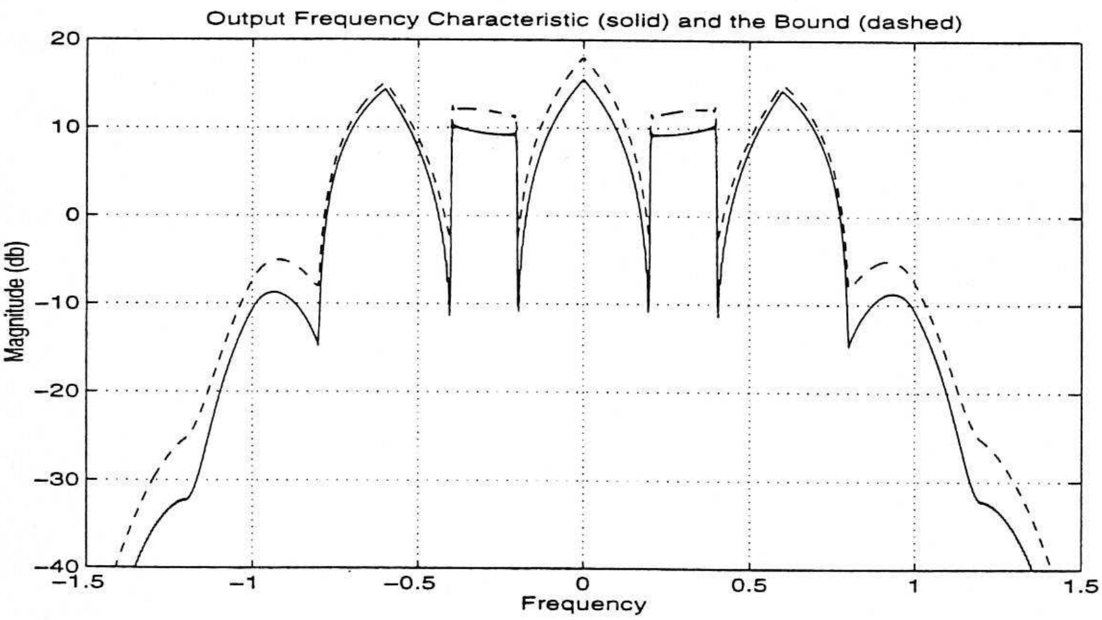

Fig.6 The comparison between the magnitude characteristic of the output frequency response and the bound in Example 2.

