Evaluation of Output Frequency Responses of Nonlinear Systems Under Multiple Inputs

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***Abstract*—In this paper, a new method for evaluating output frequency responses of nonlinear systems under multiple inputs, defined as a sum of sinusoids of different frequencies, is developed. The method circumvents difficulties associated with the existing "frequency-mix vector" based approaches and can easily be applied to investigate nonlinear behaviors of practical systems, including electronic circuits, at the system simulation and design stages. Application of the method to the analysis of nonlinear interference and distortion effects in communication receivers is studied, and specific procedures are proposed which can be directly used in practice for this analysis.**

***Index Terms*—Communication receivers, frequency-domain analysis, nonlinear systems/circuits.**

I. INTRODUCTION

MOST engineering systems including electronic circuits are intrinsically nonlinear. Although measures such as differential configurations, feedback, inverse function cancellation, etc., are often taken to reduce nonlinear effects when practical design problems are addressed, nonlinear distortion cannot usually be cancelled out completely, and it is therefore important to evaluate system behaviors to estimate how the residual nonlinearity degrades system performances.

Systems such as transistor amplifiers and operational transconductance amplifier-capacitor (OTA-C) filters, which are designed to exhibit mainly linear characteristics but still possess unavoidable residual nonlinearities, can be reasonably regarded as weakly nonlinear systems [1] and can be investigated in the frequency domain using the Volterra series theory of nonlinear systems [1]–[5], [8], [9], [12].

The frequency-domain method of nonlinear systems based on the Volterra series theory was initially established in the 1950's when the concept of generalized frequency response functions (GFRF's) of nonlinear systems was introduced [7]. GFRF's were defined as the multidimensional Fourier transformations of Volterra kernels in the Volterra series expansion of nonlinear systems which extend the frequency response function of linear systems to the nonlinear case. One of the important features of GFRF's is associated with the description of nonlinear system output responses in the frequency domain. The frequency-domain output responses of practical systems are often directly related to physical system performances especially for electronic circuits. Therefore, analysis of the responses is important for examining system behaviors. Nonlinear effects which are likely to be expected in practice can be determined at the system design and simulation stages by evaluating and analyzing the system output frequency responses.

The multiple inputs are defined as a sum of sinusoids with different frequencies which can be used at the design and simulation stage and/or laboratory testing period of systems such as communication receivers [8] to excite the systems in order to examine the system output behaviors in the frequency domain. Analysis of nonlinear systems with multiple inputs has been an important topic in the frequency-domain analysis of nonlinear systems using Volterra series theory since the 1950's. Many theories and methods have been developed to address problems associated with this topic [1]–[4], [8], [9] and applications of the associated theories and methods to circuit analyses can be found in [6], [8], [10], [11].

The presently available methods for analysis of nonlinear systems with multiple inputs are almost all based on a concept called the "frequency-mix vector" which reveals the manner by which intermodulation frequencies are generated in nonlinear systems. Intermodulation is an important nonlinear phenomenon which indicates that output frequency components of a nonlinear system could be much richer than the components in the input, while in the linear system case the possible output frequency components are exactly the same as the components in the corresponding input. Although the analyses using the "frequency-mix vector" can clearly interpret how output frequencies of nonlinear systems are produced by particular frequency mixes, the output frequency response components at frequencies of interest are generally difficult to evaluate in practice using associated methods. This is because the output component of a nonlinear system at a particular frequency actually depends on many different frequency mixes and it is generally hard to identify all frequency-mix vectors associated with these different frequency mixes.

In the present study, the above problem which is associated with the practical evaluation of output frequency responses of nonlinear systems to multiple inputs is addressed. At first, an expression for the system response is derived which can be readily used in practice to evaluate the results. Then, output frequencies of nonlinear systems under multiple inputs are analyzed and an effective algorithm is developed to determine the frequencies. The algorithm extends the concept regarding the relationship between the system input and output frequencies to the nonlinear case where systems are under an arbitrary multiple input excitation. Based on the first and second results, a new method is then

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developed to evaluate output frequency responses of nonlinear systems under multiple input excitations which provides an effective and practical means for evaluating frequency-domain effects of practical nonlinear systems, including electronic circuits at the system design and simulation stages. Finally, the application of this method to the analysis of nonlinear interference and distortion effects in communication receivers is studied and specific procedures are proposed which combine the method with our previously developed nonlinear system modeling and analysis techniques and which can be directly used in practice for this analysis.

