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A Bi-objective User Equilibrium Model of Travel Time Reliability in a Road Network

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Abstract

Travel time, travel time reliability and monetary cost have been empirically identified as the most important criteria influencing route choice behaviour. We concentrate on travel time and travel time reliability and review two prominent user equilibrium models incorporating these two factors. We discuss some shortcomings of these models and propose alternative bi-objective user equilibrium models that overcome the shortcomings. Finally, based on the observation that both models use standard deviation of travel time within their measure of travel time reliability, we propose a general travel time reliability bi-objective user equilibrium model. We prove that this model encompasses those discussed previously and hence forms a general framework for the study of reliability related user equilibrium. We demonstrate and validate our concepts on a small three-link example.

Keywords: Route choice, user equilibrium, travel time reliability, bi-objective

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user equilibrium, late arrival penalty, travel time budget.

1. Introduction

It is well known from empirical studies that the three most important factors influencing route choice behaviour are travel time, travel time reliability and monetary cost. Abdel-Aty et al. (1995) performed statistical analysis to determine which route attributes that lead to the choice of a route are considered important by road users. The three most important factors are: (1) shorter travel time (ranked as the first reason by 40% of respondents); (2) travel time reliability (32%); and (3) shorter distance (31%). Although the effect of monetary cost was not considered explicitly in this study, the third most important factor, i.e. distance, is directly related to vehicle operating cost for the trip. In more recent years, the values of travel time (VOT) and travel time reliability (VOR) were estimated in two road pricing demonstrations in southern California, on California State Route 91 (SR91) and Interstate 15 (I-15) (see Lam and Small, 2001; Liu et al., 2004; Brownstone and Small, 2005). All the analyses on these two datasets share some common observations. The estimated values of VOT and VOR from these studies are comparably high. For instance, the best fitted model in Lam and Small (2001) has a VOT of $22.87 per hour, while the VOR is $15.12 per hour for men and $31.91 for women. Note that the VOR for women is 39.5% higher than the VOT. Another common observation is that substantial heterogeneity in travellers' preference of travel time and reliability is observed but it is difficult to isolate its exact origin (Brownstone and Small, 2005). More recently, evidence from Australian case studies also indicates that drivers are willing to pay more to reduce the uncertainty of travel time than they are for the same reduction in mean travel time (Li et al., 2010).

In order to model route choice behaviour realistically, the effect of uncertainty associated with travel time needs to be incorporated in the traffic assignment procedure. The conventional user equilibrium models, namely, the user equilibrium (UE) model based on Wardrop's principle, and the stochastic user equilibrium (SUE) model (Daganzo and Sheffi, 1977), do not consider the variability of travel time explicitly in general. The UE model assumes that users are minimising their generalised costs, which is often expressed as a linear combination of time and monetary cost, while the SUE model assumes that users are minimising their perceived generalised cost, which has a randomly-distributed component.

A few reliability-based equilibrium models do, however, exist. These equilibrium models were developed based on the concepts of travel time uncertainty modelling in the empirical models. There are two main theoretical frameworks, as categorised in Li et al. (2010), namely, the mean-variance model (Jackson and Jucker, 1982) and the scheduling model (Small, 1982).

Other reliability-based equilibrium models include the travel time budget (TTB) models (Shao et al., 2006a,b; Lam et al., 2008), percentile user equilibrium (PUE) model (Nie, 2011), and mean-excess traffic equilibrium (METE) models (Zhou and Chen, 2008; Chen and Zhou, 2010; Chen et al., 2011; Xu et al., 2013). The TTB model is defined as the average travel time plus an extra time (or buffer time) such that the probability of completing the trip within the TTB is no less than a predefined reliability threshold alpha. The general TTB model is formulated as a variant of the chance constrained model (Shao et al., 2006a,b; Lam et al., 2008), where the TTB is treated as the objective function to be minimised while satisfying the chance (or on-time arrival) constraint. In essence, the TTB and PUE models are equivalent for any continuous distributions of random sources, while the TTB model of Lo et al. (2006) derived from the mean-variance model under the normal distribution assumption of route travel time is a special case. Note that the PUE model does not assume any probability distribution for modelling capacity uncertainty. It resorts to some convolution methods and solves the route percentile travel time (or route travel time budget) numerically through the application of Fourier transform (Ng and Waller, 2010; Wu and Nie, 2011).

The METE model is defined as the conditional expectation of the travel time exceeding the TTB is defined as the conditional expectation of the travel time exceeding the TTB (Zhou and Chen, 2008; Chen and Zhou, 2010). As a route choice criterion, the METE model can be regarded as a combination of the "buffer time" measure that ensures the reliability of on-time arrival, and the "tardy time" measure that represents the unreliability impacts of excessively late trips. It is a risk-averse traffic equilibrium model that seeks to address two questions: "How much time do I need to allow?" and "How bad should I expect from the worse cases?" The issue of perception error is also considered in the stochastic version of METE by explicitly modelling the stochastic perception error within the METE framework (Chen et al., 2011; Xu et al., 2013).

For other traffic equilibrium models under uncertainty, interested readers may refer to the disutility/utility-based model (Mirchandani and Soroush, 1987; Yin and Ieda, 2001; Chen et al., 2002; Di et al., 2008), game theory-based models (Bell, 2000; Bell and Cassir, 2002; Szeto et al., 2006), the expected residual minimisation approach Zhang et al. (2011), and the prospect theory-based model (Connors and Sumalee, 2009; Xu et al., 2011).

Tan et al. (2013) investigate many of the above mentioned reliability based equilibrium models and determine the shape of the mean-standard deviation indifference curves in these models. They obtain results on Pareto efficiency of the equilibrium solutions of these models in terms of their Pareto efficiency regarding expected travel time and standard deviation of travel time.

In this paper, we focus on looking at the two main theoretical frameworks, i.e. the mean-variance model and the scheduling model, from a multi-objective perspective. Now we look into these two models in more detail.

In the mean-variance model, Jackson and Jucker assume that travel time variability leads to loss of utility. Every traveller has a *prior* estimate of the mean and variance of the travel time and the objective of each traveller is expressed by Equation (1).

$$\min \left\{ E\left(T_k\right) + \lambda_m V\left(T_k\right) : k \in K_p \right\}, \tag{1}$$

where λ_m is a non-negative parameter which represents the degree to which the variability of travel time is undesirable to traveller m; $E\left(T_k\right)$ is the expected travel time on path k for O-D pair p; $V\left(T_k\right)$ is the variance of the travel time on path k; and K_p is the set of all paths for O-D pair p. Variations of the mean-variance model, such as the mean-standard deviation model, constant relative risk aversion (CRRA) model, and constant absolute risk aversion (CARA) model, have also been considered in de Palma and Picard (2005) to model different risk aversion preferences towards travel time uncertainty.

