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Robust Exponential Stability of Evolution Equations

by

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Abstract

Exponential Stability for a class of finite- and infinite-dimensional nonlinear, time-varying systems is studied by regarding the system as a perturbation from a fixed operator. A new generalized Gronwall inequality is also proved.

Keywords :Exponential Stability , Pseudo-Linear Systems, Gronwall Inequality .

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1 Introduction

Consider the initial value problem in a Banach space X

$$\dot{x}(t) = A(x(t), t)x(t) \quad , \quad x(s) = x_0 \; , \; 0 \le s \le t \le T \tag{1.1}$$

where $A(x,t) : \mathcal{D}(A(x,t)) \subseteq X \longrightarrow X$ is a linear operator in X, for each $x \in X, t \ge 0$. A (classical) solution is a function $x : [s,T] \longrightarrow X$ such that x is continuous on $[s,T]$, $x(t) \in \mathcal{D}(A(x(t),t))$ for $s < t \le T$ and x is continuously differentiable on $s < t \le T$ and satisfies (1.1). We shall be concerned here in generalizing the results of [6] to nonlinear operators, in both the finite- and infinite-dimensional cases.

Nonlinear stability theory has been developed for nonlinear perturbations of nonlinear systems in the finite-dimensional case [1].[2] and the infinite-dimensional case [3], and the stability of autonomous systems of the form

$$\dot{x}(t) = A(x(t))x(t)$$

has been studied in the finite-dimensional case by the application of Lie algebra techniques [4] to the Lie algebra generated by the set

$$\{A(x) : x \in \mathbf{R}^n\}.$$

Here we shall consider the system (1.1) as a perturbation from a fixed operator $A(x_0.t_0)$ and apply Gronwall type inequalities. We shall also derive a generalized Gronwall inequality by applying the Lie series to the norm bound integral equation. Finally the results for finite-dimensional systems are generalized to parabolic partial differential equations.

The results apply directly to robust stability; in fact, we can regard

$$\dot{x}(t) = A(0,0)x(t)$$

as the nominal system and

$$\dot{x}(t) = A(x(t), t)x(t)$$

as a system with state-dependent parametric perturbations.

2 Finite-Dimensional Systems

In this section we shall consider the case of finite-dimensional systems. The method is based on perturbations around the zero point of the system 'A' function. First we discuss autonomous systems.

Theorem 2.1 Consider the system

$$\dot{x}(t) = A(x(t))x(t) \quad , x(0) = x_0 \ , \ x_0 \in \mathbf{R}^n \tag{2.1}$$

and suppose that

$$||A(x_1) - A(x_2)|| \leq L||x_1 - x_2||^{\alpha}$$

for all $x_1, x_2 \in \mathbf{R}^n$ and for some $L, \alpha > 0$. If

$$||e^{A(0)t}|| \leq Ke^{-\delta t}$$

and

$$\max\{LK, LK^{(1+\alpha)}\} < \delta$$

then the system is asymptotically stable for any x_0 with $||x_0|| < 1$.

Proof Write (2.1) in the form

$$\dot{x}(t) = A(0)x(t) + [A(x(t)) - A(0)]x(t) \quad , \quad x(0) = x_0.$$

Thus,

$$x(t) = e^{A(0)t}x_0 + \int_0^t e^{A(0)(t-s)}[A(x(s)) - A(0)]x(s)ds$$

and so

$$||x(t)|| \le Ke^{-\delta t}||x_0|| + KL\int_0^t e^{-\delta(t-s)}||x(s)||^{\alpha}||x(s)||ds.$$

Suppose that $K > 1, ||x_0|| \le 1, ||x(t)|| \le K$ for $t \in [0, \tau)$ and $||x(\tau)|| = K$. Then

$$||x(t)|| \le Ke^{-\delta t}||x_0|| + KLK^{\alpha}\int_0^t e^{-\delta(t-s)}||x(s)||ds \quad , \quad 0 \le t \le \tau.$$

By Gronwall's inequality, we have

$$\begin{aligned} ||x(t)|| &\le Ke^{-\delta t}\exp\left(LK^{(1+\alpha)}t\right)||x_0|| \\ &= Ke^{(-\delta + K^{(1+\alpha)}L)t}||x_0|| \quad , \quad 0 \le t \le \tau. \end{aligned}$$

Hence, if

$$LK^{(1+\alpha)} < \delta$$

then

$$||x(t)|| < K$$

for all $t > 0$, which is a contradiction. Hence, if $||x_0|| \le 1$,

$$||x(t)|| < K \quad \text{for all } t \ge 0$$

if $LK^{(1+\alpha)} < \delta$ and so $||x(t)|| \longrightarrow 0$ as $t \longrightarrow \infty$.