II. Analysis of Nonlinear Systems Under Multiple Inputs

Systems such as transistor amplifiers and OTA-C filters which possess weak nonlinearities can be described by a Volterra series representation [2]–[4]. The Volterra series representation of a nonlinear system can be generally written as

$$y(t)=\sum_{n=1}^{N} y_n(t) \tag{1}$$

where

$$y_n(t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n)\prod_{i=1}^{n} u(t-\tau_i)\,d\tau_i \tag{2}$$

$u(t)$ and $y(t)$ are system input and output, respectively, $h_n(\tau_1,\cdots,\tau_n)$, $1\le n\le N$ are the Volterra kernels, and N is the maximum order of system nonlinearities which is finite for a wide class of nonlinear systems and input excitations [5].

Under the excitation of a multiple input defined by

$$u(t)=\sum_{i=1}^{R}|A_i|\cos(\omega_i t+\angle A_i)=\sum_{i=-R,\,i\neq 0}^{R}\frac{A_i}{2}e^{j\omega_i t} \tag{3}$$

where $A_{-i}=A_i^*$, A_i^* denotes the conjugate of A_i, and $\omega_{-i}=-\omega_i$, the nth-order output response of the system (1) and (2) can be described, by substituting (3) into (2), as

$$y_n(t)=\frac{1}{2^n}\sum_{i_1=-R,\,i_1\neq 0}^{R}\cdots\sum_{i_n=-R,\,i_n\neq 0}^{R} A_{i_1}\cdots A_{i_n} \cdot H_n(j\omega_{i_1},\cdots,j\omega_{i_n})e^{j(\omega_{i_1}+\cdots+\omega_{i_n})t} \tag{4}$$

where

$$H_n(j\omega_{i_1},\cdots,j\omega_{i_n})=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n(\tau_1,\cdots,\tau_n) \cdot e^{-j(\omega_{i_1}\tau_1+\cdots+\omega_{i_n}\tau_n)}d\tau_1\cdots d\tau_n \tag{5}$$

is the nth-order GFRF $H_n(j\omega_1,\cdots,j\omega_n)$ of the system evaluated at $\{\omega_1,\cdots,\omega_n\}=\{\omega_{i_1},\cdots,\omega_{i_n}\}$.

Equations (1) and (4) provide a general description for output responses of nonlinear systems under multiple inputs. Analysis of this response can, in most cases, be sufficiently performed based on (4) to investigate the nth-order portion of $y(t)$ and

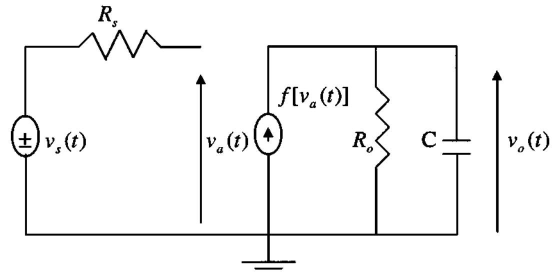

Fig. 1. The nonlinear equivalent circuit of an OTA-C integrator.

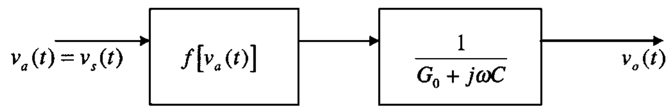

Fig. 2. Block diagram of the OTA-C integrator.

the total response is simply a summation of all $y_n(t)'s$ from $n=1$ to N.

Equation (4) indicates that when a sum of R sinusoids is applied to a nonlinear system, additional output frequencies are generated by the nth-order portion of the system output consisting of all possible combinations of the input frequencies $-\omega_R,\cdots,-\omega_1,\omega_1,\cdots,\omega_R$ taken n at a time.

In order to illustrate this general expression, consider a practical example in [6] which is an OTA with capacitor load. Fig. 1 shows the nonlinear model of the OTA-C integrator with a nonlinear current source where $f(.)$ is the characteristic function of the OTA which can be exactly determined from the circuit structure and parameters [6].