In the scheduling model, Small assumes that not arriving at the destination at the preferred arrival time (PAT) will cause disutility, and the consequence of arriving early and late could be different. Naturally one would expect that travellers would dislike being late more than being early. The utility function can be expressed as in Equation (2).

$$U\left(t_d; \text{PAT}\right) = \alpha_1 T + \alpha_2 SDE + \alpha_3 SDL + \alpha_4 D_L, \tag{2}$$

where t_d is the decision variable, the departure time choice; PAT is a preferred arrival time; T is the travel time; SDE is the scheduling delay early as defined in Equation (3); SDL is the scheduling delay late as defined in Equation (4); and D_L is a binary variable indicating whether it is a late arrival or not ($D_L = 1$ if and only if $SDL > 0$); and the estimated parameters ($\alpha_1, \alpha_2, \alpha_3$ and α_4) are assumed to be negative.

$$
\begin{aligned}
SDE &= \max\left(0, \text{PAT} - [T + t_d]\right), \quad (3) \\
SDL &= \max\left(0, [T + t_d] - \text{PAT}\right). \quad (4)
\end{aligned}
$$

Now let us look at how these concepts have been applied in equilibrium models. Based on the concept in the mean-variance model, Lo et al. (2006) formulated a multi-class equilibrium model by considering a single objective as minimising travel time budget, defined as the expected travel time plus a travel time margin (or buffer time), with the travel time margin being dependent on the level of risk aversion of each user class, as shown in Equation (5).

$$
B_k = E\left(T_k\right) + \lambda_m \sigma_{T_k}, \quad (5)
$$

for all $k \in K_p$ (the set of all paths from origin to destination of O-D pair p) and for all $p \in Z$ (the set of all O-D pairs), where B_k is the travel time budget; T_k is the random variable of travel time on route k for O-D pair p; $E\left(T_k\right)$ and σ_{T_k}, respectively, are the mean and standard deviation of T_k. λ_m is a parameter associated with the level of risk aversion of individual m. Note that although the travel time budget model shares a similar mathematical form with the mean-variance (or standard deviation) model, it has a different meaning defined by the travel time reliability chance constraint such that the probability that travel time exceeds the budget is less than a predefined confidence level specified by the traveller to represent his/her risk preference. Lo et al. (2006) called this the within budget time reliability (WBTR) or the punctuality reliability. This definition is also similar to the alpha-reliable route defined by Chen and Ji (2005) to indicate the route with the minimum travel time budget.

Based on the concept of a *schedule delay* component in the scheduling model, Watling (2006) proposed a late arrival penalised UE (LAP-UE) which assumes users minimise a composite path disutility, incorporating the generalised cost plus a late arrival penalty. Watling (2006) assumes that travellers make their route choice decision with a longest possible travel time in mind for their journey. If this is exceeded, the inconvenience incurred will be modelled by the penalty component of the utility function in Equation (6).

$$U(k;\tau_m) = \theta_0 d_k + \theta_1 E(T_k) + \theta_2 E\left[\max(0, T_k - \tau_m)\right], \tag{6}$$

where k is the decision variable, the path choice, with a longest acceptable travel time τ_m in mind. Further, $\theta_0 d_k + \theta_1 E(T_k)$ is the standard *generalised travel time* and $\theta_2 E\left[\max(0, T_k - \tau_m)\right]$ is the penalty component. In particular, d_k represents the composite of attributes (such as distance) that are independent of time and flow; $E(T_k)$ is the mean travel time on route k; θ_2 is the value of being one time unit later than acceptable; and the estimated parameters $(\theta_0, \theta_1, \theta_2)$ are assumed to be negative.

The models in Lo et al. (2006) and Watling (2006) both incorporate the effects of travel time and its uncertainty. Lo et al. (2006) use the buffer time, $\lambda_m \sigma_{T_k}$ in Equation (5), while Watling (2006) uses the penalty function, $\theta_2 E\left[\max(0, T_k - \tau_m)\right]$ in Equation (6). Although they use two different measures to model the effect of unreliability on route choice, the models share the same assumption that the effects of these two factors can be combined into a single objective with a linear disutility function. Based on the results from empirical studies as discussed earlier, one would expect that a route choice decision is in fact a multi-criteria decision based on important factors such as expected travel time and its variability. In fact, combining the two key factors into one implicitly assumes the existence of a linear (dis)utility function, and therefore pre-supposes a certain preference structure. As an effect of this, there is the possibility that some *reasonable* choices are never considered in the decision process. This can be illustrated with an example as shown in Figure 1.

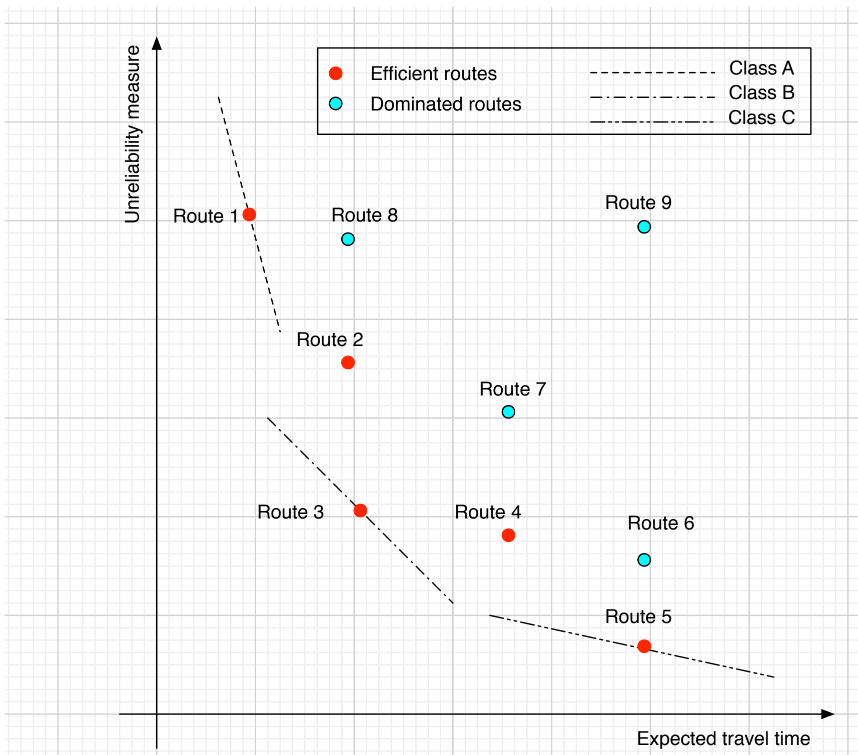

Figure 1: Trade off between expected travel time and unreliability measure

In Figure 1, the travel time reliability of nine possible routes between one origin-destination pair is plotted against their corresponding expected travel time. The measures of reliability can be, say the buffer times, $\lambda_m \sigma_{T_k}$, in Lo et al.'s formulation or the late arrival penalty in Watling's. As all travellers would want to minimise these two objectives, a set of efficient options among the nine alternatives can be identified, which are represented by Routes 1 to 5 in Figure 1. Routes 6 to 9 will not be considered by a rational traveller, as they are dominated by at least one other route, which has no worse expected travel time and buffer time, but is better in at least one of these criteria. In the equilibrium model of Lo et al. (2006), the different levels of risk aversion are modelled by different values of λ_m for different user classes in the objective function, Equation (5). Graphically, the objective functions of different user classes can be represented by the dotted lines with different slopes in Figure 1, where λ_m is the slope of the line. As a result, the optimal choices of Classes A, B and C will all be different: They are Routes 1, 3 and 5, respectively. Although Routes 2 and 4 are both *efficient* routes in this case, i.e. there are no other routes with expected travel time and travel time variability less than or equal to those of Routes 2 and 4 and at least one of these criteria better, they will never be chosen by any travellers according to this model. This is because the linear combination of $E(T_k)$ and σ_{T_k} in the objective function will not be able to completely represent a bi-objective decision process. Replacing buffer time by lateness penalty $E[\max(0, T_k - \tau_m)]$, a similar argument can be made for the LAP-UE model of Watling (2006). We note that Dial (1997) suggests a similar formulation to Lo et al. (2006), without explicitly specifying the reliability measure. Dial's model will, therefore, have the same issue as illustrated in this example.