If $K \le 1$, let $||x_0|| < 1$, then if $||x(t)|| < 1$ for $0 \le t < \tau$,

$$||x(t)|| \le Ke^{-\delta t}||x_0|| + KL\int_0^t e^{-\delta(t-s)}||x(s)||ds \quad , \quad 0 \le t \le \tau$$

and, by Gronwall's inequality,

$$||x(t)|| \leq Ke^{(-\delta+KL)t)}||x_0||$$

so $||x(t)|| \longrightarrow 0$ as $t \longrightarrow \infty$ if $KL < \delta$. □

Corollary 2.2 Under the conditions of theorem 2.1, if

$$\max\{LM^\alpha K, LM^\alpha K^{(1+\alpha)}\} < \delta$$

then the system (2.1) is asymptotically stable for all x_0 with $||x_0|| < M$.

Proof Put $y = x/M$. Then,

$$\dot{y} = A(yM)y$$

and $||y_0|| < 1$. Since

$$\begin{aligned} ||A(y_1M) - A(y_2M)|| &\leq L||My_1 - My_2||^\alpha \\ &= LM^\alpha||y_1 - y_2||^\alpha, \end{aligned}$$

the result follows from theorem 2.1. □

Now consider the case of a polynomial system

$$\dot{x} = f(x) \quad , \tag{2.2}$$

where $f(0) = 0$ and each $f_i(x)$ is a polynomial in x. Then we can write the equation in the form

$$\dot{x} = A(x)x \tag{2.3}$$

where $A(x) = (a_{ij}(x))$ and

$$a_{ij}(x) = \sum_{\mathbf{k}=\mathbf{0}}^{\ell(ij)} \alpha_{ij}^{\mathbf{k}} x^{\mathbf{k}} \quad , \quad 1 \leq i,j \leq n \tag{2.4}$$

where $\mathbf{k} = (k_1, \cdots, k_n)$ and $x^{\mathbf{k}} = x_1^{k_1} \cdots x_n^{k_n}$, for some multiindex ℓ. Then

$$A(x) - A(y) = (b_{ij})$$

where

$$\begin{aligned} b_{ij} &= \sum_{\substack{\mathbf{k}=\mathbf{0} \\ |\mathbf{k}|>0}}^{\ell(ij)} a_{ij}^{\mathbf{k}} (x^{\mathbf{k}} - y^{\mathbf{k}}) \\ &= \sum_{\substack{\mathbf{k}=\mathbf{0} \\ |\mathbf{k}|>0}}^{\ell(ij)} a_{ij}^{\mathbf{k}} (p_{k-1}(x, y))(x - y) \end{aligned}$$

where $p_{k-1}(x, y)$ is a polynomial in x and y of order $k - 1$ ($k = |\mathbf{k}|$). Thus,

$$||A(x) - A(y)|| \le L||x - y||$$

where

$$L = \left\| \left(\sum_{\substack{\mathbf{k}=\mathbf{0} \\ |\mathbf{k}|>0}}^{\ell(ij)} |a_{ij}^{\mathbf{k}}| ||p_{k-1}(x, y)|| \right) \right\|. \tag{2.5}$$

For any matrix A, it can be shown [6] that

$$||e^{At}|| \le K e^{-\delta t} \tag{2.6}$$

where

$$K = 1 + \frac{2(2^n - 1)}{\pi \delta^n} m(3m)^{n-1}, \tag{2.7}$$

if $\max\{Re\ \sigma(A)\} \le -2\delta$, and so we obtain

Corollary 2.3 Consider the system (2.2) with $f(0) = 0$ and write it in the form (2.3). Let $A = (a_{ij}^{\mathbf{0}})$ be given by (2.4) and suppose that (2.6),(2.7) hold. Then if

$$\max\{LM^{\alpha}K, LM^{\alpha}K^{(1+\alpha)}\} < \delta$$

the system is asymptotically stable in the region $\{x : \|x\| < M\}$. □

Example 2.4 Consider the system

$$\ddot{x}(t) + \mu\dot{x} + \nu x(t) + \xi x^2(t) = 0. \tag{2.8}$$

Thus, we have

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -\nu - \xi x_1(t) & -\mu \end{pmatrix} x(t) \ .$$

