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The Jacobian algebras

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Abstract

A new class of algebras (the Jacobian algebras) is introduced and studied in detail. The Jacobian algebras are obtained from the Weyl algebras by inverting (not in the sense of Ore) of certain elements. Surprisingly, the Jacobian algebras and the Weyl algebras have little in common. Moreover, they have almost opposite properties.

Key Words: the Jacobian algebras, prime ideal, prime spectrum, unique factorization of ideal, minimal primes, group of units, commutant, integro-differential operators.

Mathematics subject classification 2000: 16D25, 16S99, 16U60, 16U70, 16S60.

1 Introduction

The aim of this paper is to introduce a new class of algebras, the so-called *Jacobian algebras*. They arose in my study of the group of polynomial automorphisms and the Jacobian Conjecture, which is a conjecture that makes sense only for polynomial algebras in the class of all commutative algebras [6]. In order to solve the Jacobian Conjecture, it is reasonable to believe that one should create technique which makes sense only for polynomials; the Jacobian algebras are a step in this direction (they exist for polynomials but make no sense even for Laurent polynomials). In this Introduction, we describe the main results of the paper.

Throughout, ring means an associative ring with 1. Let K be a commutative $\mathbb{Q}$-algebra, K^* be its group of units, $P_n := K[x_1, \ldots, x_n]$ be a polynomial algebra over K; $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ be the partial derivatives (K-linear derivations) of P_n.

Definition: The *Jacobian algebra* $\mathbb{A}_n$ is the subalgebra of $\mathrm{End}_K(P_n)$ generated by the Weyl algebra $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n\rangle$ and the elements $H_1^{-1}, \ldots, H_n^{-1} \in \mathrm{End}_K(P_n)$ where $H_1 := \partial_1 x_1, \ldots, H_n := \partial_n x_n$.

Clearly, $\mathbb{A}_n = \mathbb{A}_1(1) \otimes \cdots \otimes \mathbb{A}_1(n) \simeq \mathbb{A}_1^{\otimes n}$ where $\mathbb{A}_1(i) := K\langle x_i, \partial_i, H_i^{-1}\rangle \simeq \mathbb{A}_1$. The algebra $\mathbb{A}_n$ contains all the integrations $\int_i : P_n \to P_n$, $p \mapsto \int p\, dx_i$, since $\int_i = x_i H_i^{-1}$. In particular, the algebra $\mathbb{A}_n$ contains all (formal) integro-differential operators with polynomial coefficients. This fact explains (i) the significance of the algebras $\mathbb{A}_n$ for Algebraic Geometry and the theory of integro-differential operators; (ii) why the algebras $\mathbb{A}_n$ and A_n have different properties, and (iii) why the group $\mathbb{A}_n^*$ of units of the algebra $\mathbb{A}_n$ is huge (there are many invertible integro-differential operators).

General properties of the Jacobian algebras.

Until the end of this section, K is a field of characteristic zero. When $n=1$ the group $\mathbb{A}_1^*$ is found explicitly, $\mathbb{A}_1^*\simeq K^*\times (\mathbb{Z}^{(\mathbb{Z})}\ltimes \mathrm{GL}_\infty(K))$ (Theorem 4.2) as well as an inversion formula u^{-1} for $u\in \mathbb{A}_1^*$. This gives explicitly polynomial solutions for *all invertible* integro-differential operators on an affine line: $uy=f\Rightarrow y=u^{-1}f$ where $f\in K[x_1]$ and y is an unknown. For $n\geq 2$, a description of the group $\mathbb{A}_n^*$ is given (Theorem 4.4), it looks like it is a challenging problem to find an inversion formula for $u\in \mathbb{A}_n^*$ (one should go far beyond the Dieudonné determinant). Though, a criterion of invertibility is found (Theorem 4.5). Moreover, the group $\mathbb{A}_n^*$ contains the subgroup $K^*\times ((\mathbb{Z}^n)^{(\mathbb{Z})}\ltimes \mathrm{GL}_\infty(K))$ elements of which are called *minimal integro-differential operators*. For each such an operator u one can write down an inversion formula u^{-1} in the same manner as in the case $n=1$, and, therefore, one obtains explicitly polynomial solutions for all minimal integro-differential equations $uy=f$ where $f\in P_n$.

The Weyl algebra $A_n=A_n(K)$ is a simple, Noetherian domain of Gelfand-Kirillov dimension $\mathrm{GK}\,(A_n)=2n$. The Jacobian algebra $\mathbb{A}_n$ is neither left nor right Noetherian, it contains infinite direct sums of nonzero left and right ideals. This means that the concept of the left (and right) Krull dimension makes no sense for $\mathbb{A}_n$ but the classical Krull dimension of $\mathbb{A}_n$ is n (Corollary 3.7). The algebra $\mathbb{A}_n$ is a central, prime algebra of Gelfand-Kirillov dimension $3n$ (Corollary 2.7).

The canonical involution θ of the Weyl algebra can be extended to the algebra $\mathbb{A}_n$ (see (15)). This means that the algebra $\mathbb{A}_n$ is *self-dual* ($\mathbb{A}_n\simeq \mathbb{A}_n^{op}$), and so its left and right algebraic properties are the same. Note that the *Fourier transform* on the Weyl algebra A_n can not be lifted to $\mathbb{A}_n$. Many properties of the algebra $\mathbb{A}_n=\mathbb{A}_1^{\otimes n}$ are determined by properties of $\mathbb{A}_1$. When $n=1$ we usually drop the subscript '1' in x_1, ∂_1, H_1, etc. The algebra $\mathbb{A}_1$ contains the only proper ideal $F=\oplus_{i,j\in\mathbb{N}}KE_{ij}$ where

$$E_{ij}:=\begin{cases} x^{i-j}(x^j\frac{1}{\partial^jx^j}\partial^j-x^{j+1}\frac{1}{\partial^{j+1}x^{j+1}}\partial^{j+1}) & \text{if } i\geq j,\\ (\frac{1}{\partial x}\partial)^{j-i}(x^j\frac{1}{\partial^jx^j}\partial^j-x^{j+1}\frac{1}{\partial^{j+1}x^{j+1}}\partial^{j+1}) & \text{if } i<j.\end{cases}$$

As a ring without 1, the ring F is canonically isomorphic to the ring $M_\infty(K):=\varinjlim M_d(K)=\oplus_{i,j\in\mathbb{N}}KE_{ij}$ of infinite-dimensional matrices where E_{ij} are the matrix units ($F\to M_\infty(K)$, $E_{ij}\mapsto E_{ij}$). This is a very important fact as we can apply concepts of finite-dimensional linear algebra (like trace, determinant, etc) to integro-differential operators which is not obvious from the outset. This fact is crucial in finding an inversion formula for elements of $\mathbb{A}_1^*$.

The algebra $\mathbb{A}_n=\oplus_{\alpha\in\mathbb{Z}^n}\mathbb{A}_{n,\alpha}$ is a $\mathbb{Z}^n$-graded algebra where $\mathbb{A}_{n,\alpha}:=\otimes_{k=1}^n\mathbb{A}_{1,\alpha_k}(k)$ and, for $n=1$, (Theorem 2.3)

$$\mathbb{A}_{1,i}=\begin{cases} x^i\mathbb{D}_1 & \text{if } i\geq 1,\\ \mathbb{D}_1 & \text{if } i=0,\\ \mathbb{D}_1\partial^{-i} & \text{if } i\leq -1,\end{cases}$$

where $\mathbb{D}_1:=L\oplus(\oplus_{i,j\geq 1}Kx^iH^{-j}\partial^i)$ is a *commutative, non-Noetherian* algebra and $L=K[H^{\pm 1},(H+1)^{-1},(H+2)^{-1},\ldots]$. This gives a 'compact' K-basis for the algebra $\mathbb{A}_1$ (and

$\mathbb{A}_n$). This basis 'behaves badly' under multiplication. A more conceptual ('multiplicatively friendly') basis is given in Theorem 2.5.

- (Corollary 2.7.(10)) *P_n is the only faithful, simple $\mathbb{A}_n$-module.*

Spec ($\mathbb{A}_n$). 0 is a prime ideal of $\mathbb{A}_n$.

$$\mathfrak{p}_1 := F \otimes \mathbb{A}_{n-1}, \mathfrak{p}_2 := \mathbb{A}_1 \otimes F \otimes \mathbb{A}_{n-2}, \ldots, \mathfrak{p}_n := \mathbb{A}_{n-1} \otimes F,$$

are precisely the prime ideals of height 1 of $\mathbb{A}_n$. Let Sub_n be the set of all subsets of $\{1, \ldots, n\}$.

- (Corollary 3.5) *The map $\mathrm{Sub}_n \to \mathrm{Spec}(\mathbb{A}_n)$, $I \mapsto \mathfrak{p}_I := \sum_{i \in I} \mathfrak{p}_i$, $\emptyset \mapsto 0$, is a bijection, i.e. any nonzero prime ideal of $\mathbb{A}_n$ is a unique sum of primes of height 1; $|\mathrm{Spec}(\mathbb{A}_n)| = 2^n$; the height of $\mathfrak{p}_I$ is $|I|$; and*

- (Lemma 3.6) *$\mathfrak{p}_I \subseteq \mathfrak{p}_J$ iff $I \subseteq J$.*

- (Corollary 3.15) *$\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$ is the only prime ideal of $\mathbb{A}_n$ which is completely prime; $\mathfrak{a}_n$ is the only ideal $\mathfrak{a}$ of $\mathbb{A}_n$ such that $\mathfrak{a} \neq \mathbb{A}_n$ and $\mathbb{A}_n/\mathfrak{a}$ is a Noetherian (resp. left Noetherian, resp. right Noetherian) ring.*

Ideals of $\mathbb{A}_n$ and their unique factorization. The ideal theory of $\mathbb{A}_n$ is 'very arithmetic.' Let $\mathcal{B}_n$ be the set of all functions $f : \{1, 2, \ldots, n\} \to \{0, 1\}$. For each function $f \in \mathcal{B}_n$, $I_f := I_{f(1)} \otimes \cdots \otimes I_{f(n)}$ is the ideal of $\mathbb{A}_n$ where $I_0 := F$ and $I_1 := \mathbb{A}_1$. Let $\mathcal{C}_n$ be the set of all subsets of $\mathcal{B}_n$ all distinct elements of which are incomparable (two distinct elements f and g of $\mathcal{B}_n$ are *incomparable* if neither $f(i) \leq g(i)$ nor $f(i) \geq g(i)$ for all i). For each $C \in \mathcal{C}_n$, let $I_C := \sum_{f \in C} I_f$, the ideal of $\mathbb{A}_n$. The next result classifies all the ideals of $\mathbb{A}_n$.

- (Theorem 3.1) *The map $C \mapsto I_C := \sum_{f \in C} I_f$ from the set $\mathcal{C}_n$ to the set of ideals of $\mathbb{A}_n$ is a bijection where $I_\emptyset := 0$. In particular, there are only finitely many ideals, say s_n, of $\mathbb{A}_n$. Moreover, $2 - n + \sum_{i=1}^{n} 2^{\binom{n}{i}} \leq s_n \leq 2^{2^n}$* (Corollary 3.4).

- *Each ideal I of $\mathbb{A}_n$ is an idempotent ideal, i.e. $I^2 = I$.*

- *Ideals of $\mathbb{A}_n$ commute ($IJ = JI$).*

- (Theorem 3.11) *The lattice of ideals of $\mathbb{A}_n$ is distributive.*

- (Corollary 2.7.(4,7)) *The ideal $\mathfrak{a}_n$ is the largest (hence, the only maximal) ideal of $\mathbb{A}_n$ distinct from $\mathbb{A}_n$, and $F^{\otimes n}$ is the smallest nonzero ideal of $\mathbb{A}_n$.*

- (Corollary 2.7.(11)) $\mathrm{GK}\,(\mathbb{A}_n/\mathfrak{a}) = 3n$ *for all ideals $\mathfrak{a}$ of $\mathbb{A}_n$ such that $\mathfrak{a} \neq \mathbb{A}_n$.*

For each ideal $\mathfrak{a}$ of $\mathbb{A}_n$, $\mathrm{Min}(\mathfrak{a})$ denotes the set of minimal primes over $\mathfrak{a}$. Two distinct prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ are called *incomparable* if neither $\mathfrak{p} \subseteq \mathfrak{q}$ nor $\mathfrak{p} \supseteq \mathfrak{q}$. The algebras $\mathbb{A}_n$ have beautiful ideal theory as the following unique factorization properties demonstrate.

- (Theorem 3.8) 1. *Each ideal* $\mathfrak{a}$ *of* $\mathbb{A}_n$ *such that* $\mathfrak{a} \neq \mathbb{A}_n$ *is a unique product of incomparable primes, i.e. if* $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{r}_1 \cdots \mathfrak{r}_t$ *are two such products then* $s = t$ *and* $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \dots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ *for a permutation* σ *of* $\{1, \dots, n\}$.

  2. *Each ideal* $\mathfrak{a}$ *of* $\mathbb{A}_n$ *such that* $\mathfrak{a} \neq \mathbb{A}_n$ *is a unique intersection of incomparable primes, i.e. if* $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_t$ *are two such intersections then* $s = t$ *and* $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \dots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ *for a permutation* σ *of* $\{1, \dots, n\}$.

  3. *For each ideal* $\mathfrak{a}$ *of* $\mathbb{A}_n$ *such that* $\mathfrak{a} \neq \mathbb{A}_n$, *the sets of incomparable primes in statements 1 and 2 are the same, and so* $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$.

  4. *The ideals* $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ *in statement 3 are the minimal primes of* $\mathfrak{a}$, *and so* $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p} = \cap_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p}$.

- (Corollary 3.10) $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$ *for all ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $\mathbb{A}_n$.

The next theorem gives all decompositions of an ideal as a product or intersection of ideals.

- (Theorem 3.12) *Let* $\mathfrak{a}$ *be an ideal of* $\mathbb{A}_n$, *and* $\mathcal{M}$ *be the minimal elements with respect to inclusion of a set of ideals* $\mathfrak{a}_1, \dots, \mathfrak{a}_k$ *of* $\mathbb{A}_n$. *Then*

  1. $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k$ *iff* $\mathrm{Min}(\mathfrak{a}) = \mathcal{M}$.

  2. $\mathfrak{a} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_k$ *iff* $\mathrm{Min}(\mathfrak{a}) = \mathcal{M}$.

This is a rare example of a non-commutative algebra of Krull dimension > 1 where one has a complete picture of decompositions of ideals.

The group $\mathbb{A}_1^*$ of units of $\mathbb{A}_1$. For each integer $i \geq 1$, consider the element of $\mathbb{A}_1^*$:

$$(H-i)_1^{-1} := \begin{cases} x\frac{1}{H^2}\partial + 1 - x\frac{1}{H}\partial & \text{if } i = 1 \ , \\ x\frac{1}{H\partial^i x^i}\partial + \sum_{j=0}^{i-2} \frac{1}{j+1-i}\pi_j + \pi_{i-1} & \text{if } i \geq 2 \ , \end{cases}$$

where $\pi_j := x^j \frac{1}{\partial^j x^j}\partial^j - x^{j+1}\frac{1}{\partial^{j+1}x^{j+1}}\partial^{j+1}$. Consider the following subgroup of $\mathbb{A}_1^*$,

$$\mathcal{H} := \{\prod_{i\geq 0}(H+i)^{n_i} \cdot \prod_{i \geq 1}(H-i)_1^{n_{-i}} \,|\, (n_i) \in \mathbb{Z}^{(\mathbb{Z})}\} \simeq \mathbb{Z}^{(\mathbb{Z})}$$

where $\mathbb{Z}^{(\mathbb{Z})}$ is the direct sum of $\mathbb{Z}$ copies of the group $\mathbb{Z}$, see (32) for detail. Let $\mathrm{GL}_\infty(K) := \{u \in 1 + M_\infty(K) \,|\, \det(u) \neq 0\}$. The group $(1+F)^*$ of units of the multiplicative monoid $1 + F$ is equal to $(1+F)^* = (1 + M_\infty(K))^* = \mathrm{GL}_\infty(K)$. Note that $(1+F)^* \subseteq \mathbb{A}_1^*$.

- (Theorem 4.2) 1. $\mathbb{A}_1^* = K^* \times (\mathcal{H} \ltimes (1+F)^*)$, *each unit* a *of* $\mathbb{A}_1$ *is a unique product* $a = \lambda\alpha(1+f)$ *for some elements* $\lambda \in K^*$, $\alpha \in \mathcal{H}$, *and* $f \in F$ *such that* $\det(1+f) \neq 0$.

  2. $\mathbb{A}_1^* = K^* \times (\mathcal{H} \ltimes \mathrm{GL}_\infty(K))$.

  3. *The centre of the group* $\mathbb{A}_1^*$ *is* K^*.

  4. *The commutant* $\mathbb{A}_1^{*(2)} := [\mathbb{A}_1^*, \mathbb{A}_1^*]$ *of the group* $\mathbb{A}_1^*$ *is equal to* $\mathrm{SL}_\infty(K) := \{v \in (1+F)^* = M_\infty(K) \,|\, \det(v) = 1\}$, *and* $\mathbb{A}_1/[\mathbb{A}_1^*, \mathbb{A}_1^*] \simeq K^* \times \mathcal{H} \times K^*$.

  5. *All the higher commutants* $\mathbb{A}_1^{*(i)} := [\mathbb{A}_1^*, \mathbb{A}_1^{*(i-1)}]$, $i \geq 2$, *are equal to* $\mathbb{A}_1^{*(2)}$.

The group of units $\mathbb{A}_n^*$ of $\mathbb{A}_n$.

- (Theorem 4.4) 1. $\mathbb{A}_n^* = K^* \times (\mathcal{H}_n \ltimes (1+\mathfrak{a}_n)^*)$ where $\mathcal{H}_n := \prod_{i=1}^n \mathcal{H}(i)$ and $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$.

  2. *The centre of the group $\mathbb{A}_n^*$ is K^*.*

Theorem 4.5 is a criterion of when an element of the monoid $1+\mathfrak{a}_n$ belongs to its group $(1+\mathfrak{a}_n)^*$ of units.

Question. Is the global dimension of $\mathbb{A}_n$ equal to $2n$ (or ∞)?

2 The Jacobian algebras and localizations of the Weyl algebras

In this section, two K-bases for the Jacobian algebras are found (Theorems 2.3 and 2.5), and several properties of the algebras $\mathbb{A}_n$ are proved: $\mathbb{A}_n$ is a central, prime, self-dual, non-Noetherian algebra.

We start by recalling some properties of generalized Weyl algebras. Some of these algebras are factor algebras of the Jacobian algebras.

Generalized Weyl Algebras. Let D be a ring, $\sigma = (\sigma_1, ..., \sigma_n)$ be an n-tuple of commuting automorphisms of D, and $a = (a_1, ..., a_n)$ be an n-tuple of (non-zero) elements of the centre $Z(D)$ of D such that $\sigma_i(a_j) = a_j$ for all $i \neq j$.

The *generalized Weyl algebra* $A = D(\sigma, a)$ (briefly GWA) of degree n with the *base* ring D is a ring generated by D and $2n$ indeterminates $x_1, ..., x_n, y_1, ..., y_n$ subject to the defining relations [3], [4]:

$$y_i x_i = a_i, \qquad x_i y_i = \sigma_i(a_i),$$
$$x_i \alpha = \sigma_i(\alpha) x_i, \qquad y_i \alpha = \sigma_i^{-1}(\alpha) y_i, \quad \alpha \in D,$$

$$[x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 0, \quad i \neq j,$$

where $[x, y] = xy - yx$. We say that a and σ are the sets of *defining* elements and automorphisms of A respectively. The GWAs are also known as *hyperbolic rings*, see the book of Rosenberg [20]. For a vector $k = (k_1, ..., k_n) \in \mathbb{Z}^n$, let $v_k = v_{k_1}(1) \cdots v_{k_n}(n)$ where, for $1 \leq i \leq n$ and $m \geq 0$: $v_m(i) = x_i^m$, $v_{-m}(i) = y_i^m$, $v_0(i) = 1$. It follows from the definition of the GWA that

$$A = \oplus_{k \in \mathbb{Z}^n} A_k$$

is a $\mathbb{Z}^n$-graded algebra ($A_k A_e \subseteq A_{k+e}$, for all $k, e \in \mathbb{Z}^n$), where $A_k = Dv_k = v_k D$.

The tensor product (over the base field) $A \otimes A'$ of generalized Weyl algebras of degree n and n' respectively is a GWA of degree $n + n'$:

$$A \otimes A' = D \otimes D'((\sigma, \sigma'), (a, a')).$$

Let $\mathcal{P}_n$ be a polynomial algebra $K[H_1, \dots, H_n]$ in n indeterminates and let $\sigma = (\sigma_1, ..., \sigma_n)$ be an n-tuple of commuting automorphisms of $\mathcal{P}_n$ such that $\sigma_i(H_i) = H_i - 1$ and $\sigma_i(H_j) = H_j$, for $i \neq j$. Let $A_n = K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n\rangle$ be the Weyl algebra. The algebra homomorphism

$$A_n \to \mathcal{P}_n((\sigma_1, ..., \sigma_n), (H_1, \dots, H_n)), \quad x_i \mapsto x_i, \quad \partial_i \mapsto y_i, \quad i = 1, \dots, n, \tag{1}$$

is an isomorphism. We identify the Weyl algebra A_n with the GWA above via this isomorphism. Note that $H_i = \partial_i x_i = x_i \partial_i + 1$. Denote by S_n the multiplicative submonoid of $\mathcal{P}_n$ generated by the elements $H_i + j$, $i = 1, \dots, n$, and $j \in \mathbb{Z}$. It follows from the above presentation of the Weyl algebra A_n as a GWA that S_n is an Ore set in A_n, and, using the $\mathbb{Z}^n$-grading, that the (two-sided) localization $\mathcal{A}_n := S_n^{-1}A_n$ of the Weyl algebra A_n at S_n is the *skew Laurent polynomial ring*

$$\mathcal{A}_n = S_n^{-1}\mathcal{P}_n[x_1^{\pm 1}, \dots, x_n^{\pm 1}; \sigma_1, ..., \sigma_n] \tag{2}$$

with coefficients from

$$S_n^{-1}\mathcal{P}_n = K[H_1^{\pm 1}, (H_1 \pm 1)^{-1}, (H_1 \pm 2)^{-1}, \dots, H_n^{\pm 1}, (H_n \pm 1)^{-1}, (H_n \pm 2)^{-1}, \dots],$$

the localization of $\mathcal{P}_n$ at S_n. We identify the Weyl algebra A_n with the subalgebra of $\mathcal{A}_n$ via the monomorphism,

$$A_n \to \mathcal{A}_n, \quad x_i \mapsto x_i, \quad \partial_i \mapsto H_i x_i^{-1}, \quad i = 1, \dots, n.$$

Let k_n be the n'th *Weyl skew field*, that is the full ring of quotients of the n'th Weyl algebra A_n (it exists by Goldie's Theorem since A_n is a Noetherian domain). Then the algebra $\mathcal{A}_n$ is a K-subalgebra of k_n generated by the elements x_i, x_i^{-1}, H_i and H_i^{-1}, $i = 1, \dots, n$ since, for all natural j,

$$(H_i \mp j)^{-1} = x_i^{\pm j} H_i^{-1} x_i^{\mp j}, \quad i = 1, \dots, n.$$

Clearly, $\mathcal{A}_n \simeq \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_1$ (n times).