While missing out some *rational* alternatives is a general problem that needs to be addressed, there are some other properties of this decision process that a single objective formulation might not be able to address. For instance, in the time budget equilibrium model (Lo et al., 2006), all the used routes at equilibrium will have equal travel time budget for the users in the same class. This means that the used routes even for the same user class can have different expected travel times as well as different travel time margin, as long as the sums, i.e. the travel time budgets, are equal and minimal.

This condition implicitly implies two characteristics at equilibrium. Firstly, since the travel time budget on all used routes is equal, the departure time relative to the same desired arrival time window of users in the same class will all be the same. Secondly, the choice set for users in the same class consists of routes with different expected travel time but the users are indifferent towards these different travel times as long as the travel time budget on each route is the same and minimal. In other words, a used route with a lower expected travel time but higher variability is equally attractive as another route with a higher expected travel time but lower variability as long as the travel time budgets on the two routes are the same. This might not be true as some users might prefer to spend less time in traffic on average. In that case, the route with the shortest expected travel time would be the most attractive. Once we introduce the mathematical formulation of the late arrival penalty user equilibrium model (Watling, 2006) in Section 3, it is easy to see that a similar comment applies for that model, too.

In this paper, we address the possibility that users' travel time margin not only varies between different user classes but also within the same class and users' preference is not only dependent on travel time budget but on both the expected travel time and travel time budget. We propose a new modelling framework to model such conditions with a travel time reliability bi-objective user equilibrium (TTR-BUE) model. The idea of bi-objective user equilibrium was introduced in Wang et al. (2010) in the context of tolling analysis, but can be adapted to any modelling framework in which we expect users might react differently to several objectives influencing their route choices. Our research also contributes to the growing literature that uses multi-objective methods in a variety of transportation research contexts, such as Tan and Yang (2012), who study built-operate-transfer contracts in the context of optimising social welfare and private profit; Chen and Yang (2012), who consider minimising the conflicting social costs of congestion and emissions with toll schemes and Yang et al. (2012), who consider speed limits to obtain efficient flow patterns in terms of reducing both total travel time and total emissions.

In Sections 2 and 3, we will describe the travel time budget and late arrival penalty user equilibrium models mathematically. We also introduce bi-objective versions of these models, and prove that the equilibrium solutions of the models of Lo et al. (2006) and Watling (2006) are special cases of the corresponding bi-objective user equilibrium models. In Section 4, we present a new general travel time reliability bi-objective user equilibrium model, which eliminates the need for user-class-specific parameters and preference assumptions. We prove that all four models mentioned in Sections 2 and 3 are special cases of this general model. Hence, the general model serves as a modelling framework for the study of travel time reliability. We demonstrate our concepts on a small example in Section 5 and draw some conclusions and suggestions for further work in Section 6.

2. Travel Time Budget User Equilibrium

The travel time budget user equilibrium focuses on modelling the travel behaviour of road users in response to the day-to-day variations in travel time induced by disruptions on a minor scale, caused by traffic incidents. We, therefore, adopt the results from Lo and Tung (2003), summarised as follows. Throughout the paper, the Bureau of Public Roads (1964) link performance function

$$t_a\left(f_a\right) = t_a^0 \left[1 + \beta \left(\frac{f_a}{C_a}\right)^n\right] \tag{7}$$

is adopted, where t_a^0 is the free-flow travel time and C_a is the capacity of link a. Thus, $t_a(f_a)$ is the link travel time with link flow f_a and β, n are deterministic parameters.

Lo and Tung (2003) assume that link capacity follows a uniform distribution, defined by an upper bound (the design capacity) and a lower bound (the worst-degraded capacity), which is a fraction, ϕ_a, of the design capacity, $\bar{c}_a$, i.e.

$$C_a \sim U\left(\phi_a \cdot \bar{c}_a, \bar{c}_a\right). \tag{8}$$

Hence ϕ_a serves the role as a reliability parameter for travel time: As derived in Lo and Tung (2003), the path travel time is normally distributed with mean and standard deviation that can be written as

$$\begin{aligned} T_k &\sim N\left(E\left(T_k\right), \sigma_{T_k}\right) & (9)\\ E\left(T_k\right) &= \sum_a \left[\delta_a^k \cdot E\left(t_a\right)\right] & (10)\\ \sigma_{T_k} &= \sqrt{\sum_a \left[\delta_a^k \cdot \mathrm{var}\left(t_a\right)\right]}. & (11)\end{aligned}$$

Here δ_a^k is the usual link-path incidence, i.e. $\delta_a^k = 1$ if link a belongs to path k and 0 otherwise. By applying the assumption of uniformly distributed arc capacity as expressed in Equation (8), the mean and standard deviation of the route travel time distribution are

$$E\left(T_k\right) = \sum_a \left\{\delta_a^k \cdot \left[t_a^0 + \beta t_a^0 f_a^n \frac{1-\phi_a^{1-n}}{\bar{c}_a^n\left(1-\phi_a\right)\left(1-n\right)}\right]\right\}, \tag{12}$$

$$\sigma_{T_k} = \sqrt{\sum_a \left[\delta_a^k \cdot \beta^2 \left(t_a^0\right)^2 f_a^{2n} \left\{\frac{1-\phi_a^{1-2n}}{\bar{c}_a^{2n}\left(1-\phi_a\right)\left(1-2n\right)} - \left[\frac{1-\phi_a^{1-n}}{\bar{c}_a^n\left(1-\phi_a\right)\left(1-n\right)}\right]^2\right\}\right]}. \tag{13}$$

The travel time budget model of Lo et al. (2006) is a multi-user class equilibrium model which considers both the expected travel time $E(T_k)$ and the variability of travel time, as measured by σ_{T_k} with users in class m minimising their travel time budget $B_k = E(T_k) + \lambda_m \sigma_{T_k}$. Mathematically, λ_m can be related to the probability ρ_m that a trip arrives within the travel time budget,

$$P\left\{T_k \leqslant B_k = E\left(T_k\right) + \lambda_m \sigma_{T_k}\right\} = \rho_m. \tag{14}$$

After rearranging (14), we have

$$P\left(S_{T_k} = \frac{T_k - E\left(T_k\right)}{\sigma_{T_k}} \leqslant \lambda_m\right) = \rho_m. \tag{15}$$

Note that the left hand side in Equation (15) is the standard normal variate of T_k, $S_{T_k} \sim$ N(0,1).

As pointed out in Section 1, in any solution of the travel time budget equilibrium problem, it is possible that for a given user class m, there are several paths with equal and minimal time budget. As mentioned before, users in the same class would be indifferent with respect to such paths. We believe that this might not be realistic and suggest a bi-objective user equilibrium model that overcomes this problem.

Now let us consider the formulation in Lo et al. (2006) from a bi-objective perspective. The travel time budget represents how much time needs to be allowed for the trip while the expected travel time represents how much time is expected to be spent in traffic. One would expect that users will always want: (1) to minimise the expected travel time, i.e. $\min E\left(T_k\right)$; and (2) to minimise the travel time budget, i.e. $\min B_k$, subject to an acceptable level of risk. As explained above, risk is represented by the probability of the actual travel time being longer than the travel time budget.