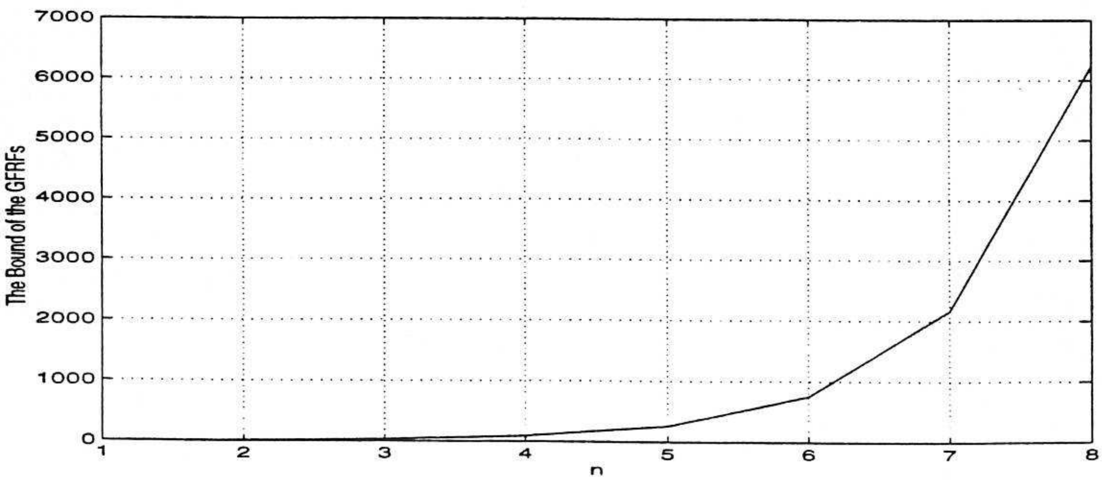

Fig.7 The bounds for the GFRFs of the system in Example 3

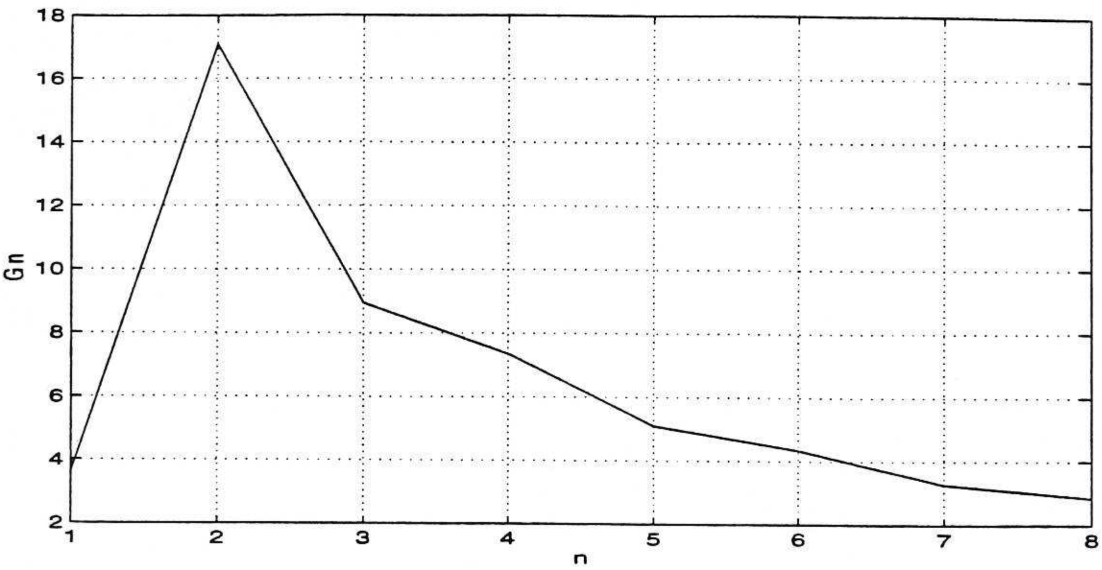

Fig.8. Illustration of the measurement for the effect of nth order nonlinearity on the system output in Example 3

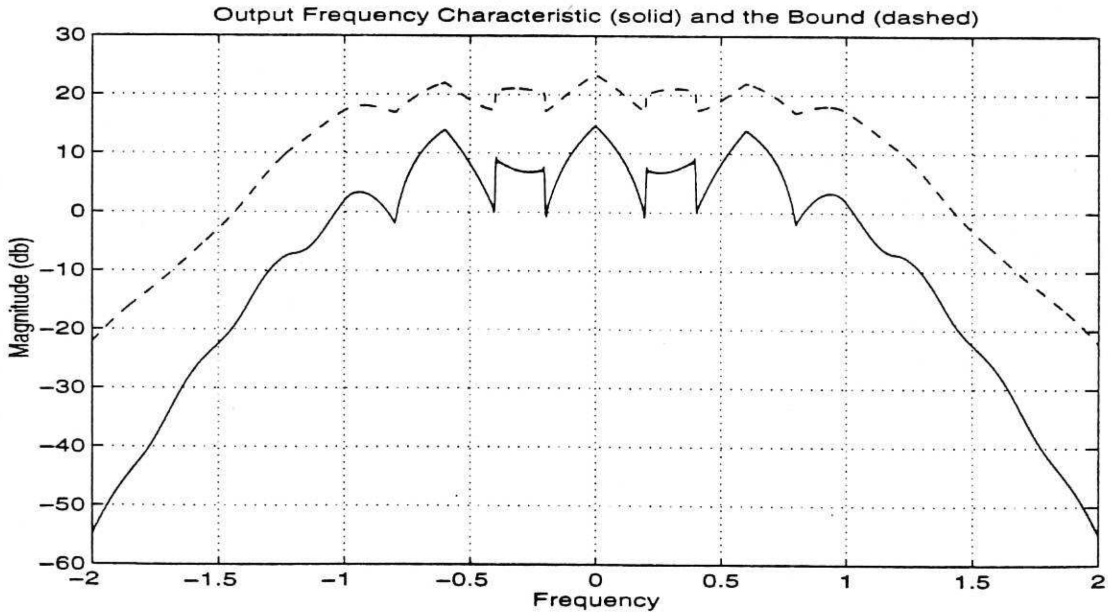

Fig.9 The comparison between the magnitude characteristic of the output frequency response and the bound in Example 3.