It can be shown from Fig. 1 that the circuit equations in the frequency domain are given by

$$\begin{cases}(V_a-V_s)G_s=0\\(G_0+j\omega C)V_0-f[v_a(t)]=0\end{cases} \tag{6}$$

where $G_s=(1/R_s)$, $G_0=(1/R_0)$, and V_s, V_a, V_o are the Fourier transforms of $v_s(t)$, $v_a(t)$, and $v_o(t)$, respectively. Thus, the block diagram description of the circuit can be shown as in Fig. 2.

Representing $f[v_a(t)]$ by the Taylor series expansion about the operation point $v_a(t)=0$ yields

$$f[v_a(t)]=\sum_{n=1}^{\infty} g_n v_a^n(t) \tag{7}$$

where $g_n=(1/n!)(df[v_a(t)]/v_a(t))|_{v_a(t)=0}$. Then the nth-order GFRF of the circuit can be obtained for $n=1,2,3,\cdots$ as [6]

$$H_n(j\omega_1,\cdots,j\omega_n)=\frac{g_nR_o}{j(\omega_1+,\cdots,+\omega_n)R_oC+1}. \tag{8}$$

Consider a case where the circuit is subject to a one-tone input, i.e., $R=1$, and examine the second-order output response $y_2(t)$ of the circuit to this input. In this specific situation

$$u(t)=\sum_{i=-R,\,i\neq 0}^{R}\frac{A_i}{2}e^{j\omega_i t}=\sum_{i=-1,\,i\neq 0}^{1}\frac{A_i}{2}e^{j\omega_i t} \tag{9}$$

and

$$\begin{aligned} y_2(t) = &\frac{(A_1^*)^2}{2^2} H_2(-j\omega_1, -j\omega_1) e^{-2j\omega_1 t} \\ &+ \frac{|A_1|^2}{2^2} H_2(-j\omega_1, j\omega_1) e^{j(\omega_1-\omega_1)t} \\ &+ \frac{|A_1|^2}{2^2} H_2(j\omega_1, -j\omega_1) e^{j(\omega_1-\omega_1)t} \\ &+ \frac{(A_1)^2}{2^2} H_2(j\omega_1, j\omega_1) e^{2j\omega_1 t}. \end{aligned} \tag{10}$$

Moreover, substituting (8) into (10) for $n = 2$ yields

$$\begin{aligned} y_2(t) = &\frac{|A_1|^2}{2} g_2 R_o + \frac{g_2 R_o |A_1|^2}{2\sqrt{(2\omega_1 R_o C)^2 + 1}} \\ &\cdot \cos\left[2\omega_1 t + \angle A_1^2 - \tan^{-1}(2\omega_1 R_o C)\right] \end{aligned} \tag{11}$$

Equation (11) indicates that the second-order response of the circuit to the one-tone input is composed of two frequency components $\omega = 0$ and $\omega = 2\omega_1$, which are the absolute values of the summations of the input frequencies $-\omega_1$ and ω_1 taken two at a time, that is

$$|\omega_1 - \omega_1| = |-\omega_1 + \omega_1| = 0,\ |-\omega_1 - \omega_1| = 2\omega_1, \text{ and } |\omega_1 + \omega_1| = 2\omega_1.$$

The specific case above is a very simple example where the output frequencies composed of all possible combinations of the input frequencies can be easily identified. For general cases where systems are subject to arbitrary multiple inputs where R could be any integer, the frequency-domain analysis of nonlinear systems under multiple inputs is more complicated and is usually carried out based on a concept called the "frequency-mix vector" [1].

Because, under a multiple input, output frequencies generated by the nth-order system nonlinearity consist of all possible combinations of the input frequencies $-\omega_R, \cdots, -\omega_1, \omega_1, \cdots, \omega_R$ taken n at a time, let m_i denote the number of times the frequency ω_i appears in a particular frequency mix, the frequency mix can then be represented by the vector

$$m = [m_{-R}, \cdots, m_{-1}, m_1, \cdots, m_R] \tag{12}$$

where the m_i's obey the constraint

$$m_{-R} + \cdots + m_{-1} + m_1 + \cdots + m_R = n. \tag{13}$$

Vector m is referred to as the nth-order frequency-mix vector and the corresponding output frequency is given by

$$\omega_m = (m_1 - m_{-1})\omega_1 + \cdots + (m_R - m_{-R})\omega_R. \tag{14}$$

Therefore, the output frequencies in $y_n(t)$ given by (4) can be interpreted as those frequencies that can be generated by all possible choices of the m_i's such that (13) is satisfied.