Definition: A K-algebra R has the *endomorphism property over* K if, for each simple R-module M, $\mathrm{End}_R(M)$ is algebraic over K.

Theorem 2.1 *[5] Let K be a field of characteristic zero.*

1. *The algebra $\mathcal{A}_n$ is a simple, affine, Noetherian domain.*
2. *The Gelfand-Kirillov dimension* $\mathrm{GK}\,(\mathcal{A}_n) = 3n$ $(\neq 2n = \mathrm{GK}\,(A_n))$.
3. *The (left and right) global dimension* $\mathrm{gl.dim}(\mathcal{A}_n) = n$.
4. *The (left and right) Krull dimension* $\mathrm{K.dim}(\mathcal{A}_n) = n$.

5. *Let* $\mathrm{d} = \mathrm{gl.dim}$ *or* $\mathrm{d} = \mathrm{K.dim}$. *Let R be a Noetherian K-algebra with* $\mathrm{d}(R) < \infty$ *such that $R[t]$, the polynomial ring in a central indeterminate, has the endomorphism property over K. Then* $\mathrm{d}(\mathcal{A}_1 \otimes R) = \mathrm{d}(R) + 1$. *If, in addition, the field K is algebraically closed and uncountable, and the algebra R is affine, then* $\mathrm{d}(\mathcal{A}_n \otimes R) = \mathrm{d}(R) + n$.

$\mathrm{GK}\,(\mathcal{A}_1) = 3$ is due to A. Joseph [12], p. 336; see also [16], Example 4.11, p. 45.

It is an experimental fact that many small quantum groups are GWAs. More about GWAs and their generalizations the interested reader can find in [1, 2, 6, 7, 8, 11, 13, 14, 15, 17, 18, 19, 21].

Projections. The polynomial algebra $P_n = \oplus_{\alpha\in\mathbb{N}^n} Kx^\alpha$ is a left $\mathbb{A}_n$-module and $\mathrm{End}_K(P_n)$-module. For each $i = 1, \ldots, n$ and $\alpha \in \mathbb{N}^n$, $H_i(x^\alpha) = (\alpha_i + 1)x^\alpha$, and so H_i is an *invertible* map with $H_i^{-1}(x^\alpha) = (\alpha_i + 1)^{-1}x^\alpha$. Let $h_i := x_i\partial_i$. Then, in P_n, $h_i(x^\alpha) = \alpha_i x^\alpha$, and so $\ker(h_i) = K[x_1, \ldots, \widehat{x}_i, \ldots, x_n]$.

Note that (in $\mathrm{End}_K(P_n)$)

$$(H^{-1}\partial)^i(xH^{-1})^i = \frac{1}{H(H+1)\cdots(H+i-1)}, \quad i \geq 1. \tag{3}$$

For each $\alpha \in \mathbb{N}^n$, the following element of $\mathrm{End}_K(P_n)$ is invertible,

$$(-\alpha, \alpha) := \partial^\alpha x^\alpha = \prod_{i=1}^n H_i(H_i + 1)\cdots(H_i + \alpha_i - 1). \tag{4}$$

Lemma 2.2 *Let K be a commutative $\mathbb{Q}$-algebra and $\alpha \in \mathbb{N}_n$. Then $x^\alpha(-\alpha, \alpha)^{-1}\partial^\alpha$ is the projection onto the ideal (x^α) of P_n in the decomposition $P_n = (\oplus_{\beta : x^\beta \notin (x^\alpha)} Kx^\beta) \oplus (x^\alpha)$.*

Proof. If $x^\beta \notin (x^\alpha)$ then $\partial^\alpha(x^\beta) = 0$, and so $x^\alpha \frac{1}{\partial^\alpha x^\alpha}\partial^\alpha(x^\beta) = 0$. If $x^\beta \in (x^\alpha)$ then $x^\alpha \frac{1}{\partial^\alpha x^\alpha}\partial^\alpha(x^\beta) = x^\alpha \frac{1}{\partial^\alpha x^\alpha}\partial^\alpha x^\alpha(x^{\beta-\alpha}) = x^\alpha(x^{\beta-\alpha}) = x^\beta$. $\square$

Lemma 2.2 is useful in producing various projections onto homogeneous K-submodules of P_n. Let S be a subset of $\mathbb{N}^n$ and S' be its complement. Then $P_n = P_{n,S} \oplus P_{n,S'}$ where $P_{n,S} := \oplus_{\alpha\in S} Kx^\alpha$ and $P_{n,S'} := \oplus_{\alpha\in S'} Kx^\alpha$. Then $\pi_S := \sum_{\alpha\in S}\pi_\alpha$ is the projection onto $P_{n,S}$.

Example. Let $a, b \in \mathbb{N}^n$ with $a \leq b$, i.e. $a_1 \leq b_1, \ldots, a_n \leq b_n$; and $C := \{\alpha \in \mathbb{N}^n \,|\, a \leq \alpha \leq b\}$ be the discrete cube in $\mathbb{N}^n$ and C' be its complement. Then $\pi_C := \sum_{\alpha\in C}\pi_\alpha$ is the projection onto $P_{n,C}$ in the decomposition $P_n = P_{n,C} \oplus P_{n,C'}$. Note that $\pi_C = \prod_{i=1}^n (x_i^{a_i}\frac{1}{\partial_i^{a_i}x_i^{a_i}}\partial_i^{a_i} - x_i^{b_i}\frac{1}{\partial_i^{b_i}x_i^{b_i}}\partial_i^{b_i})$, by Lemma 2.2. In more detail, for each $i = 1, \ldots, n$, $x_i^{a_i}\frac{1}{\partial_i^{a_i}x_i^{a_i}}\partial_i^{a_i}$ is the projection onto the ideal $x_i^{a_i}K[x_i]$ in the decomposition $K[x_i] = (\oplus_{j=0}^{a_i-1} Kx_i^j) \oplus x_i^{a_i}K[x_i]$. Therefore, $p_i := x_i^{a_i}\frac{1}{\partial_i^{a_i}x_i^{a_i}}\partial_i^{a_i} - x_i^{b_i}\frac{1}{\partial_i^{b_i}x_i^{b_i}}\partial_i^{b_i}$ is the projection onto $Kx_i^{a_i} \oplus Kx_i^{a_i+1} \oplus \cdots \oplus Kx_i^{b_i}$. Now, it is obvious that the product $p_1 \cdots p_n$ is equal to π_C.

The Jacobian algebra $\mathbb{A}_n$. Let K be a commutative $\mathbb{Q}$-algebra.

Definition: The **Jacobian algebra** $\mathbb{A}_n$ is the subalgebra of $\mathrm{End}_K(P_n)$ generated by the Weyl algebra A_n and the elements $H_1^{-1}, \ldots, H_n^{-1}$.

Surprisingly, the Weyl algebras A_n and the Jacobian algebras $\mathbb{A}_n$ have little in common. For example, the algebra $\mathbb{A}_n$ contains the infinite direct sum $K^{(\mathbb{N})}$ of rings K. In particular, $\mathbb{A}_n$ is not a domain, and we will see that $\mathbb{A}_n$ is not left or right Noetherian algebra.

By the very definition,

$$\mathbb{A}_n = \mathbb{A}_1(1) \otimes \mathbb{A}_1(2) \otimes \cdots \otimes \mathbb{A}_1(n) \simeq \mathbb{A}_1^{\otimes n}, \qquad (5)$$

where $\mathbb{A}_1(i) := K\langle x_i, \partial_i, H_i^{-1}\rangle$ and $\otimes := \otimes_K$. The algebra $\mathbb{A}_n$ contains all the integrations $\int_i = x_i H_i^{-1}$, $1 \leq i \leq n$. In the algebra $\mathbb{A}_n$, each element ∂_i has a right inverse, $\int_i$: $\partial_i \int_i = \mathrm{id}_{P_n}$; and each element x_i has a left inverse, $H_i^{-1}\partial_i$: $H_i^{-1}\partial_i x_i = \mathrm{id}_{P_n}$. So, the algebra $\mathbb{A}_n$ contains all necessary operations of Analysis (like integrations and differentiations) to deal with polynomials. The algebra $\mathbb{A}_n$ contains all integro-differential operators. By (5), properties of the algebra $\mathbb{A}_n$ is mainly determined by properties of the algebra $\mathbb{A}_1$.

We pointed out already that the multiplicative submonoid S_n of $K[H_1, \ldots, H_n]$ generated by the elements $H_i + j$, $1 \leq i \leq n$, $j \in \mathbb{Z}$, is a (left and right) Ore set of the Weyl algebra A_n and $S_n^{-1}A_n = \mathcal{A}_n$. Using the $\mathbb{Z}^n$-grading of the Weyl algebra A_n coming from its presentation as a generalized Weyl algebra one can easily verify that the multiplicative submonoid $S_{n,+}$ of $K[H_1, \ldots, H_n]$ generated by the elements $H_i + j$, $1 \leq i \leq n$, $j \in \mathbb{N}$, is not a (left and right) Ore set of the Weyl algebra A_n. This also follows from the fact that the algebra $\mathcal{A}_n$ is a domain but $\mathbb{A}_n$ is not (if $S_{n,+}$ were a left or right Ore set then $\mathbb{A}_n \subseteq \mathcal{A}_n$, a contradiction).

Consider the case $n = 1$. Let $2^{\mathbb{N}}$ be the Boolean algebra of all subsets of $\mathbb{N}$ and $\mathcal{B}_1$ be the Boolean subalgebra generated by all the finite subsets of $\mathbb{N}$. So, a subset S of $\mathbb{N}$ is an element of $\mathcal{B}_1$ iff either S is finite or co-finite (that is, its complement is finite). Note that the Jacobian algebra $\mathbb{A}_1$ contains all the projections π_S, $S \in \mathcal{B}_1$.

In order to make formulae more readable, we drop the subscript 1. So, let, for a moment, $x := x_1$, $\partial := \partial_1$, and $H := H_1$. Since $(H^{-1}\partial)^i H^{-1} x^i = (H^{-1}\partial)^i x^i (H+i)^{-1} = (H+i)^{-1}$, $i \geq 1$, the algebra $\mathbb{A}_1$ contains the subalgebra $L := K[H, H^{-1}, (H+1)^{-1}, \ldots, (H+i)^{-1}, \ldots]$. For each $i \geq 1$, $\frac{1}{\partial^i x^i} = \frac{1}{H(H+1)\cdots(H+i-1)} \in L$. Let $\mathbb{D}_1 := L + \sum_{i,j\geq 1} K x^i H^{-j}\partial^i$ and $V := \oplus_{j\geq 1} K H^{-j}$. The next theorem gives a K-basis for the algebra $\mathbb{A}_1$.

Theorem 2.3 *Let K be a commutative $\mathbb{Q}$-algebra. Then the Jacobian algebra $\mathbb{A}_1 = \oplus_{i\in\mathbb{Z}} \mathbb{A}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{A}_{1,i}\mathbb{A}_{1,j} \subseteq \mathbb{A}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where $\mathbb{A}_{1,0} = \mathbb{D}_1$, $\mathbb{D}_1 = L \oplus (\oplus_{i,j\geq 1} K x^i H^{-j}\partial^i)$; and, for each $i \geq 1$, $\mathbb{A}_{1,i} = x^i \mathbb{D}_1$ and $\mathbb{A}_{1,-i} = \mathbb{D}_1 \partial^i$.*

Proof. First, let us prove that the sum in the definition of $\mathbb{D}_1$ is the direct one. Suppose that $r := l + xv_1\partial + x^2 v_2 \partial^2 + \cdots + x^s v_s \partial^s = 0$ is a nontrivial relation for some elements $l \in L$ and $v_i \in V := \oplus_{j\geq 1} K H^{-j}$. We seek a contradiction. Since L is a subalgebra of $\mathrm{End}_K(P_n)$, the relation r is not of the type $r = l$. So, we can assume that $v_s \neq 0$ and the natural number $s \geq 1$ is called the degree of the relation r. Let r be a nontrivial relation of the least degree. The rational function $l \in K(H)$ can be written as $\frac{p}{q}$ where p and q are co-prime polynomials and the polynomial q is a finite product of the type $\prod_{i\geq 0}(H+i)^{n_i}$. Evaluating the relation r at 1: $0=r(1)=l(1)=\frac{p(1)}{q(1)}$, we see that the polynomial p is equal to $(H-1)p'$ for some polynomial $p'\in K[H]$. Suppose that $n=1$, then $0=H^{-1}\partial rx=H\sigma^{-1}(\frac{p'}{q})+v_1H$ where $\sigma: H\mapsto H-1$ is the K-automorphism of the polynomial algebra $K[H]$ (and of its field of fractions $K(H)$). It follows that $v_1=-\sigma^{-1}(\frac{p'}{q})\in V\cap\sigma^{-1}(L)=0$, a contradiction. Therefore, $s\geq 2$.

The $K[H]$-module $(V+K[H])/K[H]\simeq V$ has the K-basis $\{H^{-i}, i\geq 1\}$. In this basis, the matrix of the K-linear map $v\mapsto (H+\lambda)v$ (where $\lambda\in K$) is an upper triangular infinite matrix with λ on the diagonal. In particular, the matrices of the maps $H+1, H+2,\ldots,H+s-1$, are invertible, upper triangular. It follows from this fact that the relation

$$H^{-1}\partial rx=H\sigma^{-1}(\frac{p'}{q})+v_1H+xv_2(H+1)\partial+\cdots+x^{i-1}v_i(H+i-1)\partial^{i-1}+\cdots+x^{s-1}v_s(H+s-1)\partial^{s-1}$$

has degree $s-1$ since $s\geq 2$. Then, by induction on s, and the fact that each matrix of the map $H+i-1$, $2\leq i\leq s$, is an upper triangular, invertible matrix with $i-1\neq 0$ on the diagonal, we have $v_2=\cdots=v_s=0$ and $H\sigma^{-1}(\frac{p'}{q})+v_1H=0$. Then, $v_1=-\sigma^{-1}(\frac{p'}{q})\in V\cap\sigma^{-1}(L)=0$. This means that the relation r is a trivial one, a contradiction. This finishes the proof of the claim.

Let $(G,+)$ be an additive group (not necessarily commutative) and $U=\oplus_{\alpha\in G}U_\alpha$ be a G-graded K-module, i.e. a direct sum of K-modules U_α. A K-linear map $f:U\rightarrow U$ has degree $\beta\in G$ if $f(U_\alpha)\subseteq U_{\beta+\alpha}$ for all $\alpha\in G$. The set E_β of all K-linear maps of degree β is a K-submodule of $\mathrm{End}_K(U)$. Clearly, $E_\beta=\prod_{\beta\in G}\mathrm{Hom}_K(U_\alpha,U_{\beta+\alpha})$. In particular, $E_0=\prod_{\alpha\in G}\mathrm{End}_K(U_\alpha)$. It follows at once that the sum $E:=\sum_{\beta\in G}E_\beta\subseteq\mathrm{End}_K(U)$ is a direct one,

$$E=\oplus_{\beta\in G}E_\beta \text{ and } E_\beta E_\gamma\subseteq E_{\beta+\gamma},\quad \beta,\gamma\in G. \tag{6}$$

So, $E:=E(U)$ is a G-graded ring.

The K-module $K[x]=\oplus_{i\geq 0}Kx^i$ is naturally $\mathbb{Z}$-graded (even $\mathbb{N}$-graded). By (6), the sum $S:=\sum_{i\geq 1}\mathbb{D}_1\partial^i+\mathbb{D}_1+\sum_{i\geq 1}x^i\mathbb{D}_1$ is a direct sum since the maps $\mathbb{D}_1\partial^i$, $\mathbb{D}_1$, and $x^i\mathbb{D}_i$ have degree $-i$, 0, and i respectively. In order to prove that $\mathbb{A}_1=\oplus_{i\in\mathbb{Z}}\mathbb{A}_{1,i}$ it suffices to show that $\mathbb{A}_1\subseteq S$. It follows directly from the inclusions:

$$\begin{aligned} \mathbb{D}_1x^i &\subseteq x^i\mathbb{D}_1, & \partial^i\mathbb{D}_1\subseteq\mathbb{D}_1\partial^i, && i\geq 0,\\ x^i\partial^j &\subseteq \mathbb{D}_1\partial^{j-i}, & x^j\partial^i\subseteq x^{j-i}\mathbb{D}_1, && j\geq i,\end{aligned}$$

that $\mathbb{A}_1=\sum_{i,j\geq 0}x^i\mathbb{D}_1\partial^j=\sum_{s\in\mathbb{Z}}(\sum_{i-j=s}x^i\mathbb{D}_1\partial^j)$. It remains to show that, for $s\geq 0$, $\sum_{i-j=s}x^i\mathbb{D}_1\partial^j\subseteq x^s\mathbb{D}_1$; and, for $s<0$, $\sum_{i-j=s}x^i\mathbb{D}_1\partial^j\subseteq\mathbb{D}_1\partial^{-s}$.

Consider the case $s=0$. We have to show that $\sum_{i\geq 0}x^i\mathbb{D}_1\partial^i\subseteq\mathbb{D}_1$. By the very definition of $\mathbb{D}_1$, this is equivalent to the inclusions $x^iL\partial^i\subseteq\mathbb{D}_1$, $i\geq 0$; and, by the very definition of L, this is equivalent to the inclusions $x^i(H+j)^{-k}\partial^i\in\mathbb{D}_1$, $i,j,k\geq 0$.

If $i\leq j$ then $x^i(H+j)^{-k}\partial^i=(H+j-i)^{-k}x^i\partial^i\in L\subseteq\mathbb{D}_1$.

If $i>j$ then $x^i(H+j)^{-k}\partial^i=x^{i-j}H^{-k}x^j\partial^i=x^{i-j}H^{-k}(H-1)(H-2)\cdots(H-j)\partial^{i-j}\in x^{i-j}(V+K[H])\partial^{i-j}\subseteq\mathbb{D}_1$. This proves the case $s=0$.

If $s \geq 1$ then $\sum_{i-j=s} x^i \mathbb{D}_1 \partial^j = x^s(\sum_{k\geq 0} x^k \mathbb{D}_1 \partial^k) \subseteq x^s \mathbb{D}_1$, by the case $s = 0$.

If $s \leq -1$ then $\sum_{i-j=s} x^i \mathbb{D}_1 \partial^j = (\sum_{k\geq 0} x^k \mathbb{D}_1 \partial^k)\partial^{-s} \subseteq \mathbb{D}_1 \partial^{-s}$, by the case $s = 0$.

Thus, the equality $\mathbb{A}_1 = \oplus_{i\in\mathbb{Z}} \mathbb{A}_{1,i}$ is established. By (6), this is a $\mathbb{Z}$-graded algebra since the maps $\mathbb{A}_{1,i}$ have degree i. $\square$

Despite the fact that Theorem 2.3 provides a cute K-basis for the algebra $\mathbb{D}_1$ it is unsuitable for computations: to write down the product of the type $x^i H^{-j} \partial^i \cdot x^k H^{-l} \partial^k$ literally takes half a page. Later, in Corollary 2.4 a more conceptual K-basis is introduced, and which is more important we interpret elements of $\mathbb{D}_1$ as functions from $\mathbb{N}$ to K. We will see that the ring $\mathbb{D}_1$ is large and has analytic flavour.

The polynomial algebra $K[x] = \oplus_{i\geq 0} Kx^i$ is naturally a $\mathbb{Z}$-graded algebra. Let $E = \oplus_{i\in\mathbb{Z}} E_i$ be the algebra from (6) for $K[x]$. The E-module $K[x]$ is simple. Note that the map

$$E_0 = \{f \in \mathrm{End}_K(K[x]) \,|\, f(x^i) = f_i x^i, i \geq 0, f_i \in K\} \to K^{\mathbb{N}}, \;\; f \mapsto (f_i),$$

is an isomorphism of K-algebras. In particular, E_0 is a commutative algebra, and so $\mathbb{D}_1$ is a *commutative* algebra since $\mathbb{D}_1 \subseteq E_0$ ($K[x]$ is the faithful $\mathbb{A}_1$-module). It is obvious that, for $i \geq 0$, $E_i = x^i E_0$ and $E_{-i} = E_0 \partial^i$. The algebra $K^{\mathbb{N}}$ is the algebra of all functions from $\mathbb{N}$ to K. When we identify the set of monomials $\mathcal{M} := \{x^i\}_{i\in\mathbb{N}}$ and $\mathbb{N}$ via $x^i \mapsto i$ the algebras E_0 and $K^{\mathbb{N}}$ are identified. So, each element of E_0 can be seen as a function. This is a very nice observation indeed as we can use facts and terminology of Analysis. For a function $\varphi : \mathbb{N} \to K$, the set $\mathrm{supp}(\varphi) := \{i \in \mathbb{N} \,|\, \varphi(i) \neq 0\}$ is called the *support* of φ. The set of functions F_0 with finite support is an ideal of the algebra E_0. Clearly, $F_0 = \oplus_{i\in\mathbb{N}} K\pi_i$ where

$$\pi_i := x^i \frac{1}{\partial^i x^i} \partial^i - x^{i+1} \frac{1}{\partial^{i+1} x^{i+1}} \partial^{i+1} : K[x] \to K[x]$$

is the projection onto Kx^i (Lemma 2.2), i.e. $\pi_i(x^j) = \delta_{ij} x^j$ where δ_{ij} is the Kronecker delta.