Then

$$A(x) - A(y) = \begin{pmatrix} 0 & 0 \\ -\xi(x_1 - y_1) & 0 \end{pmatrix} ,$$

where

$$A(x) = \begin{pmatrix} 0 & 1 \\ -\nu - \xi x_1 & -\mu \end{pmatrix}$$

and

$$b_{21} = -\xi(x_1 - y_1).$$

Hence

$$L = \left\| \begin{pmatrix} 0 & 0 \\ \xi & 0 \end{pmatrix} \right\| = |\xi|,$$

and $\alpha = 1$.

Also,

$$\begin{pmatrix} 0 & 1 \\ -\nu & -\mu \end{pmatrix}$$

and

$$\sigma(A) = \left\{ -\frac{\mu}{2} + \frac{\sqrt{\mu^2 - 4\nu}}{2}, -\frac{\mu}{2} - \frac{\sqrt{\mu^2 - 4\nu}}{2} \right\} \triangleq \{\lambda_1, \lambda_2\}.$$

Rather than use (2.7) here we can diagonalize $A(0)$ and write

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = P^{-1}A(0)P$$

where

$$P = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}.$$

Then,

$$\begin{aligned} \|e^{A(0)t}\| &\leq \|P\|\,\|P^{-1}\|\,\|e^{\Lambda t}\| \\ &\leq e^{-\frac{\mu}{2}t}K \end{aligned}$$

where

$$K = \left(\sqrt{\lambda_1^2 + \lambda_2^2 + 2}\right)\left(\sqrt{\frac{\lambda_1^2}{\mu^2 - 4\nu} + \frac{\lambda_2^2}{\mu^2 - 4\nu} + \frac{2}{\mu^2 - 4\nu} +}\right) \tag{2.9}$$

if $4\nu \geq \mu^2$ and $\mu > 0$. Hence, by corollary 2.3, (2.8) is asymptotically stable in the region $\{x : \|x\| < M\}$ if

$$\max\{|\xi|MK, |\xi|MK^2\} < \mu/2$$

where K is given by (2.9). □

Consider next the case of nonautonomous systems

$$\dot{x}(t) = A(x(t), t)x(t) \quad , \quad x(0) = x_0 \ , \ x_0 \in \mathbf{R}^n. \tag{2.10}$$

This time we shall assume that A is jointly Lipschitz in x and t, i.e.

$$\|(A(x_1, t) - A(x_2, \tau)\| \leq L_x\|x_1 - x_2\|^{\alpha_x} + L_t|t - \tau|^{\alpha_t} \tag{2.11}$$

for some constants L_x, L_t, α_x and α_t.

The following result on nonautonomous linear systems given in ([6]) will be used:

Theorem 2.5 Consider the system

$$\dot{x}(t) = A(t)x(t) \quad , \quad x(0) = x_0$$

and suppose that:

$$(i) \quad \sup_t ||A(t)|| \text{ is finite}$$

$$(ii) \quad ||e^{A(t)s}|| \leq Ke^{-\delta s} \quad \forall\, t, s \geq 0$$

$$(iii) \; ||A(t_1) - A(t_2)|| \leq L|t_1 - t_2|^{\alpha} \quad \forall\, t_1, t_2 \geq 0$$

where $\alpha > 0$ and

$$L < \frac{\delta(\alpha+1)}{2K(2\ell n\, K/\delta)^{\alpha}}.$$

Then the system is exponentially stable and we have

$$||x(t)|| \leq K^2 e^{-\beta t} ||x_0|| \tag{2.12}$$

where

$$\beta = \frac{\delta}{2} - \frac{KL}{\alpha+1}\left(\frac{2\ell n\, K}{\delta}\right)^{\alpha} \quad (> 0). \quad \square$$

Now, returning to system (2.10) we write it in the form

$$\dot{x}(t) = A(0,t)x(t) + [A(x(t),t) - A(0,t)]x(t).$$

Let $\Phi(t,s)$ be the evolution operator (transition matrix) for $A(0,t)$. Then

$$x(t) = \Phi(t,0) + \int_0^t \Phi(t,s)[A(x(s),s) - A(0,s)]x(s)ds$$

and if $A(0,t)$ satisfies the assumptions of theorem 2, we have

$$||\Phi(t,0)|| \leq K^2 e^{-\beta t}$$

and so

$$||x(t)|| \leq K^2 e^{-\beta t}||x_0|| + \int_0^t K^2 e^{-\beta(t-s)} L_x ||x(s)||^{\alpha_x} ||x(s)|| ds.$$

