Frequency Domain Effects of Constant Terms in Nonlinear Models

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Abstract: When converting a time-domain model of a nonlinear system into the frequency domain to get nonlinear frequency response functions, any constant or dc term in the time-domain model can have a significant effect on the time-to-frequency domain mapping. The constant term can not therefore simply be discarded as in the linear case. This paper investigates this effect and two new algorithms are derived to compute the frequency response functions for two classes of nonlinear models, the polynomial model and the rational model when a constant term is present. The results are illustrated using several examples.

1. Introduction

The generalised frequency response functions (GFRF) of a nonlinear system are important tools for investigating the behaviour of nonlinear systems. These frequency domain representations, which are based on the Volterra series, are a natural extension of the well-known linear frequency response function for linear systems. Classical methods of estimating the GFRF from measured signals of real processes with unknown structure isolate each order of frequency response function and employ multidimensional correlation or FFT techniques together with windowing and smoothing techniques(Schetzen, 1980; Vinh et al 1987; Kim and Powers, 1988; Chua and Liao, 1989 etc). An alternative approach, which is also more powerful for higher order GFRF's, is to identify a nonlinear time-domain model, usually a NARMAX model, and then to use this model to generate the required GFRF's for the underlying system (Billings and Tsang, 1988; Peyton-Jones and Billings, 1989). The higher order GFRF's for rational NARMAX models which represent a large class of nonlinear systems have also been derived recently by Zhang, Billings and Zhu (1993). But all these methods are currently restricted to models without constant terms.

In the linear case the system models, linear differential or difference equations, usually do not include constant terms in the model expressions. This often happens because the constant term can simply be discarded if the static behaviour is not of interest, without affecting the system dynamics. Usually before system identification any non-zero mean value of the measured input or output data can be removed by subtracting off the dc level if the system is known to be linear. This has no effect on the dynamic characteristics of the system. However, this will not be the case for nonlinear systems. Nonlinear systems may produce a d.c. output component even when the input is zero-mean. Therefore, removing the mean level may modify the dynamic information contained in the data and give misleading results. Indeed, it is easy to show that the process of prefiltering the input/output data leaves the system operator unchanged only in the linear case (Peyton-Jones and Billings, 1992). For most nonlinear systems the act of prefiltering will introduce structural bias into the estimates even in the ideal noise free case.

As a consequence, a constant term should always be explicitly estimated as part of the model when the system is nonlinear (Billings and Fadzil, 1985). For linear models the constant term can simply be discarded when converting the model to a transfer function since this will not affect the frequency domain characteristics. But in the nonlinear case discarding the constant term will affect the frequency domain characteristics (Peyton-Jones and Billings, 1993). In this study, the frequency domain effects of the constant term contained in nonlinear models is examined and a new modified algorithm which generates higher order GFRF for nonlinear rational models is derived for the case when the time domain model contains a constant term. Practical aspects of the modified algorithm are discussed and a comparison with previous work is included together with several illustrative examples.

2. Harmonic Expansion With a Constant Term Included

Often for simplicity of analysis it is assumed that the system steady state response to a zero input is also zero. For this reason the Volterra series normally excludes a constant term. For the case when the system has non-zero steady state for zero input, the model expression should contain a constant term such as

$$M(t;\, \theta,\, \mathbf{y},\mathbf{u}) + c_0 = 0 \tag{1}$$

It is necessary therefore to include a constant term in the Volterra series in order to accommodate the mean level of the output. This can be achieved by adding a degree-0 term, y_0 into the Volterra series. The output can then be expressed as

$$y(t) = \sum_{n=0}^{N} y_n(t)$$

$$= y_0 + \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i \tag{2}$$

for which the nth order frequency response function is defined as the multiple Fourier transform of the nth order Volterra kernel $h_n(\cdot)$

$$H_n(j\omega_1, \ldots, j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n)\, e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, d\tau_1 \cdots d\tau_n \tag{3}$$

Because the so-defined $H_n(\cdot)$ are not unique in the sense that changing the order of the arguments may give different values of $h_n(\cdot)$ but will still yield the same output $y_n(t)$, it is common practice to define a symmetrised function by summing all the asymmetric functions over every possible permutation of the arguments and divided by the number $n!$ to give

$$H_n^{sym}(j\omega_1, \ldots, j\omega_n) = \frac{1}{n!} \sum_{\substack{\text{all permutations} \\ \text{of } \omega_1 \cdots \omega_n}} H_n(j\omega_1, \ldots, j\omega_n) \tag{4}$$

This symmetric GFRF is then unique and independent of the order of the arguments (Bussgang, Ehrman and Graham, 1974; Chua and Ng, 1979a; 1979b).

By applying the harmonic input

$$u(t) = \sum_{r=1}^{R} e^{j\omega_r t} \tag{5}$$

to the Volterra model eqn.(2), the output $y(t)$ is then given as

$$y(t) = y_0 + \sum_{n=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} \sum_{r=1}^{R} e^{j\omega_r(t-\tau_i)}\, d\tau_i$$

$$= y_0 + \sum_{n=1}^{N} \sum_{r_1, r_n=1}^{R} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} e^{j\omega_{r_i}(t-\tau_i)}\, d\tau_i$$

$$= y_0 + \sum_{n=1}^{N} \sum_{r_1, r_n=1}^{R} \left[\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} e^{-j\omega_{r_i}\tau_i}\, d\tau_i \right] e^{j(\omega_{r_1} + \ldots + \omega_{r_n})t}$$

$$= H_0 + \sum_{n=1}^{N} \sum_{r_1, r_n=1}^{R} H_n(j\omega_{r_1}, \cdots, j\omega_{r_n})\, e^{j(\omega_{r_1} + \cdots + \omega_{r_n})t} \tag{6}$$

where H_0 is associated with the 0 order output y_0 which is independent of any input. Eqn.(6) shows that $H_n(j\omega_{r_1}, \cdots, j\omega_{r_n})$ can be recognised as the nonlinear transfer function relating the output at intermodulation frequency $(\omega_{r_1} + \cdots + \omega_{r_n})$ to inputs at frequencies $\omega_{r_1}, \cdots, \omega_{r_n}$. Clearly the value of $H_n(j\omega_{r_1}, \cdots, j\omega_{r_n})$ along $\omega_{r_1} + \cdots + \omega_{r_n} = 0$ will result in a d.c. component (zero frequency) in the output, this is in addition to y_0, even with a zero-mean input. Substituting eqn's (5) and (6) into the model expression eqn.(1) yields

$$M(t;\, \theta, H, \omega_r) + c_0 = 0 \tag{7}$$

where $H \equiv \{H_0, H_1, \cdots, H_N\}$. The above expression consists of various complex exponential terms and $H_n(j\omega_1, \cdots, \omega_n)$ will only appear as the coefficient of the non-repetitious frequency exponential $e^{j(\omega_1 + \ldots + \omega_n)t}$. Following the procedure introduced in the previous paper by Zhang, Billing and Zhu (1993) the operator $\mathcal{E}_n[\cdot]$ will be used to denote the operation of extracting the coefficient of a particular term $e^{j(\omega_1 + \ldots + \omega_n)t}$ from the argument expression. Notice n starts at 0 now ($n = 0,1,\ldots,N$) and $\mathcal{E}_0[\cdot]$ is associated with the constant term, or d.c. component. Because the above equation holds for all time t and arbitrarily chosen variables $\{\omega_1, \ldots, \omega_n\}$ the coefficient of the $e^{j(\omega_1 + \ldots + \omega_n)t}$ term contained in the harmonic expression (7) should be zero. That is

$$\mathcal{E}_n\left[M(t;\, \theta, H, \omega_r) + c_0\right] = 0 \qquad \text{for } n = 0,1,\ldots,N \tag{8}$$

$H_n(\cdot)$ can now be obtained by solving eqn.(8) with $R = n$. In the presence of a constant term, it is necessary to re-consider the effects of various terms since these may be different from the case without the constant term, as well as the case $n = 0$.

To illustrate the application of the harmonic expansion method used in these circumstances, consider for example a difference equation model which contains a constant term

$$y(t) = a_1 y(t-1) + bu(t-1) + a_2 y^2(t-1) + c \tag{9}$$

The above model with $c \equiv 0$ has been used by Billings and Tsang(1989) and it has been shown that H_1 and H_2 are given by

$$H_1(j\omega) = \frac{be-j\omega}{1-a_1e^{-j\omega}}, \quad \text{and} \quad H_2(j\omega_1,j\omega_2) = \frac{a_2H_1(j\omega_1)H_1(j\omega_2)e^{-j(\omega_1+\omega_2)}}{1-a_1e^{-j(\omega_1+\omega_2)}}$$

respectively. If $c \neq 0$ H_1 and H_2 can be re-derived using the harmonic expansion method by applying the input consisting of two input exponentials,

$$u(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \tag{10}$$

The harmonic response, $y(t;\, H,\omega)$, when this input is applied to the Volterra model, is given from eqn.(2) by,

$$\begin{aligned} y(t;\, H,\omega_r) = {} & y_0 + H_1(j\omega_1)\, e^{j\omega_1 t} + H_1(j\omega_2)\, e^{j\omega_2 t} + \\ & 2!\, H_2^{sym}(j\omega_1,j\omega_2)\, e^{j(\omega_1+\omega_2)t} + H_2^{sym}(j\omega_1,j\omega_1)\, e^{2j\omega_1 t} + H_2^{sym}(j\omega_2,j\omega_2)\, e^{2j\omega_2 t} \end{aligned} \tag{11}$$

where all terms of order higher than two are ignored.

