Nonlinear Chaotic Systems: Approaches and Implications for Science and Engineering - A Tutorial

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Abstract

In the last few years a great deal of attention has been devoted to detecting and, to a certain extent, quantifying complex dynamics produced by nonlinear systems. Most techniques developed for the analysis of linear systems is totally inadequate for handling nonlinear systems in a systematic, consistent and global way. This paper presents a brief introduction to some of the main concepts and tools used in the analysis of nonlinear dynamical systems and chaos with an emphasis on the signal processing aspects. Acquaintace with such approaches usually enables a better understanding of a number of phenomena which otherwise would remain unclear or even go unnoticed. The main objectives of this tutorial are to present a readable introduction to the subject avoiding as much as possible any mathematics, to mention a few implications for real problems and to provide several references for further reading.

1 Introduction

The use of linear models has always been common practice in science and engineering. A good linear model, however, describes the dynamics of the system only in the neighbourhood of the particular operating point for which the model was derived. The need for a broader picture of the dynamics of real systems has prompted the development and use of dynamical models which include the nonlinear interactions observed in practice.

In this paper a few basic concepts related to nonlinear dynamical systems are briefly reviewed. The objective is threefold, namely i) to provide a brief introduction to complex dynamics and chaos, ii) to mention some implications of using such ideas to problems in the fields of modelling, analysis, signal processing and control of nonlinear systems, and iii) to indicate a few basic references which can be used as a starting point for a more detailed study on this subject.

The ambit of the techniques developed for nonlinear systems with chaotic dynamics can be appreciated by considering the wide range of examples in which chaos has been found. Different types of mathematical equations exhibit chaotic solutions, for instance ordinary differential equations, partial differential equations (Abhyankar et al., 1993), continued fractions (Corless, 1992) and delay equations (Farmer, 1982).

Chaos is also quite common in many fields of control systems such as nonlinear feedback systems (Baillieul et al., 1980; Genesio and Tesi, 1991), adaptive control (Mareels and Bitmead, 1986; Mareels and Bitmead, 1988; Golden and Ydstie, 1992) and digital control systems (Ushio and Hirai, 1983; Ushio and Hsu, 1987)

Chaos seems to be the rule rather than the exception in many nonlinear mechanical and electrical oscillators and pendula (Blackburn et al., 1987; Hasler, 1987; Matsumoto, 1987; Ketema, 1991; Kleczka et al., 1992).

Chaos, fractals and nonlinear dynamics are common in some aspects of human physiology (Mackey and Glass, 1977; Glass et al., 1987; Goldberger et al., 1990), population dynamics (May, 1987; Hassell et al., 1991), ecology and epidemiology (May, 1980; Schaffer, 1985), and the solar system (Wisdom, 1987; Kern, 1992; Sussman and Wisdom, 1992).

Models of electrical systems have been found to exhibit chaotic dynamics. A few examples include DC-DC converters (Hamill et al., 1992), digital filters (Lin and Chua, 1991; Ogorzalek, 1992), power electronic regulators (Tse, 1994), microelectronics (Van Buskirk and Jeffries, 1985), robotics (Varghese et al., 1991), circuits with saturable inductors (Kawakami, 1992) and power system models (Abed et al., 1993).

There seems to be some evidence of low dimensional chaos in time series recorded from the electroencephalogram (Babloyantz et al., 1985; Babloyantz, 1986; Layne et al., 1986) although such results are so far unconclusive. Other areas where there has been much debate concerning the possibility of chaotic dynamics are economics (Boldrin, 1992; Jaditz and Sayers, 1993) and the climate (Lorenz, 1963; Elgar and Kadtke, 1993).

Many other examples in which chaos has apparently been diagnosed include the models of a rotor blade lag (Flowers and Tongue, 1992), force impacting systems (Foale and Bishop, 1992), belt conveyors (Harrison, 1992), neural systems (Harth, 1983), biological networks (Lewis and Glass, 1991), spacecraft attitude control systems (Piper and Kwatny, 1991), fuzzy logic (Grim, 1993), an agricultural implement system (Sakai and Aihara, 1994) and friction force (Wojewoda et al., 1992), to mention just a few.

An advantage of focusing on chaotic systems is that chaos is ubiquitous in nature, science and engineering. Thus simple systems which exhibit chaos commend themselves as valuable paradigms and benchmarks for developing and testing new concepts and algorithms which in principle would apply to a much wider class of problems. Most of the tools and concepts reviewed in this paper are therefore also very relevant to systems which display regular dynamics.

2 Nonlinear Dynamics: Concepts and Tools

This section provides some concepts and tools for the analysis of nonlinear dynamics. Some of the tools considered in this section currently constitute active fields of research in their own right. Although no attempt has been made to give a thorough treatment on such issues, a significative number of references has been included for further reading.

2.1 Differential and difference equations

An n th-order continuous-time system can be described by the differential equation

$$\frac{dy}{dt} = \dot{y} = f(y,t) \ , \tag{1}$$

where $y(t) \in \mathbb{R}^n$ is the *state* at time t and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth function called the *vector field*. f is said to generate a *flow* $\phi_t : \mathbb{R}^n \rightarrow \mathbb{R}^n$, where $\phi_t(y,t)$ is a smooth function which satisfies the group properties $\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$, and $\phi(y,0) = y$.

Given an *initial condition*, $y_0 \in \mathbb{R}^n$ and a time t_0, a *trajectory*, *orbit* or *solution* of equation (1) passing through (or based at) y_0 at time t_0 is denoted as $\phi_t(y_0, t_0)$.

Because the time is explicit in equation (1), f is said to be *non-autonomous*. Conversely, systems in which the vector field does not contain time explicitly are called *autonomous*.

A system is said to be *time periodic* with period T if $f(y,t) = f(y,t+T)$, $\forall y,t$. An n th-order non-autonomous system with period T can be converted into an $(n+1)$ th-order autonomous system by adding an extra state $\theta = 2/T$ in which case the state space will be transformed from the Euclidean space $\mathbb{R}^{n+1}$ to the *cylindrical space* $\mathbb{R}^n \times \mathbb{S}^1$, where $\mathbb{S}^1 = \mathbb{R}/T$ is the circle of length $T = 2\pi/\omega$. It is noted that all the non-autonomous systems considered in this work will be time periodic in most situations.

A *Fixed point* of f or *equilibrium*, $\bar{y}$, is defined as $f(\bar{y}) = 0$ for continuous-time systems and as $\bar{y} = f(\bar{y})$ for discrete-time systems. $\mathrm{D}f$ is the *Jacobian* matrix of the system, defined as the matrix of first partial derivatives. Evaluating the Jacobian at a particular point on a trajectory of the system, that is $\mathrm{D}f(y_i)$ gives a local approximation of the vector field f in the neighbourhood of y_i, sometimes $\mathrm{D}f(y_i)$ is referred to as a *linearisation* of f at y_i. If $\mathrm{D}f(\bar{y})$ has no zero or purely imaginary eigenvalues, then the eigenvalues of this matrix characterise the stability of the fixed point $\bar{y}$.

An n th-order discrete-time system can be described by a difference equation of the form

$$y(k+1) = f(y(k),t) \ . \tag{2}$$

A *trajectory* or *orbit* of a discrete system is a set of points $\{y(k+1)\}_{k=0}^{\infty}$. The definitions for discrete systems are analogous to the ones described for continuous-time systems and therefore will be omitted. For details see (Guckenheimer and Holmes, 1983; Parker and Chua, 1989; Wiggins, 1990).

2.2 Numerical simulation of dynamical systems

Generating time series for a system described by a difference equation is quite straightforward since $y(k)$, $k = n_y, n_y + 1, n_y + 2, \ldots$ can be computed by simply interating an equation like (2) from a set of n_y initial conditions.

If the system is described by an ordinary differential equation, simulation cannot be performed as easily since an equation like (1) will need to be integrated. Fortunately, there are a number of well known algorithms available for performing this task (Parker and Chua, 1989) of which the fourth-order Runge-Kutta is undoubtedly the most commonly used.

An important question when integrating differential equations on a digital computer is the choice of the integration interval. In the case of linear systems or nonlinear systems with relatively slow dynamics the choice of the integration interval is not usually critical. For some nonlinear systems, however, if such an interval is not sufficiently short spurious chaotic regimes may be induced when integrating the system using, for instance, a fourth-order Runge-Kutta algorithm, whilst second-order Runge-Kutta algorithms may induce spurious dynamics even for fairly short integration intervals (Grantham and Athalye, 1990). It has also been reported that in some cases the location of the *bifurcation points* depend on the integration interval if it exceeds a critical value (Aguirre and Billings, 1994a).

Irrespective of the type of the dynamical equations or the algorithm used to solve such equations, an important question which should be answered is whether the simulated results are representative of the 'real solution'. This is a nontrivial matter, and to address it would involve a detailed look into the *shadowing lemma* (Guckenheimer and Holmes, 1983). For the purposes of this tutorial, it suffices to mention that there is abundant evidence that computer simulations are generally reliable as numerical tools for the analysis of dynamical systems (Sauer and Yorke, 1991). However, it should also be borne in mind that pitfalls do exist (Troparevsky, 1992), some of them as a consequence of the *extreme sensitivity to initial conditions conditions* exhibited by some systems. This characteristic is one of the most peculiar features of a chaotic system and will be briefly illustrated in section 2.9. Extreme sensitivity to initial conditions does not invalidate numerical computations but certainly calls for caution in analysing the results.

2.3 Spectral methods

One of the first tools used to diagnose chaos was the power spectrum (Mees and Sparrow, 1981). The appearance of a broad spectrum of frequencies of highly structured humps near the low-order resonances is usually credited to chaos in low-order systems (Blacher and Perdang, 1981). However, broad-band noise and the existence of phase coherence can make it difficult to discriminate experimentally between chaotic and periodic behaviour by means of the power spectrum (Farmer et al., 1980). More recently the *raw spectrum* (sum of the absolute values of the real and imaginary components) and the *log spectrum* (log of the raw spectrum) have been compared with more classical techniques in the context of chaotic time series analysis (Denton and Diamond, 1991).

Recently, the application of spectral techniques to the analysis of chaotic systems has concentrated on the bispectrum and trispectrum (Pezeshki et al., 1990; Subba Rao, 1992; Chandran et al., 1993; Elgar and Chandran, 1993; Elgar and Kennedy, 1993). See (Nikias and Mendel, 1993; Nikias and Petropulu, 1993) for an introduction on higher-order spectral analysis. Such techniques have been used to detect and, to a certain extent, to quantify the energy transfer among frequency modes in chaotic systems.

2.4 Embedded trajectories

One technique used in the analysis of nonlinear dynamical systems is to plot a steady-state trajectory of a system in the phase-space. Thus if $y(t)$ is a trajectory of a given system this can be achieved by plotting $\dot{y}(t)$ against $y(t)$. For low-order systems this procedure can be used to distinguish between different dynamical regimes.

In many practical situations, however, only one variable is measured. In these cases an alternative procedure is to plot $y(t - T_p)$ against $y(t)$ where T_p is a time lag. These variables can be used in the reconstruction of attractors (Packard et al., 1980; Takens, 1980) and such variables also define the so-called pseudo-phase plane. This is motivated by the fact that $y(t - T_p)$ is, in a way, related to $\dot{y}(t)$ and consequently the embedded trajectories represented in the pseudo-phase plane should have properties similar to those of the original attractor represented in the phase plane (Moon, 1987).

A further advantage of this technique is that it enables the comparison of trajectories computed from continuous systems where $\dot{y}(t)$ is usually available, and from discrete models where $\dot{y}(t)$ is often not available and would have to be estimated. Phase portraits and plots of trajectory embeddings can be used not only as a means of distinguishing different dynamical regimes, but also to demonstrate qualitative relationships between original and reconstructed attractors.

The choice of T_p for graphical representation purposes is not critical and plotting a trajectory onto the pseudo-phase plane for varying values of T_p may give some insight regarding the information flow on the attractor (Fraser and Swinney, 1986). These and other related issues will be addressed in more detail in section 3.

2.5 Dynamical attractors

If a deterministic and stable system is simulated for a sufficiently long time it reaches *steady-state.* In state space this corresponds to the trajectories of the system falling on a particular 'object' which is called the *attractor.* Asymptotically stable linear systems excited by constant inputs have point attractors which have dimension zero and correspond to a constant time series.

Nonlinear systems, on the other hand, usually display a wealth of possible attractors. To which attractor the system will finally settle depends on the system itself and also on the initial conditions.

An advantage of considering attractors in state space as alternative representations of time series is that a number of geometrical and topological results can be used. For the purposes of this tutorial, it will suffice to point out that the *shape* and *dimension* of the attractors in state space are directly related to the complexity of the dynamics of the respective time series. Thus simple low dimensional attractors correspond to simple time series dynamics whereas more complex time series lie on attractors with higher dimension. This is illustrated in figure 1.