In the same way as theorem 2.1 and corollary 2.2 were proved, by applying theorem 2.5, we obtain

Theorem 2.6 If $A(x,t)$ satisfies (2.11) and if

$$\max\{L_x M^{\alpha_x} K^2, L_x M^{\alpha_x} K^{2(1+\alpha_x)}\} < \beta \tag{2.13}$$

where

$$\beta = \frac{\delta}{2} - \frac{K L_t}{\alpha_t + 1}\left(\frac{2\ell n\ K}{\delta}\right)^{\alpha_t} \tag{2.14}$$

then the system (2.10) is asymptotically stable for all x_0 with $||x_0|| < M$.

Remark The result remains true if we know that the linear system

$$\dot{x} = A(0,t)x(t)$$

satisfies an inequality of the form (2.12) for some K, β and then we can dispence with condition (2.14). □

Example 2.7 Consider the system

$$\ddot{x}(t) + \mu\dot{x}(t) + (1 + 0.5\cos(2t))x(t) + \xi x^2 = 0. \tag{2.15}$$

Then we have

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ -1 - 0.5\cos(2t) - \xi x_1(t) & -\mu \end{pmatrix} x(t) \\ &= A(x,t)x(t) \end{aligned}$$

where

$$A(x,t) = \begin{pmatrix} 0 & 1 \\ -1 - 0.5\cos(2t) - \xi x_1(t) & -\mu \end{pmatrix}.$$

Thus,

$$\begin{aligned} ||A(x_1,t) - A(x_2,\tau)|| &\leq \left\| \begin{pmatrix} 0 & 0 \\ 0.5(\cos(2t) - \cos(2\tau)) & 0 \end{pmatrix} \right\| + |\xi| \ ||x_1 - x_2|| \\ &= 0.5|2\sin(2\eta)| \ |t-\tau| + |\xi| \ ||x_1 - x2|| \end{aligned}$$

for some $\eta \in [\tau, t]$ (if $\tau < t$). Hence, $L_t = 1$, $\alpha_t = 1$, $L_x = |\xi|$, $\alpha_x = 1$. As discussed in [6] the conditions of theorem 2.6 do not hold for L_t but by the remark above we can apply classical Floquet theory and it can be seen that

$$||x(t)|| \leq 2e^{-0.4t}||x_0||$$

for the system (2.15) with $\xi = 0$. (the numbers in the estimate are conservative–one could do much better with a more careful analysis of this system.) Hence, by (2.13), if

$$M|\xi| < 0.1$$

then the system (2.15) is asymptotically stable for $||x_0|| < M$. □

3 A Generalized Gronwall Inequality

We now show that a generalized Gronwall type inequality can be obtained by applying the Lie series. The results will be seen to be direct generalizations of the well-known inequality and lead to some new stability conditions.

Consider first the case of an autonomous nonlinear differential equation

$$\dot{x} = f(x) \ , \ x(0) = x_0 \in \mathbf{R}^n \tag{3.1}$$

where f is an analytic function. Then,

$$x(t) = x_0 + \int_0^t f(x(s))ds \tag{3.2}$$

and so

$$\|x(t)\| \leq \|x_0\| + \int_0^t \|f(x(s))\|ds. \tag{3.3}$$

We assume that f satisfies the inequality

$$\|f(x)\| \leq g(\|x\|)$$

for some analytic function g which is strictly increasing, i.e.

$$\xi < y \Rightarrow g(\xi) < g(y) \quad \forall\, \xi, y \in \mathbf{R}.$$

By (3.3) we have

$$\|x(t)\| \leq \|x_0\| + \int_0^t g(\|x(s)\|)ds. \tag{3.4}$$

We shall use a comparison argument and therefore consider the equation

$$y(t) = y_0 + \int_0^t g(y(s))ds. \tag{3.5}$$

Write $\xi(t) = \|x(t)\|$ and we have from (3.4),

$$\xi(t) \leq \xi_0 + \int_0^t g(\|\xi(s)\|)ds. \tag{3.6}$$

Lemma 3.1 Under the above assumptions, if y satisfies (3.5) and ξ satisfies (3.6), with $\xi_0 < y_0$, then