It has been shown that the output component in (4) which corresponds to a particular frequency mix m is given by [1]

$$\begin{aligned} y_n(t, m) = &\frac{n!}{2^n} \left(\prod_{i=-R, i\neq 0}^{R} \frac{A_i^{m_i}}{m_i!} \right) \\ &\cdot H_n \begin{pmatrix} m_{-R}\{j\omega_{-R}\}, \cdots, m_{-1}\{j\omega_{-1}\} \\ m_1\{j\omega_1\}, \cdots, m_R\{j\omega_R\} \end{pmatrix} e^{j\omega_m t} \end{aligned} \tag{15}$$

where the GFRF $H_n(.)$ is assumed to be symmetric $H_2(j\omega_1, j\omega_2) = H_2(j\omega_2, j\omega_1)$ in the $n = 2$ case, for example, and $m_i\{j\omega_i\}$ denotes m_i consecutive arguments in $H_n(.)$ having the same frequency $j\omega_i$. Thus the nth-order portion of $y(t)$ can be written as

$$y_n(t) = \sum_m y_n(t, m) \tag{16}$$

where the summation over m is defined to be

$$\sum_m = \underset{(13)}{\sum_{m_{-R}=0}^{n} \cdots \sum_{m_R=0}^{n}}$$

and (13) appended below the summation signs indicates that only terms for which the indices sum to n are included in the 2R-fold summation.

The above analysis for the nth-order nonlinear output response to a multiple input clearly reflects how an output frequency component is produced by a particular frequency mix and how the component can be evaluated using the associated frequency-mix vector.

Consider the above circuit example again but under a two-tone input. The frequencies in the second-order output response of the circuit are $\omega_m = (m_1 - m_{-1})\omega_1 + (m_2 - m_{-2})\omega_2$ with the m_i's obeying the constraint $m_{-2} + m_{-1} + m_1 + m_2 = 2$ The output component corresponding to a particular $m = [m_{-2}, m_{-1}, m_1, m_2]$ can be determined using (15) where the GFRF is defined by (8) for $n = 2$. In this case, it is not difficult to show the associated frequency-mix vectors are

$$\{1, 1, 0, 0\}, \{0, 1, 1, 0\}, \{0, 0, 1, 1\}, \{1, 0, 0, 1\}, \{1, 0, 1, 0\}$$
$$\{0, 1, 0, 1\}, \{2, 0, 0, 0\}, \{0, 2, 0, 0\}, \{0, 0, 2, 0\}, \{0, 0, 0, 2\}$$

and $y_2(t)$ is, therefore, the result of the summation of $y_n(t, m)$ given by (15) over all these frequency-mix vectors.

Output frequencies corresponding to these vectors can be easily obtained as $-\omega_1 - \omega_2, -\omega_1 + \omega_1 = 0 \omega_1 + \omega_2, -\omega_2 + \omega_2 = 0 -\omega_2 + \omega_1, -\omega_1 + \omega_2, -2\omega_2, -2\omega_1, 2\omega_1, 2\omega_2$.Therefore, the practical output frequencies, which are the nonnegative results of the above frequencies, are $\omega_1 + \omega_2, \omega_2 - \omega_1, 2\omega_1, 2\omega_2,$ and 0.

Although, as shown above, the frequency-mix vector is very useful in nonlinear frequency-domain response analyses under multiple inputs, an important defect with this concept is that distinct frequency-mix vectors of the same order may give rise to the same output frequency. For example, when $R = 3$, $\{\omega_1, \omega_2, \omega_3\} = \{1, 2, 4\}$ and $n = 2$, the frequency-mix vector $m = (0, 0, -1, 0, 0, 1)$ yields an output frequency $\omega_m = \omega_3 - \omega_1 = 3$, while the frequency-mix vector $m = (0, 0, 0, 1, 1, 0)$ also yields $\omega_m = \omega_2 + \omega_1 = 3$. So, in general, (15) can not be used to represent the frequency response of the system nth-order nonlinear output. Based on the concept of "frequency-mix vector," this response can only be represented as

$$y_{n\omega}(t) = \sum_{\text{all possible } m \text{ such that } \omega_m=\omega} y_n(t\colon m) \tag{17}$$

where $y_{n\omega}(t)$ denotes the total nth-order output response at frequency ω.