Let $\pi_{-1} := 0$; then

$$x\pi_i = \pi_{i+1} x, \;\; \partial\pi_i = \pi_{i-1}\partial, \;\; i \geq 0.$$

The concept of support can be extended to an arbitrary element f of the algebra E as $\mathrm{supp}(f) := \{i \in \mathbb{N} \,|\, f(x^i) \neq 0\}$. The set F of all maps $f \in E$ with finite support is a $\mathbb{Z}$-graded algebra $F = \oplus_{i\in\mathbb{Z}} F_i$ without 1 where $F_i = F \cap E_i$. For $i \geq 1$, $F_i = x^i F_0 = \oplus_{j\in\mathbb{N}} Kx^i \pi_j = \oplus_{j\in\mathbb{N}} K\pi_{j+i} x^i = F_0 x^i$ and $F_{-i} = F_0 \partial^i = \oplus_{j\in\mathbb{N}} K\pi_j \partial^i = \oplus_{j\in\mathbb{N}} K\partial^i \pi_{j+i} = \partial^i F_0$. Note that $F = \{f \in \mathrm{End}_K(K[x]) \,|\, f(x^i K[x]) = 0 \text{ for some } i \in \mathbb{N}\}$. It is obvious that F is an *ideal* of the algebra E, and so F is also an ideal of $\mathbb{A}_1$ since $F \subseteq \mathbb{A}_1 \subseteq E$.

Note that $h := x\partial \in E_0$ and $h(x^i) = ix^i$, $i \geq 0$. So, h can be identified with the function $\mathbb{N} \to K$, $i \mapsto i$. Note that $H = h + 1$. When h runs through $0, 1, 2 \ldots$, H runs through $1, 2, \ldots$. Under the identification $E_0 = K^{\mathbb{N}}$, for $i, j \geq 1$,

$$x^i H^{-j} \partial^i = \begin{cases} \frac{(H-1)(H-2)\cdots(H-i+1)}{(H-i)^{j-1}} & \text{if } H = i+1, i+2, \ldots, \\ 0 & \text{if } H = 1, 2, \ldots, i. \end{cases} \tag{7}$$

This means that the element $x^i H^{-j} \partial^i$ is a function of the discrete argument $H = 1, 2, \ldots$ which takes zero value for $H = 1, 2, \ldots, i$; and $\frac{(H-1)(H-2)\cdots(H-i+1)}{(H-i)^{j-1}}$ for $H = i+1, i+2, \ldots$.

Before the identification this simply means that

$$x^iH^{-j}\partial^i(x^k)=\begin{cases}\frac{(k+1-1)(k+1-2)\cdots(k+1-i+1)}{(k+1-i)^{j-1}}x^k & \text{if } k\geq i,\\ 0 & \text{if } k=0,1,\dots,i-1.\end{cases}$$

So, the function $x^iH^{-j}\partial^i$ is almost a rational function. The case $i=j=1$ is rather special, it yields almost a constant function

$$xH^{-1}\partial=\begin{cases}1 & \text{if } H=2,3,\dots,\\ 0 & \text{if } H=1.\end{cases}$$

Similarly, for each $i=1,2,\dots$, the element $\rho_i:=x^i\frac{1}{\partial^i x^i}\partial^i\in\mathbb{D}_1$ is the function

$$\rho_i=\begin{cases}1 & \text{if } h=i,i+1,\dots,\\ 0 & \text{if } h=0,1,\dots,i-1.\end{cases}\tag{8}$$

For each $i\geq 0$, $\pi_i=\rho_i-\rho_{i+1}\in\mathbb{D}_1$ where $\rho_0:=1$, and so $F_0=\oplus_{i\geq 0}K\pi_i\subseteq\mathbb{D}_1$. More generally, for each $i=1,2,\dots$ and $j\in\mathbb{N}$, let

$$\rho_{ji}:=x^i\frac{1}{H^j\partial^ix^i}\partial^i=\frac{1}{(H-i)^j}\rho_i=\begin{cases}\frac{1}{(H-i)^j} & \text{if } H=i+1,i+2,\dots,\\ 0 & \text{if } H=1,2,\dots,i.\end{cases}\tag{9}$$

Note that all $\rho_{ji}\in\mathbb{D}_1$ and $\rho_{0i}=\rho_i$. For $\lambda\in\mathbb{Z}$, the element $H+\lambda$ is invertible in $\mathbb{D}_1$ iff $\lambda\neq-1,-2,\dots$ iff $H+\lambda$ is invertible in E_0. For $i=1,2,\dots$, $\ker_{\mathbb{D}_1}(H-i)^j=\ker_{E_0}(H-i)^j=K\pi_i$ where $\ker_{\mathbb{D}_1}(H-i)^j$ is the kernel of the map $\mathbb{D}_1\to\mathbb{D}_1$, $d\mapsto(H-i)^jd$. Similarly, $\ker_{E_0}(H-i)^j$ is defined.

For each $i\geq 0$, let $\pi_i':=1-\pi_i$. For natural numbers $i,j\geq 1$, consider the element $\frac{1}{(H-i)^j}\pi_{i-1}'$ of E_0 which is as a linear map defined by the rule

$$\frac{1}{(H-i)^j}\pi_{i-1}'(x^k)=\begin{cases}\frac{1}{(k+1-i)^j}x^k & \text{if } k\neq i-1,\\ 0 & \text{if } k=i-1.\end{cases}$$

As a function, it is almost the rational function $\frac{1}{(H-i)^j}$ but at $H=i$ it takes value 0 rather than ∞ as the usual function $\frac{1}{(H-i)^j}$ does. All $\frac{1}{(H-i)^j}\pi_{i-1}'\in\mathbb{D}_1$ since

$$\frac{1}{(H-i)^j}\pi_{i-1}'=\begin{cases}\rho_{j1} & \text{if } i=1,\\ \rho_{ji}+\sum_{k=0}^{i-2}\frac{1}{(k+1-i)^j}\pi_k & \text{if } i\geq 2.\end{cases}$$

For $i,j,n,m\geq 1$, $\frac{1}{(H-i)^n}\pi_{i-1}'\cdot\frac{1}{(H-i)^m}\pi_{i-1}'=\frac{1}{(H-i)^{n+m}}\pi_{i-1}'$, $(H-i)^m\cdot\frac{1}{(H-i)^n}\pi_{i-1}'=\frac{1}{(H-i)^{n-m}}\pi_{i-1}'$; and for $j\geq 1$ such that $j\neq i$

$$\frac{1}{(H-i)^n}\pi_{i-1}'\cdot\frac{1}{(H-j)^m}\pi_{j-1}'\subseteq\sum_{s=1}^nK\frac{1}{(H-i)^s}\pi_{i-1}'+\sum_{t=1}^mK\frac{1}{(H-j)^t}\pi_{j-1}'+K\pi_{i-1}+K\pi_{j-1}.$$

Clearly, the set

$$\mathbb{L}_1 := L \oplus \oplus_{i,j\geq 1} K \frac{1}{(H-i)^j}\pi'_{i-1} \tag{10}$$

is a K-submodule of $\mathbb{D}_1$. $\mathbb{L}_1$ is not an algebra, though it is an algebra modulo F_0 which is isomorphic to the algebra $K[H^{\pm 1}, (H\pm 1)^{-1}, (H\pm 2)^{-1}, \dots,]$ (see Corollary 2.4 and (11)).

Corollary 2.4 *Let K be a commutative $\mathbb{Q}$-algebra, $\rho_{ji} := x^i \frac{1}{H^j \partial^i x^i}\partial^i$, $j\geq 0$, $i\geq 1$. Then $\mathbb{D}_1 = \mathbb{L}_1 \oplus F_0 = L \oplus (\oplus_{i\geq 1, j\geq 0} K\rho_{ji})$.*

Proof. By Theorem 2.3, $\mathbb{D}_1 = L \oplus (\oplus_{i,j\geq 1} Kx^i H^{-j}\partial^i)$. By (7) and $F_0 \subseteq \mathbb{D}_1$, we have $\mathbb{D}_1 \subseteq \mathbb{L}_1 + F_0$. By (9) and $F_0\subseteq \mathbb{D}_1$, we have the opposite inclusion, and so $\mathbb{D}_1 = \mathbb{L}_1 + F_0 = \mathbb{L}_1 \oplus F_0$ since $\mathbb{L}_1 \cap F_0 = 0$.

The K-module $M := L + \sum_{j\geq 0, i\geq 1} K\rho_{ji}$ contains $F_0 = \oplus_{i\geq 0}\pi_i$ since $\pi_i = \rho_i - \rho_{i+1}$ for all $i\geq 0$ where $\rho_0 := 1$; and $M = L' + F_0$ where $L' := L + \sum_{j,i\geq 1} K\rho_{ji}$. Consider the factor module M/F_0. By (9), $\rho_{ji} \equiv \frac{1}{(H-i)^j}\pi'_{i-1} \mod F_0$, hence $L' \equiv \mathbb{L}_1 \mod F_0$. Since $\mathbb{D}_1 = \mathbb{L}_1 \oplus F_0$ and $M = L' + F_0$, we must have the equality $\mathbb{D}_1 = M$. To finish the proof of the second equality of the corollary it suffices to show that $L' + F_0 = L \oplus (\oplus_{i,j\geq 1} K\rho_{ji}) \oplus F_0$ since then the equality $M = L \oplus (\oplus_{i\geq 1, j\geq 0} K\rho_{ji})$ follows as $F_0 = \oplus_{i\geq 0} K\pi_i$ and $\pi_i = \rho_i - \rho_{i+1}$. Let $l + \sum_{i,j\geq 1} \lambda_{ji}\rho_{ji} + f = 0$ for some $l\in L$, $\lambda_{ji}\in K$, and $f\in F_0$. Taking this equality modulo F_0 yields $l = 0$ and $\lambda_{ji} = 0$ since $\rho_{ji} \equiv \frac{1}{(H-i)^j}\pi'_{i-1} \mod F_0$. This implies $f = 0$, and we are done. $\square$

Note that F_0 is an ideal of $\mathbb{D}_1$ such that $F_0^2 = F_0$ and

$$\mathbb{D}_1/F_0 \simeq K[H^{\pm 1}, (H\pm 1)^{-1}, (H\pm 2)^{-1}, \dots]. \tag{11}$$

The equality $\mathbb{D}_1 = \mathbb{L}_1 \oplus F_0$ (Corollary 2.4) means that the set

$$\{H^i, \frac{1}{(H+j)^k}, \frac{1}{(H-j)^k}\pi'_{j-1}, \pi_l \,|\, i\in \mathbb{Z}, l\in \mathbb{N}, j,k\geq 1\}$$

is a K-basis for $\mathbb{D}_1$. Clearly, the set

$$\{H^i, \frac{1}{(H+j)^k}, \frac{1}{(H-j)^k}\pi'_{j-1} \,|\, i\in \mathbb{Z}, j,k\geq 1\} \tag{12}$$

is a K-basis for $\mathbb{L}_1$. Similarly, the equality $\mathbb{D}_1 = L\oplus (\oplus_{i\geq 1, j\geq 0} K\rho_{ji})$ means that the set

$$\{H^j, \frac{1}{(H+j)^k}, \rho_{ji} \,|\, j\in \mathbb{N}, i,k\geq 1\}$$

is a K-basis for $\mathbb{D}_1$.

Clearly, $F = \oplus_{i,j\geq 0} KE_{ij}$ where $E_{ij}(x^k) := \delta_{jk}x^i$, i.e. $\{E_{ij}\}$ are the 'elementary matrices' ($E_{ij}E_{kl} = \delta_{jk}E_{il}$), and

$$E_{ij} = \begin{cases} x^{i-j}\pi_j & \text{if } i\geq j, \\ (\frac{1}{H}\partial)^{j-i}\pi_j & \text{if } i<j. \end{cases} \tag{13}$$

For $k \in \mathbb{N}$, $F_{\pm k} = \oplus_{i-j=\pm k} KE_{ij}$. Note that $E_{ij} = E_{ik}\pi_k E_{kj}$ for all $i, j, k \geq 0$. Therefore, F is a *simple* ring such that $F^2 = F$, and $K[x]$ is a simple faithful F-module. The ring F is neither left nor right Noetherian as the next arguments show: for each natural $k \geq 0$, let $L_k := \oplus_{i\in\mathbb{N}, 0\leq j\leq k} KE_{ij}$ and $R_k := \oplus_{0\leq i\leq k, j\in\mathbb{N}} KE_{ij}$ then $L_0 \subset L_1 \subset \cdots$ and $R_0 \subset R_1 \subset \cdots$ are strictly ascending sequences of left and right F-modules respectively. This is the main reason why the Jacobian algebra $\mathbb{A}_1$ is also neither left nor right Noetherian (Theorem 2.5.(3)). The ring F is neither left nor right Artinian: for each natural $k \geq 1$, let $L'_k := \oplus_{i,j\in\mathbb{N}} KE_{i,j2^k}$ and $R_k := \oplus_{i,j\in\mathbb{N}} KE_{i2^k,j}$ then $L'_0 \supset L'_1 \supset \cdots$ and $R'_0 \supset R'_1 \supset \cdots$ are strictly descending sequences of left and right F-modules respectively.

Theorem 2.5 *Let K be a commutative $\mathbb{Q}$-algebra. Then*

1. $\mathbb{A}_1 = \oplus_{i\geq 1}\mathbb{L}_1\partial^i \oplus \mathbb{L}_1 \oplus (\oplus_{i\geq 1} x^i \mathbb{L}_1) \oplus F$.

2. *The set* $\{H^i\partial^l, \frac{1}{(H+j)^k}\partial^l, \frac{1}{(H-j)^k}\pi'_{j-1}\partial^l, x^m H^i, x^m\frac{1}{(H+j)^k}, x^m\frac{1}{(H-j)^k}\pi'_{j-1}, E_{st} \,|\, i \in \mathbb{Z}; j, k, l \geq 1; m, s, t \in \mathbb{N}\}$ *is a K-basis for $\mathbb{A}_1$.*

3. *The algebra $\mathbb{A}_1$ is neither a left nor right Noetherian algebra.*

4. *F is an ideal of $\mathbb{A}_1$, $F^2 = F$, and the factor algebra $\mathbb{A}_1/F$ is canonically isomorphic to the algebra $\mathcal{A}_1$ (the localization of the Weyl algebra A_1 at S_1, the multiplicative monoid generated by $H + i$, $i \in \mathbb{Z}$).*

Proof. 1. By Corollary 2.4, $\mathbb{D}_1 = \mathbb{L}_1 \oplus F_0$. For each natural number $i \geq 1$, $\mathbb{D}_1\partial^i = \mathbb{L}_1\partial^i \oplus F_{-i}$ and $x^i\mathbb{D}_1 = x^i\mathbb{L}_1 \oplus F_i$. Using these equalities together with the equalities $\mathbb{A}_1 = \oplus_{i\geq 1}\mathbb{D}_1\partial^i \oplus \mathbb{D}_1 \oplus (\oplus_{i\geq 1} x^i\mathbb{D}_1)$ (Theorem 2.3) and $F = \oplus_{i\geq 1} F_0\partial^i \oplus F_0 \oplus (\oplus_{i\geq 1} x^i F_0)$, one obtains the equality of statement 1 and (12).

2. Since, for all $i \geq 1$, the maps $\mathbb{L}_1 \to \mathbb{L}_1\partial^i$, $u \mapsto u\partial^i$, and $\mathbb{L}_1 \to x^i\mathbb{L}_1$, $u \mapsto x^i u$, are isomorphisms of K-modules, statement 2 follows from statement 1.

3. For each $i \in \mathbb{N}$, the sum $I_i := \oplus_{j=0}^i K\pi_j$ is an ideal of E_0. Since $I_i \subseteq \mathbb{D}_1 \subseteq E_0$, one has the strictly ascending chain of ideals of $\mathbb{D}_1$: $I_0 \subset I_1 \subset \cdots$. The ascending chain $\mathbb{A}_1 I_0 \subset \mathbb{A}_1 I_1 \subset \cdots$ of left homogeneous ideals of the algebra $\mathbb{A}_1$ is strictly ascending since the zero component of the left ideal $\mathbb{A}_1 I_j = \oplus_{i\geq 1}\mathbb{D}_1\partial^i I_j \oplus I_j \oplus (\oplus_{i\geq 1} x^i I_j)$ is I_j (note that $\mathbb{D}_1\partial^i I_j \subseteq F_{-i}$ and $x^i I_j \subseteq F_i$). Therefore, $\mathbb{A}_1$ is not a left Noetherian algebra.

Similarly, the ascending chain $I_0\mathbb{A}_1 \subset I_1\mathbb{A}_1 \subset \cdots$ of right homogeneous ideals of the algebra $\mathbb{A}_1$ is strictly ascending since the zero component of the right ideal $I_j\mathbb{A}_1 = \oplus_{i\geq 1} I_j\partial^i \oplus I_j \oplus (\oplus_{i\geq 1} I_j x^i \mathbb{D}_1)$ is I_j, and so $\mathbb{A}_1$ is not a right Noetherian algebra.

4. We proved already that F is an ideal of $\mathbb{A}_1$ such that $F^2 = F$. The K-module $K[x]$ is a topological K-module (even a topological K-algebra) with respect to the $\mathfrak{m}$-adic topology determined by the $\mathfrak{m}$-adic filtration $\{\mathfrak{m}^i\}_{i\geq 0}$ on $K[x]$ where $\mathfrak{m} := (x)$. Let $\mathrm{End}_{K,c}(K[x])$ be the algebra of all *continuous* K-endomorphisms of $K[x]$. Then $E \subseteq \mathrm{End}_{K,c}(K[x])$. Let $\mathcal{G}$ be the algebra of *germs* of continuous K-endomorphisms of $K[x]$ at 0. An element of $\mathcal{G}$ is an equivalence class $[f]$ of a continuous K-linear map of the type $f : \mathfrak{m}^i \to K[x]$, and two such maps are equivalent, $f \sim f'$, if they have the same restriction $f|_{\mathfrak{m}^j} = f'|_{\mathfrak{m}^j}$ for

a sufficiently large j ($\mathfrak{m}^i$ is a topological K-module with respect to the induced topology coming from the inclusion $\mathfrak{m}^i \subseteq K[x]$).

The kernel of the K-algebra homomorphism $E \to \mathcal{G}$, $f \mapsto [f]$, is F. Thus, the kernel of the K-algebra homomorphism $g : \mathbb{A}_1 \to \mathcal{G}$, $f \mapsto [f]$, is also F since $F \subseteq \mathbb{A}_1$. The image $g(\mathbb{D}_1)$ is naturally isomorphic to the algebra $S_1^{-1}K[H]$ since $g(H) = [H]$ and $g(\frac{1}{(H-j)^k}\pi'_{j-1}) = [\frac{1}{(H-j)^k}]$ for all $j, k \geq 1$. Now, it follows from statement 1 and the decomposition $\mathcal{A}_1 = \oplus_{i\geq 1} S_1^{-1}K[H]\partial^i \oplus S_1^{-1}K[H] \oplus (\oplus_{i\geq 1} x^i S_1^{-1}K[H])$ that the image $g(\mathbb{A}_1)$ is naturally isomorphic to the algebra $\mathcal{A}_1$. $\Box$

An ideal I of a ring R such that $0 \neq I \neq R$ is called a *proper* ideal of R.

Corollary 2.6 *Let K be a field of characteristic zero. Then*

1. *F is the only proper ideal of the algebra $\mathbb{A}_1$, hence F is a maximal ideal.*
2. *$\mathbb{A}_1/F \simeq \mathcal{A}_1$ is a simple Noetherian domain.*
3. $\mathrm{GK}\,(\mathbb{A}_1) = \mathrm{GK}\,(\mathcal{A}_1) = 3$.
4. *$\mathbb{A}_1$ is a prime ring.*
5. *The algebra $\mathbb{A}_1$ is central, i.e. the centre of $\mathbb{A}_1$ is K.*

Proof. 2. Statement 2 follows from Theorem 2.5.(4) and Theorem 2.1.(1).

1. By statement 2, F is a maximal ideal of $\mathbb{A}_1$. Let I be a proper ideal of $\mathbb{A}_1$. We have to show that $I = F$. Let a be a nonzero element of I. Then $0 \neq FaF \subseteq F \cap I$, and so $FaF = F$ since F is a simple algebra (i.e. a simple F-bimodule). Now, $F \subseteq I$ implies $F = I$ by the maximality of F. So, F is the only proper ideal of the algebra $\mathbb{A}_1$.

3. Since $\mathbb{A}_1/F \simeq \mathcal{A}_1$, we have $\mathrm{GK}\,(\mathbb{A}_1) \geq \mathrm{GK}\,(\mathcal{A}_1) = 3$ (Theorem 2.1.(2)). Since $\mathbb{A}_1 = \oplus_{i\geq 1}\mathbb{L}_1\partial^i \oplus \mathbb{L}_1 \oplus (\oplus_{i\geq 1} x^i \mathbb{L}_1) \oplus F$ and $\mathcal{A}_1 = \oplus_{i\geq 1} S_1^{-1}K[H]\partial^i \oplus S_1^{-1}K[H] \oplus (\oplus_{i\geq 1} x^i S_1^{-1}K[H])$, the reverse inequality $\mathrm{GK}\,(\mathbb{A}_1) \leq 3$ follows from Theorem 2.5.(1,2) using the same sort of estimates as in the proof of the inequality $\mathrm{GK}\,(\mathcal{A}_1) \leq 3$ (see [5] for details and the fact that the elements of F do not contribute to the growth of degree 3).