The same response can also be obtained by substituting the harmonic input $u(t)$ in (10) and $y(t)$ in (11) into the original difference equation (9) to yield

$$\begin{aligned} y(t;\, H,\omega_r) = {} & c + b\left[e^{j\omega_1(t-1)} + e^{(j\omega_2)(t-1)}\right] \\ & + a_1\left[\begin{matrix} y_0 + 2!\, H_2^{sym}(j\omega_1,j\omega_2)\, e^{j(\omega_1+\omega_2)(t-1)} + H_2^{sym}(j\omega_1,j\omega_1)\, e^{2j\omega_1(t-1)} \\ + H_2^{sym}(j\omega_2,j\omega_2)\, e^{2j\omega_2(t-1)} + H_1(j\omega_1)\, e^{j\omega_1(t-1)} + H_1(j\omega_2)\, e^{j\omega_2(t-1)} \end{matrix}\right] \\ & + a_2\left[\begin{matrix} y_0 + 2!\, H_2^{sym}(j\omega_1,j\omega_2)\, e^{j(\omega_1+\omega_2)(t-1)} + H_2^{sym}(j\omega_1,j\omega_1)\, e^{2j\omega_1(t-1)} \\ + H_2^{sym}(j\omega_2,j\omega_2)\, e^{2j\omega_2(t-1)} + H_1(j\omega_1)\, e^{j\omega_1(t-1)} + H_1(j\omega_2)\, e^{j\omega_2(t-1)} \end{matrix}\right]^2 \end{aligned} \tag{12}$$

Equating the constant terms across equations (11) and (12) yields

$$y_0 = a_1y_0 + a_2y_0^2 + c \tag{13}$$

and the zero-input response y_0 can be solved from the above equation. The first order frequency response function is obtained by equating the coefficients of either the term $e^{j\omega_1 t}$ or $e^{j\omega_2 t}$ across equations (11) and (12). Comparing the coefficients of $e^{j\omega_1 t}$, for instance, yields

$$H_1(j\omega_1) = b\cdot e^{-j\omega_1} + a_1H_1(j\omega_1)e^{-j\omega_1}$$

$$+ a_2[y_0 H_1(j\omega_1)e^{-j\omega_1} + H_1(j\omega_1)e^{-j\omega_1}y_0\,] \tag{14}$$

hence

$$H_1(j\omega_1) = \frac{be^{-j\omega_1}}{1 - a_1e^{-j\omega_1} - 2a_2y_0e^{-j\omega_1}} = \frac{be^{-j\omega_1}}{1-(a_1+2a_2y_0\,)e^{-j\omega_1}} \tag{15}$$

The second order frequency response function follows in the similar fashion. By equating coefficients of $e^{j(\omega_1+\omega_2)t}$ across equations (11) and (12) the desired second order response is obtained as

$$2!\,H_2^{sym}(j\omega_1,j\omega_2) = 2!a_1H_2^{sym}(j\omega_1,j\omega_2)\,e^{-j(\omega_1+\omega_2)} + 2a_2H_1(j\omega_1)H_1(j\omega_2)\,e^{-j(\omega_1+\omega_2)}$$
$$a_2y_0 2\,H_2^{sym}(j\omega_1,j\omega_2)\,e^{-j(\omega_1+\omega_2)} + a_2 2H_2^{sym}(j\omega_1,j\omega_2)\,e^{-j(\omega_1+\omega_2)} \tag{16}$$

so that

$$H_2^{sym}(j\omega_1,j\omega_2) = \frac{a_2H_1(j\omega_1)H_1(j\omega_2)\,e^{-j(\omega_1+\omega_2)}}{1-(a_1+2a_2y_0)\,e^{-j(\omega_1+\omega_2)}} \tag{17}$$

Comparing eqn.(15) and (17) with the results obtained previously by Billings and Tsang(1989) in which the constant term c was not considered, both H_1 and H_2 have changed. This simple example clearly illustrates the fact that if a constant term is present this should not be discarded from the model if the system is nonlinear. Indeed, the constant term may affect the location of the poles of the nonlinear systems and therefore it is necessary to modify the algorithm for computing the higher order GFRF to incorporate the effects of constant terms.

3. Properties of $\mathcal{E}_n[\cdot]$ in the Presence of a Constant Term

When the nonlinear model contains a constant term the mapping from the time-domain model to the frequency-domain GFRF is affected. In other words, the way in which the frequency response functions are generated will be changed in the presence of a constant term. This change is reflected in the properties of the operator $\mathcal{E}_n[\cdot]$. These are summarised in following remarks.

<u>Remark 1</u>: The operator $\mathcal{E}_n$ is a linear operator and this is the same as in the case of

no constant term. That is

$$\mathcal{E}_n\Big[c_1\mathrm{M}_1(\cdot)+c_2\mathrm{M}_2(\cdot)\Big]=c_1\mathcal{E}_n\Big[\mathrm{M}_1(\cdot)\Big]+c_2\mathcal{E}_n\Big[\mathrm{M}_2(\cdot)\Big] \tag{18}$$

Remark 2: While $\mathcal{E}_n$ represents the coefficient of all the $e^{j(\omega_1+\ldots+\omega_n)t}$ terms contained in the harmonic expansion, it would be convenient to introduce $\mathcal{E}_n^{asym}$ to denote coefficients of any individual $\mathcal{E}_n$ (there are many according to the permutation of $\{\omega_1,\ldots,\omega_n\}$). Therefore

$$\mathcal{E}_n[\cdot] = \sum_{\substack{all\ permutations\\ of\ \omega_1\cdots\omega_n}} \mathcal{E}_n^{asym}[\cdot] \tag{19}$$

Since $H_n^{sym}(\cdot)$ is also defined by taking the average of any asymmetric function over all possible permutations of its arguments (see eqn.(4)) an asymmetric GFRF $H_n^{asym}(\cdot)$ can be obtained by considering only the asymmetric coefficient of $e^{j(\omega_1+\ldots+\omega_n)t}$, that is $\mathcal{E}_n^{asym}$.

Remark 3: All the polynomial type terms contained in the model can still be classified into three classes: pure input, pure output and the input/output cross-product terms. The constant term can be treated as a zero degree pure output term. The effect of each type of term on $\mathcal{E}_n$ can then be analysed individually.

Pure Input Nonlinearities: The contribution to $\mathcal{E}_n$ arising from pure nonlinearities in the input is unchanged and is given by

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{K} u(t-k_i)\right] = \begin{cases} e^{-j(\omega_1 k_1+\cdots+\omega_n k_n)} & K=n \\ 0 & K\neq n \end{cases} \tag{20}$$

Pure Output Nonlinearities: The effect of the additional constant term occurs mainly in the contribution from various terms involving the output $y(t)$. Using the Volterra series eqn.(6) a lagged output is expressed as

$$y(t-k_i) = \sum_{\gamma=0}^{N}\alpha^{\gamma}\sum_{r_1,r_\gamma=1}^{R} H_\gamma(j\omega_{r_1},\cdots,j\omega_{r_\gamma})\; e^{j(\omega_{r_1}+\cdots+\omega_{r_\gamma})(t-k_i)} \tag{21}$$

where γ is used to remember the number of frequencies appearing as the arguments of $H_i(\cdot)$ as well as in the exponential functions, and $\alpha=1$ is a dummy variable used to keep track of all terms with the same degree (number of frequencies). Here the influence of the constant term is taken into account when $\gamma=0$.