$$\xi(t) < y(t) \quad , \quad \forall\, t \geq 0.$$

Proof Since $\xi_0 < y_0$ and g is analytic, $\xi(t) < y(t)$ for t in some interval $[0, \delta]$. Suppose that $\xi(\tau) = y(\tau)$ for some τ and that $\xi(t) < y(t)$ for all $t < \tau$. Then, from (3.5),(3.6),

$$\begin{aligned} \xi(t) - y(t) &\leq \xi_0 - y_0 + \int_0^t (g(\xi(s)) - g(y(s)))ds \\ &< \int_0^t (g(\xi(s)) - g(y(s)))ds \;. \end{aligned}$$

Consider $t = \tau$. Since $\xi(t) < y(t)$ for $t \in [0, \tau)$, the integral

$$\int_0^\tau (g(\xi(s)) - g(y(s)))ds$$

must be strictly negative. Hence, $\xi(t) < y(\tau)$, which is a contradiction. □

Corollary 3.2 If $\xi_0 \leq y_0$ we have $\xi(t) \leq y(t), \forall\, t \geq 0$.

Proof This follows from the lemma since $\xi(t; \xi_0)$ is continuous in ξ_0. □

We are thus led to consider the integral equation (3.5), or in differential form, the equation

$$\dot{y}(t) = g(y(t)) \quad , \quad y(0) = y_0.$$

The solution of this equation is given by the Lie series ([5]):

$$\begin{aligned} y(t) &= \exp\left(tg\frac{d}{dt}\right) y\Big|_{y=y_0} \\ &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(L_g^k y\right)(y_0) \end{aligned}$$

where L_g is the Lie derivative with respect to g. From corollary 3.2 we obtain the following theorem:

Theorem 3.3 If f is an analytic function for which

$$||f(x)|| \leq g(||x||)$$

for some strictly increasing analytic function g, then the solution of (3.1) is bounded by

$$||x(t)|| \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(L_g^k \xi\right)(\xi_0)$$

where $\xi_0 = ||x_0||$. □

Example 3.4 Consider the equation

$$\dot{x} = Ax + f(x)$$

where $||f(x)|| \leq M||x||$. Then

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(x(s))ds$$

so that

$$||x(t)|| \leq K e^{\omega t}||x_0|| + \int_0^t K e^{\omega(t-s)} M||x(s)||ds$$

for some K, ω. Thus,

$$y(t) \leq ||x_0|| + \int_0^t M y(s)ds$$

where

$$y(t) = \frac{1}{K}||x(t)||e^{-\omega t}.$$

By theorem 3.3, we have

$$y(t) \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_g^k \xi)(||x_0||)$$

where $g(\xi) = M\xi$. Now,

$$L_g \xi = g\frac{d\xi}{d\xi} = M\xi \quad ,$$

and so

$$y(t) \leq e^{Mt}||x_0||.$$

Hence,

$$||x(t)|| \leq K e^{(M+\omega)t}||x_0|| \quad ,$$

which is simply Gronwall's inequality for linear bounded functions. □

Consider next the nonautonomous system

$$\dot{x} = f(x,t) \quad , \quad x(0) = x_0 \in \mathbf{R}^n \quad , \tag{3.7}$$

where f is again analytic in x and t. Then

$$||x(t)|| \leq ||x_0|| + \int_0^t ||f(x(s), s)|| ds.$$

This time we assume that f satisfies the inequality

$$||f(x, t)|| \leq g(||x||, t)$$

for some analytic function g which is strictly increasing in $||x||$ for each $t \geq 0$. Lemma 3.1 and corollary 3.2 clearly generalize directly so that if

$$y(t) = y_0 + \int_0^t g(y(s), s) ds$$

and

$$\xi(t) \leq \xi_0 + \int_0^t g(\xi(s), s) ds$$

where $\xi_0 \leq y_0$, then $\xi(t) \leq y(t)$, $\forall t \geq 0$. To generalize theorem 3.3 we need to find the Lie series for the equation

$$\dot{y}(t) = g(y(t), t).$$

To do this, we write the equation in the form

$$\begin{aligned} \dot{z}_1 &= \zeta_1(z_1, z_2) \ , \quad z(0) = z_0 \\ \dot{z}_2 &= \zeta_2(z_1, z_2) \end{aligned} \tag{3.8}$$

where

$$z_1 = y \ , \ z_2 = t \ , \ \zeta_1(z_1, z_2) = g(y, t) \ , \ \zeta_2(z_1, z_2) = 1$$

and

$$z_0 = (y_0 . t_0).$$

Thus,

$$\begin{aligned} z(t) &= \exp\left(t\zeta\frac{\partial}{\partial z}\right) z\Big|_{z=z_0} \\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}(L_\zeta)^k z\Big|_{z=z_0} . \end{aligned}$$