Mathematically, the two objectives are:

$$\begin{aligned} &\min E\left(T_k\right), \\ &\min B_k = E\left(T_k\right) + \lambda_m \sigma_{T_k}, \end{aligned} \tag{16}$$

where B_k is dependent on the level of risk aversion of the individual or user class m, measured by ρ_m, which determines the value of λ_m as in Equation (15), i.e. B_k is the objective function of the travel time budget model.

Based on the objective functions in (16), we can formulate the travel time budget bi-objective user equilibrium (TTB-BUE) as follows.

> "Under *travel time budget bi-objective user equilibrium* conditions traffic arranges itself in such a way that no individual trip maker can improve either his/her expected travel time or travel time budget or both without worsening the other objective by unilaterally switching routes."

We will show that every solution of the travel time budget equilibrium model of Lo et al. (2006) is also a solution to at least the weak TTB-BUE model. To that end, we define the weak TTB-BUE model.

> "Under *weak travel time budget bi-objective user equilibrium* conditions traffic arranges itself in such a way that no individual trip maker can improve both his/her expected travel time and travel time budget by unilaterally switching routes."

Theorem 1. *Let $\mathcal{F}$ be a path flow solution to the travel time budget equilibrium model. Then $\mathcal{F}$ also satisfies the weak TTR-BUE condition.*

Proof. Assume that $\mathcal{F}$ does not satisfy the weak TTR-BUE condition. Then, for at least one user class m there must exist two used paths k and k' between some O-D pair p such that $E(T_{k'}) < E(T_k)$ and $E(T_{k'}) + \lambda_m \sigma_{T_{k'}} < E(T_k) + \lambda_m \sigma_{T_k}$. The second of these inequalities contradicts the assumption that $\mathbf{F}$ satisfies the travel time budget equilibrium condition. $\square$

3. Late Arrival Penalty User Equilibrium

Based on the concept of *schedule delay*, as introduced by Small (1982), Watling developed the idea of a schedule delay equilibrium model, known as LAP-UE

(Watling, 2006) as described earlier. The assumption behind this model is that users are concerned about expected travel time as well as the expected schedule delay given a longest possible travel time τ_m (for user class m).

Based on Watling (2006)'s derivation, the schedule delay $E[\max(0, T_k - \tau_m)]$ in Equation (6) can be simplified to Equation (17) where $L(x)$ is given in Equation (18).

$$E\left[\max\left(0, T_k - \tau_m\right)\right] = \sigma_{T_k} L\left(\frac{\tau_m - E\left(T_k\right)}{\sigma_{T_k}}\right), \tag{17}$$

$$L\left(x\right) = \int_x^{\inf} \left(u - x\right) \phi\left(u\right) du = \phi\left(x\right) + x\Phi\left(x\right) - x, \tag{18}$$

where ϕ and Φ are the probability density function and cumulative distribution function of a $N(0, 1)$ variate, respectively. In the LAP-UE model, users minimise Equation (6). In this study, we are not concerned with attributes that are independent of time or flow, hence we assume that $\theta_0 = 0$ and we can normalise θ_1 to 1. This also puts the discussion of the model of Watling (2006) in the same framework as that of Lo et al. (2006), where travel time independent factors are not considered. The user objective becomes the disutility of path k

$$\min u_k = E(T_k) + \theta_2 L\left(\frac{\tau_m - E\left(T_k\right)}{\sigma_{T_k}}\right)\sigma_{T_k}. \tag{19}$$

We have mentioned before that this model leads to a similar problem to that of Lo et al. (2006): There might be several paths with the same minimal value of u_k that have differing expected travel times (and, therefore, different late arrival penalties). The model implicitly assumes that users are indifferent to these paths. To avoid this, we can proceed in the same way as for the model of Lo et al. (2006) by considering the model from a bi-objective perspective and separate the two components of u_k out. That is, we assume users would want: (1) to minimise expected travel time; and (2) to minimise the expected schedule delay or lateness penalty.

Mathematically, the two objectives are:

$$\min E\left(T_k\right), \tag{20}$$
$$\min E\left[\max\left(0, T_k - \tau_m\right)\right].$$

With these objectives, we can define the late arrival penalty bi-objective user equilibrium (LAP-BUE) as follows.

> "Under *late arrival penalty bi-objective user equilibrium* conditions traffic arranges itself in such a way that no individual trip maker can improve either his/her expected travel time or late arrival penalty or both without worsening the other objective by unilaterally switching routes."

As for the time budget model, we now proceed to show that a solution to the LAP-UE model is always a solution to the LAP-BUE model.

Theorem 2. *Let* $\mathbf{F}$ *be a path flow solution to the late arrival penalty user equilibrium model. Then* $\mathbf{F}$ *also satisfies the LAP-BUE condition.*

Proof. Assume that $\mathbf{F}$ does not satisfy the LAP-BUE condition. Then, for at least one user class m there must exist two used paths k and k' such that $E(T_{k'}) \leqslant E(T_k)$ and $L\left(\frac{\tau_m - E(T_{k'})}{\sigma_{T_{k'}}}\right)\sigma_{T_{k'}} \leqslant L\left(\frac{\tau_m - E(T_k)}{\sigma_{T_k}}\right)\sigma_{T_k}$, with at least one of these inequalities strict. But this implies that

$$E(T_{k'}) + \theta_2 L\left(\frac{\tau_m - E\left(T_{k'}\right)}{\sigma_{T_{k'}}}\right)\sigma_{T_{k'}} < E(T_k) + \theta_2 L\left(\frac{\tau_m - E\left(T_k\right)}{\sigma_{T_k}}\right)\sigma_{T_k}$$

contradicting the LAP-UE condition. □

Under the LAP-BUE condition, if several paths with the same minimal value of u_k exist, users would always prefer the one which has lower expected travel time. We may also use this LAP-BUE model as a tie-breaker in the conventional user equilibrium model considering only (generalised) travel time: Faced with the choice between two paths with equal expected travel time, users would prefer the one which has lowest schedule delay.

4. The General Travel Time Reliability Bi-objective User Equilibrium

In Sections 2 and 3, we have briefly presented the travel time budget user equilibrium (Lo et al., 2006) and late arrival penalty user equilibrium (Watling, 2006) models as the main network equilibrium models in the literature that consider expected travel time as well as standard deviation of travel time in a network equilibrium model. We have illustrated that the implicit assumption of user indifference towards the two components of the function used in these models creates ambiguity, and that it may not be realistic to assume that users are indifferent towards the different expected travel times that used paths in an equilibrium solution may have. We have suggested bi-objective user equilibrium models to overcome these problems. In this section, we propose a general travel time reliability bi-objective user equilibrium model (TTR-BUE) that incorporates both the original TTB-UE and LAP-UE models, as well as their bi-objective counterparts (16) and (20) and other possible reliability models. From now on, we omit the assumption of normal distribution of travel time, which Watling used and which Lo and Tung (2003) obtained from the assumption of uniform distribution of capacity, and only assume that travel time follows a distribution such that expected (path) travel time as well as standard deviation of (path) travel time are continuous and positive functions of flow. Note that Equations (12) and (13) meet this assumption. Therefore, the assumptions of the travel time budget model of Lo et al. (2006) are more restrictive than the assumptions for our model.