From (17), it is hard to evaluate $y_{n\omega}(t)$ practically. This is because, given a frequency of interest ω, it is generally a very difficult job to identify all possible m's such that $\omega_m = \omega$. However, determining these m's is necessary if $y_{n\omega}(t)$ is to be evaluated from (17). In [9], a general algorithm was proposed to address this problem which transformed the problem of identifying all possible m's to the problem of sorting out all possible integers ρ's such that $\rho_1\omega_1+, \cdots, +\rho_R\omega_R = \omega$. Obviously, the difficulties with the original method of identifying all possible m's can not be bypassed when using this algorithm.

Motivated by the attempt to completely resolve the problem above with existing methods, a new method is proposed in Section V to provide a practical and effective strategy to evaluate the nonlinear frequency responses to multiple inputs and therefore to investigate possible nonlinear behaviors of systems in the frequency domain. The derivations and analyses in Sections III and IV establish the important and necessary basis for this new method.

III. Expression for the Output Frequency Responses

When a nonlinear system described by (1) and (2) is excited by a multiple input (3), the system nth-order nonlinear output is generally given by (4), which can be rewritten as

$$y_n(t) = \frac{1}{2^n} \sum_{i_1=-R,\, i_1\neq 0}^{R} \cdots \sum_{i_n=-R,\, i_n\neq 0}^{R} A(\omega_{i_1})\cdots A(\omega_{i_n}) \cdot H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}) e^{j(\omega_{i_1}+\cdots+\omega_{i_n})t} \tag{18}$$

where $A(.)$ is defined by

$$A(\omega) = \begin{cases} A_i, & \text{if } \omega \in \{\omega_i,\, i=\pm 1, \cdots, \pm R\} \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

In order to obtain a more transparent frequency-domain to time-domain relationship, consider

$$A(-\omega_1)\cdots A(-\omega_n) H_n(-j\omega_1, \cdots, -j\omega_n) = [A(\omega_1)\cdots A(\omega_n) H_n(j\omega_1, \cdots, j\omega_n)]^* \tag{20}$$

where the * denotes conjugation, and write (18) as

$$\begin{aligned} y_n(t) = {} & \frac{2}{2^n} \sum_{\text{all possible } \omega>0} \sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} \cdot \text{Re}[A(\omega_{i_1})\cdots A(\omega_{i_n}) H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}) e^{j\omega t}] \\ & + \frac{1}{2^n} \sum_{\omega_{i_1}+\cdots+\omega_{i_n}=0} A(\omega_{i_1})\cdots A(\omega_{i_n}) H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}) \\ = {} & \sum_{\text{all possible } \omega\geq 0} |\overline{Y}_n(j\omega)| \cos[\omega t + \angle \overline{Y}_n(j\omega)] \end{aligned} \tag{21}$$

where

$$\overline{Y}_n(j\omega) = \begin{cases} \frac{1}{2^{n-1}} \sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} A(\omega_{i_1})\cdots A(\omega_{i_n}) \cdot H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}), & \omega > 0 \\ \frac{1}{2^n} \sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} A(\omega_{i_1})\cdots A(\omega_{i_n}) \cdot H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}), & \omega = 0. \end{cases} \tag{22}$$

In (21) and (22)

$$\sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} A(\omega_{i_1})\cdots A(\omega_{i_n}) H_n(j\omega_{i_1}, \cdots, j\omega_{i_n})$$

denotes the summation of

$$A(\omega_{i_1})\cdots A(\omega_{i_n}) H_n(j\omega_{i_1}, \cdots, j\omega_{i_n})$$

over all the $\omega_{i_1}, \cdots, \omega_{i_n}$ which satisfy the constraint $\omega_{i_1}+, \cdots, +\omega_{i_n} = \omega$ with

$$\omega_{i_l} \in \{-\omega_R, \cdots, -\omega_1, \omega_1, \cdots, \omega_R\}, \qquad l = 1, \cdots, n.$$

$\overline{Y}_n(j\omega)$ defined by (22) is the nth-order output frequency response of the system (1) and (2) to the input (3) in terms of nonnegative frequencies, which represents the contribution of the nth-order system nonlinearity to the output frequency component ω.