4. F is the only proper ideal of $\mathbb{A}_1$, and so $F^2 = F$, hence $\mathbb{A}_1$ is a prime ring.

5. The field K belongs to the centre of $\mathbb{A}_1$. Let z be a central element of $\mathbb{A}_1$. We have to show that $z \in K$. The algebra $\mathbb{A}_1/F \simeq \mathcal{A}_1$ is central, hence $z = \lambda + f$ for some $\lambda \in K$ and $f \in F$. Then $z - \lambda = f$ belongs to the centre of F which is obviously equal to zero. Hence $z = \lambda \in K$, as required. $\Box$

For $k \geq 1$,

$$\begin{aligned} x\frac{1}{(H-j)^k}\pi'_{j-1} &= \frac{1}{(H-1-j)^k}\pi'_j x, \quad j \geq 1, \\ \partial\frac{1}{(H-j)^k}\pi'_{j-1} &= \frac{1}{(H+1-j)^k}\pi'_{j-2}\partial, \quad j \geq 2, \\ \partial\frac{1}{(H-1)^k}\pi'_0 &= \frac{1}{H^k}\partial. \end{aligned}$$

By (5), the algebra $\mathbb{A}_n$ is the tensor product of the $\mathbb{Z}$-graded algebras $\mathbb{A}_1(i) = \oplus_{j\in\mathbb{Z}}\mathbb{A}_{1,j}(i)$. Therefore, the algebra $\mathbb{A}_n$ is a $\mathbb{Z}^n$-graded algebra,

$$\mathbb{A}_n = \oplus_{\alpha\in\mathbb{Z}^n}\mathbb{A}_{n,\alpha}, \ \ \mathbb{A}_{n,\alpha} := \otimes_{i=1}^n \mathbb{A}_{1,\alpha_i}(i).$$

The $\mathbb{Z}^n$-grading on $\mathbb{A}_n$ is the tensor product of the $\mathbb{Z}$-gradings of the tensor multiples, and an element a of $\mathbb{A}_n$ belongs to $\mathbb{A}_{n,\alpha}$ iff $a(x^\beta) \in Kx^{\alpha+\beta}$ for all $\beta \in \mathbb{N}^n$. Let

$$\mathbb{D}_n := \mathbb{A}_{n,0} = \mathbb{D}_1(1)\otimes\mathbb{D}_1(2)\otimes\cdots\otimes\mathbb{D}_1(n).$$

The polynomial algebra $P_n = \oplus_{\alpha\in\mathbb{N}^n}P_{n,\alpha}$ is an $\mathbb{N}^n$-graded, hence a $\mathbb{Z}^n$-graded algebra. Let $E = E(P_n) = \oplus_{\alpha\in\mathbb{Z}^n}E_\alpha$ be the $\mathbb{Z}^n$-graded algebra as in (6). The map

$$E_0 = \{f \in \mathrm{End}_K(P_n)\,|\, f(x^\alpha) = f_\alpha x^\alpha, f_\alpha \in K, \alpha\in\mathbb{N}^n\} \to K^{\mathbb{N}^n}, \ \ f\mapsto (f_\alpha),$$

is a K-algebra isomorphism. Each element $\alpha = \sum_{i=1}^n \alpha_i e_i \in \mathbb{Z}^n = \oplus_{i=1}^n\mathbb{Z}e_i$ is a unique difference $\alpha = \alpha_+ - \alpha_-$ where $\alpha_+ = \sum_{\alpha_i\geq 0}\alpha_i e_i$ and $\alpha_- = -\sum_{\alpha_i\leq 0}\alpha_i e_i$. For each $\alpha\in\mathbb{Z}^n$, $E_\alpha = x^{\alpha_+}E_0\partial^{\alpha_-}$, and so

$$E = \oplus_{\alpha\in\mathbb{Z}_n}x^{\alpha_+}E_0\partial^{\alpha_-}. \qquad (14)$$

For each $\alpha\in\mathbb{Z}^n$, $\mathbb{A}_{n,\alpha} = \mathbb{A}_n\cap E_\alpha = x^{\alpha_+}\mathbb{D}_n\partial^{\alpha_-}$ where $\mathbb{D}_n = \mathbb{A}_{n,0} = \mathbb{A}_n\cap E_0$.

The involution θ on $\mathbb{A}_n$. Let K be a commutative $\mathbb{Q}$-algebra. The Weyl algebra A_n admits the *involution*

$$\theta : A_n \to A_n, \ \ x_i\mapsto\partial_i, \ \ \partial_i\mapsto x_i, \ \ i=1,\ldots,n,$$

i.e. it is a K-algebra anti-isomorphism ($\theta(ab) = \theta(b)\theta(a)$) such that $\theta^2 = \mathrm{id}_{A_n}$. The involution θ can be uniquely extended to the involution of $\mathbb{A}_n$ by the rule

$$\theta : \mathbb{A}_n \to \mathbb{A}_n, \ \ x_i\mapsto\partial_i, \ \ \partial_i\mapsto x_i, \ \ \theta(H_i^{-1}) = H_i^{-1} \ \ i=1,\ldots,n. \qquad (15)$$

Uniqueness is obvious: $\theta(H_i) = \theta(\partial_i x_i) = \theta(x_i)\theta(\partial_i) = \partial_i x_i = H_i$ and so $\theta(H_i^{-1}) = H_i^{-1}$. To prove existence recall that each right module over a ring R is a left module over the opposite ring R^{op}. The involution θ on A_n comes from considering the polynomial algebra P_n as the *right* A_n-module by the rule $pa := \theta(a)p$ for all $p\in P_n$ and $a\in A_n$. Since $\theta(H_i) = H_i$, $i=1,\ldots,n$, P_n is the *faithful* right $\mathbb{A}_n$-module, and this proves the existence of the involution $\theta : \mathbb{A}_n\to\mathbb{A}_n$ (θ is injective since P_n is a faithful right $\mathbb{A}_n$-module, θ is obviously surjective). So, the algebra $\mathbb{A}_n$ is *self-dual* (i.e. it is isomorphic to its opposite algebra, $\theta : \mathbb{A}_n \simeq \mathbb{A}_n^{op}$). This means that left and right algebraic properties of the algebra $\mathbb{A}_n$ are the same.

For $n=1$, F is the only proper ideal of $\mathbb{A}_1$, hence $\theta(F) = F$. Moreover,

$$\theta(E_{ij}) = \frac{i!}{j!}\,E_{ji} \qquad (16)$$

where $0! := 1$. In more detail, since $\theta(H) = H$ and $E_{ii} = \pi_i = x^i \frac{1}{(-i,i)}\partial^i - x^{i+1}\frac{1}{(-i-1,i+1)}\partial^{i+1}$, we have $\theta(E_{ii}) = E_{ii}$. For $i > j$, $E_{ij} = x^{i-j}\pi_j$, and so

$$\theta(E_{ij}) = \pi_j \partial^{i-j} = i(i-1)\cdots(j+1)E_{ji} = \frac{i!}{j!}E_{ji}.$$

For $i < j$, $E_{ij} = (\frac{1}{H}\partial)^{j-i}\pi_j$, and so $\theta(E_{ij}) = \pi_j(x\frac{1}{H})^{j-i} = \frac{1}{(i+1)(i+2)\cdots j}E_{ji} = \frac{i!}{j!}E_{ji}$.

For $n = 1$, the ring $F = \oplus_{i,j\in\mathbb{N}} KE_{ij}$ is equal to the matrix ring $M_\infty(K) := \cup_{d\geq 1} M_d(K)$ where $M_d(K) := \oplus_{0\leq i,j\leq d-1} KE_{ij}$. The ring $F = M_\infty(K)$ admits the canonical involution which is the transposition $(\cdot)^t : E_{ij} \mapsto E_{ji}$. Let $D_!$ be the infinite diagonal matrix $\mathrm{diag}(0!, 1!, 2!, \ldots)$. Then, for $u \in F = M_\infty(K)$,

$$\theta(u) = D_!^{-1}u^t D_!. \tag{17}$$

Note that $D_! \notin M_\infty(K)$.

For an arbitrary n, $F^{\otimes n} = \oplus_{\alpha,\beta\in\mathbb{N}^n} KE_{\alpha\beta} = M_\infty(K)^{\otimes n}$ where $E_{\alpha\beta} := \otimes_{i=1}^n E_{\alpha_i\beta_i}$. By (16),

$$\theta(E_{\alpha\beta}) = \frac{\alpha!}{\beta!}E_{\beta\alpha}, \tag{18}$$

$$\theta(F^{\otimes n}) = F^{\otimes n}. \tag{19}$$

Let $D_{n,!} := D_!^{\otimes n}$. Then, for $u \in F^{\otimes n}$,

$$\theta(u) = D_{n,!}^{-1}u^t D_{n,!} \tag{20}$$

where $(\cdot)^t : M_\infty(K)^{\otimes n} \to M_\infty(K)^{\otimes n}$, $E_{\alpha\beta} \mapsto E_{\beta\alpha}$, is the transposition map.

Consider the bilinear, symmetric, non-degenerate form $(\cdot,\cdot) : P_n \times P_n \to K$ given by the rule $(x^\alpha, x^\beta) := \alpha!\delta_{\alpha,\beta}$ for all $\alpha,\beta \in \mathbb{N}^n$. Then, for all $p, q \in P_n$ and $a \in \mathbb{A}_n$,

$$(p, aq) = (\theta(a)p, q). \tag{21}$$

The Weyl algebra A_n admits, so-called, the *Fourier transform*, it is the K-algebra automorphism $\mathcal{F} : A_n \to A_n$, $x_i \mapsto \partial_i$, $\partial_i \mapsto -x_i$, $i = 1, \ldots, n$. Since $\mathcal{F}(H_i) = -(H_i - 1)$, H_i is a unit of $\mathbb{A}_n$ and $H_i - 1$ is not, one *cannot* extend the Fourier transform to $\mathbb{A}_n$.

The algebra $\mathbb{A}_n$ is a prime algebra. Consider the ideals of the algebra $\mathbb{A}_n$:

$$\mathfrak{p}_1 := F \otimes \mathbb{A}_{n-1}, \mathfrak{p}_2 := \mathbb{A}_1 \otimes F \otimes \mathbb{A}_{n-2}, \ldots, \mathfrak{p}_n := \mathbb{A}_{n-1} \otimes F.$$

Then $\mathbb{A}_n/\mathfrak{p}_i \simeq (\mathbb{A}_1/F) \otimes \mathbb{A}_{n-1} \simeq \mathcal{A}_1 \otimes \mathbb{A}_{n-1}$ and $\cap_{i=1}^n \mathfrak{p}_i = F^{\otimes n}$. Let $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$. Then

$$\mathbb{A}_n/\mathfrak{a}_n \simeq (\mathbb{A}_1/F)^{\otimes n} \simeq \mathcal{A}_1^{\otimes n} = \mathcal{A}_n. \tag{22}$$

Corollary 2.7 *Let K be a field of characteristic zero. Then*

1. $\mathrm{GK}\,(\mathbb{A}_n) = 3n$.

2. $\mathrm{GK}\,(M) \geq n$ *for all nonzero finitely generated (left or right)* $\mathbb{A}_n$*-modules* M.

3. The centre of $\mathbb{A}_n$ *is* K.

4. $F^{\otimes n}$ *is the smallest nonzero ideal of the algebra* $\mathbb{A}_n$, $(F^{\otimes n})^2 = F^{\otimes n}$.

5. The algebra $\mathbb{A}_n$ *is prime.*

6. If $0 \neq a \in \mathbb{A}_n$ *and* I *is a nonzero ideal of* $\mathbb{A}_n$ *then* $aI \neq 0$, $Ia \neq 0$, *and* $IaI \neq 0$.

7. The ideal $\mathfrak{a}_n$ *is the largest ideal of* $\mathbb{A}_n$ *distinct from* $\mathbb{A}_n$; $\mathfrak{a}_n^2 = \mathfrak{a}_n$; *hence* $\mathfrak{a}_n$ *is the only maximal ideal of* $\mathbb{A}_n$.

8. ${}_{\mathbb{A}_n}F^{\otimes n} \simeq P_n^{(\mathbb{N}^n)}$ *is a faithful, semi-simple, left* $\mathbb{A}_n$*-module;* $F^{\otimes n}_{\mathbb{A}_n} \simeq P_n^{(\mathbb{N}^n)}{}_{\mathbb{A}_n}$ *is a faithful, semi-simple, right* $\mathbb{A}_n$*-module;* ${}_{\mathbb{A}_n}F^{\otimes n}_{\mathbb{A}_n} \simeq P_n^{(\mathbb{N}^n)}$ *is a faithful, simple* $\mathbb{A}_n$*-bimodule.*

9. $F^{\otimes n}$ *is the socle of* $\mathbb{A}_n$ *considered as a left* $\mathbb{A}_n$*-module, or a right* $\mathbb{A}_n$*-module, or an* $\mathbb{A}_n$*-bimodule.*

10. ${}_{\mathbb{A}_n}P_n$ *(resp.* $(P_n)_{\mathbb{A}_n}$*) is the only faithful, simple, left (resp. right)* $\mathbb{A}_n$*-module.*

11. $\mathrm{GK}\,(\mathbb{A}_n/\mathfrak{a}) = 3n$ *for all ideals* $\mathfrak{a}$ *of* $\mathbb{A}_n$ *such that* $\mathfrak{a} \neq \mathbb{A}_n$.

Proof. 1. On the one hand, $\mathrm{GK}\,(\mathbb{A}_n) \geq \mathrm{GK}\,(\mathbb{A}_n/\mathfrak{a}_n) = \mathrm{GK}\,(\mathcal{A}_n) = 3n$ (Theorem 2.1.(2)); on the other, $\mathrm{GK}\,(\mathbb{A}_n) = \mathrm{GK}\,(\mathbb{A}_1^{\otimes n}) \leq n\,\mathrm{GK}\,(\mathbb{A}_1) = 3n$. Therefore, $\mathrm{GK}\,(\mathbb{A}_n) = 3n$.

2. Statement 2 is an easy corollary of the *inequality of Bernstein*: $\mathrm{GK}\,(N) \geq n$ for all nonzero finitely generated (left or right) A_n-modules N. Let M be a nonzero finitely generated $\mathbb{A}_n$-module and $0 \neq u \in M$. Then $\mathrm{GK}_{\,\mathbb{A}_n}(M) \geq \mathrm{GK}_{\,\mathbb{A}_n}(\mathbb{A}_n u) \geq \mathrm{GK}_{\,A_n}(A_n u) \geq n$.

3. To prove that the centre $Z(\mathbb{A}_n)$ of $\mathbb{A}_n$ is K we use induction on n. The case $n = 1$ is Corollary 2.6.(5). Suppose that $n > 1$ and the algebra $\mathbb{A}_m$ is central for all $m < n$. The kernel of the algebra homomorphism $\mathbb{A}_n \to \prod_{i=1}^n \mathbb{A}_n/\mathfrak{p}_i$ is $\cap_{i=1}^n \mathfrak{p}_i = F^{\otimes n}$. Since all the algebras $\mathbb{A}_n/\mathfrak{p}_i \simeq \mathcal{A}_1 \otimes \mathbb{A}_{n-1}$ are central, we have $Z(\prod_{i=1}^n \mathbb{A}_n/\mathfrak{p}_i) = \prod_{i=1}^n Z(\mathbb{A}_n/\mathfrak{p}_i) = \prod_{i=1}^n K$ where $Z(R)$ is the centre of R. If $z \in Z(\mathbb{A}_n)$ then $z + F^{\otimes n} \in Z(\prod_{i=1}^n \mathbb{A}_n/\mathfrak{p}_i) = \prod_{i=1}^n K$, and so $z = \lambda + f$ for some $\lambda \in K$ and $f \in F^{\otimes n}$. Now, $f = z - \lambda \in Z(F^{\otimes n}) = 0$, i.e. $z = \lambda \in K$, and so the algebra $\mathbb{A}_n$ is central.

4. Clearly, $(F^{\otimes n})^2 = (F^2)^{\otimes n} = F^{\otimes n}$. It remains to prove minimality of $F^{\otimes n}$. This is obvious for $n = 1$ (Corollary 2.6.(1)). To prove the general case we use induction on n. Suppose that $n > 1$ and the result is true for all $n' < n$. Let I be a nonzero ideal of $\mathbb{A}_n$. We have to show that $F^{\otimes n} \subseteq I$. Choose a nonzero element, say a, of I. Since $a \in \mathbb{A}_n = \mathbb{A}_1 \otimes \mathbb{A}_{n-1}$, the element a can be written as a sum $a = \sum_{i=1}^s a_i \otimes b_i$ for some elements $a_i \in \mathbb{A}_1$ and $b_i \in \mathbb{A}_{n-1}$ such that the elements $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ are K-linearly independent elements of the algebras $\mathbb{A}_1$ and $\mathbb{A}_{n-1}$ respectively. Choose an element, say f of F such that $fa_1 \neq 0$. Changing a for $fa \neq 0$ (and deleting zero terms of the type $fa_i \otimes b_i$) one may assume that all $a_i \in F$. Note that $\{E_{kl}\}$ is the K-basis of F. So, the element a can be written as a finite sum $a = E_{kl} \otimes \alpha + E_{st} \otimes \beta + \cdots + E_{pq} \otimes \gamma$ for some

K-linearly independent elements $\alpha, \beta, \dots, \gamma$ of $\mathbb{A}_{n-1}$ and distinct elements $E_{kl}, E_{st}, \dots, E_{pq}$. Then $b := E_{kk}aE_{ll} = E_{kl} \otimes \alpha \in I$. By induction, $F^{\otimes(n-1)} \subseteq \mathbb{A}_{n-1}\alpha\mathbb{A}_{n-1}$, and so

$$I \supseteq \mathbb{A}_n b \mathbb{A}_n = \mathbb{A}_1 E_{kl} \mathbb{A}_1 \otimes \mathbb{A}_{n-1}\alpha\mathbb{A}_{n-1} \supseteq F \otimes F^{\otimes(n-1)} = F^{\otimes n}.$$

5. Let I and J be nonzero ideals of $\mathbb{A}_n$. By statement 4, they contain the ideal $F^{\otimes n}$. Now, $IJ \supseteq (F^{\otimes n})^2 = F^{\otimes n} \neq 0$. This means that $\mathbb{A}_n$ is a prime ring.

6. Statement 6 follows directly from statement 5. Suppose that $aI = 0$ for some nonzero element a of $\mathbb{A}_n$ and a nonzero ideal I. We seek a contradiction. Then $0 = \mathbb{A}_n aI = \mathbb{A}_n a \mathbb{A}_n I \neq 0$ since $\mathbb{A}_n$ is a prime algebra, a contradiction. Similarly, $Ia = 0$ (resp. $IaI = 0$) implies $0 = I\mathbb{A}_n a \mathbb{A}_n \neq 0$ (resp. $0 = I\mathbb{A}_n aI \neq 0$), a contradiction.

7. $\mathfrak{a}_n \supseteq \mathfrak{a}_n^2 = (\mathfrak{p}_1 + \cdots + \mathfrak{p}_n)^2 \supseteq \mathfrak{p}_1^2 + \cdots + \mathfrak{p}_n^2 = \mathfrak{p}_1 + \cdots + \mathfrak{p}_n = \mathfrak{a}_n$ since $\mathfrak{p}_1^2 = \mathfrak{p}_1, \dots, \mathfrak{p}_n^2 = \mathfrak{p}_n$, and so $\mathfrak{a}_n = \mathfrak{a}_n^2$.

It remains to show that $\mathfrak{a}_n$ is the largest ideal distinct from $\mathbb{A}_n$, that is $a \notin \mathfrak{a}_n$ implies $\mathbb{A}_n a \mathbb{A}_n = \mathbb{A}_n$ where $a \in \mathbb{A}_n$. Let B_n be the K-basis for $\mathbb{A}_n$ that is the tensor product of the K-bases from Theorem 2.5.(2). B_n is the disjoint union of its two subsets $\mathcal{M}_n := \{b \in B_n \,|\, b \in \mathfrak{a}_n\}$ and $\mathcal{N}_n := \{b \in B_n \,|\, b \notin \mathfrak{a}_n\}$. So, $b \in \mathcal{M}_n$ iff the product b contains a matrix unit $E_{st}(i) \in F(i)$. Clearly, $\mathfrak{a}_n = \oplus_{\mu \in \mathcal{N}_n} K\nu$ and $\mathbb{A}_n = \mathfrak{a}'_n \oplus \mathfrak{a}_n$ where $\mathfrak{a}'_n := \oplus_{\mu \in \mathcal{M}_n} K\mu$. Elements of the set $\mathbb{A}_n \backslash \mathfrak{a}_n$ are called *generic*. So, an element of $\mathbb{A}_n$ is generic iff it has at least one nonzero μ-coordinate for some $\mu \in \mathcal{M}_n$. We have to show that $\mathbb{A}_n a \mathbb{A}_n = \mathbb{A}_n$ for all generic elements $a \in \mathbb{A}_n$. This is true when $n = 1$ (Corollary 2.6.(1)). To prove general case we use induction on n. So, let $n \geq 2$ and we assume that the claim is true for all $n' < n$. Let a be a generic element of $\mathbb{A}_n$. Then $a = a_1 \otimes b_1 + \cdots + a_s \otimes b_s \in \mathbb{A}_1 \otimes \mathbb{A}_{n-1}$ where a_i are nonzero elements of $\mathbb{A}_1$ such that a_1 is generic; b_i are distinct elements of the basis B_{n-1} such that $b_1 \in \mathcal{N}_{n-1}$. By statement 6, $Fa_1F \neq 0$, and so $E_{ij}a_1E_{kl} \neq 0$ for some $i, j, k, l \in \mathbb{N}$. Then, for each $t = 1, \dots, s$, $E_{ij}a_tE_{kl} = \lambda_t E_{il}$ for some $\lambda_t \in K$, necessarily $\lambda_1 \neq 0$. Now, $E_{ij}aE_{kl} = E_{il} \otimes u$ where $u := \lambda_1 b_1 + \cdots + \lambda_s b_s$ is a generic element of $\mathbb{A}_{n-1}$ since $\lambda_1 \neq 0$ and $b_1 \in \mathcal{N}_{n-1}$. By induction, $\mathbb{A}_{n-1}u\mathbb{A}_{n-1} = \mathbb{A}_{n-1}$, and so

$$\mathbb{A}_n a \mathbb{A}_n \supseteq \mathbb{A}_n(E_{il} \otimes u)\mathbb{A}_n = \mathbb{A}_1 E_{il} \mathbb{A}_1 \otimes \mathbb{A}_{n-1}u\mathbb{A}_{n-1} = F \otimes \mathbb{A}_{n-1} = \mathfrak{p}_1.$$

By symmetry, all $\mathfrak{p}_i \subseteq \mathbb{A}_n a \mathbb{A}_n$, and so $\mathfrak{a}_n = \mathfrak{p}_1 + \cdots + \mathfrak{p}_n \subseteq \mathbb{A}_n a \mathbb{A}_n$. $\mathfrak{a}_n$ is the maximal ideal of $\mathbb{A}_n$ that is properly contained in the ideal $\mathbb{A}_n a \mathbb{A}_n$ (since a is generic), and so $\mathbb{A}_n a \mathbb{A}_n = \mathbb{A}_n$, as required.