The common feature of all models discussed so far is that they consider expected travel time $E(T_k)$ as well as a reliability component, with the reliability component modelled as either travel time margin in Lo et al. (2006) or lateness penalty in Watling (2006).

We observe that both Equations (5) from Lo et al. (2006) and (6) from Watling (2006) with the reformulation (17) contain the standard deviation of travel time σ_{T_k} weighted by either a constant λ_m or the constant θ_2 multiplied by function L, which itself depends on $E(T_k)$ and σ_{T_k}. Clearly, both λ_m and L are user (class) dependent. Recall that λ_m is derived from the level of risk aversion of user m (see Equations (14) and (15)), and that L in (19) contains τ_m as the maximum conceivable travel time of user m as a parameter.

We now postulate that the essential components of travel time reliability equilibrium models are expected travel time $E(T_k)$ and standard deviation of travel time σ_{T_k}. We will not make any further assumptions on how to combine these two factors into a single objective function such as Equations (5) and (19) do. Hence, we do not assume the existence of a value λ_m that allows a weighting of travel time reliability (standard deviation) relative to expected travel time nor do we assume that users make their path choice based on the schedule delay model. Instead, we only assume that users will always want: (1) to minimise the expected travel time, i.e. $\min E\left(T_k\right)$; and (2) to maximise travel time reliability, or alternatively, to minimise the standard deviation of travel time, i.e. $\min \sigma_{T_k}$. Note that based on this assumption, we are modelling users who are either risk neutral or risk averse, but not risk prone. As a result, the value of λ_m will always be greater than zero.

In this way, we consider the problem from a multi-objective point of view and we can formulate a general TTR-BUE model with the two objectives

$$\begin{aligned} &\min E\left(T_k\right), \\ &\min \sigma_{T_k}. \end{aligned} \tag{21}$$

We consider this formulation *general* in the sense that we assume that travellers perceive *unreliability* solely based on the variability of travel time, which is measurable as the standard deviation. The general TTR-BUE condition reads as follows.

> "Under *travel time reliability bi-objective user equilibrium* conditions traffic arranges itself in such a way that no individual trip maker can improve either his/her expected travel time or standard deviation of travel time or both without worsening the other objective by unilaterally switching routes."

Based on this definition, all the used routes between a given O-D pair are *efficient*. For an efficient route, there does not exist any alternative route that has lower expected travel time or lower standard deviation unless the other component is bigger. This means every route *dominated* by an efficient route, i.e. one which has at least the same or higher expected travel time as well as at least the same or higher standard deviation of travel time, as compared with the efficient route should have zero flow. This assumption appears to be realistic for rational users.

Next we give a mathematical statement of the TTR-BUE model as an equilibrium problem. For notational simplicity, we only state it for a single user class. Let us first introduce the necessary notation. Let $G = (N, A)$ be a network, where N is a finite set of $|N|$ nodes and $A \subset N \times N$ is a set of $|A|$ arcs or links. Let $Z \subset N \times N$ be a set of origin-destination pairs (O-D pairs) and for all $p \in Z$, let D_p denote the demand for travel between O-D pair p. The set of all paths between

O-D pair p is denoted K_p and $K := \cup_{pinZ} K_p$ is the set of all paths. Let $\mathbf{F} \in \mathbb{R}^{|K|}$ be a path flow vector that satisfies demand, i.e. $\sum_{k \in K_p} F_k = D_p$ for all $p \in Z$. Finally, let $C_k(\mathbf{F}) := (E(T_k), \sigma_{T_k})^T$ be the vector containing the expected travel time and standard deviation of travel time of path k.

Definition 1. *Path flow vector* $\mathbf{F}$ *is a travel time reliability bi-objective user equilibrium flow if* $\mathbf{F}$ *is feasible, i.e.* $\mathbf{F} \geq 0$, $\sum_{k \in K_p} F_k = D_p$ *for all* $p \in Z$, *and the following conditions hold.*

1. *If for any* $p \in Z$ *and any* $k, k' \in K_p$ *it holds that* $C_{k'}(\mathbf{F}) \leq C_k(\mathbf{F})$ *and* $C_{k'}(\mathbf{F}) \neq C_k(\mathbf{F})$ *then* $F_k = 0$.
2. *If for any* $p \in Z$ *and* $k \in K_p$ *it holds that* $F_k > 0$ *then there is no* $k' \in K_p$ *with* $F_{k'} > 0$ *such that* $C_{k'}(\mathbf{F}) \leq C_k(\mathbf{F})$ *and* $C_{k'}(\mathbf{F}) \neq C_k(\mathbf{F})$.

Notice that the TTB-BUE and LAP-BUE solutions in Sections 2 and 3 are formally defined in the same way as TTR-BUE in Definition 1, but with the cost functions of Equations (16) and (20) rather than (21). We now show that under our assumptions that $E(T_k)$ and σ_{T_k} are positive and continuous functions of flow, travel time reliability bi-objective user equilibrium flows exist.

Theorem 3. *Let* $G = (N, A)$ *be a network,* $Z \subset N \times N$ *be a set of O-D pairs and for all* $p \in Z$, *let* D_p *be the demand of O-D pair* p. *Assume that both cost functions* $C_k^{(i)}(\mathbf{F}), i = 1, 2$ *are positive and continuous. Then a travel time reliability bi-objective user equilibrium flow exists.*

Proof. Because of the assumption that $E(T_k)$ and σ_{T_k} are positive and continuous functions of flow, we know that the time budget function $B_k(\mathbf{F}) := E(T_k) + \lambda \sigma_{T_k}$ for positive λ is positive and continuous. Hence an equilibrium flow $\mathbf{F}^*$ with respect to B_k exists. We show that this equilibrium flow $\mathbf{F}^*$ is a TTR-BUE flow. Assume to the contrary that there is an O-D pair p and two paths $k, k' \in K_p$

with positive flow such that $C_{k'}(\mathbf{F}^*) \leq C_k(\mathbf{F}^*)$ and $C_{k'}(\mathbf{F}) \neq C_k(\mathbf{F})$. Then $B_{k'}(\mathbf{F}^*) < B_k(\mathbf{F}^*)$ contradicting the fact that $\mathbf{F}^*$ is an equilibrium flow with respect to B_k. □

This model can capture all the possible equilibria based on our definition of TTR-BUE without specifying how travellers might respond to the uncertainty in travel time associated with each route as modelled by standard deviation of travel time. We now prove that both the TTB-BUE model (and hence the TTB-UE model) and the LAP-BUE model (and hence the LAP-UE model) are special cases of our new general TTR-BUE model, see Figure 2, which summarises the results of Theorems 1, 2 and 4.

Theorem 4. *The following two statements hold.*

1. *Let* $\mathbf{F}$ *be a path flow solution of the TTB-BUE model. Then* $\mathbf{F}$ *also satisfies the TTR-BUE condition.*
2. *Let* $\mathbf{F}$ *be a path flow solution of the LAP-BUE model. Then* $\mathbf{F}$ *also satisfies the TTR-BUE model.*

Proof. We prove both statements separately.