The most common attractors are the *point attractor* (dimension zero), *limit cycles* (dimension one) and *tori* (dimension two). Another type of attractor which has recently attracted a great deal of attention are the so-called *strange* or *chaotic attractors* which are *fractal* objects. Such attractors will be introduced in section 2.7.

2.6 Bifurcation diagrams

Most nonlinear systems have more than one attractor. To which attractor the system will ultimately converge depends on the system parameters and on the initial conditions. A useful tool for assessing how a given nonlinear system 'moves' from one attractor to another over a range of parameter values is the bifurcation diagram. This reveals how the system bifurcates as a certain parameter, called the bifurcation parameter, is varied. Roughly, a system is said to undergo a bifurcation when there is a *qualitative* change in the trajectory (or attractor) of the system. At the bifurcation point, the Jacobian of the system has at least one eigenvalue with the real part equal to zero for continuous-time systems or on the unit circle for discrete-time systems.

There are a number of known bifurcations. The most common co-dimension one bifurcations are the *pitchfork*, the *saddle-node*, the *transcritical*, the *Hopf* bifurcation, and the *flip* or *period doubling*, which only occur in discrete maps or periodically driven systems. For an introduction to bifurcation and a description of the aforementioned types see (Guckenheimer and Holmes, 1983; Mees, 1983; Thompson and Stewart, 1986).

Approaches to calculate bifurcation diagrams include the *brute force*, *path following* (Parker and Chua, 1989), the *cell-to-cell mapping* technique (Hsu, 1987) and frequency domain methods (Moiola and Chen, 1993). For reasons of simplicity, the brute force approach is described in what follows. This approach is simple and robust but in general it is computationally intensive.

A point r of a bifurcation diagram of a nonautonomous systems driven by $A\ \cos(\omega\, t)$ with A as the bifurcation parameter is defined as

$$r = \{\ (y, A)\ \in\ \mathbb{R} \times I \mid y = y(t_i),\ A = A_0;\ t_i = t_o + K_{ss} \times 2\pi/\omega\}\ , \tag{3}$$

where I is the interval $I = [A_i\ A_f] \subset \mathbb{R}$, $0 \leq t_o \leq 2\pi/\omega$ and K_{ss} is a constant. This means that the point r is chosen by simulating the system for a sufficiently long time $K_{ss} \times 2\pi/\omega$ with $A = A_0$ to ensure that transients have died out before plotting $y(K_{ss} \times 2\pi/\omega)$ against A_0. In practice for each value of the parameter A, n_b points are taken at the instants

$$t_i = t_o + (K_{ss} + \ i\ ) \times 2\pi/\omega, \qquad i = 0, 1, ..., n_b - 1\ . \tag{4}$$

Clearly, the input frequency ω can also be used as a bifurcation parameter. For autonomous systems a bifurcation diagram can be obtained in an analogous way by choosing

$$t_i = t_o + (K_{ss} + \ i), \qquad i = 0, 1, ..., n_b - 1\ . \tag{5}$$

A bifurcation diagram will therefore reveal at which values of the parameter $A \in I$ the solution of the system bifurcates and how it bifurcates. When studying chaos such diagrams are also useful in detecting parameter ranges for which the system behaviour is chaotic.

As an example of a bifurcation diagram consider figure 2. Througout this tutorial, bifurcation parameters are denoted by A. Thus, figure 2 shows some of the different types of attractors displayed by the system as A is varied. In particular, for $A = 4.5$, 9 and 11 the system displays period-one, period-three and chaotic dynamics, respectively. For clarity the respective attractors represented in the cylindrical state-space (see section 2.1) are also shown.

2.7 Chaotic or strange attractors

Despite some attempts, there is no widely accepted definition of chaos or chaotic attractors. In this section, a rather intuitive introduction to some aspects of strange attractors will be briefly given based on the ideas presented above.

Firstly it should be realised that chaos is not a pathological dynamical regime which is only exhibited by carefully designed paradigms. Secondly, chaos is not a dynamical regime which lacks order or pattern. On the contrary, chaotic systems have well defined patterns of behaviour and are full of order and beauty.

The terms *strange* and *chaotic* were coined because in the genesis of chaos, the (now well known) attractors which scientists came accross were totally different from what was known at the time (thus the term *strange*), and the time series produced by such systems would not follow any predictable path (thus the term *chaotic*). Although there exist some pathological cases in which strange attractors are nonchaotic (Grebogi et al., 1984; Ding et al., 1989; Kapitaniak, 1993), the terms 'strange' and 'chaotic' are usually used interchangeably.

In order to suggest an *image* of chaos, let us consider a system which originally has a simple and well known attractor and then, via bifurcations, becomes increasingly complex. There are many ways, of course, in which a system may become more complex. Figure 3 illustrates two such paths. Figures 3a-b-c-d shows a two dimensional projection of the state space of a system which originally (3a) follows a periodic motion of period one, that is, the trajectory takes one revolution to repeat itself. After a bifurcation, the system is still periodic but now (3b) the periodicity is the double of what it was before. After a sequence of such bifurcations, the resulting trajectory is aperiodic but still resembles the overall shape of the preceeding periodic attractors. Thus figure 3d is the projection of a chaotic attractor and the sequence of bifurcations illustrated by 3a-b-c-d is called the *period-doubling route to chaos*.

Similarly, the templates shown in 3e-f-g-h illustrate another sequence of bifurcations. Here, each bifurcation increases not the periodicity but rather the dimension of the attractor. It is worth pointing out that from figure 3g to figure 3h the dimension increases from two to a number which is between two and three because of two main reasons. First, the trajectories on the strange attractor in figure 3h cannot cross (if they did the motion would be periodic) and in second place the volume of this attractor is zero.

In the study of chaotic systems it is somewhat instructive to consider the different routes to chaos in order to gain further insight about the dynamics of the system under investigation. As pointed out "the benefit in identifying a particular prechaos pattern of motion with one of these now *classic* models is that a body of mathematical work on each exists which may offer better understanding of the chaotic phenomenon under study" (Moon, 1987, page 62).

Because a thorough study of the routes to chaos is beyond the immediate scope of this work, some of the most well-known patterns will be listed with some references for further reading. Some of the routes to chaos reported in the literature include *period doubling cascade* (Feigenbaum, 1983; Wiesenfeld, 1989), *quasi-periodic route to chaos* (Moon, 1987), *itermittency* (Manneville and Pomeau, 1980; Kadanoff, 1983), *frequency locking* (Swinney, 1983). For other routes to chaos see (Robinson, 1982) and references therein.

In many cases it would be of interest to see more detail in graphical representations than provided by figures 3d and 3h. In the next section a mathematical device is described which reveals the fine strucute of atttractors. Also, quantitave measures of chaotic attractors will be described in sections 2.10 and 2.11.

2.8 Poincaré sections

A bifurcation diagram shows the different types of attractors to which the system settles to as the bifurcation parameter is varied. However, a bifurcation diagram provides very little information concerning the shape of the attractors in state-space. In order to gain further insight into the geometry of attractors one may use the so-called Poincaré map. Such a map is a cross section of the attractor and can be obtained by defining a plane which should be transversal to the flow in state space as shown in figure 4.

More precisely, consider a periodic orbit γ of some flow ϕ_t in $\mathbb{R}^n$ arising from a nonlinear vector field. Let $\Sigma \subset \mathbb{R}^n$ be a hypersurface of dimension $n-1$ which is transverse to the flow ϕ_t. Thus the first return or Poincaré map $\mathbf{P} = \Sigma \rightarrow \Sigma$ is defined for a point $q \in \Sigma$ by

$$\mathbf{P}(q) = \phi_{\tau_p}(q) \ , \tag{6}$$

where τ_p is the time taken for the orbit $\phi_t(q)$ based at q to first return to Σ.

This map is very useful in the analysis of nonlinear systems since it takes place in a space which is of lower dimension than the actual system. It is therefore easy to see that a fixed point of $\mathbf{P}$ corresponds to a periodic orbit of period $2\pi/\omega$ for the flow. Similarly, a subharmonic of period $K \times 2\pi/\omega$ will appear as K fixed points of $\mathbf{P}$. Quasiperiodic and chaotic regimes can also be readily recognised using Poincaré maps. For instance, the first-return map of a chaotic solution is formed by a well-defined and finely-structured set of points for noise-free dissipative systems.

From the above definition it is clear that if a system has $n > 3$, the Poincaré map would require more than two dimensions for a graphical presentation. In order to restrict the plots to two-dimensional figures, $y(t - T_p)$ is plotted against $y(t)$ at a constant period. For periodically driven systems the input period is a natural choice and the resulting plot is called a Poincaré section.

This procedure amounts to defining the Poincaré plane Σ_p in the pseudo-phase-space and then sampling the orbit represented in such a space. The choice of T_p is not critical but it should not be chosen to be too small nor too large compared to the correlation time of the trajectory. Otherwise the geometry and fine structure of the attractor would not be well represented. The qualitative information conveyed by both Poincaré maps and sections are equivalent as demonstrated by the theory of embeddings (Takens, 1980; Sauer et al., 1991).

Although the Poincaré sections are usually obtained by means of numerical simulation, it is possible, although not always feasible, to determine Poincaré maps analytically (Guckenheimer and Holmes, 1983; Brown and Chua, 1993).

2.9 Sensitivity to initial conditions

Probably the most fundamental property of chaotic systems is the sensitive dependence on initial conditions. This feature arises due to the local divergence of trajectories in state space in at least one 'direccion'. This will be also addressed in the next section.

In order to illustrate sensitivity to initial conditions and one of its main consequences, it will be helpful to consider the map

$$y(k) = A\,[1 - y(k-1)]\,y(k-1)\ . \tag{7}$$

In order to iterate equation (7) on a digital computer, an initial condition $y(0)$ is required. Using this value, the right hand side of equation (7) can be evaluated for any value of A. This produces $y(1)$ which should be 'fedback' and used as the initial condition in the following iteration. This procedure can be then repeated as many times as necessary to generate a time series $y(0), y(1), y(2), \ldots$.

A graphical way of seeing this is illustrated in figure 5. It should be noted that the right hand side of equation (7) is a parabola, as shown in figure 5a. Thus to evaluate equation (7) is equivalent to finding the value on the parabola which corresponds to the initial condition. This is represented in figure 5a by the first vertical line. The feeding back of the new value is then represented by projecting the value found on the parabola on the bisector. This completes one iteration.

Chosing the initial condition $y(0) = 0.22$ and $A = 2.6$, figure 5a shows the iterative procedure and reveals that after a few iterations the equation settles to a point attractor. The respective time series is shown in figure 5b. The same procedure was followed for the same initial condition and $A = 3.9$. The results are shown in figures 5c-d. Clearly, the equation does not settle onto any fixed point and not even onto a limit cycle. In fact, it is known that equation (7) displays chaos for $A = 3.9$.

What happens if instead of a single initial condition an interval of initial conditions is iterated? This is shown in figures 5e-f. For $A = 2.6$, the map will eventually settle to the same point attractor as before. This is a typical result for regular stable systems and it illustrates how all the trajectories based on the initial conditions taken from the original interval converge to the same attractor.

Considering a much narrower interval of initial conditions and proceeding as before yielded the results shown in figure 5f for which $A = 3.9$. Clearly, the interval of initial conditions was widened at each iteration. Such an interval can be interpreted as an error in the original initial condition, $y(0) = 0.22$. In practice errors in initial conditions will be always present due to a number of factors such as noise, digitalisation effects, round-off errors, finite wordlength precision, etc. It is this effect of amplifying errors in initial conditions which is known as the sensitive dependence on initial conditions and an immediate consequence of this feature is the impossibility of making long-term predictions for chaotic systems. The next section describes indices which quantify the sensitivity to initial conditions.

2.10 Lyapunov exponents

Lyapunov exponents measure the average divergence of nearby trajectories along certain 'directions' in state space. A chaotic attracting set has at least one positive Lyapunov exponent and no Lyapunov exponent of a non-chaotic attracting set can be positive. Consequently such exponents have been used as a criterion to determine if a given attracting set is or is not chaotic (Wolf, 1986). Recently the concept of *local* Lyapunov exponents has been investigated (Abarbanel, 1992). The local exponents describe orbit instabilities a fixed number of steps ahead rather than an infinite number. The (global) Lyapunov exponents of an attracting set of length N can be defined as [1]

$$\lambda_i = \lim_{N\to\infty} \frac{1}{N} \log_e j_i(N), \qquad i = 1, 2, ..., n \ , \tag{8}$$

where $\log_e = \ln$ and the $\{j_i(N)\}_{i=1}^{n}$ are the absolute values of the eigenvalues of

$$[\mathrm{D}f(y_N)][\mathrm{D}f(y_{N-1})]...[\mathrm{D}f(y_1)] \ , \tag{9}$$

where $\mathrm{D}f(y_i) \in \mathbb{R}^{n\times n}$ is the Jacobian matrix of the n-dimensional differential equation (or discrete map) evaluated at y_i, and $\{y_k\}_{k=1}^{N}$ is a trajectory on the attractor. Note that n is the dynamical order of the system.