Hence, we have

$$y(t) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\left(g(y,t)\frac{\partial}{\partial y}+\frac{\partial}{\partial t}\right)^k y\Big|_{y=y_0} .$$

Theorem 3.3 now generalizes as follows:

Theorem 3.5 If f is an analytic function for which

$$\|f(x,t)\| \le g(\|x\|,t)$$

for some analytic function g which is stricly increasing in $\|x\|$ for each t, then the solution of equation (3.7) is bounded by

$$\|x(t)\| \le \sum_{k=0}^{\infty}\left[\frac{t^k}{k!}\left(g(\xi,t)\frac{\partial}{\partial \xi}+\frac{\partial}{\partial t}\right)^k \xi\right](\xi_0) \tag{3.9}$$

where $\xi_0 = \|x_0\|$. □

Example 3.6 Consider the system

$$\dot{x} = Ax + f(x,t) \quad , \quad x(0) = x_0$$

and suppose that

$$\begin{aligned} \|e^{At}\| &\le Me^{\omega t} , \\ \|f(x,t)\| &\le g(\|x\|)p(t) \end{aligned}$$

where g is homogeneous of degree k, i.e.

$$g(a\|x\|) = a^k g(\|x\|) \text{ for any } a \in \mathbf{R}.$$

Then

$$\begin{aligned} ||x(t)|| &\leq Me^{\omega t}||x_0|| + \int_0^t Me^{\omega(t-s)} g(||x(s)||)p(s)ds \\ &= Me^{\omega t}||x_0|| + Me^{\omega t}\int_0^t g\left(\frac{e^{-\omega s}}{M}||x(s)||\right) M^k p(s) e^{(k-1)\omega s} ds \ . \end{aligned}$$

Put

$$y(t) = \frac{||x(t)||}{M} e^{-\omega t} \ .$$

Then we obtain

$$y(t) \leq ||x_0|| + \int_0^t g(y(s)) M^k p(s) e^{(k-1)\omega s} ds$$

and so

$$\dot{y} = M^k g(y) p(t) e^{(k-1)\omega t} \ .$$

Note first that if g is linear, so that $k = 1$, we have

$$\dot{y} = Myp(t)$$

so that

$$y = y_0 e^{\int_0^t Mp(s)ds}$$

and

$$||x(t)|| \leq Me^{\omega t} e^{\int_0^t Mp(s)ds}$$

which is just Gronwall's inequality. As a second example, suppose that $g(y) = y^3$, $k = 3$, so that

$$\dot{y} = M^3 y^3 p(t) e^{2\omega t} \ . \qquad (3.10)$$

Then,

$$y^2 = \frac{y_0^2}{1 - 2y_0^2 \int_0^t M^2 p(s) e^{2\omega s} ds}$$

i.e.

$$||x||^2 \leq \frac{||x_0||^2 M^2 e^{2\omega t}}{1 - 2||x_0||^2 \int_0^t M^2 p(s) e^{2\omega s} ds}$$

provided

$$2||x_0||^2 \int_0^t M^2 p(s) e^{2\omega s} ds \leq \delta < 1 \quad , \quad 0 \leq t < \infty \tag{3.11}$$

for some δ. Thus, if this condition holds, then $||x(t)|| \longrightarrow 0$ as $t \longrightarrow \infty$ if $\omega < 0$. Note that condition (3.10) is required in order that the Lie series in (3.8) converges, so that equation (3.9) has a well-defined real solution.

4 Nonlinear Parabolic Systems

In this section we shall extend the previous results to the case of nonlinear parabolic systems. Thus, consider the nonlinear evolution equation

$$\dot{u}(t) = A(u,t)u(t) \quad , \quad u(0) \in L^2(\Omega) \tag{4.1}$$

where $\Omega \subseteq \mathbf{R}^n$ is open and for each $u \in L^2(\Omega)$, $t \geq 0$, $A(u,t)$ is a sectorial operator (see [7]), i.e. $A(u,t)$ is closed, densely defined and the sector

$$S_{a,\phi} = \{\lambda : \phi \leq |\arg(\lambda - a)| \leq \pi \ , \ \lambda \neq a\}$$

is in the resolvent set of $A(u,t)$. Moreover,

$$||(\lambda - A(u,t))^{-1}|| \leq \frac{M}{|\lambda - a|} \ , \ \forall \lambda \in S_{a,\phi}$$

where a, ϕ and M are independent of (u,t).