1. If $\mathbf{F}$ does not satisfy the TTR-BUE condition, there must exist a user class m and two paths k and k' between an O-D pair p such that $E(T_{k'}) \leqslant E(T_k)$ and $\sigma_{T_{k'}} \leqslant \sigma_{T_k}$ with at least one strict inequality. Then, because λ_m is positive in the TTB-BUE model, we must have $E(T_{k'}) + \lambda_m \sigma_{T_{k'}} < E(T_k) + \lambda_m \sigma_{T_k}$. This combined with $E(T_{k'}) \leqslant E(T_k)$ shows that $\mathbf{F}$ would then also violate the TTB-BUE condition.
2. Assume $\mathbf{F}$ satisfies the LAP-BUE but not the TTR-BUE conditions. Then, as in the proof of the first statement, there must exist a user class m and two paths k and k' between an O-D pair p such that $E(T_{k'}) \leqslant E(T_k)$ and

$\sigma_{T_{k'}} \leqslant \sigma_{T_k}$ with at least one strict inequality. It is well known that $L(x)$is a decreasing function of x. Hence $L\left(\frac{\tau_m - E(T_k)}{\sigma_{T_k}}\right)$ increases as both $E(T_k)$ and σ_{T_k} increase and therefore

$$L\left(\frac{\tau_m - E(T_{k'})}{\sigma_{T_{k'}}}\right)\sigma_{T_{k'}} \leqslant L\left(\frac{\tau_m - E(T_k)}{\sigma_{T_k}}\right)\sigma_{T_k},$$

which, with an analogous argument as in the proof of the first statement, together with $E(T_{k'}) \leqslant E(T_k)$ and the fact that at least one of the inequalities must be strict, contradicts the LAP-BUE condition.

□

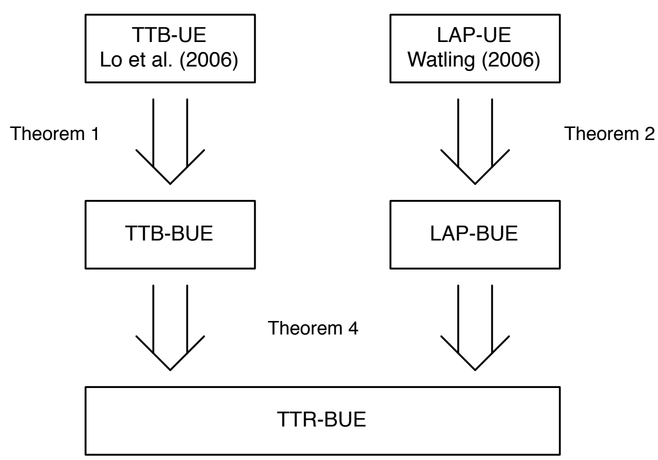

Figure 2: The relationship between single objective and bi-objective user equilibrium models for travel time reliability.

At this stage, we need to point out that the TTR-BUE model is not in itself suitable to derive a particular equilibrium solution, but only serves as a framework, identifying a range of solution within which any equilibrium based on expected travel time and standard deviation of travel time as the route choice criteria must fall. The computation of this range of solutions is difficult, and the development of algorithms to do this is the subject of further research.

5. A Three-link Example

In this section, we demonstrate and validate our concepts with a simple three-link example as follows.

5.1. Network Specification

Our test three-link network is shown in Figure 3, where the link parameters are specified in Table 1. The parameters of the travel time function, Equation (7), are $\beta = 0.15$ and $n = 4$. The total demand is assumed to be fixed at 15,000 vehicles per hour. For simplicity, we consider a single user class.

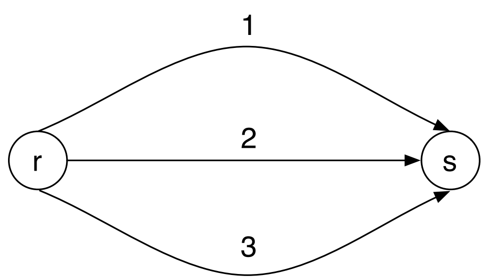

Figure 3: A three-link example network.

Note that in Table 1, we specify a travel time reliability parameter of ϕ_a for route a as defined in Equation (8). The $\phi-$value for the expressway is the lowest, meaning that it is the route that could be most degradable although it is the shortest, while the arterial route is assumed to be the most reliable with the highest $\phi-$value.

5.2. The TTR-BUE Solution Space

As the demand is fixed, the solution space for this three-link network can be represented two-dimensionally with the horizontal axis and the vertical axis rep-

Table 1: Route characteristics of the three-link network.

Route	Type	Distance	Free flow travel time	Capacity	Reliability
a		(km)	(mins)	(veh/hr)	ϕ_a
1	Expressway	20	12	4000	0.5
2	Highway	50	30	5400	0.7
3	Arterial	40	40	4800	0.9

resenting the flows on Routes 1 and 2, respectively. In order to illustrate the set of solutions of the three bi-objective user equilibrium models in this three-link example, we first discretise the two-dimensional solution space and identify the solutions for each of the three cases as formulated in Sections 2, 3 and 4. For each feasible solution, we can evaluate the corresponding travel time and travel time reliability on each of the three routes. We can then determine whether all the three data points are efficient based on the concept illustrated in Figure 1. If all three routes are efficient, the solution is within the BUE region.

5.2.1. Travel Time Budget (TTB) Versus General (TTR) BUE

The solution sets of the TTB-BUE formulation for different levels of risk aversion (with $\rho-$values of 0.8 and 0.9) are compared with that of the general TTR-BUE formulation in Figure 4. As predicted by Theorem 4, comparing Figures 4 (a) & (b) with Figure 4 (c), the TTB-BUE solution sets are within the general TTR-BUE region. By comparing Figures 4 (a) and (b), a higher level of risk aversion leads to a bigger solution set.

5.2.2. *Late Arrival Penalty (LAP) Versus General (TTR) BUE*

The solution sets of the LAP-BUE formulation for different levels of risk aversion (with $\tau-$values of 40 and 50 minutes) are compared with that of the general TTR-BUE formulation in Figure 5. As stated by Theorem 4, comparing Figures 5 (a) & (b) with Figure 5 (c), the LAP-BUE solution sets are within the general TTR-BUE region. By comparing Figures 5 (a) and (b), a higher time allowance leads to a bigger solution set.

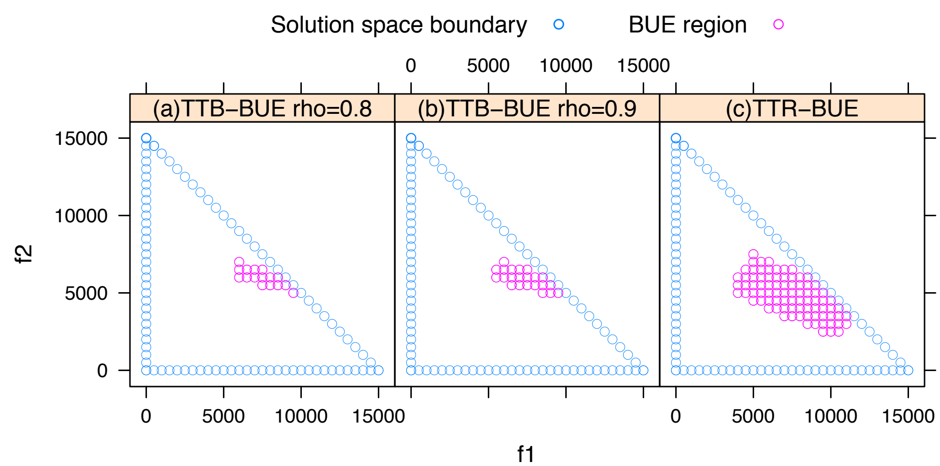

Figure 4: Travel time budget (TTB)-BUE versus general (TTR)-BUE solutions.