The p degree pure output nonlinearities can then be expressed as

$$\prod_{i=1}^{p} y(t-k_i) = \prod_{i=1}^{p} \sum_{\gamma=0}^{N} \alpha^{\gamma} \sum_{r_1,r_\gamma=1}^{R} H_\gamma(j\omega_{r_1}, \cdots j\omega_{r_\gamma})\, e^{j(\omega_{r_1}+\cdots+\omega_{r_\gamma})(t-k_i)} \tag{22}$$

$$= \sum_{\gamma_1,\gamma_p=0}^{N} \alpha^{\gamma_1+\ldots+\gamma_p} \prod_{i=1}^{p} \sum_{r_1,r_\gamma=1}^{R} H_{\gamma_i}(j\omega_{r_1},\ldots,j\omega_{r_{\gamma_i}})\, e^{j(\omega_{r_1}+\ldots+\omega_{r_{\gamma_i}})(t-k_i)}$$

From the power of the dummy variable α, it is seen that eqn.(22) contains terms from order 0 up to Np. This is the major difference compared with the result obtained previously for the case without the constant component (i.e. when γ starts from 1) where the terms run from order p up to Np. This suggests that in the absence of a constant term a p-th order nonlinearity in the output contributes only to the Volterra transfer functions of order p and above but with the addition of a constant term the output nonlinearity contributes to all Volterra transfer functions including those of order less than p. This consequence may be seen more clearly by dividing the leftmost summation of eqn.(22) into terms of like order n, giving,

$$\prod_{i=1}^{p} y(t-k_i) = \sum_{n=0}^{Np} \alpha^n \sum_{\substack{\gamma_1,\gamma_p=0 \\ \mid \sum\gamma_i=n}}^{n} \prod_{i=1}^{p} \sum_{r_1,r_\gamma=1}^{R} H_{\gamma_i}(j\omega_{r_1}, \cdots, j\omega_{r_{\gamma_i}})\, e^{j(\omega_{r_1}+\ldots+\omega_{r_{\gamma_i}})(t-k_i)} \tag{23}$$

where the constraint that $\sum\gamma_i=n$ also lowers the limit N to n.

Now the coefficient of $e^{j(\omega_1+\ldots+\omega_n)t}$ can be extracted from the right hand side of the eqn.(23) in a similar way as before. The difference only occurs at the lower and upper limits of the summations. An 'asymmetric' coefficient of $e^{j(\omega_1+\ldots+\omega_n)t}$ is given by

$$\varepsilon_n^{asym}\left[\prod_{i=1}^{p} y(t-k_i)\right] = \sum_{\substack{\gamma_1,\gamma_p=0 \\ \mid \sum\gamma_i=n}}^{n} \prod_{i=1}^{p} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots j\omega_{r_{X+\gamma_i}})\, e^{-j(\omega_{r_{X+1}}+\cdots+\omega_{r_{X+\gamma_i}})k_i} \tag{24}$$

where $X = \sum\gamma_x$, $x = 1..i-1$. For the sake of simplicity denote

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{p} y(t-k_i)\right] = H_{n,p}^{asym}(\cdot) \tag{25}$$

But here $H_{n,p}^{asym}$ is different from that when the constant term is not considered. In the same way as before $H_{n,p}^{asym}$ can also be computed in recursive form by expanding the last term of the product to yield

$$H_{n,p}^{asym}(\cdot) = \sum_{\gamma_p=0}^{n} H_{\gamma_p}(j\omega_{n-\gamma_p+1}, \cdots j\omega_n)\, e^{-j(\omega_{n-\gamma_p+1}+\cdots+\omega_n)k_p} \tag{26}$$

$$\times \sum_{\substack{\gamma_1,\gamma_{p-1}=0 \\ \sum\gamma_i=n-\gamma_p}}^{(n-\gamma_p)} \prod_{i=1}^{p-1} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots, j\omega_{r_{X+\gamma_i}})\, e^{-j(\omega_{r_{X+1}}+\cdots+\omega_{r_{X+\gamma_i}})k_i}$$

$$= \sum_{\gamma_p=0}^{n} H_{\gamma_p}(j\omega_{n-\gamma_p+1}, \cdots j\omega_n)\, e^{-j(\omega_{n-\gamma_p+1}+\cdots+\omega_n)k_p}\, H_{n-\gamma_p,p-1}^{asym}(j\omega_1, \cdots, j\omega_{n-\gamma_p})$$

The above equation can be written more conveniently using new subscripts, and a different (asymmetric) permutation of frequencies, as,

$$H_{n,p}^{asym}(\cdot) = \sum_{i=0}^{n} H_i^{asym}(j\omega_1, \cdots, j\omega_i) H_{n-i,p-1}(j\omega_{i+1}, \cdots, j\omega_n)\, e^{-j(\omega_1+\cdots+\omega_i)k_p} \tag{27}$$

The recursion also finishes at $p=1$ (although the computation starts from 0), and $H_{n,1}(j\omega_1, \cdots, j\omega_n)$ is given by

$$H_{n,1}(j\omega_1, \cdots, j\omega_n) = H_n(j\omega_1,\ldots,j\omega_n)\, e^{-j(\omega_1+\cdots+\omega_n)k_1} \tag{28}$$

Notice that the right hand side of eqn.(27) contains H_0, H_1 up to H_n. This suggests that a p degree pure output term may well produce or contribute to any order of GFRF's ranging from 0 to n, including those of lower order than p. Fortunately there is no higher than n order GFRF included in the recursive formulae, otherwise it would be impossible to obtain the solution recursively starting from lower order to higher order. Also it is important to know that eqn.(27) holds for $n=0$ and $p=0$. The case $n=0$ implies the contribution to the constant term such as

$$\mathcal{E}_0\left[\prod_{i=1}^{p} y(t-k_i)\right] = H_{0,p} = H_0^p$$

which suggests that a p degree pure output term will generate a constant term H_0^p. This result is fairly obvious after discarding the dynamics of $y(t)$. The case $p=0$

implies the constant term is included in the model expression. In the harmonic expansion of the model, the constant term will make no contribution to any $\mathcal{E}_n$ term (that is, will not generate a term $e^{j(\omega_1+...+\omega_n)t}$ with $n>0$), except $\mathcal{E}_0$, hence

$$\mathcal{E}_n[\,c_0] = H_{n,0} = \begin{cases} 0 & \text{for } n>0 \\ H_0 & \text{for } n=0 \end{cases}$$

Apart from the initial value $H_{n,1}$ given in eqn.(28), some other special cases can also be derived.

$$H_{1,p}(j\omega) = H_0^{p-1} H_1(j\omega) \times \sum_{i=1}^{p} e^{-j\omega k_i}$$

$$H_{2,2}(j\omega_1,j\omega_2) = H_0 H_2(j\omega_1,j\omega_2) \times \sum_{i=1}^{2} e^{-j(\omega_1+\omega_2)k_i} + H_1(j\omega_1)H_1(j\omega_2)e^{-j(\omega_1 k_2+\omega_2 k_1)}$$

$$H_{2,3}(j\omega_1,j\omega_2) = H_0^2 H_2(j\omega_1,j\omega_2) \times \sum_{i=1}^{3} e^{-j(\omega_1+\omega_2)k_i} + H_0 H_1(j\omega_1)H_1(j\omega_2)[e^{-j(\omega_1 k_2+\omega_2 k_1)}$$

$$+ e^{-j(\omega_2 k_1+\omega_1 k_3)} + e^{-j(\omega_2 k_2+\omega_1 k_3)}]$$

and so on. Clearly expanding the recursions for the higher order cases is complex but fortunately the computations can be implemented with relative ease in a computer programme and the values can be generated automatically.

Input/Output Cross-Products: Having obtained expressions for the contribution generated by pure input and pure output nonlinearities, the contribution from pure input/output cross-product terms is relatively easy to evaluate. The previous analysis on the effect of pure cross-product terms still holds, the contribution from such terms will be jointly produced by the input part and the output part. This can be written as

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i)\right] = \mathcal{E}_{n-q}^{asym}\left[\prod_{i=1}^{p} y(t-k_i)\right] \cdot \mathcal{E}_q^{asym}\left[\prod_{i=p+1}^{p+q} u(t-k_i)\right] \quad (29)$$

Substituting eqn.(20) and (25) yields

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i)\right] = e^{-j(\omega_{n-q+1}k_{p+1} + ... + \omega_n k_{p+q})} H_{n-q,p}(j\omega_1,...,j\omega_{n-q}) \quad (30)$$

where the exponential factor relates to the input part of the nonlinearity, and the recursive factor $H_{n-q,p}(\cdot)$ to the output part. Because the p degree pure output in the presence of d.c. term can produce a $e^{j(\omega_1+...+\omega_n)t}$ term of whatever order, eqn.(29) and (30) hold true even for $p+q>n$ which would be zero in the absence of a constant term. But $\mathcal{E}_n[\cdot] = 0$ if $q>n$ since q degree pure input terms can not produce a $e^{j(\omega_1+...+\omega_n)t}$ with $n<q$.

In summary, the contributions from all three types of polynomial terms in the presence of a constant term are given as

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{K} u(t-k_i)\right] = \begin{cases} e^{-j(\omega_1 k_1+\cdots+\omega_n k_n)} & K=n \\ 0 & K\neq n \end{cases} \tag{31}$$

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{K} y(t-k_i)\right] = \begin{cases} H_{n,p}(j\omega_1,...,j\omega_n) & K\leq n \\ 0 & K>n \end{cases} \tag{32}$$

and

$$\mathcal{E}_n^{asym}\left[\prod_{i=1}^{p} y(t-k_i)\prod_{i=p+1}^{p+q} u(t-k_i)\right] =$$

$$\begin{cases} \mathcal{E}_{n-q}^{asym}\left[\prod_{i=1}^{p} y(t-k_i)\right]\cdot\mathcal{E}_q^{asym}\left[\prod_{i=p+1}^{p+q} u(t-k_i)\right] & p+q\leq n \\ 0 & p+q>n \end{cases} \tag{33}$$

where $H_{n,p}$ is given recursively by eqn.(27)

$$H_{n,p}^{asym}(\cdot) = \sum_{i=0}^{n} H_i^{asym}(j\omega_1,\cdots,j\omega_i)H_{n-i,p-1}(j\omega_{i+1},\cdots,j\omega_n)\, e^{-j(\omega_1+\cdots+\omega_i)k_p} \tag{34}$$

Again notice that this is different from the case when a constant term is not considered in the model.