In many situations the reconstructed or identified models may have a dimension which is larger than that of the original systems and therefore such models have more Lyapunov exponents. These 'extra' exponents are called *spurious Lyapunov exponents*. The estimation of Lyapunov exponents is known to be a nontrivial task. The simplest algorithms (Wolf et al., 1985; Moon, 1987) can only reliably estimate the largest Lyapunov exponent (Vastano and Kostelich, 1986). Estimating the entire spectrum is a typically ill-conditioned problem and requires more sophisticated algorithms (Parker and Chua, 1989). Further problems arise when it comes to deciding which of the estimated exponents are *true* and which are *spurious* (Stoop and Parisi, 1991; Parlitz, 1992; Abarbanel, 1992). The estimation of Lyapunov exponents is currently an active field of research as can be verified from the following references (Sano and Sawada, 1985; Eckmann et al., 1986; Bryant et al., 1990; Brown et al., 1991; Parlitz, 1992; Kadtke et al., 1993; Nicolis and Nicolis, 1993; Chialina et al., 1994). For application of Lyapunov exponents in the quantification of real data see (Brandstäter et al., 1983; Wolf and Bessoir, 1991; Vastano and Kostelich, 1986).

In view of such difficulties and the fact that the largest Lyapunov exponent, λ_1, is in many cases the only positive exponent[2] and that this gives an indication of how far into the future accurate predictions can be made, it seems appropriate to use λ_1 to characterise a chaotic attracting set (Rosenstein et al., 1993). Indeed, the largest Lyapunov exponent has been used in this way and to compare several identified models (Abarbanel et al., 1989; Abarbanel et al., 1990; Principe et al., 1992).

The algorithm suggested in (Moon, 1987) for estimating λ_1 is described below. A similar algorithm which simultaneously estimates the correlation dimension to be defined in section 2.11 has been recently investigated in (Rosenstein et al., 1993).

Consider a point x_0 on the trajectory $x(k)$ (for the moment it is assumed that such a trajectory is available *a priori*), say $x_0 = x(0)$, and a nearby point $x_0 + \delta_0$. For simplicity it is assumed that $x(k) \in \mathbb{R}$, but in general higher-dimensional systems will be the case. The largest Lyapunov exponent of an attracting set of length N can be defined as (see also equation (8))

$$\lambda_1 \doteq \frac{1}{N} \lim_{N\to\infty} \sum_{k=1}^{N} \log_e \frac{\| \delta_{k+1} \|}{\| \delta_k \|} \ , \tag{10}$$

[1]Many authors use $\log_2$ in this definition

[2]In this case $\lambda_1 \geq h$, where h is the Kolmogorov-Sinai or metric entropy. Note that for dissipative systems (chaotic and non-chaotic) $\sum_{i=1}^{n} \lambda_i < 0$ (Eckmann and Ruelle, 1985; Wolf, 1986).

where δ_k is the distance between two points on nearby trajectories at time k. The estimation of λ_1 is a simulation-based calculation (Moon, 1987; Parker and Chua, 1989). The main idea is to be able to determine the ratio

$$\frac{\| x_1 - (x_1 + \delta_1) \|}{\| x_0 - (x_0 + \delta_0) \|} = \frac{\| \delta_1 \|}{\| \delta_0 \|} , \tag{11}$$

where x_1 is another point on the trajectory $x(k)$, namely $x(\Delta L)$, $x_1+\delta_1$ is a point obtained by following the evolution of the randomly chosen initial condition $x_0+\delta_0$ over the interval ΔL where ΔL will be referred to as the Lyapunov interval.

From the last equation it is clear that one only needs to follow the evolution of perturbations δ_i along the reference trajectory $x(k)$. It is well known that the Jacobian matrix $\mathrm{D}f(x_i)$ describes the dynamics of the system for small perturbations in the neighbourhood of x_i. Thus the computation of the largest Lyapunov exponent, λ_1, consists in solving the variational equations

$$\dot{\delta} = \mathrm{D}f(x_i)\,\delta , \tag{12}$$

where $\mathrm{D}f(x_i)$ is the Jacobian matrix of $f(\cdot)$ evaluated at x_i, and also of simulating the system

$$\dot{x} = f(x) \tag{13}$$

if the trajectory $x(k) = \{x_i\}_{i=0}^{N}$ is not available in advance. Equations (12) and (13) are simulated and the ratio $\| \delta_{k+1} \| / \| \delta_k \|$ is calculated once at each ΔL interval. Therefore estimating λ_1 consists in successively predicting the systems governed by $\mathrm{D}f(\cdot)$ and $f(\cdot)$ ΔL seconds into the future and assessing the expansion of the perturbations δ_i.

Some of the ideas described above are illustrated in figures 6a-b. The former figure is the bifurcation diagram of the logistic equation (7). Figure 6b shows the largest Lyapunov exponent of such an equation for a range of values of A. The largest Lyapunov exponent was calculated as described above. Note that $\lambda_1 = 0$ at bifurcation points and that $\lambda_1 > 0$ for chaotic regimes as predicted by the theory. These figures also reveal the narrow windows of regular dynamics which are surrounded by chaos.

2.11 Correlation dimension

Another quantitative measure of an attracting set is the fractal dimension. In theory, the fractal dimension of a chaotic (non-chaotic) attracting set is non-integer (integer). An exception to this rule are *fat fractals* which have integer fractal dimension which is consequently inadequate to describe the properties of such fractals (Farmer, 1986). Nontheless, like the largest Lyapunov exponent, the fractal dimension can, in principle, be used not only to diagnose chaos but also to provide some further dynamical information (Grassberger et al., 1991). A deeper treatment can be found in (Russell et al., 1980; Farmer et al., 1983; Grassberger and Procaccia, 1983a; Atten et al., 1984; Caputo et al., 1986) for raw data and in (Badii and Politi, 1986; Badii et al., 1988; Mitschke, 1990; Brown et al., 1992; Sauer and Yorke, 1993) for filtered time series.

The fractal dimension is related to the amount of information required to characterise a certain trajectory. If the fractal dimension of an attracting set is $D + \delta$, $D \in \mathbb{Z}^+$, where

$0 < \delta < 1$, then the smallest number of first-order differential equations required to describe the data is $D+1$.

There are several types of fractal dimension such as the *pointwise dimension*, *correlation dimension*, *information dimension*, *Housdorff dimension*, *Lyapunov dimension*, for a comparison of some of these dimensions see (Farmer, 1982; Hentschel and Procaccia, 1983; Moon, 1987). For many strange attractors, however, such measures give roughly the same value (Moon, 1987; Parker and Chua, 1989). The correlation dimension[3] (Grassberger and Procaccia, 1983b), however, is clearly the most widely used measure of fractal dimension employed in the literature.

A time series $\{y_i\}_{i=1}^{N}$ can be embedded in the phase space where it is represented as a sequence of d_e-dimensional points $\mathbf{y}_j = [y_j \; y_{j-1} \; \cdots \; y_{j-d_e+1}]$. Suppose the distance between two such points is[4] $S_{ij} = | \mathbf{y}_i - \mathbf{y}_j |$ then a correlation function is defined as (Grassberger and Procaccia, 1983b)

$$C(\epsilon) = \lim_{N \to \infty} \frac{1}{N} (\text{number of pairs } (i,j) \text{ with } S_{ij} < \epsilon) \ . \tag{14}$$

The correlation dimension is then defined as

$$D_c = \lim_{\epsilon \to \infty} \frac{\log_e C(\epsilon)}{\log_e \epsilon} \ . \tag{15}$$

For many attractors D_c will be (roughly) constant for values of ϵ within a certain range. In theory, the choice of d_e does not influence the final value of D_c if d_e is greater than a certain value. In particular, it has been shown that provided there are sufficient noise-free data, $d_e = \text{Ceil}(D_c)$, where $\text{Ceil}(\cdot)$ is the smallest integer greater than or equal to D_c (Ding et al., 1993) and that this result remains true in the case the data have been filtered using *finite impulse response* (FIR) filters (Sauer and Yorke, 1993). In practice, due to the lack of data and to the presence of noise, $d_e > \text{Ceil}(D_c)$, thus several estimates of the correlation dimension are obtained for increasing values of d_e. If the data were produced by a low-dimensional system, such estimates would eventually converge. Of course, these results depend largely on both the amount and quality of the data available. For a brief account of data requirements, see section 4.1 below.

In order to illustrate the estimation of D_c a time series with $N = 15000$ data points was obtained by simulating Chua's circuit (Chua and Hasler, 1993) operating on the double scroll attractor. The correlation function $C(\epsilon)$ was then calculated for $2 \leq d_e \leq 10$ and plotted in figure 7. For small embedding dimensions ($d_e = 2$) the correlation dimension is $D_c \approx 1.8$ but as d_e is increased the scaling region converges to the correct value $D_c \approx 2.0$ for $d_e \geq 5$.

Some fractals have the properties of *self-similarity*. This is illustrated in figure 8 which shows the well known Hénon attractor (Hénon, 1976) and an amplification of a small section of one of its legs. It should be observed that what appears to be a single 'line' in the attractor turns out to be two lines (see zoom in figure 8). However, if each of these lines were zoomed again it would become apparent that they are composed of two lines each and this continues *ad infinitum*. This particular fractal structure is sometimes referred to as having a Cantor set structure.

[3]This measure can be seen as a *generalised dimension* and is considered to be the easiest to estimate reliably (Grassberger, 1986b) and thus remains the most popular procedure so far.

[4]Several norms can be used here such as Euclidean, ℓ_1, etc.

Probably the greatest application of the correlation dimension is to diagnose if the underlying dynamics of a time series have been produced by a low-order system (Grassberger, 1986a; Lorenz, 1991). Because this is an important problem, the estimation of correlation dimension has attracted much attention in recent years. Many papers have focussed on determining the causes of bad estimates (Theiler, 1986), estimating error-bounds (Holzfuss and Mayer-Kress, 1986; Judd and Mees, 1991) and suggesting improvements on the original algorithm described in (Grassberger and Procaccia, 1983b).

2.12 Other invariants

There are a number of less used invariants of strange attractors reported in the literature such as the Kolmogorov or metric entropy, topological entropy, generalised entropies and dimensions, partial dimensions, mutual information, etc. (Grassberger and Procaccia, 1984; Eckmann and Ruelle, 1985; Fraser, 1986; Grassberger, 1986b).

With few exceptions (Hsu and Kim, 1985), statistics have received little attention as invariant measures of strange attractors. Apparently, the most useful such measure is the *probability density function* (Packard et al., 1980; Moon, 1987; Vallée et al., 1984; Kapitaniak, 1988)

The estimation of unstable limit cycles has also been put forward as a way of characterising strange attractors. The motivation behind this approach is that because a strange attractor can be viewed as a bundle of infinite unstable limit cycles, the number of the periodic orbits, the respective distribution and properties should be representative of the attractor dynamics. Indeed, from such information other invariants such as entropies and dimensions can be estimated (Auerbach et al., 1987). For more information on this subject, see (Grebogi et al., 1987; Cvitanović, 1988; Lathorp and Kostelich, 1989; Lathorp and Kostelich, 1992)

3 Embedding Techniques

An n th-order dynamical system such as the one in equation (1) can be represented as a set of n first-order ordinary differential equations each governed by a *state variable*. The global system would therefore have n time variables $\{y_1,\ y_2,\ \ldots,\ y_n\}$ and the solution of such a system could be thought of as n time series.

In a sense, the n time series mentioned above are obtained from the original n th-order system by decomposition. Also, given the n times series it is possible to recover the original n-dimensional solution by taking each state variable to be a coordinate of a 'reconstruction space' and to represent each time series in such a space. Thus n time series can be used to *compose* or *reconstruct* the system solution or trajectory. This is illustrated in figure 9.

A difficulty encountered in practice with this approach is that the order of the system n is seldom known and even when an accurate estimate of this variable exists the number of measurements will not be as large as n. Take for for instance the atmosphere which is usually thought of as a high-order system, but monitoring and weather forecasting stations only measure a very limited number of variables of this system.

This can be described in a more mathematical way by considering the action of a *measuring* function $h(y) : \mathbb{R}^n \rightarrow \mathbb{R}$ which operates on the entire state or phase space but which yields just a scalar which is called the *measured variable*. The question which naturally arises at this stage is the following: given $f : \mathbb{R}^n \to \mathbb{R}^n$ and $h(y) : \mathbb{R}^n \to \mathbb{R}$ is it possible to reconstruct a trajectory or solution of f from the scalar measurement $h(y)$?

Fortunately, it turns out that this question has an affirmative answer if certain requirements are met (Takens, 1980; Packard et al., 1980; Sauer et al., 1991). Thus *embedology* is concerned with how to reconstruct the phase space of a dynamical system of order n from a limited set of measurements q where $q < n$, and more often than not $q = 1$. In other words, the objective is to reconstruct the phase space of a system from a single time series. The resulting phase space is usually referred to as *embedded phase space*, *embedding space* or just *embedding*.