We shall assume that equation (4.1) has local solutions in u and t, i.e. for each $u(0) \in L^2(\Omega)$, there exists a solution $u(\cdot) \in L^\infty([0,\tau), L^2(\Omega))$ of (4.1) for some $\tau > 0$. The following result is well-known ([7]):

Lemma 4.1 If the resolvent $\mathcal{R}(\lambda; A(u,t))$ exists for all $\lambda \in S_{a,\phi}$ and

$$||\mathcal{R}(\lambda; A(u,t))|| \le \frac{M}{|\lambda+\delta|+1} \quad , \quad \lambda \in S_{a,\phi} \ , \ t \ge 0 \ , \ u \in L^2(\Omega)$$

then there exists $K > 0$ such that

$$(i) \ \ ||T_{(u,s)}(t)|| \le K\epsilon^{-\delta t} \quad , \quad t > 0$$
$$(ii) \ ||A^\alpha(u,s)T_{(u,s)}(t)|| \le \frac{K\epsilon^{-\delta t}}{t^\alpha}$$

for all $s \ge 0, 0 < \alpha, u \in L^2(\Omega)$, where $T_{(u,s)}(t)$ is the semigroup generated by $A(u,s)$. □

Now we assume that $A(u,t)$ satisfies the inequality

$$||T_{(0,n)}(t-s)[A(u_1,t_1) - A(u_2,t_2)]|| \le \frac{K}{(t-s)^\alpha}\epsilon^{-\delta(t-s)}\left(L_u||u_1-u_2||^{\beta_1} + L_t|t_1-t_2|^{\beta_2}\right)$$

for some α with $0 < \alpha < 1$ and some numbers $L_u, L_t, \beta_1, \beta_2 > 0$. Then, from (4.1), we have

$$\dot{u}(t) = A(0,0)u(t) + [A(u,t) - A(0,0)]u(t)$$

so that

$$u(t) = T_{(0,0)}(t)u(0) + \int_0^t T_{(0,0)}(t-s)[A(u,s) - A(0,0)]u(s)ds$$

and

$$||u(t)|| \le K\epsilon^{-\delta t}||u(0)|| + \int_0^t K\frac{\epsilon^{-\delta(t-s)}}{(t-s)^\alpha}\left(L_u||u||^{\beta_1} + L_t s^{\beta_2}\right)||u(s)||ds. \tag{4.2}$$

We need the following generalization of Gronwall's inequality ([7]):

Lemma 4.2 Suppose that $\xi(t)$ satisfies the inequality

$$\xi(t) \le a + b\int_0^t \frac{1}{(t-s)^\alpha}\xi(s)ds$$

for some constants a, b, where $0 < \alpha < 1$. Then,

$$\xi(t) \leq aE_\alpha\left([b\Gamma(1-\alpha)]^{1/(1-\alpha)}t\right)$$

where

$$E_\alpha(z) = \sum_{n=0}^{\infty} z^{n(1-\alpha)}/\Gamma(n(1-\alpha)+1).$$

□

Note that $E_\alpha(z) \approx \frac{1}{1-\alpha}e^z$ as $z \longrightarrow \infty$. Now let $u(0)$ be such that $||u(0)|| < M$ and suppose that $||u(t)|| < M$ for $0 \leq t < \tau$ where $\tau \leq 1$. Then by (4.2) we have

$$||u(t)|| \leq MKe^{-\delta t} + \int_0^t K\frac{e^{-\delta(t-s)}}{(t-s)^\alpha}\left(L_u M^{\beta_1} + L_t s^{\beta_2}\right)||u(s)||ds$$

for $0 \leq t < \tau \leq 1$, and so

$$y(t) \leq a + b\int_0^t \frac{1}{(t-s)^\alpha}y(s)ds,$$

where

$$y(t) = e^{\delta t}||u(t)|| \ , \ a = MK \ , \ b = K(L_u M^{\beta_1} + L_t \cdot 1) \ .$$