Substituting (21) into (1) gives

$$y(t) = \sum_{\text{all possible } \omega\geq 0} |\overline{Y}(j\omega)| \cos[\omega t + \angle \overline{Y}(j\omega)] \tag{23}$$

where

$$\overline{Y}(j\omega) = \sum_{n=1}^{N} \overline{Y}_n(j\omega). \tag{24}$$

So, the output frequency response of a nonlinear system under a multiple input is given, in terms of nonnegative frequencies, by

$$\begin{cases} \overline{Y}(j\omega) = \sum_{n=1}^{N} \overline{Y}_n(j\omega), \quad \omega \geq 0 \\ \overline{Y}_n(j\omega) = \begin{cases} \frac{1}{2^{n-1}} \sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} A(\omega_{i_1})\cdots A(\omega_{i_n}) \cdot H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}), & \omega > 0 \\ \frac{1}{2^n} \sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} A(\omega_{i_1})\cdots A(\omega_{i_n}) \cdot H_n(j\omega_{i_1}, \cdots, j\omega_{i_n}), & \omega = 0 \end{cases} \\ \omega_{i_l} \in \{-\omega_R, \cdots, -\omega_1, \omega_1, \cdots, \omega_R\}, \quad l = 1, \cdots, n \end{cases} \tag{25}$$

Notice that the relationship between the system output frequency spectrum $Y(j\omega)$ and $\overline{Y}(j\omega)$ is

$$Y(j\omega) = \begin{cases} \frac{\overline{Y}(j\omega)}{2}, & \omega \neq 0 \\ \overline{Y}(j\omega), & \omega = 0. \end{cases} \tag{26}$$

Because of (23), $\overline{Y}(j\omega)$ can be more easily related to the system time domain response.

It can be observed from (25) that the possible output frequencies in the nth-order nonlinear output are $\omega = \omega_{i_1} + \cdots + \omega_{i_n}$ with

$$\omega_{i_l} \in \{-\omega_R, \cdots, -\omega_1, \omega_1, \cdots, \omega_R\}, \quad l = 1, \cdots, n.$$

This clearly reflects how the output frequencies are composed in this situation. In addition, from the definition of

$$\sum_{\omega_{i_1}+\cdots+\omega_{i_n}=\omega} A(\omega_{i_1}) \cdots A(\omega_{i_n}) H_n(j\omega_{i_1}, \cdots, j\omega_{i_n})$$

the terms which compose the nth-order output frequency component $\overline{Y}_n(j\omega)$ can be readily identified. Therefore, the evaluation of $\overline{Y}_n(j\omega)$ and, moreover, of the total frequency response $\overline{Y}(j\omega)$ can easily be achieved using (25). This is because the evaluation of $\overline{Y}_n(j\omega)$ can be simply implemented in the following way

$$\begin{aligned}\overline{Y}_n(j\omega) = \frac{1}{2^{n_1}} \sum_{i_1=-R, i_1\neq 0}^{R} \cdots \sum_{i_{n-1}=-R, i_{n-1}\neq 0}^{R} A(\omega_{i_1}) \cdots \\ \cdot A(\omega_{i_{n-1}}) A(\omega - \omega_{i_1} - \cdots - \omega_{i_{n-1}}) \\ \cdot H_n[j\omega_{i_1}, \cdots, j\omega_{i_{n-1}}, j(\omega - \omega_{i_1} -, \cdots, -\omega_{i_{n-1}})]\end{aligned} \tag{27}$$

where $n_1 = 0$ for $\omega = 0$, $n_1 = n - 1$, for $\omega > 0$, and the terms in which

$$\omega - \omega_{i_1} - \cdots - \omega_{i_{n-1}} \neq \omega_l, \quad l \in \{-R, \cdots, -1, 1, \cdots, R\}$$

are zeros according to the definition of $A(.)$ given by (19), and, moreover, $\overline{Y}(j\omega)$ can be obtained by just making a summation of the results determined from (27) from $n = 1$ to N.

In order to illustrate how to evaluate $\overline{Y}(j\omega)$ using the above idea, consider an example where the OTA-C circuit in Section II is excited by a two-tone input $u(t) = \cos 2t + \cos 3t$ and the second-order output frequency response of the circuit at frequencies $\omega = 1$ and $\omega = 3$ is to be examined.