Another question which should be addressed is: why should we be concerned in reconstructing the trajectories of a dynamical system? In sections 2.5 and 2.7 it was seen that in state (or phase) space the steady state dynamics of a system are represented by geometrical figures which are called attractors. A stable autonomous linear system only has one kind of attractor, a point attractor. However, nonlinear systems may have more complicated attractors such as limit cycles, tori and even strange attractors.

Therefore if time series are used to reconstruct the phase space of dynamical systems via embedding techniques, it is possible to use results from differential geometry and topology to analyse the resulting attractors which are *geometrical objects* in the reconstructed space. Moreover, if the embedding was successful, both the reconstructed and the original attractors are equivalent from a topological point of view, or in other words, they are said to be diffeomorphic.

The practical consequences of this are obvious. No matter how complex a dynamical system might be, even if only one variable of such a system is measured, it is possible to reconstruct the original phase space via embedding techniques. It is also possible to estimate qualitative and quantitative invariants of the original attractor, such as Poincaré maps, fractal dimension and Lyapunov exponents, directly from the reconstructed attractor which is topologically equivalent to the original one. These ideas are illustrated in figure 10.

A convenient but by no means unique way of reconstructing phase spaces from scalar measurements is achieved by using *delay coordinates* (Packard et al., 1980; Takens, 1980; Sauer et al., 1991). Other coordinates include the *singular value* (Broomhead and King, 1986; Albano et al., 1988) and *derivatives* (Baake et al., 1992; Gouesbet and Maquet, 1992). A framework for the comparison of several reconstructions has been developed in (Casdagli et al., 1991) and three of the most common methods have been studied in (Gibson et al., 1992).

A *delay vector* has the following form

$$\mathbf{y}(k) = [y(k) \;\; y(k-\tau) \;\; \dots \;\; y(k-(d_e-1)\tau)]^{\mathrm{T}} \,, \tag{16}$$

where d_e is the *embedding dimension* and τ is the *delay time*. Clearly, $\mathbf{y}(k)$ can be represented as a point in the d_e-dimensional *embedding space*. Takens has shown that embeddings with $d_e > 2n$ will be faithful generically so that there is a smooth map $f_T : \mathbb{R}^{d_e} \to \mathbb{R}$ such that (Takens, 1980)

$$y(k+T) = f_T(\mathbf{y}(k)) \tag{17}$$

for all integers k, and where the *forecasting time* T and τ are also assumed to be integers.

A consequence of Taken's theorem is that the attractor reconstructed in $\mathbb{R}_e^d$ is diffeomorphic to the original attractor in state space and therefore the former retains dynamical and topological characteristics of the latter.

In the case of delay reconstructions, the choice of the reconstruction parameters, that is, the embedding dimension d_e and the delay time τ is of the greatest importance since such parameters strongly affect the quality of the embedded space. The selection of d_e has been investigated in (Cenys and Pyragas, 1988; Aleksić, 1991; Cheng and Tong, 1992; Kennel et al., 1992). The choice of the delay time has been discussed in (Albano et al., 1991; Buzug et al., 1990; Fraser, 1989; Kember and Fowler, 1993; Liebert and Schuster, 1989; Billings and Aguirre, 1995). Many authors have suggested that in some applications it is more meaningful to estimate these parameters simultaneously, this is tantamount to estimating the embedding window defined as $(d_e - 1)\tau$ (Albano et al., 1988; Buzug and Pfister, 1992; Martinerie et al., 1992). Some of these methods have recently been compared in (Rosenstein et al., 1994). Dynamical reconstructions from nonuniformly sampled data has been addressed in (Breedon and Packard, 1992) and phase space reconstruction of symmetric attractors has been considered in (King and Stewart, 1992).

Taken's theorem gives sufficient conditions for equation (17) to hold, that is, in order to be able to infer dynamical invariants of the original system from the time series of a single variable, however no indication is given as to how to estimate the map f_T. A number of papers have been devoted to this goal and such methods can be separated into two major groups, namely *local* and *global* approximation techniques.

The local approaches usually begin by partitioning the embedding space into neighbourhoods $\{\mathcal{U}_i\}_{i=1}^{N_n}$ within which the dynamics can be appropriately described by a linear map $g_T : \mathbb{R}_e^d \rightarrow \mathbb{R}$ such that

$$y(k+T) \approx g_{Ti}(\mathbf{y}(k)) \quad \text{for } \mathbf{y}(k) \in \mathcal{U}_i, \ i = 1, \ldots, N_n \ . \tag{18}$$

Several choices for g_T have been suggested in the literature such as linear polynomials (Farmer and Sidorowich, 1987; Casdagli, 1991) which can be interpolated to obtain an approximation of the map f_T (Abarbanel et al., 1990). Simpler choices include *zeroth-order approximations*, also known as *local constant predictors* (Farmer and Sidorowich, 1987; Kennel and Isabelle, 1992; Wayland et al., 1993) and a *weighted predictor* (Linsay, 1991).

Global approximators overcome some of the difficulties faced by local maps. Although global models have problems of their own, some attention has been devoted to the investigation of such models (Cremers and Hübler, 1987; Crutchfield and McNamara, 1987; Kadtke et al., 1993; Aguirre and Billings, 1995e).

4 Diagnosing Chaos

In general, the problem of diagnosing chaos can be reduced to estimating invariants which would suggest that the data are chaotic. For instance, positive Lyapunov exponents, non-integer dimensions and fractal structures in Poincaré sections would suggest the presence of chaos. The main question is how to confidently estimate such properties from the data, especially when the available records are relatively short and possibly noisy. The techniques that have been suggested in the literature may be divided in two major groups.

Non-parametric methods. These include the use of tools which take the data and estimate dynamical invariants which, in turn, will give an indication of the presence of chaos. Such tools include power spectra, the largest Lyapunov exponent, the correlation dimension, reconstructed trajectories, Poincaré sections, relative rotation rates etc. Detailed description and application of these techniques can be found in the literature (Moon, 1987; Tufillaro et al., 1990; Denton and Diamond, 1991). For a recent comment of the practical difficulties in using Lyapunov exponents and dimensions for diagnosing chaos see (Mitschke and Dämmig, 1993).

Two practical difficulties common to most of these approaches are the number of data points available and the noise present in the data. These aspects are briefly discussed in the following section.

Poincaré sections are very popular for detecting chaos because for a chaotic system the Poincaré section reveals the fractal structure of the attractor. However, in order to be able to distinguish between a fractal object and a fuzzy cloud of points a certain amount of data is necessary. Moon (1987) has suggested that a Poincaré section should consist of at least 4000 points before declaring a system chaotic. For non-autonomous systems this means 4×10^3 forcing periods which could amount to 4×10^5 data points.

Prediction-based techniques. Some methods try to diagnose chaos in a data set based upon prediction errors (Sugihara and May, 1990; Casdagli, 1991; Elsner, 1992; Kennel and Isabelle, 1992). Thus predictors are estimated from, say, the first half of the data records and used to predict over the last half. Chaos can, in principle, be diagnosed based on how the prediction errors behave as the prediction time is increased (Sugihara and May, 1990), or based on how the prediction errors related to the true data compare to the prediction errors obtained from 'faked' data which are random but have the same length and spectral magnitude as the original data (Kennel and Isabelle, 1992). A related approach has been termed *the method of surrogate data* (Theiler et al., 1992a; Theiler et al., 1992b).

Regardless of which criterion is used to decide if the data are chaotic or not, predictions have to be made. Clearly, the viability of these approaches depends on how easily predictors can be estimated and on the convenience of making predictions. Once a predictor is estimated criteria and statistics such as the ones presented in (Sugihara and May, 1990; Kennel and Isabelle, 1992) can be used to diagnose chaos.

4.1 Data requirements

The length and quality of the data records are crucial in the problem of characterisation of strange attractors. At present, there seems to be no general rule which determines the amount of data required to learn the dynamics, to estimate Lyapunov exponents and the correlation dimension of attractors. However it is known that "in general the detailed diagnosis of chaotic dynamical systems requires long time series of high quality" (Ruelle, 1987).

Typical values of data length for learning the dynamics are $2{\times}10^4$ (Farmer and Sidorowich, 1987; Abarbanel et al., 1990) for systems of dimension 2 to 3, $1.2\times10^4 - 4\times10^4$ (Casdagli, 1991).

It has been argued that to estimate the Lyapunov exponents $10^3 - 10^4$ forcing periods should be used (Denton and Diamond, 1991). Other estimates are $N > 10^D$ (quoted in (Rosenstein et al., 1993) and $N > 30^D$ where D is the dimension of the system (Wolf et al.,

1985) but in some cases at least 2×30^D was required (Abarbanel et al., 1990). Typical examples in the literature use $4 \times 10^4 - 6.4 \times 10^4$ (Eckmann et al., 1986) 1.6×10^4 (Wolf and Bessoir, 1991) and 2×10^4 data points (Ellner et al., 1991).

Fairly long time series are also required for estimating the correlation dimension. In fact, it has been pointed out that dimension calculations generally require larger data records (Wolf and Bessoir, 1991). For a strange attractor, if insufficient data is used the results would indicate the dimension of certain parts of the attractor rather than the dimension of the entire attractor (Denton and Diamond, 1991). However, results have been reported which suggest that consistent estimates of the correlation dimension can be obtained from data sequences with less than 1000 points (Abraham et al., 1986). On the other hand, there seems to be evidence that "spuriously small dimension estimates can be obtained from using too few, too finely sampled and too highly smoothed data" (Grassberger, 1986a). Moreover, the use of short and noisy data sets may cause the correct scaling regions to become increasingly shorter and may cause the estimate of the correlation dimension to converge to the correct result for relatively large values of the embedding dimension (Ding et al., 1993). Thus typical examples use $1.5 \times 10^4 - 2.5 \times 10^4$ (Grassberger and Procaccia, 1983b) and $0.8 \times 10^4 - 30 \times 10^4$ data points (Atten et al., 1984). Thus there seems to be no agreed upon rule to determine the amount of data required to estimate dimensions with confidence but it appears that at least a few thousand points for low dimensional attractors are needed (Theiler, 1986; Havstad and Ehlers, 1989; Ruelle, 1990; Essex and Nerenberg, 1991). In particular, $N \geq 10^{D_c/2}$ has been quoted in (Ding et al., 1993).

It should be realised that the difficulties in obtaining long time series goes beyond problems such as storage and computation time. Indeed, it has been pointed out that for some real systems, stationarity cannot always be guaranteed even over relatively short periods of time. Examples of this include biological systems (May, 1987; Denton and Diamond, 1991), ecological and epidemological data (Schaffer, 1985; Sugihara and May, 1990). A test for stationarity has been recently suggested in (Isliker and Kurths, 1993).

5 Applications

The investigation of mathematical tools for analysing nonlinear dynamical systems is still in its infancy. Nevertheless a great number of techniques are available and a few have been applied to real problems sometimes with promissing results. Some tools were discussed in section 2. Many concepts discussed in that section used the idea of phase space. Thus the practical reconstruction of dynamical trajectories in such a space was discussed in section 3. The important question of how to detect chaos is intimately related to the data requirements of such techniques and has been briefly addressed in section 4.

In this section a few applications will be mentioned. In fact most of what will be cited could be labelled as *benefits and implications* of the application of the concepts and tools described in the first part of the paper. It goes without saying that the list of 'applications' discussed in this work is a small fraction of a much wider universe.

5.1 Signal processing

Signal processing was probably one of the fields most affected by the advent of chaos. Traditionally, a signal was always thought of being composed by a deterministic component which could be predicted once a 'model' of the underlying dynamics was available and by an unpredictable component which would usually be modelled as a stochastic process. In most cases, the best that could be hoped for was to infer a good model for the deterministic part and to use it in forecasting problems while the unpredictable component was most of the time ignored.

The concept of chaos and in particular the reality of *sensitivity to initial conditions* has strongly influenced the way models are conceived. The fact that deterministic chaos does appear random at times has uncovered the fact that *'random' behaviour can be modelled, analysed and predicted to a certain extent using purely deterministic models.* Moreover, because chaos seems to be ubiquitous it appears that most of the randomness which scientist find in data records and have to deal with is, after all, produced by a deterministic mechanism. This mechanism is chaos.

Thus when it comes to processing and modelling of signals and systems, one can dare to model randomness as long it is of low order (Crutchfield and McNamara, 1987; Casdagli, 1989; Haynes and Billings, 1992; Mees, 1993; Aguirre and Billings, 1995e). If the randomness in the data is of high-order, the best option still seems to be stochastic modelling. The question of how to detect if the randomness in the data is of low or high order has been addressed in section 4.