<u>Remark 4</u> In the presence of a constant term, nonlinear terms with m'th degree of nonlinearity may produce an $e^{j(\omega_1+...+\omega_n)t}$ term with $n<m$ in the harmonic expansion. In other words, an m'th order nonlinear term in $y(t)$ and $u(t)$ may contribute directly to $\mathcal{E}_n$ with $n<m$. This is not the case when a constant term is not considered.

Remark 5 (Extracting H_n) By inspecting the effect of the three types of terms, it is found that after applying the operator $\mathcal{E}_n$ only the pure output terms will generate $H_n(\cdot)$. All the other terms can only produce lower order $H_i(\cdot)$, $i<n$. In eqn.(24) the terms which contain H_n are most easily divided out by expanding the uppermost values of γ in the multiple summation of eqn.(24). This gives

$$H_{n,p}^{asym}(\cdot) = H_0^{p-1}\; H_n(j\omega_1,...,j\omega_n) \times \sum_{i=1}^{p} e^{-j(\omega_1+...+\omega_n)k_i} + $$

$$\sum_{\substack{\gamma_1,\gamma_p=0 \\ |\sum\gamma_i=n}}^{n-1} \prod_{i=1}^{p} H_{\gamma_i}(j\omega_{r_{X+1}}, \cdots, j\omega_{r_{X+\gamma_i}})\; e^{-j(\omega_{r_{X+1}}+\cdots+\omega_{r_{X+\gamma_i}})k_i}$$

$$= H_0^{p-1}\; H_n(j\omega_1, \cdots, j\omega_n) \times \sum_{i=1}^{p} e^{-j(\omega_1+\cdots+\omega_n)k_i} + \bar{H}_{n,p}(\cdot) \tag{35}$$

Thus the contribution from degree-p nonlinearity in the output can be divided into two parts, one part containing H_n and another part containing only the lower order H_i, $i<n$. $\bar{H}_{n,p}$ is used to denote the second part which is simply $H_{n,p}$ given in eqn.(27) but excluding all the terms containing $H_n(\cdot)$. In programming this can be implemented by using the same recursive relation eqn.(27) but setting $H_n \equiv 0$. From the analysis in Remark 3, it is easy to conclude that

$$\bar{H}_{1,p} = 0$$

$$\bar{H}_{n,1} = 0$$

and

$$\bar{H}_{2,2}(j\omega_1, j\omega_2) = H_1(j\omega_1) H_1(j\omega_2)\; e^{-j(\omega_1 k_2 + \omega_2 k_1)}$$

All those terms containing H_n can be brought over to the left hand side of eqn.(8) and be computed recursively from $n=1$.

4. Algorithm Derivation for the Polynomial Model

Before deriving the GFRF for the nonlinear rational model with a constant term consider initially the polynomial model which is a subset of the rational model. By means of a traditional probing method, Peyton-Jones and Billings (1993) have derived a mapping from the time-domain to the frequency domain for the polynomial NARX model in the presence of a constant term. In this section it is shown that the same mapping can be easily derived by using the operator $\mathcal{E}_n[\cdot]$ defined above. The computation of the zero order GFRF H_0 will also be discussed. The expression for the general class of nonlinear polynomial models containing a constant is given by

$$y(t) = \sum_{m=0}^{M} \sum_{p=0}^{m} \sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1, \cdots, k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \tag{36}$$

Each term is seen to contain a p-th order factor in $y(t-k_i)$ and a q-th order factor in $u(t-k_i)$ (such that $p+q=m$), and each is multiplied by a coefficient $c_{p,q}(k_1, \cdots, k_{p+q})$, while the multiple summation over the k_i, $(k_i = 1, \cdots, K)$, generates all the possible permutations of lags which might appear in these terms. All the terms can be divided into three main types as usual. Applying the linearity of $\mathcal{E}_n$ and the results presented in Remarks 1-5 above yields

$$H_n(\cdot) = \sum_{k_1,k_n=1}^{K} c_{0,n}(k_1, \cdots, k_n)\, e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)}$$

$$+ \sum_{q=1}^{n} \sum_{p=1}^{M-1} \sum_{k_1,k_n=1}^{K} c_{p,q}(k_1, \ldots, k_{p+q})\, e^{-j(\omega_{n-q+1}k_{p+1} + \ldots + \omega_n k_{p+q})} H_{n-q,p}(j\omega_1, \ldots, j\omega_{n-q})$$

$$+ \sum_{p=1}^{n} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p) \left[H_0^{p-1}\, H_n(\cdot) \times \sum_{i=1}^{p} e^{-j(\omega_1 + \ldots + \omega_n)k_i} + \overline{H}_{n,p}(\cdot) \right]$$

Bringing over all the terms which contain H_n to the left hand side of the equation gives the following modified algorithm for computing the n'th order GFRF ($n > 0$) for the polynomial model with a constant term as

$$\left[1 - \sum_{p=1}^{M} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p)\, H_0^{p-1} \times \sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i} \right] H_n^{asym}(j\omega_1, \cdots, j\omega_n) =$$

$$\sum_{k_1,k_n=1}^{K} c_{0,n}(k_1, \cdots, k_n)\, e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)}$$

$$+ \sum_{q=1}^{n} \sum_{p=1}^{M-1} \sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1, \ldots, k_{p+q})\, e^{-j(\omega_{n-q+1}k_{p+1} + \ldots + \omega_n k_{p+q})} H_{n-q,p}(j\omega_1, \ldots, j\omega_{n-q})$$

$$+ \sum_{p=2}^{M} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \cdots, k_p)\, \overline{H}_{n,p}(j\omega_1, \cdots, j\omega_n) \tag{37}$$

where the recursive relations $H_{n,p}$ are given by eqn.(27) and $\overline{H}_{n,p}$ in eqn.(35) is a subset of $H_{n,p}$ which excludes all terms containing H_n. The symmetric GFRF can be obtained by taking the average of all the permutations of an asymmetric GFRF obtained above.

It is important to point out that eqn.(37) is only valid for H_n with $n \geq 0$. In order to get the zero order kernel H_0 consider the input to be zero corresponding to the case $R=n=0$. This follows since H_0 is independent of any input. In eqn.(8) only the d.c. component arising from each term after substituting $u=0$ and $y=H_0$ needs to be considered. For the standard polynomial model, or NARX model for example it is clear that only the pure output terms can make contributions to the d.c. component after the substitutions. So that

$$H_0 - c_0 - \sum_{p=1}^{M} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1,\ldots,k_p)H_0^p = 0 \tag{38}$$

In the above equation H_0 is the only unknown which can therefore be obtained by solving the equation. Notice that the equation may have many roots but only the real roots will make sense. If there is more than one real root for eqn.(38) this suggests that the underlying system has multiple steady states. In comparison, linear systems can only have one steady state.

Using the above algorithm to derive H_1 and H_2 for the previous example given by eqn.(9), identical results are obtained. For this specific example the parameters specified in eqn.(36) are

$$M=2; \quad K=1; \quad c_{1,0}(1) = a_1; \quad c_{0,1}(1) = b;$$

$$c_{2,0}(1,1) = a_2; \quad c_0 = c; \quad else \quad c_{p,q} = 0.$$

Then

$$[1 - a_1 e^{-j\omega_1 k_1} - a_2 H_0(e^{-j\omega_1 k_1} + e^{-j\omega_1 k_2})]H_1(j\omega_1) = be^{-j\omega_1}$$

$$\Rightarrow \quad H_1(j\omega_1) = \frac{be^{-j\omega_1}}{1-(a_1+2a_2y_0)e^{-j\omega_1}}$$

$$[1 - a_1 e^{-j(\omega_1+\omega_2)k_1} - a_2 H_0(e^{-j(\omega_1+\omega_2)k_1} + e^{-j(\omega_1+\omega_2)k_2})]H_2(j\omega_1, j\omega_2) = a_2 H_1(j\omega_1)\overline{H}_{1,1}(j\omega_2)e^{-j\omega_1 k_1}$$

$$\Rightarrow H_2^{sym}(j\omega_1,j\omega_2) = \frac{a_2 H_1(j\omega_1)H_1(j\omega_2)\, e^{-j(\omega_1+\omega_2)}}{1-(a_1+2a_2y_0)\,e^{-j(\omega_1+\omega_2)}}$$