5.3. *Travel Time Reliability BUE Versus Travel Time Budget and Late Arrival Penalty UE Models*

To compare our proposed bi-objective model with the single-objective formulations of Lo et al. (2006) and Watling (2006), we first locate the single objective solutions by applying the algorithm in Lo and Chen (2000). The objective function in Lo et al. (2006) is given in Equation (5), i.e.

$$\min B_k = E\left(T_k\right) + \lambda\sigma_{T_k}. \tag{22}$$

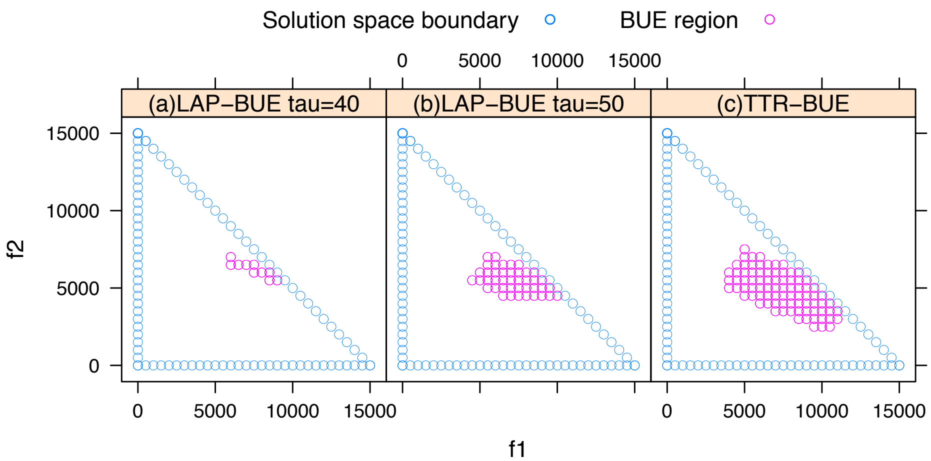

Figure 5: Late arrival penalty (LAP)-BUE versus general (TTR)-BUE solutions.

We tested a range of λ values corresponding to $\rho-$values of 0.50 to 0.95 in steps of 0.05 in Equation (14).

On the other hand, as mentioned before, we simplify the objective function for the LAP-UE formulation in Watling (2006) to include only the two components corresponding to our two objectives in Section 3, i.e. the expected travel time and the late penalty function:

$$\min U_k = E\left(T_k\right) + \theta_2 E\left[\max\left(0, T_k - \tau\right)\right]. \tag{23}$$

Here θ_2 represents the penalty weighting as the relative importance of the schedule delay to the expected travel time. We tested a range of this penalty weighting θ_2 to be between 10 and 50 in steps of 10, i.e. the extent of being late would be 10 to 50 times more important than the expected travel time, with the maximum time fixed at $\tau = 50$ minutes. We also tested a range of the maximum time τ to be between 40 and 50 minutes in steps of one minute, keeping θ_2 constant with value equals 30.

The resulting solutions are depicted in Figure 6. As implied by Theorems 1, 2 and 4, the solutions based on the single-objective formulations are all within the general TTR-BUE model solution set. Each set of parameters in either Lo et al. (2006)'s or Watling (2006)'s formulation corresponds to one identified solution. By varying the model parameters, a curve can be located in the TTR-BUE solution set as the possible solution region for each formulation.

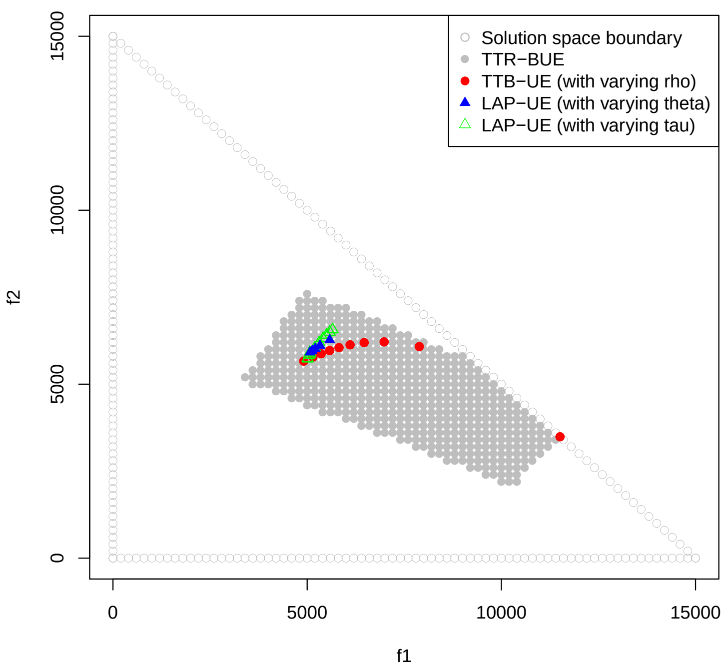

Figure 6: Single-objective solutions in TTR-BUE solution space

6. Conclusion and Outlook

In this paper, we discussed two network equilibrium models for travel time reliability, namely, the travel time budget model (Lo et al., 2006) and the late arrival penalty model (Watling, 2006). We first pointed out some properties and assumptions of these models that may not be realistic. We then adapted the bi-objective user equilibrium formulation of Wang et al. (2010) and proposed bi-objective versions of the two models to overcome the issues outlined before. Next, we elaborated on the common features of the models (namely the use of expected travel time and standard deviation of travel time as reliability measure) and proposed a general travel time reliability bi-objective user equilibrium model. We proved that this model encompasses the single-objective as well as the bi-objective versions of the TTB and LAP user equilibrium models.

The essence of our proposed model is to represent rational route choice behaviour with a BUE model but without a predetermined preference model. Based on the two objectives, the efficient routes become the natural choice set that a rational user will choose from and naturally only routes in this set should have positive flow at equilibrium. The TTR-BUE condition identifies the region that represents possible equilibrium solutions under rational behaviour with no specific preference model such as the additive utility function in Lo et al.'s, Watling's or Dial's model. The advantage of this modelling framework is that it can identify a range of possible solutions under rational behaviour rather than one solution under the assumption of preferences following a restrictive functional form. Once preferences of users are known, a preference model can then be developed that singles out one (or a set) of the solutions satisfying the TTR-BUE conditions as the one that is compatible with the preference model.

Furthermore, if observations show a traffic pattern that does not lie within the TTR-BUE solution set, then it is impossible to find a user preference model based on expected travel time and travel time standard variation that agrees with the observed behaviour. This in turn implies that users do not make decisions based on these criteria, necessitating the consideration of different models of reliability or the inclusion of other criteria, e.g. those related to monetary expenses.

In future research, we will also develop methods to compute the TTR-BUE solution set in general networks. We also intend to extend our work to include the third of the criteria mentioned at the beginning of our paper, namely, monetary cost. Furthermore, we will investigate the use of criteria other than standard deviation to measure reliability of travel time. This will allow us to compare new variants of TTR-BUE equilibrium models with reliability based equilibrium models in the literature as discussed in Section 1. This is of particular interest, because standard deviation/variance may be a convenient, but not necessarily good measure of “risk” in route choice decisions.