Chaos has also influenced the way models are validated (Haynes and Billings, 1994). Probably the most common way of validating models (and certainly the most naïve) is via simulation or prediction over the *validation data set.* This procedure is based on the accepted fact that if the model is correct then it should provide very accurate predictions. If the model fails to give good forecasts it is promply dismissed as inaccurate and a better model is searched for. It should be realised however that if the data happen to be chaotic, no matter how accurate the model is, because of some inevitable noise on the data, the predictions will *never* coincide with the data records. It thus becomes apparent that alternative ways of validating models should be sought. Some of the tools described in section 2 provide ways of verifying if a model reproduces the original dynamical features of the system even if the the predictions are not as accurate as expected (Aguirre and Billings, 1994c). Therefore whenever possible, nonlinear invariants should be used in the validation.

In the field of nonlinear system identification, the tools described insection 2 have been found very useful in characterizing some relationships which exist between the structure of nonlinear models and the respective dynamical behaviour. It should be realised that a purely statistical approach to system identification does not in general reveal how the dynamics of the final model is influenced by the various variables involved (Billings and Haynes, 1993; Haynes and Billings, 1994) although attempts have been made to link statistics and dynamics (Tong, 1992). Nonlinear invariants and in particular bifurcation diagrams have been useful in investigating the effects of overparametrization (Aguirre and Billings, 1995b) and the sampling time (Billings and Aguirre, 1995) on the dynamics of nonlinear models.

One of the truths which the chaos advent uncovered was that simple models can (and often do) produce complex dynamics. This has been highlighted by the use of very simple paradigms which produce very complex dynamics. The consequences of this to signal and system modelling is clear, namely that the resulting models do *not have to be* complex no matter how complicated and intricate the data may be. This has prompted some authors to investigate the important issue of structure selection for nonlinear models (Billings et al., 1989; Kadtke et al., 1993; Mees, 1993; Aguirre and Billings, 1995d).

Another field in which chaos has direct implications is *filtering* and *suboptimal estimation* schemes. Many techniques as for instance Kalman filtering and least squares suboptimal estimation use estimated instead of measured variables as a way of reducing the effect of noise or in order to compensate for the lack of measurements. When the data are chaotic, the estimated values will no longer be close to the real variables because of the sensitivity to initial conditions. Filtering and noise reduction of chaotic data has in fact attracted much attention over the last few years (Brown et al., 1992; Chen et al., 1990; Davies, 1992; Farmer and Sidorowich, 1991; Grassberger et al., 1993; Holzfuss and Kadtke, 1993; Kostelich and Yorke, 1988; Kostelich and Yorke, 1990; Mitschke, 1990; Schreiber and Grassberger, 1991; Sauer, 1992; Aguirre et al., 1995).

Motivated by the fact that low dimension chaotic randomness can be modelled and predicted by deterministic models, a number of techniques have been developed which are based on deterministic forecasting (Crutchfield and McNamara, 1987; Farmer and Sidorowich, 1987; Farmer and Sidorowich, 1988; Linsay, 1991; Principe et al., 1992; Smith, 1992). A good introduction to this subject is provided in (Casdagli et al., 1992).

Finally, an issue which needs furhter investigation is the effects of the sensitivity to initial conditions on the statistical and dynamical properties of parameters estimation algorithms. A few preliminary results in this direction have been discussed in (Aguirre and Billings, 1995c). This subjects seems relevant because there is some evidence that chaotic systems are not only sensitive to initial conditions but are also sensitive to parameters (Farmer, 1985; Brown et al., 1992).

5.2 Quantification of systems with complex dynamics

The investigation of chaotic systems revealed the need to characterize complex dynamics, or in other words to measure and quantify complexity. One of the most relevant concepts in this particular field is that of *fractal dimension*. There are a number of 'slightly' different indices which have been designed to measure the fractal structure of attractors reconstructed directly from data records. One of the most popular methods is the correlation dimension described in section 2.11.

Roughly, the integer part of the correlation dimension of an attractor is an indication of the degrees of freedom of the attractor and the fractional part indicates how 'complex' the signal/system is. The closer the fractional part is to unity, the more complex the signal is in the sense that it has a greater ability to occupy the state space. This property is sometimes referred to as *space filling*.

The concepts of sensitivity to initial conditions and fractal geometry usually come together as most systems which are sensitive to initial conditions often display attractors which are fractals. The sensitivity to initial conditions is measured by positive Lyapunov exponents whereas the fractal structure of attractors is somewhat characterised by the correlation dimension. Consequently, such indices can be used to measure and quantify complex systems and signals. One of the main limitations however is not only the amount but also the quality of the data available. For details see section 4.

The analysis of real signals has been investigated by many authors in the light of these new concepts. Speech signals have been studied and evidence found that such signals are nonlinear. Short-term forecasting of such signals are of potential comercial use in transmission coding (Casdagli, 1992).

Electroencephalogram data have been considered in (Babloyantz et al., 1985; Layne et al., 1986; Casdagli, 1992; Theiler et al., 1992a; Fuchs et al., 1992; Theiler, 1995). The main objective in most papers concerned with these data is to be able to establish possible relationships between dynamical properties of the reconstructed attractors and the degree of unconsciousness of the patient which could be either sleeping or under the effects of anesthesia. Any conclusive results in this direction would be welcome in particular in anesthesia control problems where one of the main problems is to feedback the degree of unconsciousness.

Data from many fields of natural sciences have been analysed by many authors. A few examples include: the monthly New York measles (Casdagli, 1992; Sugihara and May, 1990), epidemics (Rand and Wilson, 1991), postural sway data (Collins and J., 1994), the focal accommodation system of human eyes (Sumida et al., 1994), sunspots (Casdagli, 1992; Theiler et al., 1992a; Feudel et al., 1993; Mujndt et al., 1991) paleoclimatic data (Elgar and Kadtke, 1993), atmospheric data (Lorenz, 1991; Yang et al., 1994). Econometric and finance series have been analysed in (Gilmore, 1993; Jaditz and Sayers, 1993; Larsen and Lam, 1992; Barnett and Chen, 1988; Brock and Sayers, 1988).

To be able to quantify complexity in real data is a great achievement *per se*. However, having characterised complex systems to some extent is of help in developing a model for such systems and with a model at hand one can think of taming complex systems. It seems fair to say that a long-term goal of a great portion of what has been investigated in the field of complex dynamics is to find ways of controlling complex problems such as demographical growth and population spatial distributions, the spread of contagious deseases, and controlling physiological signals as heart beats, etc. The possibilities are fascinating and consequently much attention has been paid to the control of chaotic systems (Chen and Dong, 1993a). This is briefly discussed in what follows.

5.3 Control and synchronization of chaos

Is chaos a beneficial dynamical steady state? this is a central question in the control of chaotic systems. Of course, if the answer to the above question is yes, applied scientists and control engineers would be investigating ways of provoking chaos rather than suppressing it. A negative answer, on the other hand, would prompt researchers in the opposite direction.

Because of the sensitive dependence on initial conditions, displayed by chaotic systems, it is impossible to make accurate long-term predictions of such systems. In many situations, however, it is desirable that the system under investigation be predictable. Furthermore, the appearance of chaotic dynamics is not always welcome because in some situations it has been associated with abnormal behaviour (Glass and Mackey, 1988, pages 177, 179).

In other applications the onset of chaos seems to have several advantages. For instance, it has been argued that "a cognitive system *must* be chaotic in order to perform effective signal processing" (Nicolis, 1984). Further, chaos enhances heat transfer (Chang, 1992), improves mixing in chemical reactions (Ottino, 1992), reduces idle-channel tones in modulators (Schreier, 1994) and seems to have a promising future in secure communication systems (Cuomo et al., 1993; Wu and Chua, 1993; Parlitz et al., 1992). In addition, some authors have suggested that chaotic dynamics indicate a healthy state as opposed to the diseases which manifest as physiological periodic signals (Glass et al., 1987; Goldberger et al., 1990). The matter of how healthy chaos is, however, is far from settled (Pool, 1989). Consequently, techniques for controlling nonlinear dynamics are required in order to provoke or suppress chaos or any other dynamical regime according to the particular application at hand.

Most of the works concerned with the control of chaos are devoted to stabilising a chaotic system to regular dynamics, that is, fixed points, periodic orbits or quasiperiodic regimes. The related problem of driving a system from a regular to a chaotic regime has received less attention. This type of control could be important in situations where chaos is not only welcome but also desirable (Goldberger et al., 1990; Chang, 1992; Ottino, 1992; Cuomo et al., 1993; Wu and Chua, 1993).

Clearly, chaos is *per se* neither beneficial nor harmful as described by James Gleick "In some applications, turbulence is desirable — inside a jet engine, for example, where efficient burning depends on rapid mixing. But in most, turbulence means disaster. Turbulent airflow over a wing destroys lift. Turbulent flow in an oil pipe creates stupefying drag" (Gleick, 1987, p. 122). Therefore it seems appropriate to search for control schemes which would perform well in both situations.

If on the one hand sensitivity to initial conditions hampers prediction-based control schemes, on the other hand such a property might turn out to be greatly advantageous from a control point of view. To see this it should be recalled that if a system is sensitive to initial conditions, a small perturbation at time t_0 can provoke relatively large effects at time $t > t_0$. This means that to achieve a certain control objective may require a much smaller control action if the system were chaotic. The problem of course is to determine how and when should the control action be applied. Some works in this direction have appeared in the literature (Ott et al., 1990; Ditto et al., 1990; Garfinkel et al., 1992; Nitsche and Dressler, 1992; Romeiras et al., 1992; Shinbrot et al., 1990; Spano et al., 1991).

Many different techniques have been investigated in the context of controlling chaos and it seems inappropriate to try to categorize such methods here. As pointed out before, however, most methods can be grouped into two categories. When it is desired that chaos be suppressed the approaches are labelled under *control of chaos* and when the main objective is to make a system follow a chaotic trajectory the problem at hand is referred to as *synchronization of chaos*.

Chaos can be suppressed by the addition of small amplitude perturbations (Braiman and Goldhirsch, 1991; Aguirre and Billings, 1995a), random perturbations (Kapitaniak, 1991), by parametric driving (Dorning et al., 1992; Fronzoni et al., 1991; Lima and Pettini, 1990), by means of feedback (Liu et al., 1994).

The problem of synchronization has been investigated in (Chua et al., 1993; Kocarev et al., 1993; Ogorzalek, 1993; Pecora, 1990; Wu and Chua, 1993; Aguirre and Billings, 1994b).

The stabilization of chaotic systems has been achieved by applying feedback (Chen and Dong, 1993b; Dedieu and Ogorzalek, 1994; Hunt, 1991; Pyragas, 1992; Roy et al., 1992), frequency harmonic balance techniques (Genesio and Tesi, 1993; Genesio and Tesi, 1992), conventional control techniques (Hartley and Mossayebi, 1993), open plus closed loop control (Jackson and Grosu, 1994), dynamical vibration absorbers (Kapitaniak et al., 1993), adaptive control (Sinha et al., 1990; Vassiliadis, 1993; Qammar and Mossayebi, 1994) and quantitative feedback design (QFD) (Yau et al., 1993). The control of multiple attractor systems has been investigated in (Jackson, 1990).

Most of the references above are concerned with systems which are chaotic *before* control is applied. However, chaos has been detected in control systems in which the plant was not chaotic. Conditions for the occurrence of chaos in feedback systems (Genesio and Tesi, 1991), adaptive control (Mareels and Bitmead, 1986; Mareels and Bitmead, 1988; Golden and Ydstie, 1992) and in digital systems (Ushio and Hsu, 1987) have been reported in the literature. The use of estimated models in control problems has been investigated in connection with synchronization (Aguirre and Billings, 1994b) and suppression of chaos (Aguirre and Billings, 1995a).

Informative introductions to the vast field of control of chaos can be found in (Ditto and Pecora, 1993; Hunt and Johnson, 1993).

6 Discussion and Further Reading

The analysis and quantification of chaotic dynamics is a relatively recent area. Nevertheless there is an immense collection of scientific papers and books devoted to this subject and any attempt to produce a survey on nonlinear dynamics and chaos, no matter how thorough, would be, in all certainty, just a rough sketch on this fascinating subject.

The main objective of this paper has been to review in a very pragmatic way a few concepts which are believed to be basic. Since it would be inappropriate to produce an in-depth review, a rather generous number of references has been cited for further reading. Needless to say, the reference list does not exhaust the wealth of papers and books currently available.

The following references seem to be a good starting point. The books (Gleick, 1987) and (Stewart, 1989) are a good introduction for the average reader. A more formal coverage is given by (Thompson and Stewart, 1986) and (Moon, 1987). For a mathematical exposition on the subject see (Guckenheimer and Holmes, 1983) and (Wiggins, 1990). Some practical aspects of bifurcation and chaos are discussed in (Matsumoto et al., 1993) and a good account on computer algorithms for nonlinear systems applications can be found in (Parker and Chua, 1989). See also (Abraham and Shaw, 1992) for a beautifully illustrated introduction to nonlinear dynamics and bifurcations. The following papers are also good introductions to nonlinear dynamics and chaos (Mees and Sparrow, 1981; Shaw, 1981; Mees, 1983; Eckmann and Ruelle, 1985; Crutchfield et al., 1986; Mees and Sparrow, 1987; Parker and Chua, 1987; Argyris et al., 1991; Thompson and Stewart, 1993; Chen and Moiola, 1995). Good surveys on modelling and analysis of chaotic signals can be found in (Grassberger et al., 1991; Abarbanel et al., 1993). Finally, see (Hayashi, 1964; Atherton and Dorrah, 1980) for a rather 'classical' approach to the analysis of nonlinear oscillations.