8. Let, for a moment, $n = 1$. One can easily verify that, for all $i, j, k \in \mathbb{N}$,

$$x^k E_{ij} = E_{i+k,j}, \;\; \partial^k E_{ij} = i(i-1)\cdots(i-k+1)E_{i-k,j}, \tag{23}$$

$$E_{ij}x^k = E_{i,j-k}, \;\; E_{ij}\partial^k = (j+k)(j+k-1)\cdots(j+1)E_{i,j+k}, \tag{24}$$

where $E_{st} := 0$ if either $s < 0$ or $t < 0$. By (23), for each $j \in \mathbb{N}$, the left $\mathbb{A}_1$-module $C_j := \oplus_{i \in \mathbb{N}} KE_{ij}$ is isomorphic to the left $\mathbb{A}_1$-module $K[x]$, and so C_j is a faithful, simple, left $\mathbb{A}_1$-module. The left $\mathbb{A}_1$-module

$$F = \oplus_{i \in \mathbb{N}} C_i \simeq K[x]^{(\mathbb{N})} \tag{25}$$

is a direct sum of $\mathbb{N}$ copies of $K[x]$, and so ${}_{\mathbb{A}_1}F$ is the *faithful, semi-simple, left* $\mathbb{A}_1$-module.

For an arbitrary n, the left $\mathbb{A}_n$-module $C_{i_1} \otimes \cdots \otimes C_{i_n}$ is isomorphic to P_n. Therefore,

$$ {}_{\mathbb{A}_n}F^{\otimes n} \simeq \oplus_{i_1,\ldots,i_n\in\mathbb{N}} C_{i_1} \otimes \cdots \otimes C_{i_n} \simeq P_n^{(\mathbb{N}^n)} \quad (26) $$

is a semi-simple, left $\mathbb{A}_n$-module which is faithful, by statement 6.

Recall that the structure of the right $\mathbb{A}_n$-module P_n is given by the rule: $p * a := \theta(a)p$ where $p \in P_n$ and $a \in \mathbb{A}_n$. By (19), $\theta(F^{\otimes n}) = F^{\otimes n}$. Then, by (26), the K-module $F^{\otimes n} \simeq P_n^{(\mathbb{N}^n)}$ has the natural structure of *right* $\mathbb{A}_n$-module, namely, $f * a := \theta(a)f$ where $f \in F^{\otimes n}$ and $a \in \mathbb{A}_n$. In order to distinguish this structure of right $\mathbb{A}_n$-module from the obvious structure (as a right ideal of $\mathbb{A}_n$) we write $F_\theta^{\otimes n} \simeq P_n^{(\mathbb{N}^n)}$. The map

$$ \theta : F^{\otimes n} \to F_\theta^{\otimes n} \simeq P_n^{(\mathbb{N}^n)}, \quad f \mapsto \theta(f), \quad (27) $$

is an *isomorphism* of right $\mathbb{A}_n$-modules (since $\theta(fa) = \theta(a)\theta(f) = \theta(f) * a$). Therefore, $F^{\otimes n}_{\mathbb{A}_n} \simeq F_\theta^{\otimes n} \simeq P_n^{(\mathbb{N}^n)}$ is the faithful, semi-simple, right $\mathbb{A}_n$-module by the proved left version of this fact.

The $\mathbb{A}_n$-bimodule $F^{\otimes n}$ is simple since the ring $F^{\otimes n} \simeq M_\infty(K)^{\otimes n} \simeq M_\infty(K)$ is simple. The map $1 \otimes \theta : \mathbb{A}_n \otimes \mathbb{A}_n^{op} \to \mathbb{A}_{2n}$ is an isomorphism of K-algebras such that $1 \otimes \theta(F^{\otimes n} \otimes F^{\otimes n}) = F^{\otimes 2n}$. Using this equality and ${}_{\mathbb{A}_n}F^{\otimes n}_{\mathbb{A}_n} \simeq {}_{\mathbb{A}_n\otimes\mathbb{A}_n^{op}}F^{\otimes n} \simeq {}_{\mathbb{A}_{2n}}F^{\otimes n}$, we see that the $\mathbb{A}_n$-bimodule $F^{\otimes n}$ is faithful: if $\mathfrak{a}F^{\otimes n} = 0$ for some nonzero ideal $\mathfrak{a}$ of $\mathbb{A}_{2n}$ then $F^{\otimes 2n} \subseteq \mathfrak{a}$, and so $0 = \mathfrak{a}F^{\otimes n} \supseteq F^{\otimes 2n} \cdot F^{\otimes n} = F^{\otimes n}F^{\otimes n}F^{\otimes n} = F^{\otimes n} \neq 0$, a contradiction.

9. Let $\mathrm{soc}(\mathbb{A}_n)$ be the socle of the module ${}_{\mathbb{A}_n}\mathbb{A}_n$ (resp. ${}_{\mathbb{A}_n}\mathbb{A}_{n\mathbb{A}_n}$). By statement 8, $F^{\otimes n} \subseteq \mathrm{soc}(\mathbb{A}_n)$. Suppose that $F^{\otimes n} \neq \mathrm{soc}(\mathbb{A}_n)$. Then $\mathrm{soc}(\mathbb{A}_n) = F^{\otimes n} \oplus M$ for a nonzero module ${}_{\mathbb{A}_n}M$ (resp. ${}_{\mathbb{A}_n}M_{\mathbb{A}_n}$). On the one hand, $F^{\otimes n}\mathrm{soc}(\mathbb{A}_n) \subseteq F^{\otimes n}$ and so $F^{\otimes n}M = 0$. On the other hand, $F^{\otimes n}M \neq 0$, by statement 6, a contradiction. Therefore, $\mathrm{soc}(\mathbb{A}_n) = F^{\otimes n}$.

The algebra $\mathbb{A}_n$ admits the involution θ such that $\theta(F^{\otimes n}) = F^{\otimes n}$. Therefore, $\mathrm{soc}(\mathbb{A}_{n\mathbb{A}_n}) = F^{\otimes n}$ since $\mathrm{soc}({}_{\mathbb{A}_n}\mathbb{A}_n) = F^{\otimes n}$.

10. Let M be a faithful, simple, left (resp. right) $\mathbb{A}_n$-module. Then $F^{\otimes n}M \neq 0$ (resp. $MF^{\otimes n} \neq 0$). Choose a nonzero element, say m, of M such that $F^{\otimes n}m \neq 0$ (resp. $mF^{\otimes n} \neq 0$). Then $M = F^{\otimes n}m$ (resp. $M = mF^{\otimes n}$), by simplicity of M. There is the epimorphism $F^{\otimes n} \to F^{\otimes n}m = M$, $f \mapsto fm$ (resp. $F^{\otimes n} \to mF^{\otimes n} = M$, $f \mapsto mf$) of left (resp. right) $\mathbb{A}_n$-modules. Now, the result follows from statement 8.

11. $3n = \mathrm{GK}\,(\mathbb{A}_n) \geq \mathrm{GK}\,(\mathbb{A}_n/\mathfrak{a}) \geq \mathrm{GK}\,(\mathbb{A}_n/\mathfrak{a}_n) = \mathrm{GK}\,(\mathcal{A}_n) = 3n$, and so $\mathrm{GK}\,(\mathbb{A}_n/\mathfrak{a}) = 3n$. $\square$

3 Unique factorization of ideals of $\mathbb{A}_n$ and $\mathrm{Spec}(\mathbb{A}_n)$

In this section, all the results on ideals that are mentioned in the Introduction are proved.

Let $\mathcal{B}_n$ be the set of all functions $f : \{1, 2, \ldots, n\} \to \mathbb{Z}_2 := \{0, 1\}$ where $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ is a field. $\mathcal{B}_n$ is a commutative ring with respect to addition and multiplication of functions. For $f, g \in \mathcal{B}_n$, we write $f \geq g$ iff $f(i) \geq g(i)$ for all $i = 1, \ldots, n$ where $1 > 0$. Then $(\mathcal{B}_n, \geq)$ is a partially ordered set. For each function $f \in \mathcal{B}_n$, I_f denotes the ideal $I_{f(1)} \otimes \cdots \otimes I_{f(n)}$

of $\mathbb{A}_n$ which is the tensor product of the ideals $I_{f(i)}$ of the tensor components $\mathbb{A}_1(i)$ in $\mathbb{A}_n = \mathbb{A}_1(1) \otimes \cdots \otimes \mathbb{A}_1(n)$ where $I_0 := F$ and $I_1 := \mathbb{A}_1$. $f \geq g$ iff $I_f \supseteq I_g$. For $f, g \in \mathcal{B}_n$, $I_f I_g = I_f \cap I_g = I_{fg}$. By induction on the number of functions one immediately proves that, for $f_1, \ldots, f_s \in \mathcal{B}_n$,

$$\prod_{i=1}^{s} I_{f_i} = \cap_{i=1}^{s} I_{f_i} = I_{f_1 \cdots f_s}.$$

Let $\mathcal{C}_n$ be the set of all subsets of $\mathcal{B}_n$ all distinct elements of which are incomparable (two distinct elements f and g of $\mathcal{B}_n$ are *incomparable* iff $f \not\leq g$ and $g \not\leq f$). For each $C \in \mathcal{C}_n$, let $I_C := \sum_{f \in C} I_f$, the ideal of $\mathbb{A}_n$. The next result classifies ideals of the algebra $\mathbb{A}_n$.

Theorem 3.1 *Let K be a field of characteristic zero. Then*

1. *The map $C \mapsto I_C := \sum_{f \in C} I_f$ from the set $\mathcal{C}_n$ to the set of ideals of $\mathbb{A}_n$ is a bijection where $I_\emptyset := 0$. In particular, there are only finitely many ideals of $\mathbb{A}_n$.*

2. *Each ideal I of $\mathbb{A}_n$ is an idempotent ideal, i.e. $I^2 = I$.*

3. *Ideals of $\mathbb{A}_n$ commute ($IJ = JI$).*

Proof. 1. Statement 1 follows from Lemma 3.2.

2. The result is obvious for $I = 0$. So, let $I \neq 0$. By statement 1, $I = \sum_{f \in C} I_f$ for some $C \in \mathcal{C}_n$. Then $I^2 = \sum_{f \in C} I_f^2 + \sum_{f \neq g} I_f I_g = \sum_{f \in C} I_f + \sum_{f \neq g} I_f I_g = \sum_{f \in C} I_f = I$.

3. $I_f I_g = I_{fg} = I_{gf} = I_g I_f$ for all $f, g \in \mathcal{B}_n$. The result is obvious if either $I = 0$ or $J = 0$. So, let $I \neq 0$ and $J \neq 0$. By statement 1, $I = I_C$ and $J = I_D$ for some $C, D \in \mathcal{C}_n$. Then $IJ = (\sum_{f \in C} I_f)(\sum_{g \in D} I_g) = (\sum_{g \in D} I_g)(\sum_{f \in C} I_f) = JI$. $\square$

Let B_n be the K-basis for the algebra $\mathbb{A}_n$ that is the tensor product of the K-bases from Theorem 2.5.(2) (see the proof of Corollary 2.7.(7) for details). For each element $b = b_1 \otimes \cdots \otimes b_n$ of B_n, one can attach an element f_b of $\mathcal{B}_n$ by the rule

$$f_b(i) := \begin{cases} 1 & \text{if } b_i \notin F, \\ 0 & \text{if } b_i \in F. \end{cases}$$

Let $a = \sum_{b \in B_n} \lambda_b b$ be a nonzero element of $\mathbb{A}_n$ where $\{\lambda_b\}$ are the coordinates of a with respect to the basis B_n. One has the well-defined map $\mathbb{A}_n \to \mathcal{C}_n$, $a \mapsto \mathrm{Max}(a)$, where $\mathrm{Max}(a)$ are the maximal elements (in $\mathcal{B}_n$) of the subset $\{f_b \,|\, \lambda_b \neq 0\}$ of $\mathcal{B}_n$ where $\mathrm{Max}(0) := \emptyset$.

Lemma 3.2 *Let K be a field of characteristic zero and $0 \neq a = \sum_{b \in B_n} \lambda_b b \in \mathbb{A}_n$. Then $\mathbb{A}_n a \mathbb{A}_n = \sum_{b \in \mathrm{Max}(a)} I_{f_b} = \sum_{\{b \,|\, \lambda_b \neq 0\}} I_{f_b}$.*

Proof. It suffices to prove only the first equality since the second follows from the first: for any $b \in B_n$ such that $\lambda_b \neq 0$ there exists $c \in \mathrm{Max}(a)$ such that $\lambda_c \neq 0$ and $f_c \geq f_b$, and so $I_{f_c} \supseteq I_{f_b}$. Hence, $\sum_{b \in \mathrm{Max}(a)} I_{f_b} = \sum_{\{b \,|\, \lambda_b \neq 0\}} I_{f_b}$.

Fix $b \in \mathrm{Max}(a)$. Up to order of the tensor multiples in the tensor product $\mathbb{A}_n = \otimes_{i=1}^n \mathbb{A}_1(i)$, one may assume that

$$f_b(i) := \begin{cases} 1 & \text{if } 1 \leq i \leq s, \\ 0 & \text{if } s+1 \leq i \leq n. \end{cases}$$

We have to show that $\mathbb{A}_n a \mathbb{A}_n \supseteq \mathbb{A}_s \otimes F^{\otimes(n-s)}$. $a = \lambda_1 b_1 + \cdots + \lambda_t b_t + \lambda_{t+1} b_{t+1} + \cdots + \lambda_r b_r$ where $b := b_1$, all $\lambda_i \neq 0$, $f_{b_1} = \cdots = f_{b_t}$ and $f_{b_j} \neq f_b$ for $j = t+1, \ldots, r$. By the choice of b, for each j such that $t+1 \leq j \leq r$, either $f_{b_j} < f_b$ or, otherwise, the functions f_{b_j} and f_b are incomparable. $b_1 = c_1 \otimes f_1$ for unique elements $c_1 \in \mathcal{M}_s$ and $f_1 = E_{p_1,q_1}(s+1) \cdots E_{p_{n-s},q_{n-s}}(n) \in \mathcal{N}_{n-s}$ where $\mathcal{M}_s$ and $\mathcal{N}_{n-s}$ were defined in the proof of Corollary 2.7.(7). Let $E_{pp} := E_{p_1,p_1}(s+1) \cdots E_{p_{n-s},p_{n-s}}(n)$ and $E_{qq} := E_{q_1,q_1}(s+1) \cdots E_{q_{n-s},q_{n-s}}(n)$. Then $E_{pp} f_1 E_{qq} = f_1$. Note that $E_{pp} \mathbb{A}_{n-s} E_{qq} = K E_{pq} = K f_1$. Changing the element a for the element $E_{pp} a E_{qq}$ and deleting zero terms of the type $E_{pp} \lambda_\nu b_\nu E_{qq}$, one may assume that $a = (\sum_{i=1}^l \mu_i c_i) \otimes f_1$ where all $\mu_i \neq 0$ and $\mu_i \in K$; all $c_i \in B_s$; $f_{c_1} = \cdots = f_{c_k} = 1$ for some k such that $1 \leq k \leq l$, i.e. $c_1, \ldots, c_k \in \mathcal{M}_s$; and $f_{c_j} < f_{c_1}$ where $k+1 \leq j \leq l$. The element $c := \sum_{i=1}^l \mu_i c_i \notin \mathfrak{a}_s$, hence $\mathbb{A}_s c \mathbb{A}_s = \mathbb{A}_s$. Now, $\mathbb{A}_n a \mathbb{A}_n = \mathbb{A}_n (c \otimes f) \mathbb{A}_n = \mathbb{A}_s c \mathbb{A}_s \otimes \mathbb{A}_{n-s} f_1 \mathbb{A}_{n-s} = \mathbb{A}_s \otimes F^{\otimes(n-s)}$, as required. $\square$

The next result is a useful criterion of when one ideal contains another.

Corollary 3.3 *Let K be a field of characteristic zero and $C, C' \in \mathcal{C}_n$. Then $I_C \subseteq I_{C'}$ iff $C \leq C'$ (this means that, for each $f \in C$, there exists $f' \in C'$ such that $f \leq f'$).*

Proof. This follows at once from Lemma 3.2 and Theorem 3.1.(1). $\square$

Corollary 3.4 *Let K be a field of characteristic zero and s_n be the number of ideals of $\mathbb{A}_n$. Then $2 - n + \sum_{i=1}^n 2^{\binom{n}{i}} \leq s_n \leq 2^{2^n}$.*

Proof. Let Sub_n be the set of all subsets of $\{1, \ldots, n\}$. Sub_n is a partially ordered set with respect to '$\subseteq$'. For each $f \in \mathcal{B}_n$, the subset $\mathrm{supp}(f) := \{i \mid f(i) = 1\}$ of $\{1, \ldots, n\}$ is called the *support* of f. The map $\mathcal{B}_n \to \mathrm{Sub}_n$, $f \mapsto \mathrm{supp}(f)$, is an isomorphism of posets. Let SSub_n be the set of all subsets of Sub_n. An element $\{X_1, \ldots, X_s\}$ of SSub_n is called *incomparable* if for all $i \neq j$ such that $1 \leq i, j \leq s$ neither $X_i \subseteq X_j$ nor $X_i \supseteq X_j$. An empty set and one element set are called incomparable by definition. Let Inc_n be the subset of SSub_n of all incomparable elements of SSub_n. Then the map

$$\mathcal{C}_n \to \mathrm{Inc}_n, \quad \{f_1, \ldots, f_s\} \mapsto \{\mathrm{supp}(f_1), \ldots, \mathrm{supp}(f_s)\}, \tag{28}$$

is a bijection. For each $i = 1, \ldots, n$, there are precisely $\binom{n}{i}$ subsets of $\{1, \ldots, n\}$ that contain exactly i elements. Any non-empty collection of these is an incomparable set, hence $s_n \geq 2 + \sum_{i=1}^n (2^{\binom{n}{i}} - 1) = 2 - n + \sum_{i=1}^n 2^{\binom{n}{i}}$ where 2 'represents' the zero ideal and the ideal $F^{\otimes n}$ which corresponds to an empty set. Clearly, $s_n = |\mathrm{Inc}_n| \leq 2^{2^n}$. $\square$

The next corollary classifies all the prime ideals of $\mathbb{A}_n$.

Corollary 3.5 *Let K be a field of characteristic zero. Then the map*

$$\mathrm{Sub}_n \to \mathrm{Spec}(\mathbb{A}_n), \;\; I \mapsto \mathfrak{p}_I := \sum_{i\in I} \mathfrak{p}_i, \;\; \emptyset \mapsto 0,$$

is a bijection, i.e. any nonzero prime ideal of $\mathbb{A}_n$ is a unique sum of height 1 primes; $|\mathrm{Spec}(\mathbb{A}_n)| = 2^n$; the height of $\mathfrak{p}_I$ is $|I|$.

Proof. 0 is the prime ideal since $\mathbb{A}_n$ is a prime ring. Let I be a nonzero subset of $\{1,\ldots,n\}$ that contains, say s, elements. Then $\mathbb{A}_n/\mathfrak{p}_I \simeq \mathcal{A}_s \otimes \mathbb{A}_{n-s}$. The ring $\mathcal{A}_s$ is a central, simple K-algebra, hence the map $\mathfrak{a} \mapsto \mathcal{A}_s \otimes \mathfrak{a}$ is a bijection from the set of ideals of $\mathbb{A}_{n-s}$ to the set of ideals of $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ (this is an easy consequence of the Density Theorem). Then, $\mathcal{A}_s \otimes F^{\otimes(n-s)}$ is the smallest nonzero ideal of $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ and it is idempotent, and so $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ is a prime ring. This means that $\mathfrak{p}_I$ is a prime ideal. By Theorem 3.1.(1), the map $\mathrm{Sub}_n \to \mathrm{Spec}(\mathbb{A}_n)$, $I \mapsto \mathfrak{p}_I$, is an injection. It remains to prove that it is a surjection. Let $\mathfrak{p}$ be a prime nonzero ideal of $\mathbb{A}_n$. Then $\mathfrak{p} = \sum_{f\in C} I_f$ for some $C \in \mathcal{C}_n$ (Theorem 3.1.(1)). If $|\mathrm{supp}(f)| = n-1$ for all $f \in C$ then $I_f = \mathfrak{p}_i$ where $i = i(f)$ is a unique element of the set $\{1,\ldots,n\}$ such that $f(i) = 0$, and so $\mathfrak{p} = \sum \mathfrak{p}_i$.

Suppose that $|\mathrm{supp}(f)| \neq n-1$ for some $f \in C$. Let us show that this case is not possible since then $\mathfrak{p}$ would *not* be a prime ideal. One can choose two distinct functions, say $g, h \in \mathcal{B}_n$, such that $g > f$, $h > f$, and $gh = f$. Then $I_f = I_{gh} = I_g I_h$. Let $\mathfrak{a} := I_g + \mathfrak{c}$ and $\mathfrak{b} := I_h + \mathfrak{c}$ where $\mathfrak{c} := \sum_{f \neq f' \in C} I_{f'}$. The ideals $\mathfrak{a}$ and $\mathfrak{b}$ *strictly* contain the ideal $\mathfrak{p}$ and

$$\mathfrak{a}\mathfrak{b} = (I_g + \mathfrak{c})(I_h + \mathfrak{c}) = I_g I_h + \mathfrak{c} I_h + I_g \mathfrak{c} + \mathfrak{c}^2 = I_{gh} + \mathfrak{c} I_h + I_g \mathfrak{c} + \mathfrak{c}^2 \subseteq I_f + \mathfrak{c} = \mathfrak{p}.$$

This contradicts to the fact that $\mathfrak{p}$ is a prime ideal, and we are done.