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References

Abdel-Aty, M.A., Kitamura, R., Jovanis, P.P., 1995. Investigating effect of travel time variability on route choice using repeated-measurement stated preference data. Transportation Research Record 1493, 39–45.

Bell, M., 2000. A game theory approach to measuring the performance reliability of transport networks. Transportation Research Part B 34 (6), 533–545.

Bell, M., Cassir, C., 2002. Risk-averse user equilibrium traffic assignment: An application of game theory. Transportation Research Part B 36 (8), 671–681.

Brownstone, D., Small, K., 2005. Valuing time and reliability: Assessing the evidence from road pricing demonstrations. Transportation Research Part A 39 (4), 279–293.

Bureau of Public Roads, 1964. Traffic Assignment Manual. U.S. Department of Commerce, Urban Planning Division, Washington D.C.

Chen, A., Ji, Z., 2005. Path finding under uncertainty. Journal of Advanced Transportation 39 (1), 19–37.

Chen, A., Ji, Z., Recker, W., 2002. Travel time reliability with risk sensitive travelers. Transportation Research Record 1783, 27–33.

Chen, A., Zhou, Z., 2010. The α-reliable mean-excess traffic equilibrium model with stochastic travel times. Transportation Research Part B 44 (4), 493–513.

Chen, A., Zhou, Z., Lam, W., 2011. Modeling stochastic perception error in the mean-excess traffic equilibrium model with stochastic travel times. Transportation Research Part B 45 (10), 1619–1640.

Chen, L., Yang, H., 2012. Managing congestion and emissions in road networks with tolls and rebates. Transportation Research Part B 46 (8), 933–948.

Connors, R., Sumalee, A., 2009. A network equilibrium model with travellers perception of stochastic travel times. Transportation Research Part B 43 (6), 614–624.

Daganzo, C.F., Sheffi, Y., 1977. On stochastic models of traffic assignment. Transportation Science 11 (3), 253–274.

Di, S., Pan, C., Ran, B., 2008. Stochastic multiclass traffic assignment with consideration of risk-taking behaviors. Transportation Research Record 2085, 111–123.

Dial, R., 1997. Bicriterion traffic assignment: Efficient algorithms plus examples. Transportation Research Part B 31 (5), 357–379.

Jackson, W., Jucker, J.V., 1982. Empirical study of travel time variability and travel choice behaviour. Transportation Science 16 (4), 460–475.

Lam, T., Small, K., 2001. The value of time and reliability: Measurement from a value pricing experiment. Transportation Research Part E 37 (2-3), 231–251.

Lam, W.H.K., Shao, H., Sumalee, A., 2008. Modeling impacts of adverse weather conditions on a road network with uncertainties in demand and supply. Transportation Research Part B 42 (10), 890–910.

Li, Z., Hensher, D.A., Rose, J.M., 2010. Willingness to pay for travel time reliability in passenger transport: A review and some new empirical evidence. Transportation Research Part E 46 (3), 384 – 403.

Liu, H., Recker, W., Chen, A., 2004. Uncovering the contribution of travel time reliability to dynamic route choice using real-time loop data. Transportation Research Part A 38 (6), 435–453.

Lo, H.K., Chen, A., 2000. Traffic equilibrium problem with route-specific costs: Formulation and algorithms. Transportation Research Part B 34 (6), 493–513.

Lo, H.K., Luo, X.W., Siu, B.W.Y., 2006. Degradable transport network: travel time budget of travellers with heterogeneous risk aversion. Transportation Research Part B 40 (9), 792–806.

Lo, H.K., Tung, Y.K., 2003. Network with degradable links: capacity analysis and design. Transportation Research Part B 37 (4), 345 – 363.

Mirchandani, P., Soroush, H., 1987. Generalized traffic equilibrium with probabilistic travel times and perceptions. Transportation Science 21 (3), 133–152.

Ng, M., Waller, S., 2010. A computationally efficient methodology to characterize travel time reliability using the fast fourier transform. Transportation Research Part B 44 (10), 1202–1219.

Nie, Y., 2011. Multi-class percentile user equilibrium with flow-dependent stochasticity. Transportation Research Part B 45 (10), 1641–1659.

de Palma, A., Picard, N., 2005. Route choice decision under travel time uncertainty. Transportation Research Part A 39 (4), 295–324.

Shao, H., Lam, W., Meng, Q., Tam, M., 2006a. Demand-driven traffic assignment problem based on travel time reliability. Transportation Research Record 1985, 220–230.

Shao, H., Lam, W., Tam, L., 2006b. A reliability-based stochastic traffic assignment model for network with multiple user classes under uncertainty in demand. Networksand Spatial Economics 6, 173–204.

Small, K., 1982. The scheduling of consumer activities: work trips. American Economic Review 72 (3), 467–479.

Szeto, W.Y., O'Brien, L., O'Mahony, M., 2006. Risk-averse traffic assignment with elastic demands: Ncp formulation and solution method for assessing performance reliability. Networks and Spatial Economics 6, 313–332.

Tan, Z., Yang, H., 2012. Flexible build-operate-transfer contracts for road franchising under demand uncertainty. Transportation Research Part B 46 (10), 1419–1439.

Tan, Z., Yang, H., Guo, R., 2013. Pareto efficiency and risk-taking behavior of reliability-based traffic equilibria. Transportation Research Part B this special issue, TBA.

Wang, J.Y.T., Raith, A., Ehrgott, M., 2010. Tolling analysis with bi-objective traffic assignment, in: Ehrgott, M., Naujoks, B., Stewart, T., Wallenius, J. (Eds.), Multiple Criteria Decision Making for Sustainable Energy and Transportation Systems. Springer Verlag, Berlin, pp. 117–129.

Wardrop, J.G., 1952. Some theoretical aspects of road traffic research. Proceedings of the Institution of Civil Engineers: Engineering Divisions 1, 325–362.

Watling, D., 2006. User equilibrium traffic network assignment with stochastic travel times and late arrival penalty. European Journal of Operational Research 175 (3), 1539–1556.

Wu, X., Nie, Y., 2011. Application of discrete fourier transform to find reliable shortest paths. Transportation Research Record 2263, 82–91.

Xu, H., Lou, Y., Yin, Y., Zhou, J., 2011. A prospect-based user equilibrium model with endogenous reference points and its application in congestion pricing. Transportation Research Part B 45 (2), 311–328.

Xu, X., Chen, A., Cheng, L., 2013. Assessing the effects of stochastic perception error under travel time variability. Transportation 40 (3), 525–548.

Yang, H., Wang, X., Yin, Y., 2012. The impact of speed limits on traffic equilibrium and system performance in networks. Transportation Research Part B 46 (10), 1295–1307.

Yin, Y., Ieda, H., 2001. Assessing performance reliability of road networks under non-recurrent congestion. Transportation Research Record 1771, 148–155.

Zhang, C., Chen, X., Sumalee, A., 2011. Robust Wardrop's user equilibrium assignment under stochastic demand and supply: Expected residual minimization approach. Transportation Research Part B 45 (3), 534–552.

Zhou, Z., Chen, A., 2008. Comparative analysis of three user equilibrium models under stochastic demand. Journal of Advanced Transportation 42 (3), 239–263.