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Figure 1: Time series and respective attractors. (a) damped oscillations settling onto a (b) point attractor. (c) quasi-periodic oscillations lie on a (d) torus in state space. Attractors with higher dimensions and more complicated shapes correspond to time series with greater complexity.

Figure 2: Bifurcation diagram for the Duffing-Ueda oscillator $\ddot{y}+0.1\dot{y}+y^3 = A\cos(t)$, where A, the amplitude of the input, is the bifurcation parameter.

Figure 3: Routes to chaos. Sequence a-b-c-d shows the well known period-doubling route to chaos. In this sequence the periodicity doubles each time the system bifurcates until it becomes chaotic which is an aperiodic regime. Sequence e-f-g-h is sometimes called the torus-breakdown route to chaos. In this sequence the dimension increases each time the system bifurcates. In e-f-g the dimension is incremented by one. From g to h the increase is fractional.

Figure 4: A Poincaré section is obtained by defining a plane in state space which is transversal to the flow. The image formed on such a plane is the Poincaré section of the attractor and will display fractal structure if such an attractor is chaotic.

Figure 5: Graphical iteration of the logistic equation (7), (a) regular motion ($A = 2.6$) and (b) respective time series, (c) chaotic motion ($A = 3.9$), and (d) respective time series. In these figures the same initial condition has been used, namely $y(0) = 0.22$. In figures (e) and (f) an interval of initial conditions has been iterated for the same values of A as above. The intervals used were $y(0) \in [0.22\ 0.24]$ and $y(0) \in [0.220\ 0.221]$, respectively. Note how such an interval is amplified when the system is chaotic, (f). This is due to the sensitive dependence on initial conditions.

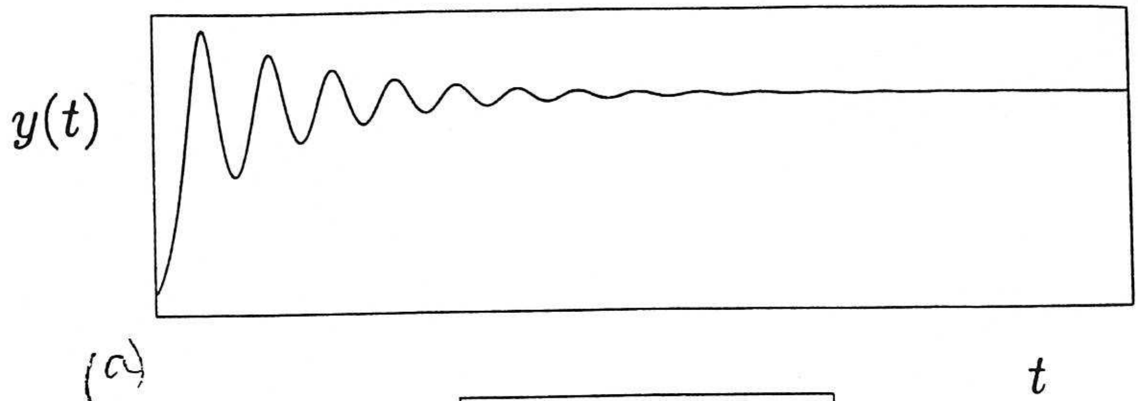
Figure 6: (a) Bifurcation diagram of the logistic map, and (b) respective largest Lyapunov exponent, λ_1. Note that $\lambda_1 = 0$ at bifurcation points and that $\lambda_1 > 0$ for chaotic regimes.

Figure 7: Logarithm of the correlation function $C(\epsilon)$ plotted against $\log(\epsilon)$ for embedding dimensions $d_e = 2$ to $d_e = 10$. The correct value, $D_c \approx 2.0$ is attained for $d_e \geq 5$.

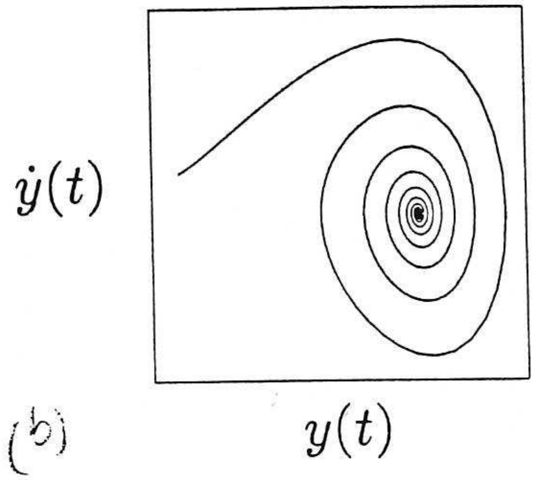
Figure 8: Fractal structure of the Hénon attractor.

Figure 9: The n time series defined by the state variables of an $n$tn-order dynamical system can be used to compose the trajectory in state space.

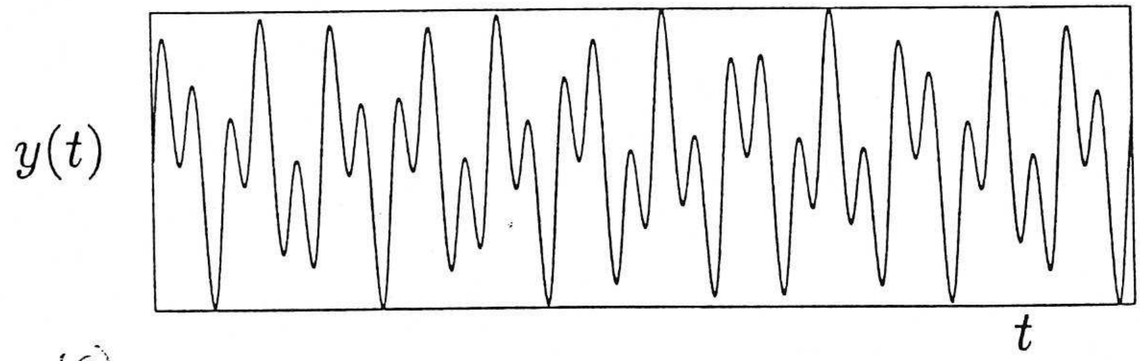
Figure 10: In many practical situations the number of measured variables is limited. Embedding techniques enable the reconstruction of the state space even from a single measurement. The reconstructed (or embedded) and the original state spaces are equivalent.



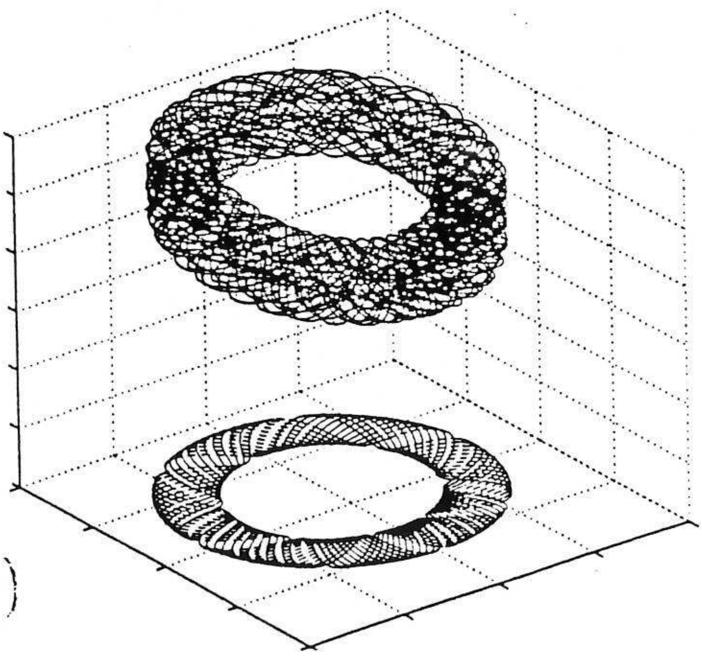
(a)



(b)



(c)



(d)

FIGURE 1.