Hence, by lemma 4.2,

$$||U(t)|| \leq aE_\alpha\left([b\Gamma(1-\alpha)]^{1/1-\alpha)}t\right)e^{-\delta t} \quad , \quad 0 \leq t < \tau \leq 1.$$

Put

$$\lambda = [b\Gamma(1-\alpha)]^{1/1-\alpha)}$$

and assume that

$$\frac{\delta}{2} > \sup_{0 \leq t < 1} \ln\left(KE_\alpha(\lambda t)\right).$$

Then,

$$||u(t)|| < Me^{(-\delta/2)t} \quad , \quad 0 \leq t \leq 1.$$

Next consider the system starting at $t = 1$. Then we have

$$u(t) = T_{(0,1)}(t)u(1) + \int_1^t T_{(0,1)}(t-s)[A(u,s) - A(0,1)]u(s)ds$$

and so

$$||u(t)|| \leq Ke^{-\delta t}||u(1)|| + \int_1^t K\frac{e^{-\delta(t-s)}}{(t-s)^\alpha}\left(L_u||u||^{\beta_1} + L_t s^{\beta_2}\right)||u(s)||ds.$$

i.e.

$$y(t) \leq a_1 + b\int_1^t \frac{1}{(t-s)^\alpha}y(s)ds \ , \ 1 \leq t \leq 2$$

where $a_1 = MKe^{-\delta/2}$ and b is as before. Thus,

$$\xi(t') \leq a_1 + b\int_0^{t'} \frac{1}{(t'-s')^\alpha}\xi(s')ds' \ , \ 0 \leq t' \leq 1$$

where $s' = s - 1$, $t' = t - 1$, $\xi(t') = y(t' + 1)$. Hence, as before,

$$||u(t)|| < Me^{-(\delta/2)t} \ , \ 0 \leq t \leq 2.$$

Continuing in this way we obtain

Theorem 4.3 Suppose that $A(u,t)$ is a sectorial operator for each $u \in L^2(\Omega)$ and $t \geq 0$ and assume that the semigroup $T_{(u,s)}(t)$ generated by this operator satisfies

$$\left\|T_{(0,n)}(t-s)\left(A(u_1,t_1) - A(u_2,t_2)\right)\right\| \leq K\frac{e^{-\delta(t-s)}}{(t-s)^\alpha}\left(L_u||u_1 - u_2||^{\beta_1} + L_t|t_1 - t_2|^{\beta_2}\right)$$

for some α with $0 < \alpha < 1$ and some numbers $L_u, L_t, \beta_1, \beta_2 > 0$, where $n \in \mathbf{N}$. Then if the resolvent $\mathcal{R}(\lambda; A(u,t))$ exists and satisfies

$$||\mathcal{R}(\lambda; A(u,t))|| \leq \frac{M}{|\lambda + \delta| + 1} \quad , \quad \lambda \in S_{a,\phi} \ , \ t \geq 0 \ , \ u \in L^2(\Omega)$$

and

$$\frac{\delta}{2} > \sup_{0 \le t \le 1} \ln(KE_\alpha(\lambda t))$$

where E_α and λ are as defined above, the system is asymptotically stable for $||u(0)|| \le M$.

□

Example 4.4 The example

$$A(t) = (1 + 0.5\sin(t))\frac{\partial^2}{\partial x^2}$$

considered in [6] is not a valid application of the results presented in that paper, since the inequality

$$||(A(t_1) - A(t_2))A(t_3)^{-\alpha}|| \le L|t_1 - t_2|^\beta$$

is required with $\alpha < 1$ (strictly less than 1–theresult is not true for $\alpha = 1$), and the example requires the inequality for $\alpha = 1$, so it is not true. However, we can consider the nonlinear operator

$$A(t, u) = \frac{\partial^2}{\partial x^2} + (1 + 0.5\sin(t)u^2)\frac{\partial}{\partial x}$$

for example, and a simple argument then shows that the equation associated with this operator is stable for small enough initial value.

5 Conclusions

In this paper we have generalized some results in [6] to the case of nonlinear evolution equations. The method is based on representing the equation as a perturbation about a fixed value of the operator and then applying Gronwall's inequality. Moreover, by using Lie series we have given a direct generalization of Gronwall's inequality. Finally, the theory has been extended to nonlinear parabolic partial differential equations by successive expansions of the system semigroup about integer temporal values.

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