5. GFRF for Nonlinear Rational Models With a Constant in the Numerator

For a nonlinear rational model, which is defined as the ratio of two polynomials, the constant term can be present either as an isolated d.c. output term or contained in the numerator polynomial. Notice that the former case can be included in the latter case when the d.c. output is merged into the numerator/denominator form. Only the latter case will therefore be considered. Consider a general form of rational model

$$y(t) = \frac{\mathbf{Y}_a(t;\theta_a,\mathbf{y},\mathbf{u})}{\mathbf{Y}_b(t;\theta_b,\mathbf{y},\mathbf{u})} \tag{39}$$

where $\mathbf{Y}_a(t;\theta_a,\mathbf{y},\mathbf{u})$ and $\mathbf{Y}_b(t;\theta_b,\mathbf{y},\mathbf{u})$ are used to denote polynomials in the numerator and denominator, respectively, which are defined as

$$\mathbf{Y}_a(t;\theta_a,\mathbf{y},\mathbf{u}) = \sum_{m=0}^{M_a}\left[\sum_{p=0}^{m}\ \sum_{k_1,k_{p+q}=1}^{K_a} \alpha_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p} y(t-k_i)\prod_{i=p+1}^{p+q} u(t-k_i)\right] \tag{40}$$

and

$$\mathbf{Y}_b(t;\theta_b,\mathbf{y},\mathbf{u}) = \sum_{m=0}^{M_b}\left[\sum_{p=0}^{m}\ \sum_{k_1,k_{p+q}=1}^{K_b} \beta_{p,q}(k_1,\cdots,k_{p+q})\prod_{i=1}^{p} y(t-k_i)\prod_{i=p+1}^{p+q} u(t-k_i)\right] \tag{41}$$

where M_a and M_b are the maximum degrees of nonlinearities, K_a and K_b are the maximum lags in the input and output, $\alpha(\cdot)$ and $\beta(\cdot)$ are the parameters associated with the various terms in the two polynomials (corresponding to the parameter sets θ_a and θ_b respectively), $p+q = m$; and

$$\sum_{k_1,k_n=1}^{K} \equiv \sum_{k_1=1}^{K}\cdots\sum_{k_n=1}^{K}$$

Notice that the lower limits on the first summations of $\mathbf{Y}_a$ and $\mathbf{Y}_b$ are zero, which implies that both the denominator and numerator may include a non-zero constant term ($\alpha_{0,0}$ and $\beta_{0,0}$, respectively). Now decomposing the two polynomials as pure input terms, pure output terms, cross-product terms and constant terms, the rational model eqn.(39) can be denoted as

$$y(t) = \frac{\alpha_{0,0} + \mathbf{Y}_a(t;\, \theta_u, \mathbf{u}) \;+\; \mathbf{Y}_a(t;\, \theta_y, \mathbf{y}) \;+\; \mathbf{Y}_a(t;\, \theta_{uy}, \mathbf{y},\mathbf{u})}{\beta_{0,0} + \mathbf{Y}_b(t;\, \theta_u, \mathbf{u}) \;+\; \mathbf{Y}_b(t;\, \theta_y, \mathbf{y}) \;+\; \mathbf{Y}_b(t;\, \theta_{uy}, \mathbf{y},\mathbf{u})}$$

After multiplying out the model becomes

$$\beta_{0,0}y(t) + y(t)\mathbf{Y}_b(t;\, \theta_u, \mathbf{u}) + y(t)\mathbf{Y}_b(t;\, \theta_y, \mathbf{y}) + y(t)\mathbf{Y}_b(t;\, \theta_{uy}, \mathbf{y},\mathbf{u})$$

$$- \mathbf{Y}_a(t;\, \theta_u, \mathbf{u}) - \mathbf{Y}_a(t;\, \theta_y, \mathbf{y}) - \mathbf{Y}_a(t;\, \theta_{uy}, \mathbf{y},\mathbf{u}) - \alpha_{0,0} = 0$$

which is the standard form given by eqn.(1). All the terms in the above model can still be classified into three parts. The pure output parts, namely, $\beta_{0,0}y(t)$, $\mathbf{Y}_a(t;\, \theta_y, y)$ and $y(t)\mathbf{Y}_b(t;\, \theta_y, y)$ contain the unknown nth order GFRF $H_n(\cdot)$ and these need to be extracted:

i) The term to be extracted from $\beta_{0,0}y(t)$ is

$$\beta_{0,0}\, H_n(\cdot)$$

ii) The terms to be extracted from $\mathbf{Y}_a(t;\, \theta_y, y)$ are

$$\sum_{p=1}^{M_a} \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1, \cdots, k_p)\, H_0^{p-1}\, H_n(\cdot) \times \sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i}$$

iii) The terms to be extracted from $y(t)\mathbf{Y}_b(t;\, \theta_y, y)$ are

$$\sum_{p=1}^{M_b} \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1, \cdots, k_p)\, H_0^{p}\, H_n(\cdot) \times \left[\sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i} + 1 \right]$$

The effects of all the other terms will remain basically in the same form except for some modifications to the summation limits. After moving all the terms containing H_n over to the left hand side of the equation, the final algorithm, for computing the nth order GFRF for the general class of nonlinear rational model with a constant term takes the form

$$\left[\beta_{0,0} - \sum_{p=1}^{M_a} \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1, \cdots, k_p)\, H_0^{p-1} \times \sum_{i=1}^{p} e^{-j(\omega_1 + \cdots + \omega_n)k_i} + \right.$$

$$\left. \sum_{p=1}^{M_b} \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1,\ldots,k_p)\, H_0^{p} \times \left[\sum_{i=1}^{p} e^{-j(\omega_1 + \ldots + \omega_n)k_i} + 1 \right] \right] H_n^{asym}(j\omega_1,\ldots,j\omega_n) =$$

$$\sum_{k_1,k_n=1}^{K_a} \alpha_{0,n}(k_1,\ldots,k_n)\, e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)}$$

$$+\sum_{p=2}^{M_a}\ \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1,\cdots,k_p)\ \overline{H}_{n,p}(\cdot)$$

$$+\sum_{q=1}^{n}\ \sum_{p=1}^{M-1}\ \sum_{k_1,k_{p+q}=1}^{K_a} \alpha_{p,q}(k_1,\ldots,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{p+1}+\ldots+\omega_n k_{p+q})}\ H_{n-q,p}(\cdot)$$

$$-\sum_{q=1}^{n}\ \sum_{k_1,k_p=1}^{K_b} \beta_{0,q}(k_1,\ldots,k_q)e^{-j(\omega_{n-q+1}k_1+\ldots+\omega_n k_q)}H_{n-q}(\cdot)$$

$$-\sum_{p=1}^{M_b}\ \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1,\cdots,k_p)\overline{H}^{asym}_{n,p+1}(\cdot)$$

$$-\sum_{q=1}^{n}\ \sum_{p=1}^{M_b-1}\ \sum_{k_1,k_{p+q}=1}^{K_b} \beta_{p,q}(k_1,\ldots,k_{p+q})\ e^{-j(\omega_{n-q+1}k_{p+1}+\ldots+\omega_n k_{p+q})}\ \overline{H}_{n-q,p+1}(\cdot) \quad (42)$$

where the recursive relations $H_{n,p}$ and $\overline{H}_{n,p}$ are given by eqn's (27) and (35), respectively. Also notice that k_{p+1} which occurs in $\overline{H}_{n,p+1}$ is zero because it is associated with $y(t)$ where the lag is zero.

Finally the initial value H_0 can be obtained by setting the input to zero and substituting the steady state output $y_0 = H_0$ into the model (discarding all the dynamics). H_0 can then be computed from the resultant steady state equation.

$$\beta_{0,0}H_0 - \alpha_{0,0} - \sum_{p=1}^{M_a}\ \sum_{k_1,k_p=1}^{K_a} \alpha_{p,0}(k_1,\ldots,k_p)\ H_0^p + \sum_{p=1}^{M_b}\ \sum_{k_1,k_p=1}^{K_b} \beta_{p,0}(k_1,\ldots,k_p)\ H_0^{p+1} = 0 \quad (43)$$

It is seen that $\alpha_{0,0}$ is the only constant term in the above equation. If $\alpha_{0,0} = 0$, that is, if the numerator polynomial does not contain a constant, H_0 or one solution of will be zero and the whole algorithm eqn.(42) will reduce to the procedure derived earlier in Zhang, Billings and Zhu (1993). Notice that more than one real solution may exist for H_0. This simply implies that the underlying system possesses multiple steady states. It is important to emphasise that the Volterra series model can not exactly match the behaviour of such multiple steady state systems, although it is always possible to use Volterra series to approximate the system near one specific steady state or equilibrium point.