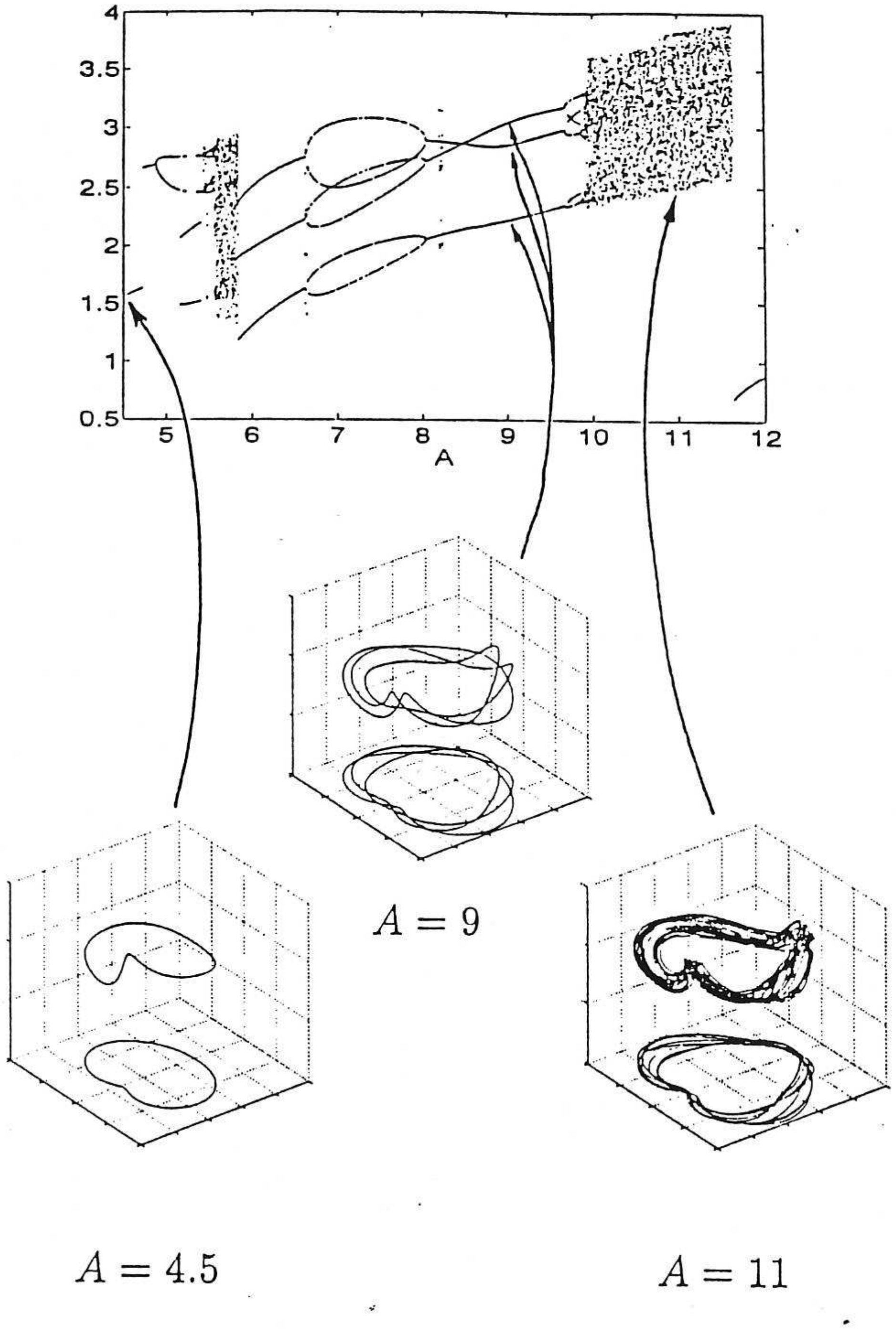


FIGURE 2

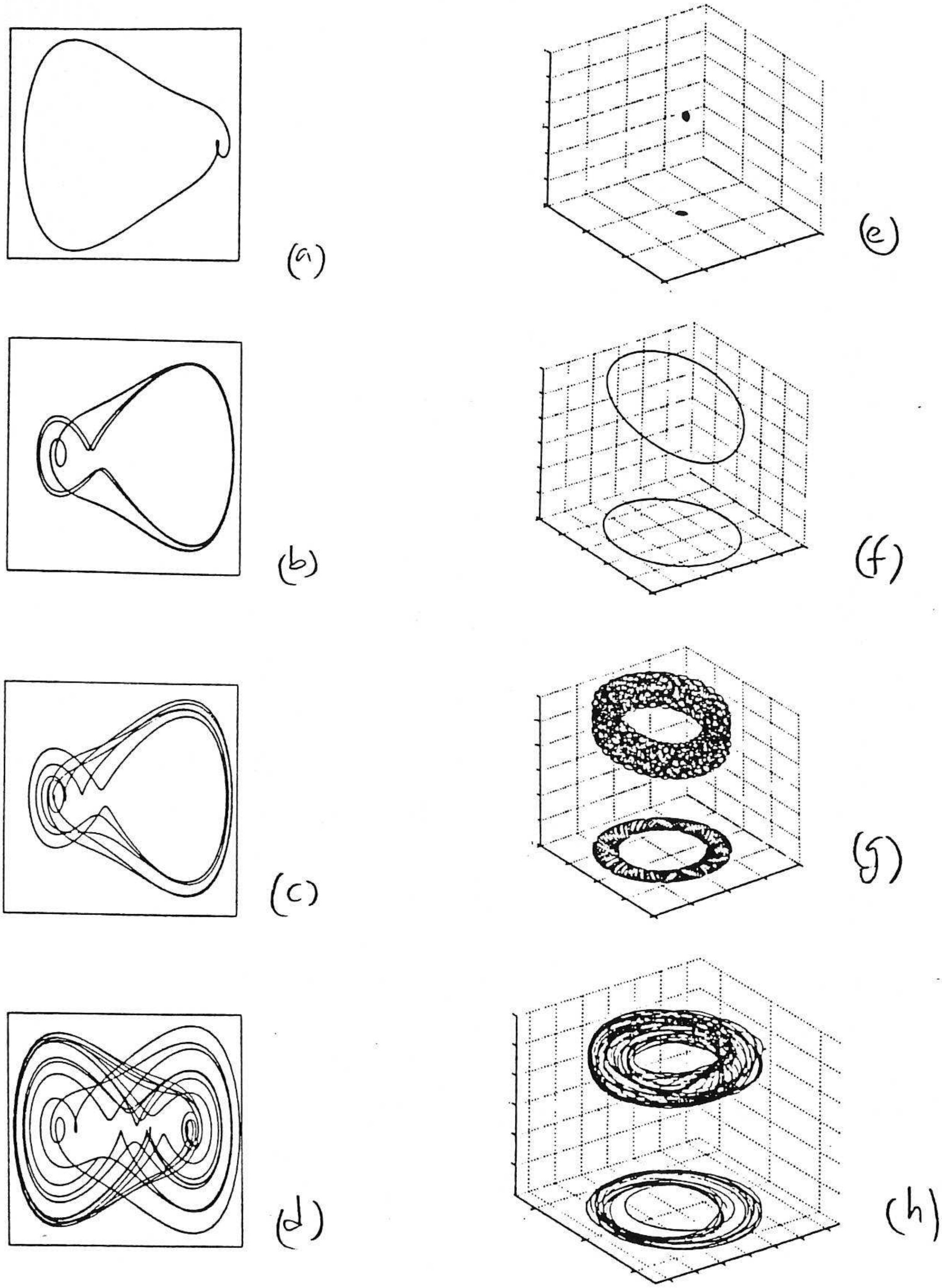


FIGURE 3

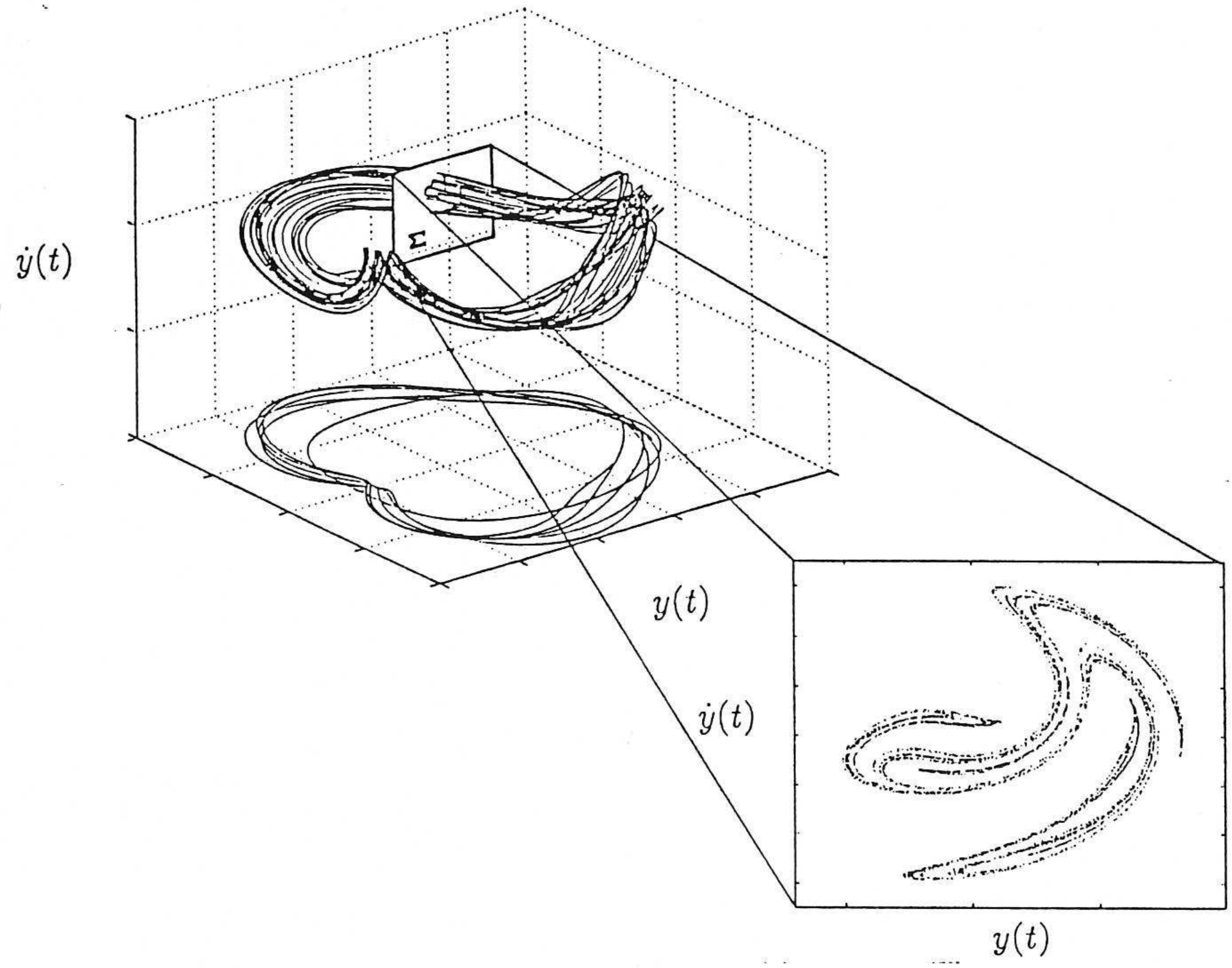


FIGURE 4

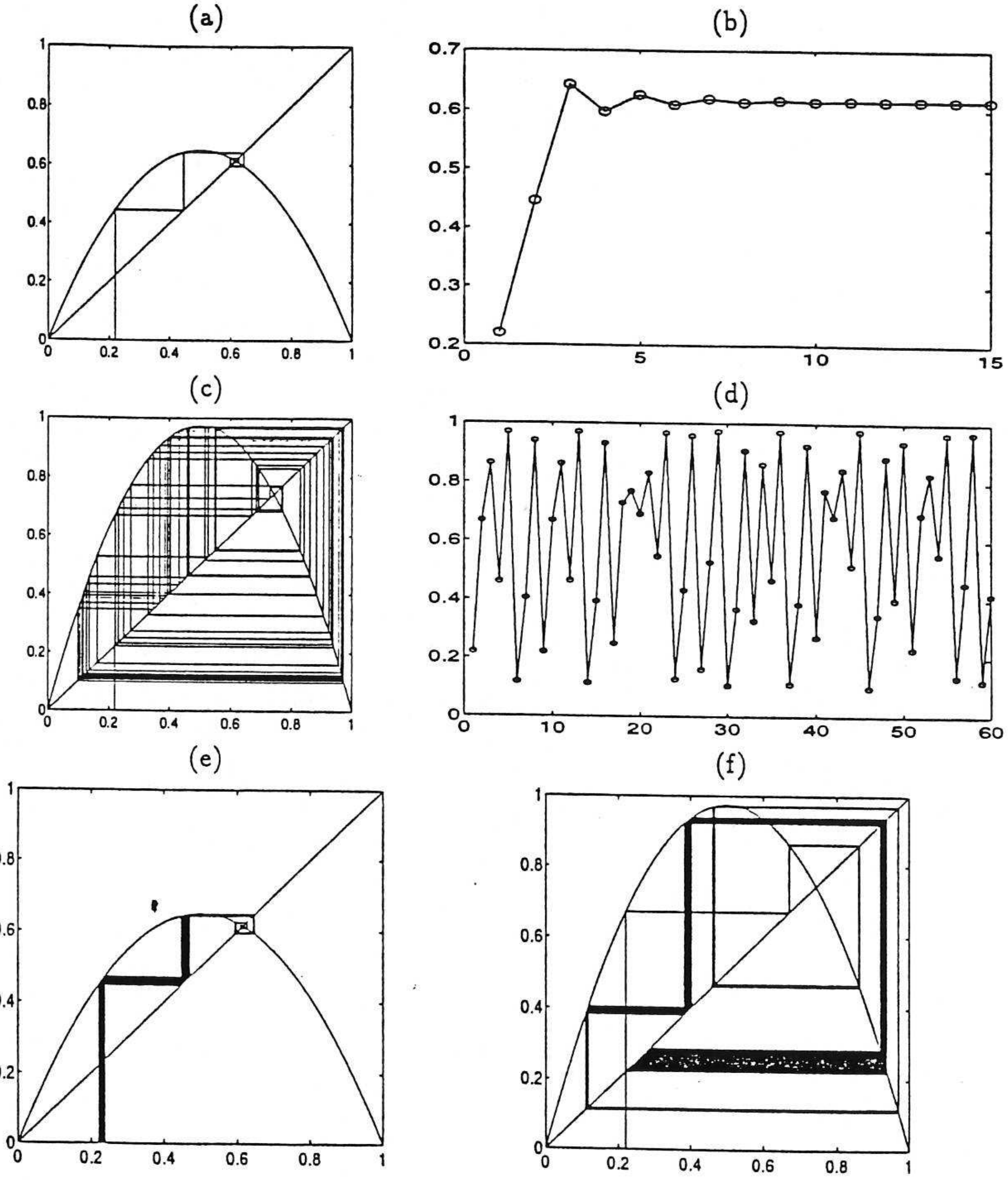
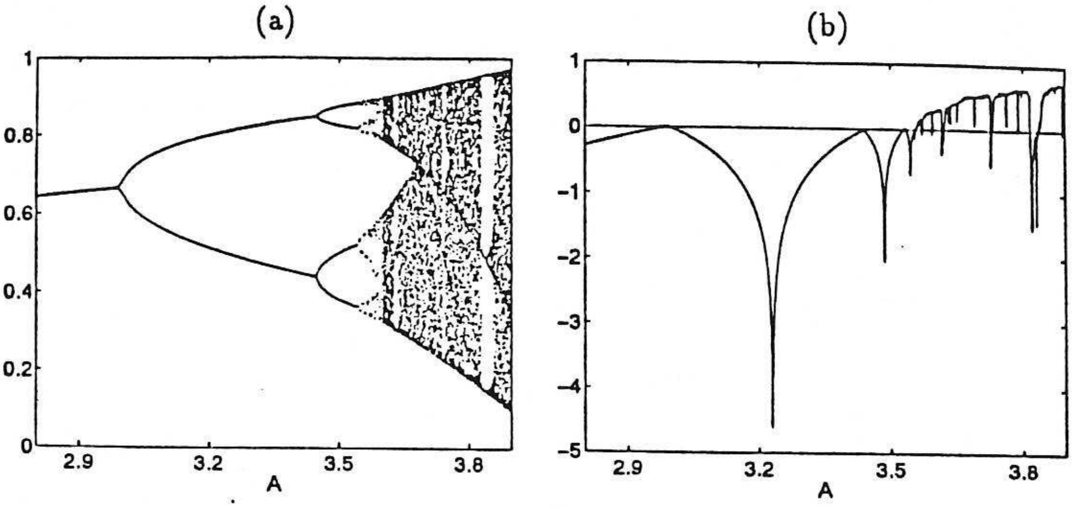
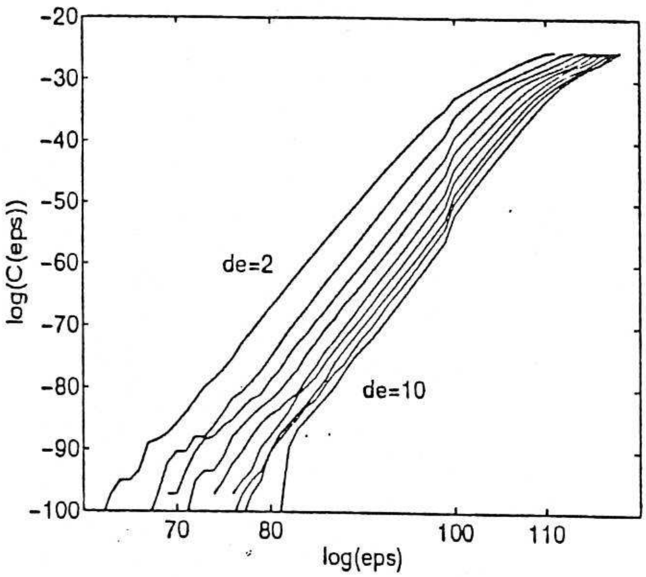