It is obvious that $|\mathrm{Spec}(\mathbb{A}_n)| = 2^n$. The fact that the height of the ideal $\mathfrak{p}_I$ is $|I|$ follows from Lemma 3.6.(1). $\Box$

The next criterion of when a prime ideal contains another prime is used in finding the classical Krull dimension of the algebra $\mathbb{A}_n$ (Corollary 3.7).

Lemma 3.6 *Let K be a field of characteristic zero; $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(\mathbb{A}_n)$; $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s}$ and $\mathfrak{q} = \mathfrak{p}_{j_1} + \cdots + \mathfrak{p}_{j_t}$ be their decompositions as in Corollary 3.5. Then*

1. *$\mathfrak{p} \subseteq \mathfrak{q}$ iff $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_s}\} \subseteq \{\mathfrak{p}_{j_1}, \ldots, \mathfrak{p}_{j_t}\}$.*
2. *If $\mathfrak{p} \subseteq \mathfrak{q}$ then $\mathfrak{p}\mathfrak{q} = \mathfrak{p}$.*
3. *The poset $(\mathrm{Spec}(\mathbb{A}_n), \subseteq)$ is an isomorphic to the set Sub_n of all subsets of $\{1, \ldots n\}$.*

Proof. 1. ($\Rightarrow$) $(\mathfrak{q} + \mathfrak{p})/\mathfrak{p}$ is the ideal of the algebra $\mathbb{A}_n/\mathfrak{p} \simeq \mathcal{A}_s \otimes \mathbb{A}_{n-s}$. The algebra $\mathcal{A}_s$ is central and simple. By the Density Theorem, each ideal of the algebra $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ is of the type $\mathcal{A}_s \otimes \mathfrak{a}$ for some ideal $\mathfrak{a}$ of $\mathbb{A}_{n-s}$. By Corollary 3.5 (applied to the algebra $\mathbb{A}_{n-s}$), we have the inclusion $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_s}\} \subseteq \{\mathfrak{p}_{j_1}, \ldots, \mathfrak{p}_{j_t}\}$.

($\Leftarrow$) If $\{\mathfrak{p}_{i_1}, \ldots, \mathfrak{p}_{i_s}\} \subseteq \{\mathfrak{p}_{j_1}, \ldots, \mathfrak{p}_{j_t}\}$ then $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s} \subseteq \mathfrak{p}_{j_1} + \cdots + \mathfrak{p}_{j_t} = \mathfrak{q}$.

2. By statement 1, if $\mathfrak{p} \subseteq \mathfrak{q}$ then $\mathfrak{q} = \mathfrak{p} + \mathfrak{r}$ for an ideal $\mathfrak{r}$. Then $\mathfrak{p}\mathfrak{q} = \mathfrak{p}^2 + \mathfrak{p}\mathfrak{r} = \mathfrak{p} + \mathfrak{p}\mathfrak{r} = \mathfrak{p}$.

3. Statement 3 follows from Corollary 3.5 and statement 1. $\Box$

For each $s = 0, 1, \ldots, n$, there are precisely $\binom{n}{s}$ prime ideals of height s, namely,

$$\{\mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s} \,|\, 1 \leq i_1 < \cdots < i_s \leq n\}.$$

Corollary 3.7 *Let K be a field of characteristic zero. Then the classical Krull dimension of $\mathbb{A}_n$ is n.*

Proof. By Lemma 3.6.(1), $0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_1 + \mathfrak{p}_2 \subset \cdots \subset \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$ is a longest chain of primes, and so $\mathrm{cl.K.dim}(\mathbb{A}_n) = n$. $\Box$

$(\mathrm{Spec}(\mathbb{A}_n), \subseteq)$ is a poset. Two primes $\mathfrak{p}$ and $\mathfrak{q}$ are called *incomparable* if neither $\mathfrak{p} \subseteq \mathfrak{q}$ nor $\mathfrak{p} \supseteq \mathfrak{q}$.

For each ideal $\mathfrak{a}$ of $\mathbb{A}_n$ such that $\mathfrak{a} \neq \mathbb{A}_n$, let $\mathrm{Min}(\mathfrak{a})$ be the set of all minimal primes over $\mathfrak{a}$. The set $\mathrm{Min}(\mathfrak{a})$ is a non-empty set since the ring $\mathbb{A}_n$ has only finitely many primes.

For each $f \in \mathcal{B}_n$, the set $\mathrm{csupp}(f) := \{i \,|\, f(i) = 0\}$ is called the *co-support* of f. Clearly, $\mathrm{csupp}(f) = \{1, \ldots, n\} \backslash \mathrm{supp}(f)$.

Theorem 3.8 *Let K be a field of characteristic zero. Then*

1. *Each ideal $\mathfrak{a}$ of $\mathbb{A}_n$ such that $\mathfrak{a} \neq \mathbb{A}_n$ is a unique product of incomparable primes, i.e. if $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{r}_1 \cdots \mathfrak{r}_t$ are two such products then $s = t$ and $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \ldots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ for a permutation σ of $\{1, \ldots, n\}$.*

2. *Each ideal $\mathfrak{a}$ of $\mathbb{A}_n$ such that $\mathfrak{a} \neq \mathbb{A}_n$ is a unique intersection of incomparable primes, i.e. if $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_t$ are two such intersections then $s = t$ and $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \ldots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ for a permutation σ of $\{1, \ldots, n\}$.*

3. *For each ideal $\mathfrak{a}$ of $\mathbb{A}_n$ such that $\mathfrak{a} \neq \mathbb{A}_n$, the sets of incomparable primes in statements 1 and 2 are the same, $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$.*

4. *The ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ in statement 3 are the minimal primes of $\mathfrak{a}$, and so $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p}$.*

Proof. 1. For each ideal $\mathfrak{a}$ of $\mathbb{A}_n$, we have to prove that $\mathfrak{a}$ is a product of incomparable primes and that this product is unique. Since the ring $\mathbb{A}_n$ is prime these two statements are obvious when $\mathfrak{a} = 0$. So, let $\mathfrak{a} \neq 0$.

Existence: Let $f \in \mathcal{B}_n$; then $I_f = \prod_{i \in \mathrm{csupp}(f)} \mathfrak{p}_i$. Let $\mathfrak{b}$ be any ideal of $\mathbb{A}_n$. Since $\mathfrak{b}^2 = \mathfrak{b}$, it follows at once that

$$I_f + \mathfrak{b} = \prod_{i \in \mathrm{csupp}(f)} (\mathfrak{p}_i + \mathfrak{b}). \qquad (29)$$

By Theorem 3.1.(1), $\mathfrak{a} = I_{f_1} + \cdots + I_{f_s}$ for some $f_i \in \mathcal{B}_n$. Repeating s times (29), we see that

$$\mathfrak{a} = \prod_{i_1 \in \mathrm{csupp}(f_1), \ldots, i_s \in \mathrm{csupp}(f_s)} (\mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s}) \qquad (30)$$

is the product of primes, by Corollary 3.5. Note that ideals of $\mathbb{A}_n$ commute; each ideal is an idempotent ideal; and if $\mathfrak{p} \subseteq \mathfrak{q}$ is an inclusion of primes then $\mathfrak{p}\mathfrak{q} = \mathfrak{p}$. Using these three facts and (30), we see that $\mathfrak{a}$ is a product of *incomparable* primes.

Uniqueness follows from the next lemma which will be used several times in the proof of this theorem.

Lemma 3.9 *Let $\{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ and $\{\mathfrak{r}_1, \dots, \mathfrak{r}_t\}$ be two sets of incomparable ideals of a ring such that each ideal from the first set contains an ideal from the second and each ideal from the second set contains an ideal from the first. Then $s = t$ and $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \dots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ for a permutation σ of $\{1, \dots, n\}$.*

Proof Lemma 3.9. For each $\mathfrak{q}_i$, there are ideals $\mathfrak{r}_j$ and $\mathfrak{r}_k$ such that $\mathfrak{r}_j \subseteq \mathfrak{q}_i \subseteq \mathfrak{r}_k$, hence $\mathfrak{q}_i = \mathfrak{r}_j = \mathfrak{r}_k$ since the ideals $\mathfrak{r}_j$ and $\mathfrak{r}_k$ are incomparable if distinct. This proves that for each ideal $\mathfrak{q}_i$ there exists a unique ideal, say $\mathfrak{r}_{\sigma(i)}$, such that $\mathfrak{q}_i = \mathfrak{r}_{\sigma(i)}$. By symmetry, for each ideal $\mathfrak{r}_j$ there exists a unique ideal, say $\mathfrak{q}_{\tau(j)}$, such that $\mathfrak{r}_j = \mathfrak{q}_{\tau(j)}$. Then, $s = t$ and $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \dots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ for the permutation σ of $\{1, \dots, n\}$. $\square$

Uniqueness: Let $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{r}_1 \cdots \mathfrak{r}_t$ be two products of incomparable primes. Each ideal $\mathfrak{q}_i$ contains an ideal $\mathfrak{r}_j$, and each ideal $\mathfrak{r}_k$ contains an ideal $\mathfrak{q}_l$. By Lemma 3.9, $s = t$ and $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \dots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ for a permutation σ of $\{1, \dots, n\}$.

2. *Uniqueness*: Suppose that an ideal $\mathfrak{a}$ has two presentations $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_t$ of incomparable primes. The sets $\{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ and $\{\mathfrak{r}_1, \dots, \mathfrak{r}_t\}$ of incomparable primes satisfy the conditions of Lemma 3.9, and so uniqueness follows.

Existence: Let $\mathcal{I}$ be the set of all the ideals of $\mathbb{A}_n$, and $\mathcal{I}'$ be the set of ideals of $\mathbb{A}_n$ that are intersection of incomparable primes. Then $\mathcal{I}' \subseteq \mathcal{I}$. The map

$$\mathcal{I} \to \mathcal{I}', \quad \mathfrak{q}_1 \cdots \mathfrak{q}_s \mapsto \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s,$$

is a bijection since $|\mathcal{I}| < \infty$ and by uniqueness of presentations $\mathfrak{q}_1 \cdots \mathfrak{q}_s$ (statement 1) and $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ (see above) where $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ are incomparable primes. Then $\mathcal{I} = \mathcal{I}'$. This proves that each ideal $\mathfrak{a}$ of $\mathbb{A}_n$ is an intersection of incomparable primes.

3. Let $\mathfrak{a}$ be an ideal of $\mathbb{A}_n$ and $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_t$ where $S := \{\mathfrak{q}_1, \dots, \mathfrak{q}_s\}$ and $T := \{\mathfrak{r}_1, \dots, \mathfrak{r}_t\}$ are sets of incomparable primes. The sets S and T satisfy the conditions of Lemma 3.9, and so $s = t$ and $\mathfrak{q}_1 = \mathfrak{r}_{\sigma(1)}, \dots, \mathfrak{q}_s = \mathfrak{r}_{\sigma(s)}$ for a permutation σ of $\{1, \dots, n\}$. This means that $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$.

4. Let $\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$ be as in statement 3 and let $\mathrm{Min}(\mathfrak{a}) = \{\mathfrak{r}_1, \dots, \mathfrak{r}_t\}$ be the set of minimal primes over $\mathfrak{a}$. Then $\mathrm{Min}(\mathfrak{a}) \subseteq S := \{\mathfrak{q}_1, \dots \mathfrak{q}_s\}$ ($\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_s \subseteq \mathfrak{r}_i$ implies $\mathfrak{q}_j \subseteq \mathfrak{r}_i$ for some j, and so $\mathfrak{q}_j = \mathfrak{r}_i$ by the minimality of $\mathfrak{r}_i$). Up to order, let $\mathfrak{r}_1 = \mathfrak{q}_1, \dots, \mathfrak{r}_t = \mathfrak{q}_t$. It remains to show that $t = s$. Suppose that $t < s$, we seek a contradiction. This means that each prime $\mathfrak{q}_i$, $i = t+1, \dots, s$, contains $\mathfrak{a}$ and is *not* minimal over $\mathfrak{a}$. Hence, $\mathfrak{q}_i$ contains a minimal prime, say $\mathfrak{q}_{\tau(i)}$, a contradiction (the ideal $\mathfrak{q}_i$ and $\mathfrak{q}_{\tau(i)}$ are incomparable). $\square$

Corollary 3.10 *Let K be a field of characteristic zero, $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $\mathbb{A}_n$ distinct from $\mathbb{A}_n$ in statement 1, 2 and 5. Then*

1. $\mathfrak{a} = \mathfrak{b}$ *iff* $\mathrm{Min}(\mathfrak{a}) = \mathrm{Min}(\mathfrak{b})$.

2. $\mathrm{Min}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{Min}(\mathfrak{a}\mathfrak{b}) =$ *the set of minimal elements (with respect to inclusion) of the set* $\mathrm{Min}(\mathfrak{a}) \cup \mathrm{Min}(\mathfrak{b})$.

3. $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

4. *If* $\mathfrak{a} \subseteq \mathfrak{b}$ *then* $\mathfrak{a}\mathfrak{b} = \mathfrak{a}$.

5. $\mathfrak{a} \subseteq \mathfrak{b}$ *iff* $\mathrm{Min}(\mathfrak{a}) \leqslant \mathrm{Min}(\mathfrak{b})$ *(the* $\leqslant$ *means that and each* $\mathfrak{q} \in \mathrm{Min}(\mathfrak{b})$ *contains some* $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$*).*

Proof. 1. Statement 1 is obvious due to Theorem 3.8.(4).

2. Let $\mathcal{M}$ be the set of minimal elements of the union $\mathrm{Min}(\mathfrak{a}) \cup \mathrm{Min}(\mathfrak{b})$. The elements of $\mathcal{M}$ are incomparable, and (by Theorem 3.8.(4))

$$\mathfrak{a} \cap \mathfrak{b} = \cap_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \cap \cap_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{b})} \mathfrak{q} = \cap_{\mathfrak{r} \in \mathcal{M}} \mathfrak{r}.$$

By Theorem 3.8.(2), $\mathrm{Min}(\mathfrak{a} \cap \mathfrak{b}) = \mathcal{M}$. By Lemma 3.6.(2),

$$\mathfrak{a}\mathfrak{b} = \prod_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p} \cdot \prod_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{b})} \mathfrak{q} = \prod_{\mathfrak{r} \in \mathcal{M}} \mathfrak{r} = \mathfrak{a} \cap \mathfrak{b}.$$

3. The result is obvious if one the ideals is equal to $\mathbb{A}_n$. So, let the ideals are distinct from $\mathbb{A}_n$. By statement 2, $\mathrm{Min}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{Min}(\mathfrak{a}\mathfrak{b})$, then, by statement 1, $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$.

4. If $\mathfrak{a} \subseteq \mathfrak{b}$ then, by statement 3, $\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}$.

5. ($\Rightarrow$) If $\mathfrak{a} \subseteq \mathfrak{b}$ then $\mathrm{Min}(\mathfrak{a}) \leqslant \mathrm{Min}(\mathfrak{b})$ since $\mathfrak{a} = \prod_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p} \subseteq \prod_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{b})} \mathfrak{q} = \mathfrak{b}$.

($\Leftarrow$) Suppose that $\mathrm{Min}(\mathfrak{a}) \leqslant \mathrm{Min}(\mathfrak{b})$. For each $\mathfrak{q} \in \mathrm{Min}(\mathfrak{b})$, let $S(\mathfrak{q})$ be the set (necessarily nonempty) of $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Then $\mathrm{Min}(\mathfrak{a}) \supseteq S := \cup_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{b})} S(\mathfrak{q})$ and

$$\mathfrak{a} = \cap_{\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{p} \subseteq \cap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq \cap_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{a})} \mathfrak{q} = \mathfrak{b}. \quad \square$$

Theorem 3.11 *Let K be a field of characteristic zero. Then the lattice of ideals of the algebra $\mathbb{A}_n$ is distributive, i.e.* $(\mathfrak{a} \cap \mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c} \cap \mathfrak{b}\mathfrak{c}$ *for all ideals* $\mathfrak{a}$, $\mathfrak{b}$, *and* $\mathfrak{c}$.

Proof. By Corollary 3.10.(3), $(\mathfrak{a} \cap \mathfrak{b})\mathfrak{c} = \mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c} = (\mathfrak{a} \cap \mathfrak{c}) \cap (\mathfrak{b} \cap \mathfrak{c}) = \mathfrak{a}\mathfrak{c} \cap \mathfrak{b}\mathfrak{c}$. $\square$

Theorem 3.12 *Let K be a field of characteristic zero,* $\mathfrak{a}$ *be an ideal of* $\mathbb{A}_n$, *and* $\mathcal{M}$ *be the minimal elements with respect to inclusion of a set of ideals* $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ *of* $\mathbb{A}_n$. *Then*

1. $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k$ *iff* $\mathrm{Min}(\mathfrak{a}) = \mathcal{M}$.

2. $\mathfrak{a} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_k$ *iff* $\mathrm{Min}(\mathfrak{a}) = \mathcal{M}$.

Proof. By Corollary 3.10.(3), it suffices to prove, say, the first statement.

($\Rightarrow$) Suppose that $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k$ then, by Theorem 3.8.(4) and Corollary 3.10.(4),

$$\mathfrak{a} = \prod_{i=1}^{k} \prod_{\mathfrak{q}_{ij} \in \mathrm{Min}(\mathfrak{a}_i)} \mathfrak{q}_{ij} = \prod_{\mathfrak{q} \in \mathcal{M}} \mathfrak{q},$$

and so $\mathrm{Min}(\mathfrak{a}) = \mathcal{M}$, by Theorem 3.8.(4).

($\Leftarrow$) If $\mathrm{Min}(\mathfrak{a}) = \mathcal{M}$ then, by Corollary 3.10.(4), $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_k$. $\Box$

The involution c. Let K be a field of characteristic zero and $\mathcal{I}(\mathbb{A}_n)$ be the set of all ideals of the algebra $\mathbb{A}_n$. Consider the map $\mathcal{C}_n \backslash \{\emptyset\} \to \mathcal{C}_n \backslash \{\emptyset\}$, $C \mapsto C+1$, where for $C = \{f_1, \ldots, f_s\}$, $C+1 := \{f_1+1, \ldots, f_s+1\}$. The map is well-defined: $C \in \mathcal{C}_n$ iff $\{\mathrm{supp}(f_1), \ldots, \mathrm{supp}(f_s)\} \in \mathrm{Inc}_n$ iff $\{\mathrm{csupp}(f_1), \ldots, \mathrm{csupp}(f_s)\} \in \mathrm{Inc}_n$ where $\mathrm{csupp}(f_i) := \{1, \ldots, n\} \backslash \mathrm{supp}(f_i)$ iff $C+1 \in \mathrm{Inc}_n$. Consider the map

$$c : \mathcal{I}(\mathbb{A}_n) \to \mathcal{I}(\mathbb{A}_n), \quad I_C \mapsto I_{C+1}, \quad C \in \mathcal{C}_n,$$

where $c(0) := 0$. Then, for $C \in \mathcal{C}_n$, $c(\sum_{f \in C} I_f) = \sum_{f \in C} c(I_f)$. Note that $c(\mathbb{A}_n) = F^{\otimes n}$, $c(\mathfrak{p}_i) = \prod_{j \neq i} \mathfrak{p}_j$, and $c(\mathfrak{a}_n) = \sum_{i=1}^{n} c(\mathfrak{p}_i)$.

Let $C, C' \in \mathcal{C}_n$, we write $C \preceq C'$ if for each $f \in C$ there exists $f' \in C'$ such that $f \leq f'$, and for each $g' \in C'$ there exists $g \in C$ such that $g \leq g'$.

Lemma 3.13 *Let K be a field of characteristic zero. Then*

1. $c : \mathcal{I}(\mathbb{A}_n) \to \mathcal{I}(\mathbb{A}_n)$ *is an involution* ($c^2 = \mathrm{id}$) *such that $f \leq g$ implies $c(I_f) \supseteq c(I_g)$.*
2. $c(\mathfrak{a}) = \mathfrak{a}$ *iff* $\mathfrak{a} = I_C$ *for some* $C = \{f_1, f_1+1, \ldots, f_s, f_s+1\}$.
3. *If* $C, C' \in \mathcal{C}_n$ *and* $C \preceq C'$ *then* $c(I_C) \supseteq c(I_{C'})$.
4. $c(\mathfrak{a} + \mathfrak{b}) \subseteq c(\mathfrak{a}) + c(\mathfrak{b})$ *for all ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $\mathbb{A}_n$. *If* $\mathfrak{a} = I_C$ *and* $\mathfrak{b} = I_{C'}$ *for some* $C, C' \in \mathcal{C}_n$ *such that* $C \cup C' \in \mathcal{C}_n$ *then* $c(\mathfrak{a} + \mathfrak{b}) = c(\mathfrak{a}) + c(\mathfrak{b})$.
5. $c(\mathfrak{a}\mathfrak{b}) \supseteq c(\mathfrak{a}) + c(\mathfrak{b})$ *for all nonzero ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$ *of* $\mathbb{A}_n$.

Proof. 1. $c^2 = \mathrm{id}$ since, for all $\emptyset \neq C \in \mathcal{C}_n$, $C+1+1 = C$. The rest is obvious.

2. $c(I_C) = I_C$ iff $C+1 = C$ iff $C = \{f_1, f_1+1, \ldots, f_s, f_s+1\}$.

3. If $C \preceq C'$ then $C'+1 \preceq C+1$, and so $c(I_C) \supseteq c(I_{C'})$.

4. The second statement is obvious: If $C \cup C' \in \mathcal{C}_n$ then

$$c(\mathfrak{a} + \mathfrak{b}) = c(I_{C \cup C'}) = \sum_{f \in C \cup C'} I_{f+1} = \sum_{f \in C} I_{f+1} + \sum_{f \in C'} I_{f'+1} = \mathfrak{a} + \mathfrak{b}.$$

For arbitrary $\mathfrak{a}$ and $\mathfrak{b}$, let $\mathfrak{a} = I_C$ and $\mathfrak{b} = I_D$, and $\mathfrak{a} + \mathfrak{b} = I_E$ for some $C, D, E \in \mathcal{C}_n$. Then, for each $e \in E$, either $e \geq f$ for some $f \in C$ or $e \geq g$ for some $g \in D$. Then, either $c(I_e) \subseteq c(I_f)$ or $c(I_e) \subseteq c(I_g)$. Hence, $c(\mathfrak{a} + \mathfrak{b}) \subseteq c(\mathfrak{a}) + c(\mathfrak{b})$.