6. Examples

The model which is analysed as example E_1 was identified from an unknown non-linear circuit in a sealed box representing different structures of a single degree of freedom system. The input/output data of the system were sampled at 1600 Hz and after structure detection, nonlinear identification, and model validation, the following model was obtained (for more details see Billings, Tsang and Tomlinson, 1990)

$$y(t) = 0.13597 + 1.6021y(t-1) - 0.94726y(t-2) + 0.061490u(t-1)$$
$$- 0.013829y(t-1)y(t-1) - 0.0025225y(t-1)y(t-1)y(t-1) \quad (44)$$

Clearly a constant term is included in the model. When the input $u(t)$ is zero, the steady state form of the model is

$$0.0025225\, y_0^3 + 0.013829\, y_0^2 + 0.3452\, y_0 - 0.13597 = 0$$

The above polynomial equation has three roots at 0.3874 and $-2.9349 \pm j\ 11.4241$. The output mean level, or the degree-zero kernel of the system is therefore the real root

$$H_0 = y_0 = 0.3874$$

If the constant term is discarded from the model then the frequency response functions H_1 and H_2 would be

$$H_1(j\omega) = \frac{0.06149\, e^{-j\omega}}{1 - 1.6021e^{-j\omega} + 0.94726e^{-2j\omega}} \quad (45)$$

and

$$H_2(j\omega_1, j\omega_2) = -\frac{0.013829H_1(j\omega_1)H_1(j\omega_2)\, e^{-j(\omega_1+\omega_2)}}{1 - 1.6021e^{-j(\omega_1+\omega_2)} + 0.94726e^{-2j(\omega_1+\omega_2)}} \quad (46)$$

The first order GFRF $H_1(j\omega)$ is plotted as the solid line in Fig.1(a) and (b). The system has a bandpass type characteristic with a resonant peak at a normalised frequency of 0.96 (corresponding to 154 Hz). The gain and phase of the second order GFRF $H_2(j\omega_1, j\omega_2)$ is given in Fig.2 (a) and (b). However, if the constant term is taken into account, using the modified algorithm eqn.(42) then H_1 and H_2 will be given as

$$[1 - 1.6021e^{-j\omega} + 0.94726e^{-2j\omega} + 0.013829H_0 \cdot 2e^{-j\omega} + 0.0025225H_0^2 \cdot 3e^{-j\omega}]H_1(j\omega)$$
$$= 0.06149\, e^{-j\omega} \quad (47)$$

hence

$$H_1(j\omega) = \frac{0.06149\, e^{-j\omega}}{1 - [1.6021 + 0.027658H_0 + 0.0075675H_0^2]e^{-j\omega} + 0.94726e^{-2j\omega}} \quad (48)$$

and in the same way

$H_2(j\omega_1, j\omega_2)\ =$

$$= \frac{0.013829H_1(j\omega_1)H_1(j\omega_2)\, e^{-j(\omega_1+\omega_2)}}{1 - [1.6021 + 0.027658H_0 + 0.0075675H_0^2 \cdot 3]e^{-j(\omega_1+\omega_2)} + 0.94726e^{-2j(\omega_1+\omega_2)}} \quad (49)$$

where $H_0 = 0.3874$. Comparing eqn.(48) and (49) with eqn.(45) and (46) shows that the denominator, or the poles of H_1 and H_2 are changed because of the effect of the constant term, because $H_0 \neq 0$. The new $H_1(j\omega)$ is plotted as a dotted line in Fig.1(a) and (b). It is seen that the frequency response plot shifts slightly towards the low frequencies because of the constant term. The shape of H_2 has also changed, although only slightly.

It should be pointed out that the mean level of the output or zero input steady state response is not very significant for this specific system and therefore the effect of the constant term appears to be small. But if the system had a larger output mean level the effect would be significant and can not simply be ignored. Consider the case where the constant term 0.13597 is replaced by a larger term 2.0 say. The output mean level would then be obtained by solving the steady state equation

$$0.0025225\, y_0^3 + 0.013829\, y_0^2 + 0.3452\, y_0 - 2 = 0$$

From which the real root is found as 4.3976. Substituting H_0=4.3976 into eqn's (48) and (49) yields the new H_1 and H_2. H_1 is plotted as a dashed-line in Fig.3 while H_2 is plotted in Fig.4. It is seen that the frequency response functions have been significantly affected.

Consider the denominator, or characteristic equation of H_1 when a constant term is present

$$1 - [1.6021 + 0.013829H_0 \cdot 2 + 0.0025225H_0^2 \cdot 3]e^{-j\omega} + 0.94726e^{-2j\omega} = 0$$

If the constant term is ignored, that is H_0=0, eqn.(49) has two conjugate roots (poles) at

$$z_{1,2} = 0.8011 \pm j0.5528 = 0.9733e^{\pm 34.6}$$

which has an undamped natural frequency ω_n given by

$$\omega_n = \cos^{-1}[\frac{1+r^2}{2r}\cos\omega_0]\cdot\omega_s = 965.7 \ rad/s = 153.7 \ Hz$$

But when a constant term c=0.13597 is considered, the poles move to

$$z_{1,2} = 0.8077 \pm j0.5441 = 0.9733 \ e^{\pm 33.9}$$

with a natural frequency ω_n = 948.3 rad/sec. or 151 Hz. Furthermore, if c=2.0 is considered, the poles locate at

$$z_{1,2} = 0.9350 \pm j0.2702 = 0.9733e^{\pm 16.1}$$

and the natural frequency becomes ω_n=448 rad/sec. or 71.3 Hz. The pole locations are marked in Fig.5 within the unit circle. The constant term contained in the system model can therefore move the pole locations of the system and this can certainly be a significant effect.

As a further example consider a simple nonlinear rational model which is expressed as

$$y(t) = \frac{a_1 y(t-1) + a_2 u(t-1) y(t-1) + c}{1 + b_1 y^2(t-1)}$$

Using the recursive formulae eqn.(42) yields

$$H_1(j\omega) = \frac{a_0 e^{-j\omega}}{1 - a_1 e^{-j\omega} - b_1 H_0^2 (1+2e^{-j\omega})} = \frac{a_0 e^{-j\omega}}{(1-b_1 H_0^2) - (a_1+2b_1 H_0^2)e^{-j\omega}}$$

and

$$H_2(j\omega_1, j\omega_2) = \frac{a_2 e^{-j\omega_2} H_{1,1}(j\omega_1)}{(1-b_1 H_0^2) - (a_1+2b_1 H_0^2)e^{-j(\omega_1+\omega_2)}}$$

$$= \frac{a_2 H_1(j\omega_1) e^{-j(\omega_1+\omega_2)}}{(1-b_1 H_0^2) - (a_1+2b_1 H_0^2)e^{-j(\omega_1+\omega_2)}}$$

The degree-0 kernel $H_0 = y_0$ can be obtained by solving the zero-input steady state equation as follows

$$y_0(1 + b_1 y_0^2) = a_1 y_0 + c$$

7. Conclusions

When identifying a nonlinear system from measured input/output data, a constant term often appears in the resulting nonlinear model. This reflects the zero-input response of the underlying system. In this study it has been shown that the constant term in the nonlinear system model may have a significant effect on the frequency domain behaviour of the system by for example altering the system poles. Therefore constant terms can not simply be ignored. Many important conclusions formulated assuming a zero constant term do not hold when such a term is present. Consequently previously developed algorithms to compute higher order frequency response functions from nonlinear time-domain models do not apply in this case. New modified algorithms which avoid these restrictions were therefore derived in this paper for the nonlinear polynomial and the nonlinear rational model. Although the new algorithm is more complicated compared to earlier versions with no constant term it is suitable for a wider class of nonlinear models, and includes the previous algorithms as a special case.

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References

BILLINGS, S.A., TSANG, K.M., AND TOMLINSON, G.R., (1988). *Application of the NARMAX Method to Nonlinear Frequency Response Estimation*, 6th IMAC, Orlando.

BILLINGS, S.A., TSANG, K.M., AND TOMLINSON, G.R., (1990). "Spectral Analysis for Nonlinear Systems. Part III - Case Study Examples," *J. Mechanical Systems and Signal Processing*, vol. 4, no. 1, pp. 3-21.

BOYD, S., TANG, Y.S., AND CHUA, L.O., (1983). "Measuring Volterra Kernels," *IEEE Trans. CAS-30*, no. 8, pp. 571-577.

BUSSGANG, J.J., EHRMAN, L., AND GRAHAM, J.W., (1974). "Analysis of Nonlinear Systems with Multiple Inputs," *Proc. IEEE*, vol. 62, pp. 1088-1119.

CHUA, L.O. AND NG, C.Y., (1979). "Frequency Domain Analysis of Nonlinear Systems: General Theory," *IEE Journal Electronic Circuits and Systems*, vol. 3, no. 4, pp. 165-185.

CHUA, L.O. AND LIAO, Y., (1989). "Measuring Volterra kernels II," *Int. J. Circuits Theory and Appl. (UK)*, vol. 17, pp. 151-190.