FIGURE 5



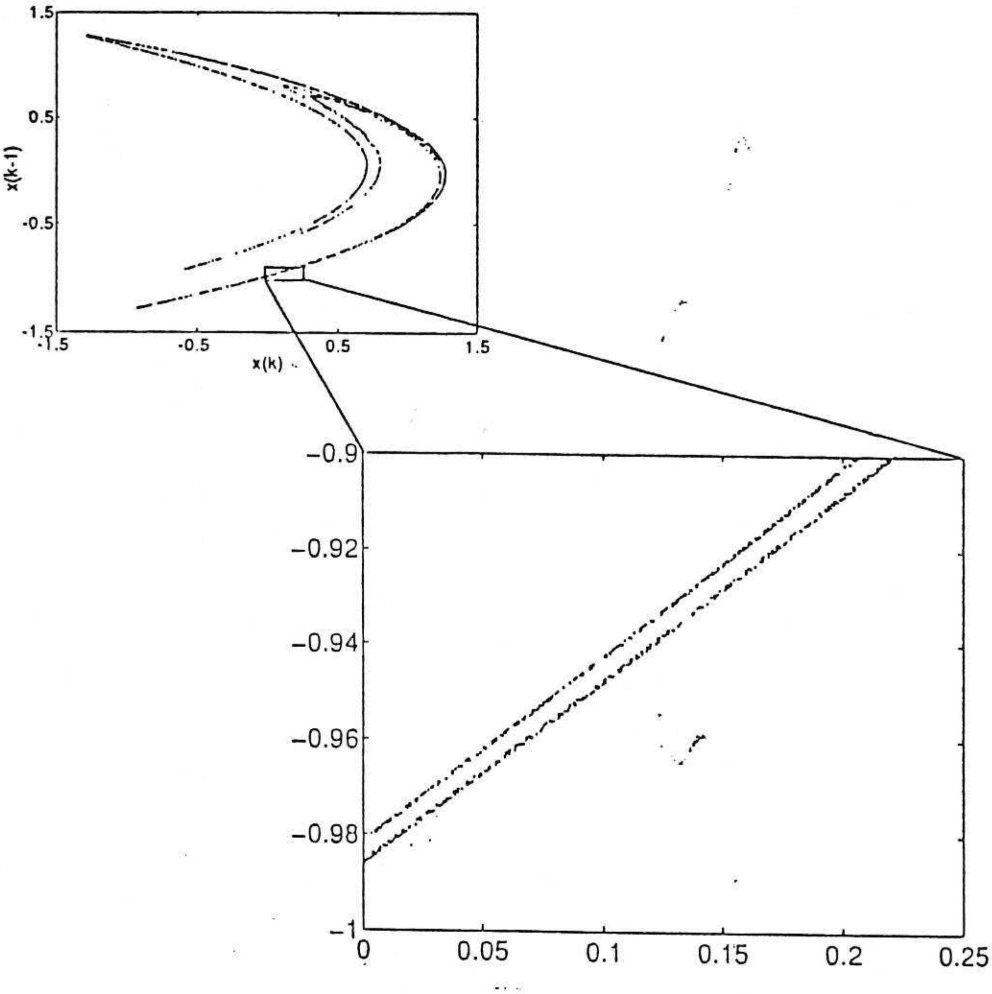
(a)
(b)
1
0.8
0.6
0.4
0.2
0
2.9
3.2
3.5
3.8
A
-1
-2
-3
-4
-5

FIG. 6



-20
-30
-40
-50
-60
-70
-80
-90
-100
log(C(eps))
de=2
de=10
70
80
100
110
log(eps)

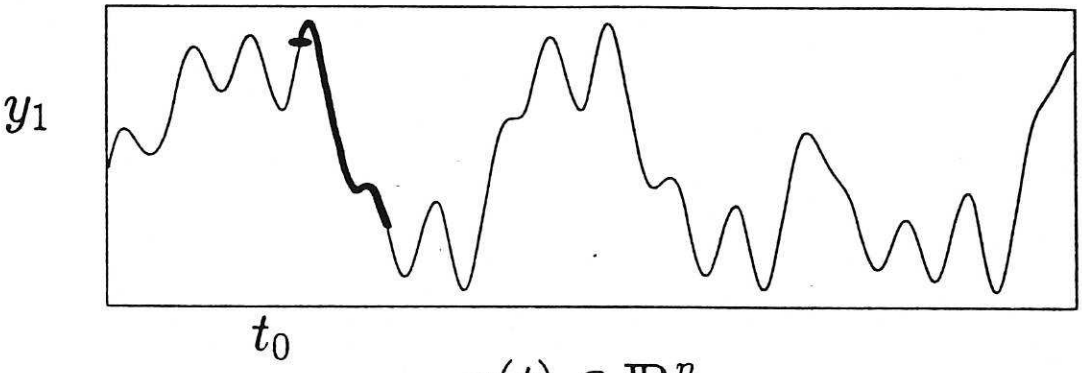
FIG. 7



1.5
0.5
-0.5
-1.5
x(k-1)
-1.5
-0.5
0.5
1.5
x(k)
-0.9
-0.92
-0.94
-0.96
-0.98
-1
0
0.05
0.1
0.15
0.2
0.25

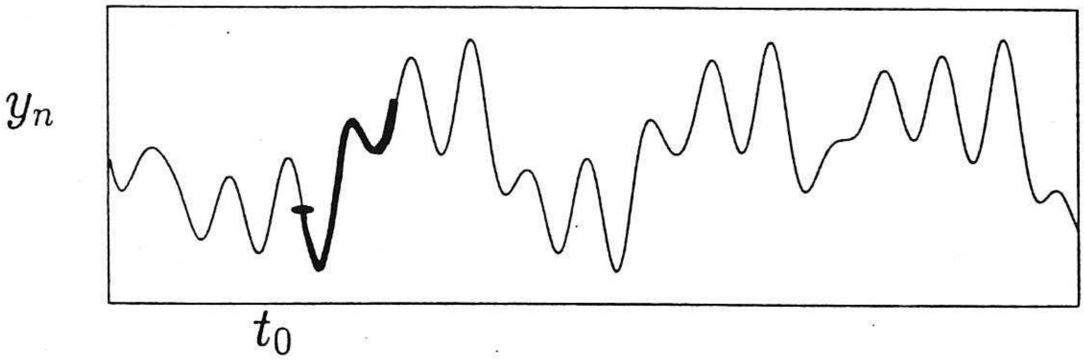
Fig 8

$$\dot{y}(t) = f_{\mathrm{c}}(y(t)) \quad \text{or} \quad y(k) = f_{\mathrm{d}}(y(k-1))$$



$$y(t) \in \mathbb{R}^n$$

$\vdots$



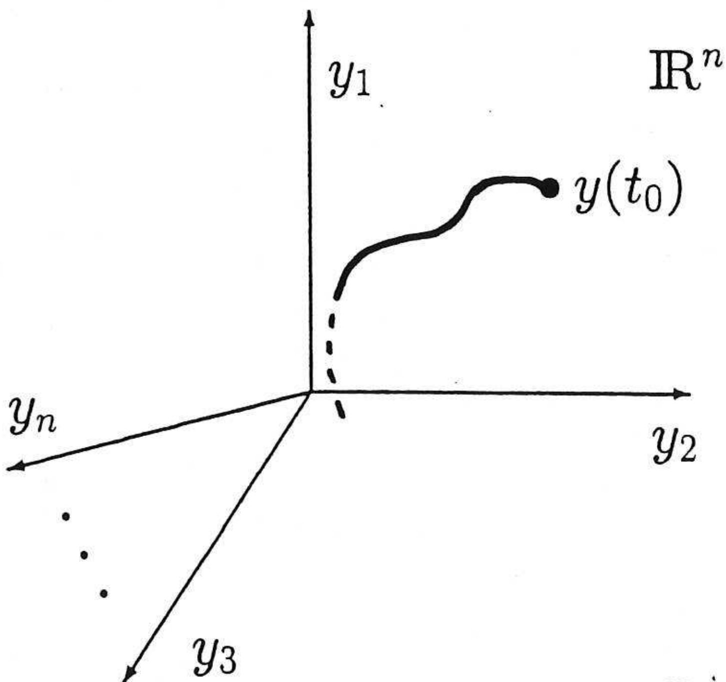


FIGURE 9

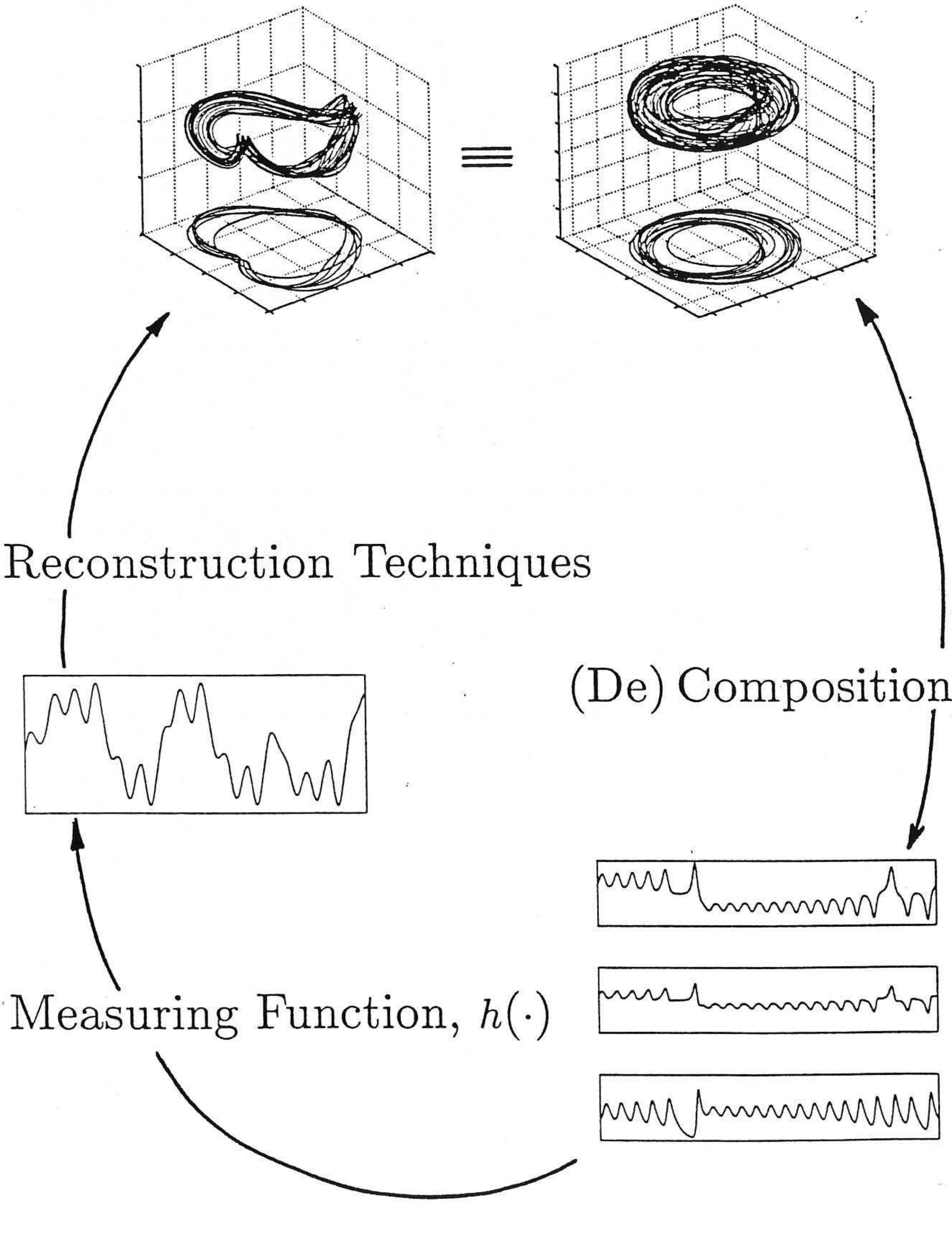

Reconstruction Techniques
(De) Composition
Measuring Function, $h(\cdot)$

FIGURE 1Ø