In this case, $R = 2$, $A_1 = A_2 = 1$, $\omega_1 = 2$, $\omega_2 = 3$

$$A(\omega) = \begin{cases} A_i = 1, & \text{if } \omega \in \{\omega_i, i = \pm 1, \pm 2\} \\ & \quad = \{-3, -2, 2, 3\} \\ 0, & \text{otherwise} \end{cases}$$

and the second-order GFRF

$$H_2(j\omega_1, j\omega_2) = \frac{g_2 R_o}{j(\omega_1 + \omega_2) R_o C + 1}.$$

Therefore

$$\begin{aligned}\overline{Y}_2(j\omega) &= \frac{1}{2^{n_1}} \sum_{i_1=-2, i_1\neq 0}^{2} A(\omega_{i_1}) A(\omega - \omega_{i_1}) \\ &\quad \cdot H_2[j\omega_{i_1}, j(\omega - \omega_{i_1})] \\ &= \frac{1}{2^{n_1}} \left\{ \begin{array}{l} A(\omega+3) + A(\omega+2) \\ +A(\omega-2) + A(\omega-3) \end{array} \right\} \frac{g_2 R_o}{j\omega R_o C + 1}.\end{aligned} \tag{28}$$

Thus $\overline{Y}_2(j1)$ and $\overline{Y}_2(j3)$ can be immediately obtained as

$$\overline{Y}_2(j1) = \frac{1}{2^{2-1}} \{0 + 1 + 0 + 1\} \frac{g_2 R_o}{jR_o C + 1} = \frac{g_2 R_o}{jR_o C + 1}$$

and

$$\overline{Y}_2(j3) = \frac{1}{2^{2-1}} \{0 + 0 + 0 + 0\} \frac{g_2 R_o}{j3R_o C + 1} = 0$$

and the corresponding output components are therefore

$$y_{2(\omega=1)}(t) = \frac{|g_2| R_o}{\sqrt{(R_o C)^2 + 1}} \cos(t - \tan^{-1} R_o C)$$

and $y_{2(\omega=3)}(t) = 0$.

Clearly, compared to the evaluation of an nth-order output response at a particular frequency using (17), the computation of this response based on (25) is much more straightforward and the difficulty with determining all possible frequency-mix vectors for a specific frequency, which is necessary when (17) is used, is circumvented. Notice that the expression for $\overline{Y}_2(j\omega)$ given in (25) or (27) accommodates all possible terms which could make contributions to frequency ω and, when given a specific value of ω, the terms which actually have no effect on the response at the specific frequency automatically become zero due to the definition of $A(\omega)$.

The analyses and examples above indicate that based on (25) the output frequency responses of nonlinear systems under multiple inputs can easily be evaluated at any frequencies of interest. However, it is obviously unnecessary to evaluate the response components at frequencies which are beyond the range of system output frequencies since these components are definitely zero. To address this issue involves determining possible output frequencies of nonlinear systems subject to multiple input excitations.

IV. Determination of the Output Frequencies

For linear systems, it is well known that the possible output frequencies are exactly the same as the frequencies in the corresponding input. However, this property does not hold if the system is nonlinear. When a nonlinear system is subject to a multiple input, it has been shown from the analyses in previous sections that output frequencies generated by the nth-order system nonlinearity consist of all possible combinations of the input frequencies $-\omega_R, \cdots, -\omega_1, \omega_1, \cdots, \omega_R$ taken n at a time. This result can be analytically described as a set given by

$$\left\{ \omega \,\middle|\, \begin{array}{l} \omega = \omega_{i_1} +, \cdots, +\omega_{i_n} \\ \omega_{i_l} \in \{-\omega_R, \cdots, -\omega_1, \omega_1, \cdots, \omega_R\} l = 1, \cdots, n \end{array} \right\}. \tag{29}$$

The problem to be addressed initially here is to develop an algorithm to determine the frequencies composed of the nonnegative part of the result given by (29).