PEYTON JONES, J.C. AND BILLINGS, S.A., (1989). "A Recursive Algorithm for Computing the Frequency Response of a Class of Nonlinear Difference Equation Models," *Int J. Control*, vol. 50, no. 5, pp. 1925-1940.

PEYTON JONES, J.C. AND BILLINGS, S.A., (1993). "Mean levels in nonlinear analysis and identification," *Int. J. Control*, vol. 58, no. 5, pp. 1033-52.

SCHETZEN, M., (1980). *The Volterra & Weiner Theories of Nonlinear Systems*, J. Wiley & Sons Inc, New York.

VINH, T., CHOUYCHAI, T., LIU, H., AND DJOUDER, M., (1987). *Second Order Transfer Function: Computation and Physical Interpretation*, 5th IMAC, London.

VOLTERRA, V., (1930). *Theory of Functionals and of Integral and Integrodifferential Equations*, Blackie and Sons Ltd., London.

ZHANG, H., BILLINGS, S.A., AND ZHU, Q.M., (1993). "Frequency response functions for nonlinear rational models," *Int. J. Control* , (to be published).

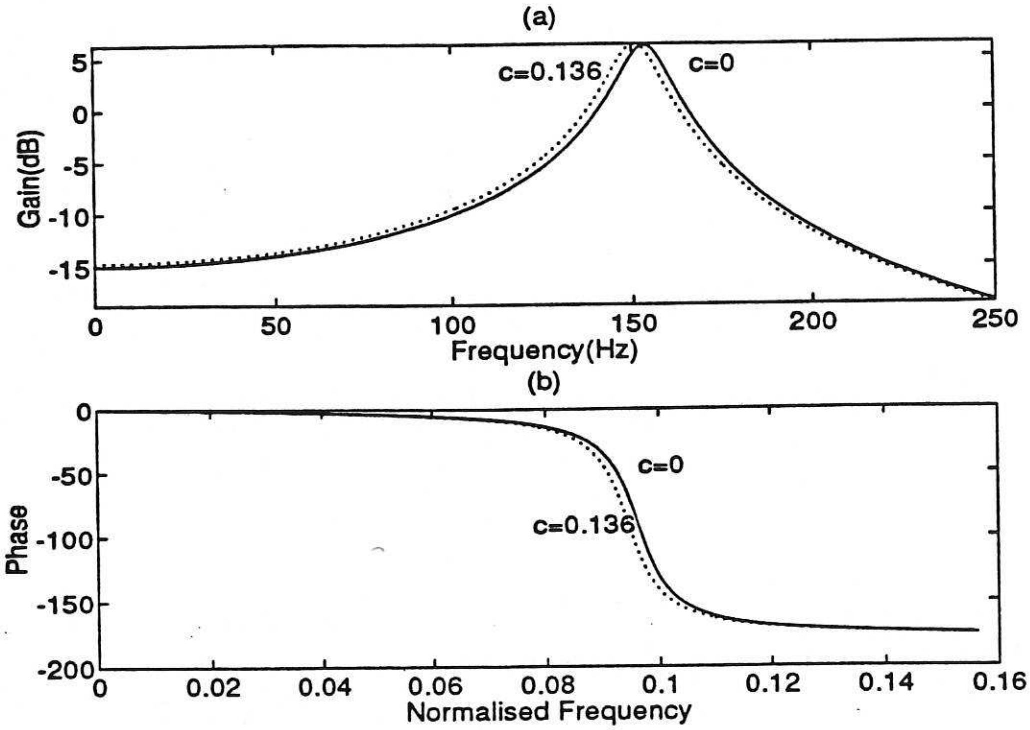

Fig.1 The gain and phase of H_1 of the example E_1 with c=0 and c=0.136.

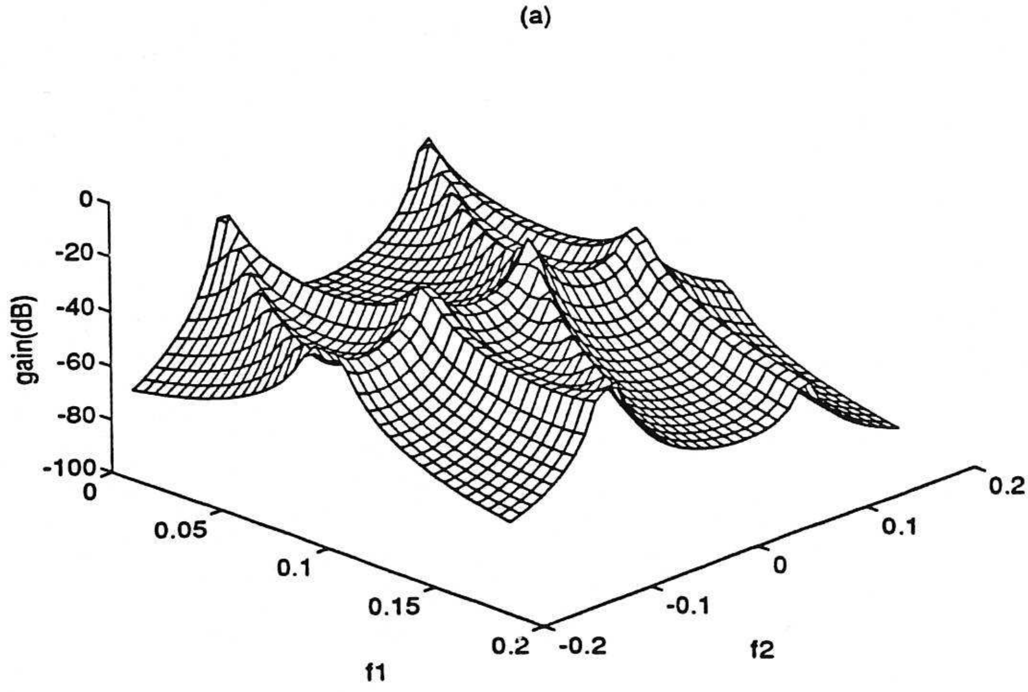

Fig.2(a) The gain of H_2 of the example E_1 with c=0.

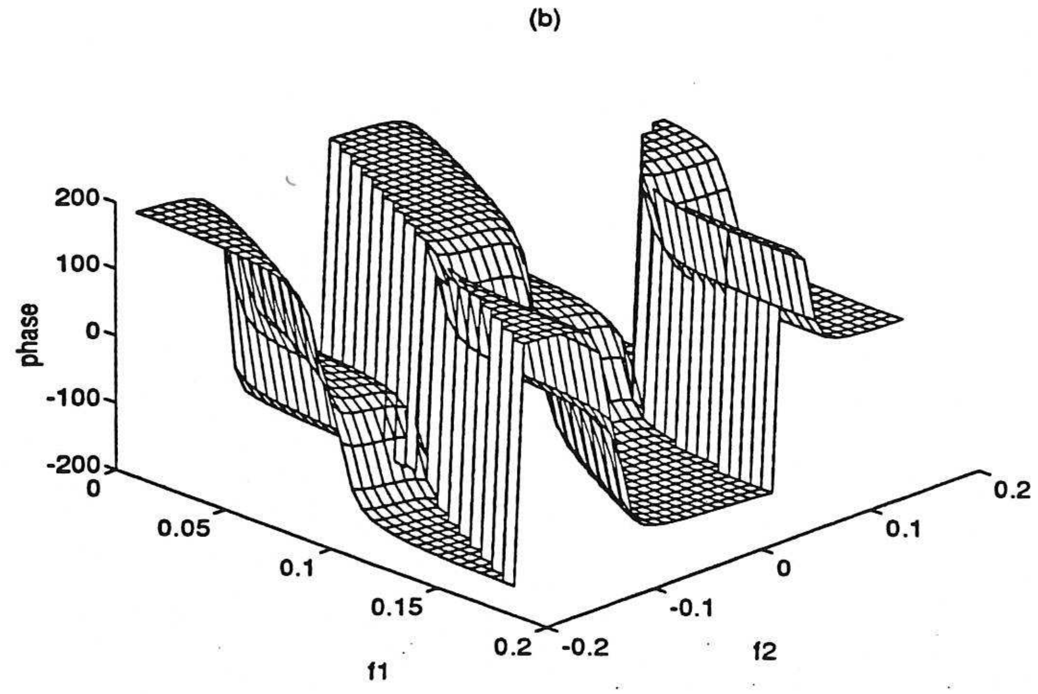

Fig.2b The phase of H_2 of the example E_1 with c=0.