5. Let $\mathfrak{a} = I_C$, $\mathfrak{b} = I_D$, and $\mathfrak{ab} = I_E$ for some $C, D, E \in \mathcal{C}_n$. Then $E \preceq C$ and $E \preceq D$. By statement 3, $c(\mathfrak{ab}) \supseteq c(\mathfrak{a}) + c(\mathfrak{b})$. $\Box$

The involution τ. Let K be a field of characteristic zero. For each $\mathfrak{p} \in \mathrm{Spec}(\mathbb{A}_n)$, there exists a unique prime ideal $\tau(\mathfrak{p})$ such that $\mathfrak{a}_n = \mathfrak{p} \oplus \tau(\mathfrak{p})$. In more detail, if $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s}$ then $\tau(\mathfrak{p}) = \mathfrak{p}_{j_1} + \cdots + \mathfrak{p}_{j_t}$ where $\{j_1, \ldots, j_t\} := \{1, \ldots, n\} \backslash \{i_1, \ldots, i_s\}$. The map $\tau : \mathrm{Spec}(\mathbb{A}_n) \to \mathrm{Spec}(\mathbb{A}_n)$ is an order reversion involution, i.e. $\mathfrak{p} \subseteq \mathfrak{q}$ implies $\tau(\mathfrak{p}) \supseteq \tau(\mathfrak{q})$ for $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(\mathbb{A}_n)$; and $\tau^2 = \mathrm{id}$. In particular, τ is an anti-automorphism of the poset $\mathrm{Spec}(\mathbb{A}_n)$. $\tau(\mathfrak{a}_n) = 0$ and $\tau(\mathfrak{p}_i) = \sum_{j \neq i} \mathfrak{p}_j$. Let $\mathcal{I}_n$ be the set of ideals of $\mathbb{A}_n$ distinct from $\mathbb{A}_n$. The map τ can be extended to the map

$$\tau : \mathcal{I}_n \to \mathcal{I}_n, \quad \mathfrak{a} = \cap_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{a})} \mapsto \tau(\mathfrak{a}) := \cap_{\mathfrak{q} \in \mathrm{Min}(\mathfrak{a})} \tau(\mathfrak{q}).$$

Lemma 3.14 *Let K be a field of characteristic zero, and $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_n$. Then*

1. *$\tau : \mathcal{I}_n \to \mathcal{I}_n$ is the involution ($\tau^2 = \mathrm{id}$).*

2. *$\tau(\mathfrak{a}) = \mathfrak{a}$ iff $\tau(\mathrm{Min}(\mathfrak{a})) = \mathrm{Min}(\mathfrak{a})$.*

Proof. 1. The elements of the set $\mathrm{Min}(\mathfrak{a})$ are incomparable, then so are the elements of the set $\tau(\mathrm{Min}(\mathfrak{a}))$, hence $\tau(\mathrm{Min}(\mathfrak{a})) = \mathrm{Min}(\tau(\mathfrak{a}))$. Then $\tau^2 = \mathrm{id}$.

2. By Corollary 3.10.(1), $\tau(\mathfrak{a}) = \mathfrak{a}$ iff $\tau(\mathrm{Min}(\mathfrak{a})) = \mathrm{Min}(\mathfrak{a})$. $\Box$

A prime ideal of a ring R is called a *completely prime* if $R/\mathfrak{p}$ is a domain.

Corollary 3.15 *Let K be a field of characteristic zero. Then*

1. *$\mathfrak{a}_n$ is the only completely prime ideal of $\mathbb{A}_n$.*

2. *$\mathfrak{a}_n$ is the only ideal $\mathfrak{a}$ of $\mathbb{A}_n$ such that the factor ring $\mathbb{A}_n/\mathfrak{a}$ is Noetherian (resp. left Noetherian, resp. right Noetherian).*

Proof. 1. By Corollary 3.5, any prime ideal $\mathfrak{p}$ of $\mathbb{A}_n$ is a unique sum $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_s}$. Then $\mathbb{A}_n/\mathfrak{p} \simeq \mathcal{A}_s \otimes \mathbb{A}_{n-s}$. The ring $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ is a domain iff $s = n$, that is $\mathfrak{p} = \mathfrak{a}_n$.

2. The factor ring $\mathbb{A}_n/\mathfrak{a}_n \simeq \mathcal{A}_n$ is Noetherian. It remains to show that $\mathbb{A}_n/\mathfrak{a}$ is not left and right Noetherian for all ideals $\mathfrak{a}$ distinct from $\mathfrak{a}_n$. By Theorem 3.8.(4), if $\mathfrak{a} \neq \mathfrak{a}_n$ then $\mathfrak{a}_n \notin \mathrm{Min}(\mathfrak{a})$. Choose $\mathfrak{p} \in \mathrm{Min}(\mathfrak{a})$. Then $\mathbb{A}_n/\mathfrak{p} \simeq \mathcal{A}_s \otimes \mathbb{A}_{n-s}$ for some $s \geq 1$. The ring $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ is not left or right Noetherian. The ring $\mathcal{A}_s \otimes \mathbb{A}_{n-s}$ is a factor ring of $\mathbb{A}_n/\mathfrak{a}$, and the result follows. $\Box$

4 The group of units $\mathbb{A}_n^*$ of $\mathbb{A}_n$

In this section, K be a field of characteristic zero. Let $\mathbb{A}_n^*$, $\mathcal{A}_n^*$ and K^* be the groups of units of the algebras $\mathbb{A}_n$, $\mathcal{A}_n$ and K respectively. Using the $\mathbb{Z}^n$-grading of the skew polynomial algebra $\mathcal{A}_n$ (see (2)), it follows that

$$\mathcal{A}_n^* = (S_n^{-1}\mathcal{P}_n)^* = \{K^* \prod_{i \in \mathbb{Z}} \prod_{j=1}^{n} (H_j + i)^{n_{ij}} \,|\, (n_{ij}) \in (\mathbb{Z}^n)^{(\mathbb{Z})}\} \simeq K^* \times (\mathbb{Z}^n)^{(\mathbb{Z})} \tag{31}$$

where the abelian group $(\mathbb{Z}^n)^{(\mathbb{Z})}$ is the direct sum of $\mathbb{Z}$ copies of $\mathbb{Z}^n$.

The group $K^* \times \mathcal{H}_n$. Let, for a moment, $n = 1$. In this case we usually drop the subscript 1. For each integer $i \geq 1$ and $\lambda \in K^*$, the element $(H-i)_\lambda := H - i + \lambda \pi_{i-1}$ is a unit of the algebra $\mathbb{D}_1$ and its inverse is equal to

$$(H-i)_\lambda^{-1} := \begin{cases} \rho_{11} + \lambda^{-1}\pi_0 & \text{if } i = 1, \\ \rho_{1i} + \sum_{j=0}^{i-2} \frac{1}{j+1-i}\pi_j + \lambda^{-1}\pi_{i-1} & \text{if } i \geq 2. \end{cases}$$

As a function of the discrete argument H, $(H-i)_\lambda^{-1}$ coincides with $\frac{1}{H-i}$ but instead of having pole at $H = i$ it takes the value λ^{-1}. Consider the following subgroup of $\mathbb{D}_1^*$,

$$\mathcal{H} := \{\prod_{i \geq 0}(H+i)^{n_i} \cdot \prod_{i \geq 1}(H-i)_1^{n_{-i}} \,|\, (n_i) \in \mathbb{Z}^{(\mathbb{Z})}\} \simeq \mathbb{Z}^{(\mathbb{Z})}. \tag{32}$$

For an arbitrary $n \geq 1$, recall that $\mathbb{A}_n = \otimes_{i=1}^n \mathbb{A}_1(i) = \mathbb{A}_1^{\otimes n}$. For each tensor multiple $\mathbb{A}_1(i) = \mathbb{A}_1$, let $\mathcal{H}(i)$ be the corresponding group $\mathcal{H}$. Their product $\mathcal{H}_n := \mathcal{H}(1)\cdots\mathcal{H}(n)$ is a subgroup of $\mathbb{D}_n^*$ and $\mathcal{H}_n \simeq \mathcal{H}^n \simeq (\mathbb{Z}^n)^{(\mathbb{Z})}$. The natural inclusion $\mathcal{H}_n \simeq (\mathcal{H}_n + \mathfrak{a}_n)/\mathfrak{a}_n \subset \mathbb{A}_n/\mathfrak{a}_n \simeq \mathcal{A}_n$ and (31) yield the isomorphism of groups

$$K^* \times \mathcal{H}_n \to \mathcal{A}_n^*, \;\; \lambda \mapsto \lambda, \;\; H_s + i \mapsto H_s + i, \;\; (H_s - j)_1 \mapsto H_s - j, \tag{33}$$

where $\lambda \in K^*$, $1 \leq s \leq n$, $i \in \mathbb{N}$ and $1 \leq j \in \mathbb{N}$. $K^*\mathcal{H}_n$ is the subgroup of $\mathbb{D}_n^*$ such that $K^*\mathcal{H}_n \simeq K^* \times \mathcal{H}_n$.

The group $(1+F^{\otimes n})^*$ **of units of the monoid** $1 + F^{\otimes n}$. We are going to find the group $(1+F^{\otimes n})^*$ of units of the multiplicative (noncommutative) monoid $1+F^{\otimes n}$. Let, for a moment, $n = 1$. The ring $F = \oplus_{i,j \in \mathbb{N}} KE_{ij}$ is the union $M_\infty(K) := \cup_{d \geq 1} M_d(K) = \varinjlim M_d(K)$ of the matrix algebras $M_d(K) := \oplus_{1 \leq i,j \leq d-1} KE_{ij}$, i.e. $F = M_\infty(K)$.

For each $d \geq 1$, consider the (usual) determinant $\det_d = \det : 1 + M_d(K) \to K$, $u \mapsto \det(u)$. These determinants determine the (global) determinant

$$\det : 1 + M_\infty(K) = 1 + F \to K, \;\; u \mapsto \det(u),$$

where $\det(u)$ is the common value of all determinants $\det_d(u)$, $d \gg 1$. The (global) determinant has usual properties of the determinant. In particular, for all $u, v \in 1 + M_\infty(K)$, $\det(uv) = \det(u) \cdot \det(v)$. It follows from Cramer's formula that the group $\mathrm{GL}_\infty(K) := (1 + M_\infty(K))^*$ of units of the monoid $1 + M_\infty(K)$ is equal to

$$GL_\infty(K) = \{u \in 1 + M_\infty(K) \,|\, \det(u) \neq 0\}. \tag{34}$$

Therefore,

$$(1+F)^* = \{u \in 1 + F \,|\, \det(u) \neq 0\} = GL_\infty(K). \tag{35}$$

The kernel

$$\mathrm{SL}_\infty(K) := \{u \in \mathrm{GL}_\infty(K) \,|\, \det(u) = 1\}$$

of the group epimorphism $\det : \mathrm{GL}_\infty(K) \to K^*$ is a *normal* subgroup of $\mathrm{GL}_\infty(K)$.

For any $n \geq 1$,

$$\begin{aligned} F^{\otimes n} &= \otimes_{i=1}^n F(i) = \otimes_{i=1}^n (\cup_{d_i \geq 1} M_{d_i}(K)) = \cup_{d_1,\ldots,d_n\geq 1} \otimes_{i=1}^n M_{d_i}(K) \\ &= \cup_{d_1,\ldots,d_n \geq 1} M_{d_1\cdots d_n}(K) = M_\infty(K). \end{aligned}$$

Consider the determinant

$$\det : 1 + F^{\otimes n} = 1 + M_\infty(K) \to K, \quad u \mapsto \det(u),$$

as in the case $n = 1$. Hence,

$$(1 + F^{\otimes n})^* = \{u \in 1 + F^{\otimes n} \,|\, \det(u) \neq 0\} = (1 + M_\infty(K))^* = \mathrm{GL}_\infty(K). \tag{36}$$

For each element $u \in (1 + F^{\otimes n})^*$, using Cramer's formula one can easily find a formula for the inverse u^{-1}, it is Cramer's formula.

Lemma 4.1 *Let K be a field of characteristic zero and $u \in 1 + F^{\otimes n}$. The following statements are equivalent.*

1. $u \in \mathbb{A}_n^*$.
2. *The element u has left inverse in $\mathbb{A}_n$ ($vu = 1$ for some $v \in \mathbb{A}_n$).*
3. *The element u has right inverse in $\mathbb{A}_n$ ($uv = 1$ for some $v \in \mathbb{A}_n$).*
4. $\det(u) \neq 0$.

Proof. Using the $\mathbb{Z}^n$-grading on $\mathbb{A}_n$, it is obvious that the first three statements are equivalent to the fourth. $\square$

Since $F^{\otimes n}$ is an ideal of $\mathbb{A}_n$, the subgroup $(1 + F^{\otimes n})^*$ of $\mathbb{A}_n^*$ is a *normal* subgroup: For all $a \in \mathbb{A}_n^*$, $a(1+F^{\otimes n})a^{-1} = 1 + aF^{\otimes n}a^{-1} \subseteq 1 + F^{\otimes n}$, and so $a(1+F^{\otimes n})^*a^{-1} \subseteq (1+F^{\otimes n})^*$.

The subgroup $K^* \times (\mathcal{H}_n \ltimes (1 + F^{\otimes n})^*)$ of $\mathbb{A}_n^*$. Let $\mathbb{A}_n'$ be the subgroup of the group $\mathbb{A}_n^*$ generated by its subgroups K^*, $\mathcal{H}_n$ and $(1 + F^{\otimes n})^*$. Let us prove that

$$\mathbb{A}_n' = K^* \times (\mathcal{H}_n \ltimes (1 + F^{\otimes n})^*). \tag{37}$$

The subgroup $(1 + F^{\otimes n})^*$ of $\mathbb{A}_n^*$ (and of $\mathbb{A}_n'$) is normal and the subgroup K^* belongs to the centre of $\mathbb{A}_n^*$, hence $\mathbb{A}_n' = K^*\mathcal{H}_n(1 + F^{\otimes n})^*$, i.e. each element a of $\mathbb{A}_n'$ is a product $a = \lambda\alpha u$ for some elements $\lambda \in K^*$, $\alpha \in \mathcal{H}_n$ and $u \in (1 + F^{\otimes n})^*$. In order to prove (37) it suffices to show uniqueness of the decomposition $a = \lambda\alpha u$. Since $a + \mathfrak{a}_n = \lambda\alpha + \mathfrak{a}_n \in (\mathbb{A}_n/\mathfrak{a}_n)^* \simeq \mathcal{A}_n^*$, the uniqueness of λ and α follows from (31) and (33). Then $u = (\lambda\alpha)^{-1}a$ is unique as well. This finishes the proof of (37).

The group $\mathbb{A}_1^*$ and its commutants.

Theorem 4.2 *Let K be a field of characteristic zero. Then*

1. $\mathbb{A}_1^* = K^* \times (\mathcal{H} \ltimes (1+F)^*)$, *each unit* a *of* $\mathbb{A}_1$ *is a unique product* $a = \lambda\alpha(1+f)$ *for some elements* $\lambda \in K^*$, $\alpha \in \mathcal{H}$, *and* $f \in F$ *such that* $\det(1+f) \neq 0$.
2. $\mathbb{A}_1^* = K^* \times (\mathcal{H} \ltimes \mathrm{GL}_\infty(K))$.
3. *The centre of the group* $\mathbb{A}_1^*$ *is* K^*.
4. *The commutant* $\mathbb{A}_1^{*(2)} := [\mathbb{A}_1^*, \mathbb{A}_1^*]$ *of the group* $\mathbb{A}_1^*$ *is equal to* $\mathrm{SL}_\infty(K) := \{v \in (1+F)^* = M_\infty(K) \,|\, \det(v) = 1\}$, *and* $\mathbb{A}_1/[\mathbb{A}_1^*, \mathbb{A}_1^*] \simeq K^* \times \mathcal{H} \times K^*$.
5. *All the higher commutants* $\mathbb{A}_1^{*(i)} := [\mathbb{A}_1^*, \mathbb{A}_1^{*(i-1)}]$, $i \geq 2$, *are equal to* $\mathbb{A}_1^{*(2)}$.

Proof. 1. By (37), $\mathbb{A}_1' = K^* \times (\mathcal{H} \ltimes (1+F)^*) \subseteq \mathbb{A}_1^*$. It suffices to show the reverse inclusion. By (33), there is the exact sequence of groups

$$1 \to (1+F)^* \to \mathbb{A}_1^* \to (\mathbb{A}_1/F)^* \simeq \mathcal{A}_1^* \to 1 \quad (38)$$

which, using again (33), yields the inclusion $\mathbb{A}_1^* \subseteq \mathbb{A}_1'$. The rest of statement 1 follows from (36).

2. Statement 2 is equivalent to statement 1 since $(1+F)^* = \mathrm{GL}_\infty(K)$, see (35).

3. Let Z be the centre of the group $\mathbb{A}_1^*$. Since $K^* \subseteq Z$, we have

$$Z = Z \cap \mathbb{A}_1^* = Z \cap (K^*\mathcal{H}(1+F^*) = K^*(Z \cap \mathcal{H}(1+F)^*).$$

We have to show that $Z \cap \mathcal{H}(1+F)^* = \{1\}$. Let $z = \alpha u \in Z \cap \mathcal{H}(1+F)^*$ where $\alpha = \alpha(H) \in \mathcal{H}$ and $u \in (1+F)^*$. It remains to show that $z = 1$. $\beta z = z\beta$ for all $\beta \in \mathcal{H}$ iff $\beta u = u\beta$ for all $\beta \in \mathcal{H}$ (since $\mathcal{H}$ is an abelian group) iff $[\beta]u = u[\beta]$ (the equality of infinite matrices) for all $\beta = \beta(H) \in \mathcal{H}$ where $[\beta]$ is the diagonal matrix $\mathrm{diag}(\beta(1), \beta(2), \ldots)$ iff u is a *diagonal* matrix of $(1+F)^*$. The diagonal entries of the matrix u, say u_i, $i \in \mathbb{N}$, are elements of K^* such that $u_i = 1$ for all $i \geq d$ for some natural number $d = d(u)$. For all distinct $i, j \in \mathbb{N}$, $1 + E_{ij} \in (1+F)^*$. Now, it follows from the equalities

$$z + \alpha(i+1)u_i E_{ij} = z(1+E_{ij}) = (1+E_{ij})z = z + E_{ij}\alpha(j+1)u_j$$

that $\alpha(i+1)u_i = \alpha(j+1)u_j$ where $\alpha(i+1) := \alpha(H)|_{H=i+1}$ (we have used that $\alpha E_{ij} = \alpha(i+1)E_{ij}$ and $E_{ij}\alpha = E_{ij}\alpha(j+1)$). For all distinct natural numbers i and j such that $i, j > d$, we have $\alpha(i+1) = \alpha(j+1)$. This means that the function $\alpha(H) \in \mathcal{H}$ is a constant, i.e. $\alpha = 1$. This gives $u_i = u_j$ for all $i, j \in \mathbb{N}$ such that $i \neq j$, i.e. all $u_i = 1$. Therefore, $z = 1$, as required.

4. The determinant can be extended from the subgroup $(1+F)^*$ of $\mathbb{A}_1^*$ to the whole group by the rule

$$\det : \mathbb{A}_1^* \to K^* \times \mathcal{H} \times K^*, \quad \lambda\,\alpha\,u \mapsto \lambda\,\alpha\,\det(u) := (\lambda, \alpha, \det(u)), \quad (39)$$

where $\lambda \in K^*$, $\alpha \in \mathcal{H}$, and $u \in (1+F)^*$. It turns out that det is a group *epimorphism*. By the very definition, det is a surjection. It remains to show that $\det(aa') = \det(a)\det(a')$ for all $a := \lambda\alpha u, a' := \lambda'\alpha' u' \in \mathbb{A}_1^*$. This follows from the following equality

$$\det(\alpha^{-1}u\alpha) = \det(u), \quad \alpha \in \mathcal{H}, \quad u \in (1+F)^*. \quad (40)$$

Indeed,

$$\begin{array}{rcl}\det(aa') & = & \det(\lambda\lambda'\alpha\alpha'\cdot(\alpha')^{-1}u\alpha'u') = \lambda\lambda'\alpha\alpha'\cdot\det((\alpha')^{-1}u\alpha')\det(u')\\ & = & \lambda\lambda'\alpha\alpha'\cdot\det(u)\det(u') = \det(a)\det(a').\end{array}$$

The proof of (40):

$$\begin{array}{rcl}\det(\alpha^{-1}u\alpha) & = & \det([\alpha^{-1}]u[\alpha]) \quad (\text{where } \ [\alpha] := \mathrm{diag}(\alpha(1),\alpha(2),\dots))\\ & = & \det([\alpha^{-1}]_d u_d[\alpha]_d) \quad \text{for all } \ d\gg 1\\ & = & \det(u_d) = \det(u) \quad \text{for all } \ d\gg 1\end{array}$$

where, for an infinite matrix $X=\sum_{i,j\in\mathbb{N}} x_{ij}E_{ij}$, $X_d := \sum_{0\leq i,j\leq d-1} x_{ij}E_{ij}$ is the sub-matrix of X of size $d\times d$.

The kernel of the epimorphism det, (39), is $\mathrm{SL}_\infty(K) := \{u\in(1+F)^* = \mathrm{GL}_\infty(K)\,|\,\det(u) = 1\}$. In particular, $\mathrm{SL}_\infty(K)$ is a normal subgroup of $\mathbb{A}_1^*$ such that the factor group $\mathbb{A}_1^*/\mathrm{SL}_\infty(K)\simeq K^*\times\mathcal{H}\times K^*$ is abelian. Hence, $[\mathbb{A}_1^*,\mathbb{A}_1^*]\subseteq \mathrm{SL}_\infty(K)$ and there is the short exact sequence of groups

$$1\to \mathrm{SL}_\infty(K)\to\mathbb{A}_1^*\xrightarrow{\det} K^*\times\mathcal{H}\times K^*\to 1. \tag{41}$$

The group $\mathrm{SL}_\infty(K)$ is generated by the *transvections* $t_{ij}(\lambda) := 1+\lambda E_{ij}$, $\lambda\in K^*$, $i,j\in\mathbb{N}$, $i\neq j$. Note that

$$[H,t_{ij}(\lambda)] = t_{ij}(\frac{i-j}{j+1}\,\lambda) \tag{42}$$

where $[a,b] := aba^{-1}b^{-1}$ is the commutator of two elements of a group. In more detail,

$$[H,t_{ij}(\lambda)] = Ht_{ij}(\lambda)H^{-1}t_{ij}(\lambda)^{-1} = t_{ij}(\frac{i+1}{j+1}\,\lambda)t_{ij}(-\lambda) = t_{ij}(\frac{i-j}{j+1}\,\lambda).$$

Since $H\in\mathcal{H}$, we have the inclusion $\mathrm{SL}_\infty(K)\subseteq[\mathbb{A}_1^*,\mathbb{A}_1^*]$, i.e. $[\mathbb{A}_1^*,\mathbb{A}_1^*] = \mathrm{SL}_\infty(K)$.

5. Statement 5 follows from (42) and the fact that the group $\mathrm{SL}_\infty(K)$ is generated by the transvections. $\square$

An inversion formula for $u\in\mathbb{A}_1^*$. Let K be a field of characteristic zero. By Theorem 4.2.(1), each element u of $\mathbb{A}_1^*$ can written as $u=\lambda a(1+f)$. The inverse $(1+f)^{-1}$ can be found using Cramer's formula for the inverse of matrix. Then $u^{-1} = \lambda^{-1}(1+f)^{-1}a^{-1}$. Let $f\in K[x]$ be a given polynomial and $y\in K[x]$ is an unknown. Then the integro-differential equation $uy=f$ can be solved explicitly: $y=u^{-1}f$.

In contrast to differential operators on an affine line, in general, the space of solutions for integro-differential operators is infinite-dimensional: Example. $E_{ij}y=0$.

For an ideal I of $\mathbb{A}_n$ such that $I\neq\mathbb{A}_n$, let $(1+I)^*$ be the group of units of the multiplicative monoid $1+I$.

Lemma 4.3 *Let K be a commutative $\mathbb{Q}$-algebra, I and J be ideals of $\mathbb{A}_n$ which are distinct from $\mathbb{A}_n$. Then*

1. $\mathbb{A}_n^*\cap(1+I) = (1+I)^*$.

2. $(1+I)^*$ *is a normal subgroup of* $\mathbb{A}_n^*$.

Proof. 1. The inclusion $\mathbb{A}_n^* \cap (1+I) \supseteq (1+I)^*$ is obvious. To prove the reverse inclusion, let $1+a \in \mathbb{A}_n^* \cap (1+I)$ where $a \in I$, and let $(1+a)^{-1} = 1+b$ for some $b \in \mathbb{A}_n$. The equality $1 = (1+a)(1+b)$ can be written as $b = -a(1+b) \in I$, i.e. $1+a \in (1+I)^*$. This proves the reverse inclusion.

2. For all $a \in \mathbb{A}_n^*$, $a(1+I)a^{-1} = 1 + aIa^{-1} = 1+I$, and so $a(1+I)^*a^{-1} = a(\mathbb{A}_n^* \cap (1+I))a^{-1} = a\mathbb{A}_n^* a^{-1} \cap a(1+I)a^{-1} = \mathbb{A}_n^* \cap (1+I) = (1+I)^*$. Therefore, $(1+I)^*$ is a normal subgroup of $\mathbb{A}_n^*$. $\square$

Let K be a field of characteristic zero. By (33), the group homomorphism $\mathbb{A}_n^* \rightarrow (\mathbb{A}_n/\mathfrak{a}_n)^* \simeq \mathcal{A}_n^*$ is an epimorphism. By Lemma 4.3.(1), its kernel is $\mathbb{A}_n^* \cap (1+\mathfrak{a}_n) = (1+\mathfrak{a}_n)^*$, and we have the short exact sequence of groups

$$1 \rightarrow (1+\mathfrak{a}_n)^* \rightarrow \mathbb{A}_n^* \rightarrow \mathcal{A}_n^* \rightarrow 1 \qquad (43)$$

which together with (33) proves the first statement of the next theorem.

Theorem 4.4 *Let* K *be a field of characteristic zero. Then*

1. $\mathbb{A}_n^* = K^* \times (\mathcal{H}_n \ltimes (1+\mathfrak{a}_n)^*)$.

2. *The centre of the group* $\mathbb{A}_n^*$ *is* K^*.

Proof. 2. Let Z be the centre of the group $\mathbb{A}_n^*$. Then $K^* \subseteq Z$ and, by statement 1,

$$Z = Z \cap \mathbb{A}_n^* = Z \cap (K^* \mathcal{H}_n (1+\mathfrak{a}_n)^*) = K^*(Z \cap \mathcal{H}_n(1+\mathfrak{a}_n)^*).$$

It remains to show that $Z' := Z \cap \mathcal{H}_n(1+\mathfrak{a}_n)^* = \{1\}$.

Let us show first that $Z' = Z \cap (1+\mathfrak{a}_n)^*$. Let $z = \varphi u \in Z'$ for some $\varphi \in \mathcal{H}_n$ and $u \in (1+\mathfrak{a}_n)^*$. It suffices to show that $\varphi = 1$. Note that, for each element $a \in \mathfrak{a}_n$, there exists a natural number $c = c(a)$ such that $aE_{\alpha\beta} = E_{\alpha\beta}a = 0$ for all $\alpha, \beta \in \mathbb{N}^n$ such that all $\alpha_i, \beta_i \geq c$. For short, we write $\alpha, \beta \gg 0$. So, $uE_{\alpha,\beta} = E_{\alpha\beta}u = E_{\alpha\beta}$ for all $\alpha, \beta \gg 0$. Note that $E_{\alpha\beta}^2 = 0$ for all $\alpha \neq \beta$, and so $1 + E_{\alpha\beta} \in \mathbb{A}_n^*$. Now, for all $\alpha, \beta \gg 0$ such that $\alpha \neq \beta$, $z(1+E_{\alpha\beta}) = (1+E_{\alpha\beta})z \Leftrightarrow z + \varphi(\alpha+1)E_{\alpha\beta} = z + E_{\alpha\beta}\varphi(\beta+1) \Leftrightarrow \varphi(\alpha+1) = \varphi(\beta+1)$ $\Leftrightarrow \varphi = 1$, as required, where $\varphi(\alpha+1)$ is the value of the function $\varphi = \varphi(H_1, \ldots, H_n)$ at $H_1 = \alpha_1 + 1, \ldots, H_n = \alpha_n + 1$. This proves the claim.

So, it remains to show that $Z' := Z \cap (1+\mathfrak{a}_n)^* = \{1\}$. The result is true for $n = 1$ (Theorem 4.2.(3)). So, we may assume that $n \geq 2$. Consider the descending chain $\mathfrak{f}_1 \supset \cdots \supset \mathfrak{f}_i \supset \cdots \supset \mathfrak{f}_n \supset \mathfrak{f}_{n+1} := 0$ of ideals

$$\mathfrak{f}_i := \sum_{1 \leq j_1 < \cdots < j_i \leq n} \mathfrak{p}_{j_1}\mathfrak{p}_{j_2}\cdots\mathfrak{p}_{j_i}$$

of $\mathbb{A}_n$. Note that $\mathfrak{f}_1 = \mathfrak{a}_n$, $\mathfrak{f}_n = F^{\otimes n}$, and

$$(\mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i} + \mathfrak{f}_{i+1})/\mathfrak{f}_{i+1} \simeq \mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i}/\mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i} \cap \mathfrak{f}_{i+1} \simeq F^{\otimes i} \otimes \mathcal{A}_{n-i} \simeq F \otimes \mathcal{A}_{n-i} \simeq M_\infty(\mathcal{A}_{n-i}).$$

In the proof of the above series of ring isomorphisms without 1 we have used the facts that $F = M_\infty(K)$ and $F^{\otimes i} \simeq F$.

$$\mathfrak{f}_i/\mathfrak{f}_{i+1} \simeq \prod_{1\leq j_1<\cdots<j_i\leq n} (\mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i} + \mathfrak{f}_{i+1})/\mathfrak{f}_{i+1} \simeq (F^{\otimes i}\otimes \mathcal{A}_{n-i})^{\binom{n}{i}} \simeq M_\infty(\mathcal{A}_{n-i})^{\binom{n}{i}}$$

are isomorphisms of rings without 1. Consider the isomorphisms of groups

$$\begin{aligned}(1+\mathfrak{f}_i/\mathfrak{f}_{i+1})^* &\simeq (1+\prod_{1\leq j_1<\cdots<j_i\leq n} (\mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i} + \mathfrak{f}_{i+1})/\mathfrak{f}_{i+1})^* \\ &\simeq \prod_{1\leq j_1<\cdots<j_i\leq n} (1+(\mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i} + \mathfrak{f}_{i+1})/\mathfrak{f}_{i+1})^* \simeq (1+F^{\otimes i}\otimes\mathcal{A}_{n-i})^{*\binom{n}{i}} \\ &\simeq (1+M_\infty(\mathcal{A}_{n-i}))^{*\binom{n}{i}} \simeq \mathrm{GL}_\infty(\mathcal{A}_{i-1})^{\binom{n}{i}}\end{aligned}$$

where $\mathrm{GL}_\infty(\mathcal{A}_{i-1}) := (1+M_\infty(\mathcal{A}_{n-i}))^*$. The descending chain above yields the descending chain of normal subgroups of $\mathbb{A}_n^*$:

$$(1+\mathfrak{f}_1)^* = (1+\mathfrak{a}_n)^* \supset \cdots \supset (1+\mathfrak{f}_i)^* \supset \cdots \supset (1+\mathfrak{f}_n)^* = (1+F^{\otimes n})^* \supset (1+\mathfrak{f}_{n+1})^* = \{1\}.$$

For each $i = 1,\dots,n$, there is the natural homomorphism of groups $\varphi_i : (1+\mathfrak{f}_i)^* \to (1+\mathfrak{f}_i/\mathfrak{f}_{i+1})^*$, the kernel of which is $(1+\mathfrak{f}_{i+1})^*$. So, $(1+\mathfrak{f}_i)^*/(1+\mathfrak{f}_{i+1})^*$ is a subgroup of $(1+\mathfrak{f}_i/\mathfrak{f}_{i+1})^*$. To prove that $Z' = \{1\}$ is equivalent to show that $Z\cap(1+\mathfrak{f}_i)^* = \{1\}$ for all $i = 1,\dots,n$. To prove this we use a *downward* induction at i starting with $i = n$. In this case, $\mathfrak{f}_n = F^{\otimes n}$, and the fact that $Z\cap(1+F^{\otimes n})^* = \{1\}$ follows from the inclusion

$$Z\cap(1+F^{\otimes n})^* \subseteq Z((1+F^{\otimes n})^*) = Z((1+M_\infty(K)^*) = \{1\}$$

since $Z(M_\infty(K)) = 0$.

Suppose that $i < n$ and $Z\cap(1+\mathfrak{f}_{i+1})^* = \{1\}$. Using a downward induction on i it remains to show that $Z_i := Z\cap(1+\mathfrak{f}_i)^* = \{1\}$. Note that in any ring elements $1+a$ and $1+b$ commute iff the elements a and b commute. Using this observation we see that, for any ring R, the centre $Z(1+M_\infty(R))$ of the multiplicative monoid $1+M_\infty(R)$ is 1 since 1 is the only element of $1+M_\infty(K)$ that commute with all the elements $1+E_{kl}$, $k\neq l$. All these elements belong to the group $\mathrm{GL}_\infty(R) := (1+M_\infty(R))^*$, and so it has the trivial centre

$$Z(\mathrm{GL}_\infty(R)) = \{1\}. \tag{44}$$

In particular, $Z(\mathrm{GL}_\infty(\mathcal{A}_{n-i})^{\binom{n}{i}}) = \{1\}$. For each subset $J = \{j_1,\dots,j_i\}$ of $\{1,\dots,n\}$ that contains exactly i elements, we have seen above that

$$(\mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i} + \mathfrak{f}_{i+1})/\mathfrak{f}_{i+1} \simeq F^{\otimes i}\otimes\mathcal{A}_{n-i} \simeq M_\infty(\mathcal{A}_{n-i}).$$

For each matrix unit $E_{\alpha\beta} = E_{\alpha_1\beta_1}(j_1)\cdots E_{\alpha_i\beta_i}(j_i) \in \mathfrak{p}_{j_1}\cdots\mathfrak{p}_{j_i}$ where $\alpha,\beta\in\mathbb{N}^i$ and $\alpha\neq\beta$, the elements $1+E_{\alpha\beta}$ belongs to $(1+\mathfrak{f}_i)^*$ since $E_{\alpha\beta}^2 = 0$, and its image under the map φ_i is equal to the element $1+E_{\alpha\beta}+\mathfrak{f}_{i+1}$. This means that an element $\varphi_i(z)$ commutes with all elements $1+E_{\alpha\beta}+\mathfrak{f}_{i+1}$ for all possible choices of J, i.e. $\varphi_i(z)\in Z(\mathrm{GL}_\infty(\mathcal{A}_{n-i})^{\binom{n}{i}})=\{1\}$. This means that $z\in Z\cap(1+\mathfrak{f}_{i+1})^*=\{1\}$, i.e. $z=1$, as required. By induction, $\{1\}=Z\cap(1+\mathfrak{f}_1)^*=Z\cap(1+\mathfrak{a}_n)^*$. This proves that $Z(\mathbb{A}_n^*)=K^*$. $\square$

The subgroup $(1+\mathfrak{a}_n)^*$ **of** $\mathbb{A}_n^*$. Let K be a commutative $\mathbb{Q}$-algebra. For each $i=1,\dots,n$,

$$\begin{aligned}(\mathfrak{p}_i+\mathfrak{f}_2)/\mathfrak{f}_2 &\simeq \mathfrak{p}_i/\mathfrak{p}_i\cap\mathfrak{f}_2\simeq\mathbb{A}_{i-1}\otimes F\otimes\mathbb{A}_{n-i}/(\mathfrak{a}_{i-1}\otimes F\otimes\mathbb{A}_{n-i}+\mathbb{A}_{i-1}\otimes F\otimes\mathfrak{a}_{n-i})\\ &\simeq \mathbb{A}_{i-1}/\mathfrak{a}_{i-1}\otimes F\otimes\mathbb{A}_{n-i}/\mathfrak{a}_{n-i}\simeq\mathcal{A}_{i-1}\otimes F\otimes\mathcal{A}_{n-i}\simeq M_\infty(\mathcal{A}_{n-1})\end{aligned}$$

is the series of isomorphisms of rings without 1. The factor ring without 1

$$\mathfrak{a}_n/\mathfrak{f}_2\simeq\prod_{i=1}^n(\mathfrak{p}_i+\mathfrak{f}_2)/\mathfrak{f}_2\simeq\prod_{i=1}^n\mathcal{A}_{i-1}\otimes F\otimes\mathcal{A}_{n-i}\simeq M_\infty(\mathcal{A}_{n-1})^n$$

is the direct product of its subrings without 1. It is a semi-simple $\mathbb{A}_n$-bimodule where $(\mathfrak{p}_i+\mathfrak{f}_2)/\mathfrak{f}_2$, $1\le i\le n$, are the *simple* isotypic components of the $\mathbb{A}_n$-bimodule $\mathfrak{a}_n/\mathfrak{f}_2$. Fix a section $s:\mathfrak{a}_n/\mathfrak{f}_2\to\mathfrak{a}_n$ of the K-module epimorphism $\mathfrak{a}_n\to\mathfrak{a}_n/\mathfrak{f}_2$, $a\mapsto a+\mathfrak{f}_2$. Then $\mathfrak{a}_n=\mathrm{im}(s)\oplus\mathfrak{f}_2$ is a direct sum of K-submodules. Using the K-basis B_n for $\mathbb{A}_n$ considered in the proof of Corollary 2.7 one can easily find such a section which even satisfies the additional property that $\mathrm{im}(s)$ is a free K-module.

Consider the ring $K\oplus(\mathfrak{a}_n/\mathfrak{f}_2)$ with 1 and the subgroup $(1+(\mathfrak{a}_n/\mathfrak{f}_2))^*$ of its group $(K\oplus(\mathfrak{a}_n/\mathfrak{f}_2))^*$ of units. There are canonical group isomorphisms

$$\begin{aligned}(1+\mathfrak{a}_n/\mathfrak{f}_2)^* &\simeq (1+\prod_{i=1}^n(\mathfrak{p}_i+\mathfrak{f}_2)/\mathfrak{f}_2)^*\\ &\simeq \prod_{i=1}^n(1+(\mathfrak{p}_i+\mathfrak{f}_2)/\mathfrak{f}_2)^*,\ \ 1+\sum_{i=1}^n p_i\mapsto\prod_{i=1}^n(1+p_i),\\ &\simeq \prod_{i=1}^n(1+M_\infty(\mathcal{A}_{n-1}))^*=\prod_{i=1}^n\mathrm{GL}_\infty(\mathcal{A}_{n-1})=\mathrm{GL}_\infty(\mathcal{A}_{n-1})^n.\end{aligned}$$

We have the group monomorphism

$$(1+\mathfrak{a}_n)^*/(1+\mathfrak{f}_2)^*\to(1+\mathfrak{a}_n/\mathfrak{f}_2)^*,\ \ (1+a)(1+\mathfrak{f}_2)^*\mapsto 1+a+\mathfrak{f}_2.$$

An invertibility criterion. The next theorem is a criterion of when an element of the monoid $1+\mathfrak{a}_n$ is invertible.

Theorem 4.5 *Let K be a commutative $\mathbb{Q}$-algebra, $a\in\mathfrak{a}_n$. Then $1+a\in(1+\mathfrak{a}_n)^*$ iff*

1. $1+a+\mathfrak{f}_2\in(1+\mathfrak{a}_n/\mathfrak{f}_2)^*$ $(\simeq\mathrm{GL}_\infty(\mathcal{A}_{n-1})^n)$, *and*
2. $a+(1+a)c\in(1+a)\mathfrak{f}_2$ *and* $a+c(1+a)\in\mathfrak{f}_2(1+a)$ *where* $c:=s((1+a+\mathfrak{f}_2)^{-1}-1)$ *(the value of the section $s:\mathfrak{a}_n/\mathfrak{f}_2\to\mathfrak{a}_n$ at the element $(1+a+\mathfrak{f}_2)^{-1}-1\in\mathfrak{a}_n/\mathfrak{f}_2$).*

Suppose that conditions 1 and 2 hold and $a+(1+a)c = (1+a)r$ (resp. $a+c(1+a) = l(1+a)$) for some $r \in \mathfrak{f}_2$ (resp. $l \in \mathfrak{f}_2$) then $a^{-1} = 1 + c - r$ (resp. $a^{-1} = 1 + c - l$).

Proof. ($\Rightarrow$) Suppose that $(1 + a) \in (1 + \mathfrak{a}_n)^*$. Due to the group homomorphism $(1 + \mathfrak{a}_n)^* \to (1 + \mathfrak{a}_n/\mathfrak{f}_2)^*$, we have $1 + a \mapsto 1 + a + \mathfrak{f}_2 \in (1 + \mathfrak{a}_n/\mathfrak{f}_2)^*$, i.e. the first condition holds. $(1 + \mathfrak{a}_n)^* \ni (1 + a)^{-1} = 1 + b$ for some element $b \in \mathfrak{a}_n = \mathrm{im}(s) \oplus \mathfrak{f}_2$ which can be written as $b = c + d$ where $c := s((1 + a + \mathfrak{f}_2)^{-1} - 1) \in \mathrm{im}(s)$ and $d := b - c \in \mathfrak{f}_2$. The equalities $(1 + a)(1 + c + d) = 1$ and $(1 + c + d)(1 + a) = 1$ can be rewritten as follows $a + (1 + a)c = -(1 + a)d \in (1 + a)\mathfrak{f}_2$ and $a + c(1 + a) = -d(1 + a) \in \mathfrak{f}_2(1 + a)$, and so the second statement holds.

($\Leftarrow$) Suppose that conditions 1 and 2 hold, we have to show that $1 + a \in (1 + \mathfrak{a}_n)^*$, i.e. the element $1 + a$ has a left and a right inverse. Condition 2 can be written as follows $a + (1 + a)c = (1 + a)r$ and $a + c(1 + a) = l(1 + a)$ for some elements $r, l \in \mathfrak{f}_2$. These two equalities can we rewritten as $(1+a)(1+c-r) = 1$ and $(1+c-l)(1+a) = 1$. This means that $1 + a \in (1 + \mathfrak{a}_n)^*$ and $a^{-1} = 1 + c - r = 1 + c - l$. $\square$

An inversion formula for $u \in K^* \times (\mathcal{H}_n \ltimes (1 + F^{\otimes n})^*)$. Let K be a field of characteristic zero. By Theorem 4.4.(1), each element $u \in \mathbb{A}_n^*$ is a unique product $u = \lambda h(1 + a)$ for some $\lambda \in K^*$, $h \in \mathcal{H}_n$, and $a \in \mathfrak{a}_n$ such that $1 + a \in (1 + \mathfrak{a}_n)^*$. Clearly, $u^{-1} = \lambda^{-1}(1 + a)^{-1}h^{-1}$. So, to write down explicitly an inversion formula for u boils down to finding $(1 + a)^{-1}$. As a first step, one should know an inversion formula for elements of $\mathrm{GL}_\infty(\mathcal{A}_{n-1})$ which is not obvious how to do, at the moment. It should not be entirely trivial since as a result one would have a formula for solutions of all invertible integro-differential equations (for all $n \geq 2$). Nevertheless, for elements $u = \lambda h(1+a) \in K^* \times (\mathcal{H}_n \ltimes (1+F^{\otimes n})^*)$ one *can* write down the inversion formula exactly in the same manner as in the case $n = 1$. Hence, one obtains explicitly solutions to the equation $uy = f$ where $f \in P_n$. Elements of the group $K^* \times (\mathcal{H}_n \ltimes (1 + F^{\otimes n})^*)$ are called *minimal integro-differential operators*.

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