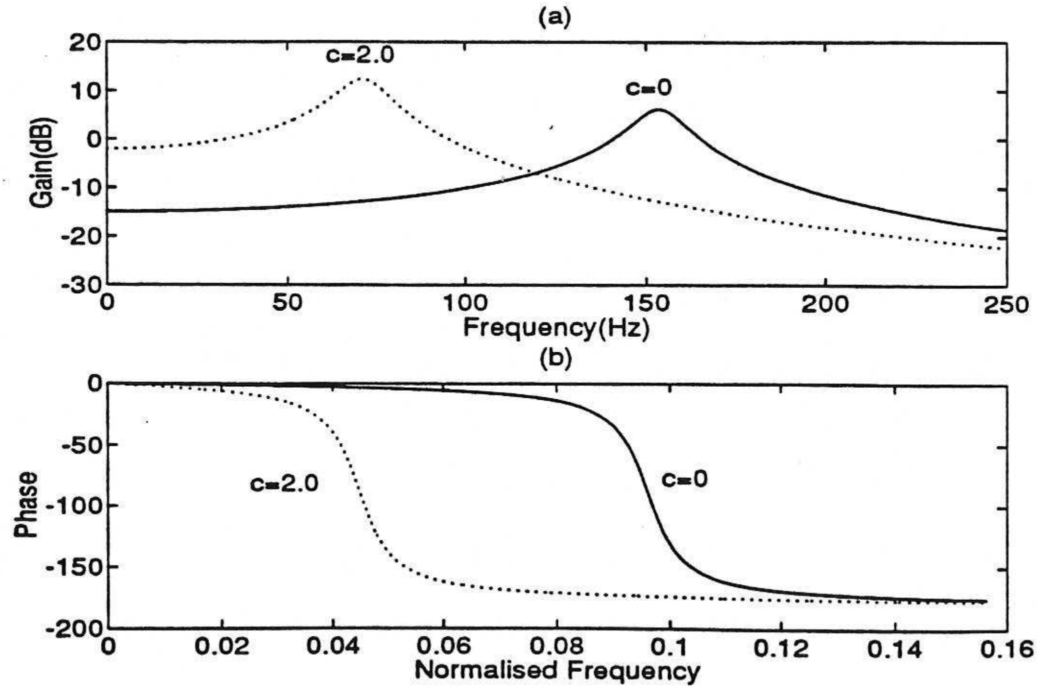

Fig.3 The gain and phase of H_1 of the example E_1 with larger constant c=2.

(a)

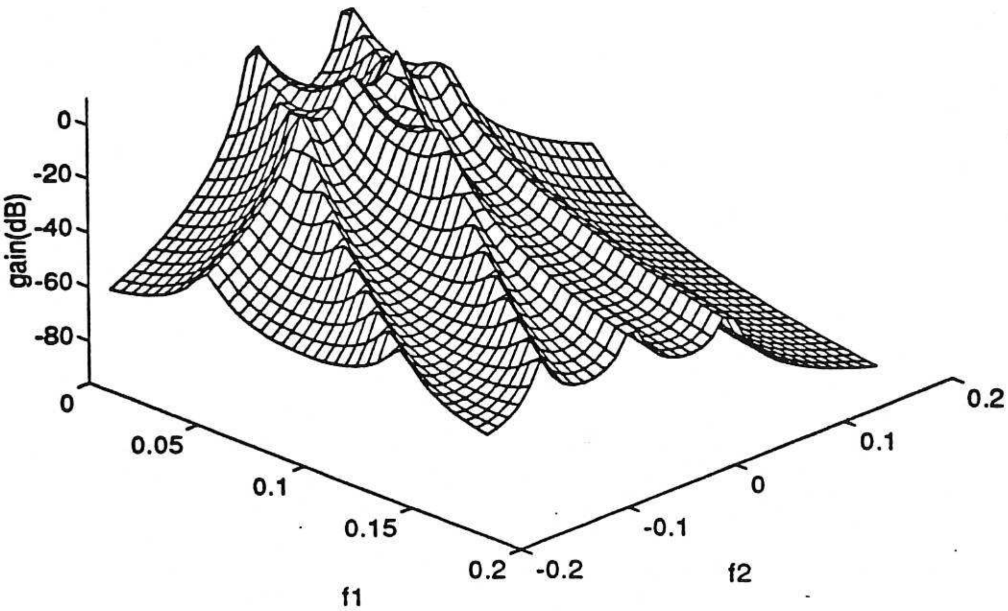

Fig.4(a) The gain of H_2 of the example E_1 with constant term c=2.0.

(b)

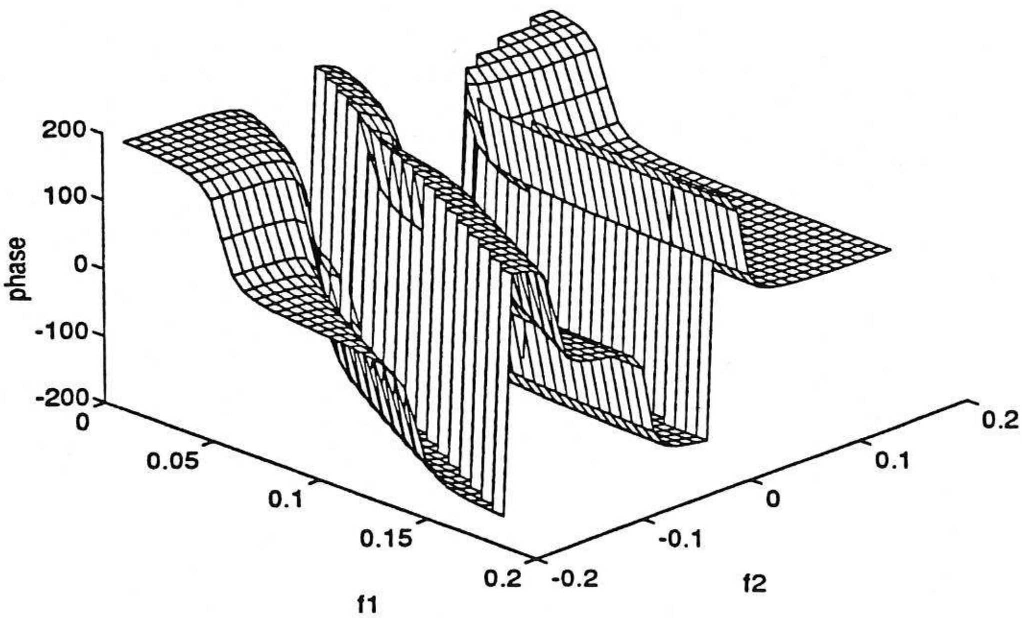

Fig.4(b) The phase of H_2 of the example E_1 with constant term c=2.0.

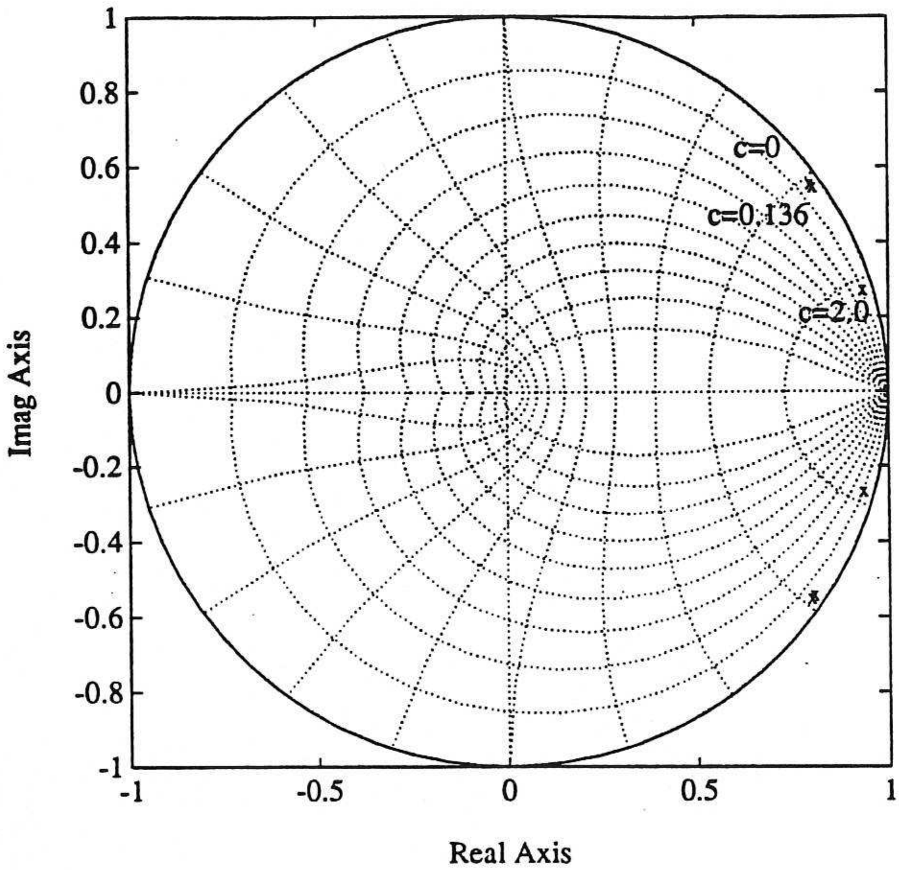

Fig.5 The movement of the systems poles caused by constant terms.