For the simplest case of $n = 1$, it is obvious that these frequencies are $\omega_1, \cdots, \omega_R$, which can be rewritten in a vector form as

$$W_1 = \begin{bmatrix} \left|\sum \overline{W}_1(1, :)\right| \\ \vdots \\ \left|\sum \overline{W}_1(R, :)\right| \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_R \end{bmatrix} \tag{30}$$

where

$$\overline{W}_1 = [\omega_1 \cdots \omega_R]^T \tag{31}$$

and $\sum \overline{W}_1(l, :)$, $1 \leq l \leq R$ denote the summation of the elements in the lth-row of matrix $\overline{W}_1$.

The output frequencies in the case of $n = 2$ can be determined from

$$\begin{cases} |\omega_1 + \omega_l|, & l = -R, \cdots, -1, 1, \cdots, R \\ \vdots & \\ |\omega_R + \omega_l|, & l = -R, \cdots, -1, 1, \cdots, R. \end{cases} \tag{32}$$

Define two vectors

$$I = [1, \cdots, 1]^T \tag{33}$$

of length 2R and

$$W = [-\omega_R \cdots - \omega_1 \omega_1 \cdots \omega_R]^T \tag{34}$$

to express (32) in terms of a vector as

$$W_2 = \left[\left|\sum \overline{W}_2(1, :)\right|, \cdots, \left|\sum \overline{W}_2(2R^2, :)\right|\right]^T \tag{35}$$

where

$$\overline{W}_2 = \begin{bmatrix} I\overline{W}_1(1, :) & W \\ \vdots & \vdots \\ I\overline{W}_1(R, :) & W \end{bmatrix}. \tag{36}$$

For $n = 3$, it is easy to show that the vector representing the output frequencies produced by the third-order nonlinearity is

$$W_3 = \left[\left|\sum \overline{W}_3(1, :)\right|, \cdots, \left|\sum \overline{W}_3(R(2R)^2, :)\right|\right]^T \tag{37}$$

where

$$\overline{W}_3 = \begin{bmatrix} I\overline{W}_2(1, :) & W \\ \vdots & \vdots \\ I\overline{W}_2(2R^2, :) & W \end{bmatrix}. \tag{38}$$

Consequently the algorithm for computing the vector representing the (nonnegative) frequencies in the nth-order nonlinear output is given by

$$\begin{cases} W_n = \begin{bmatrix} \left|\sum \overline{W}_n(1, :)\right| \\ \vdots \\ \left|\sum \overline{W}_n(R(2R)^{n-1}, :)\right| \end{bmatrix} \\ \overline{W}_n = \begin{bmatrix} I\overline{W}_{n-1}(1, :) & W \\ \vdots & \vdots \\ I\overline{W}_{n-1}(R(2R)^{n-2}, :) & W \end{bmatrix} \\ n \geq 2 \; \overline{W}_1 = [\omega_1, \cdots, \omega_R]^T. \end{cases} \tag{39}$$

Many of the elements in W_n may be the same. Therefore, the final result of this algorithm is a set composed of all different elements of W_n . Denote this set as Ω, then

$$\Omega_n = \{\{W_n\}\} \tag{40}$$

where $\{\{X\}\}$ means a set composed of all the different elements of vector X.

In order to illustrate the application of this algorithm, consider an example where $\omega_1 = 1$, $\omega_2 = 3$ and the frequencies in the second-order nonlinear output is to be determined.

In this case, $R = 2$, $n = 2$, $W_1 = [\omega_1, \cdots, \omega_R]^T = [1, 3]^T$ $I = [1, 1, 1, 1]^T$, $W = [-3, -1, 1, 3]^T$, $\overline{W}_1 = W_1$

$$\overline{W}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ -3 & -1 & 1 & 3 & -3 & -1 & 1 & 3 \end{bmatrix}^T$$

$W_2 = [2, 0, 2, 4, 0, 2, 4, 6]^T$, therefore $\Omega_2 = \{\{W_2\}\} = \{0, 2, 4, 6\}$.

From the above algorithm, Ω_n can be determined for any n. Therefore, the frequencies in the system output represented by Ω can be obtained as $\Omega = \bigcup_{n=1}^{N} \Omega_n$.

However, there is actually no need to obtain all Ω_n's and then to determine Ω as shown above. This is because, for any n, there is a deterministic relationship between the frequencies in the nth-order nonlinear output and the frequencies in the $(n+2)$th-order nonlinear output.

It can be shown from (39) that

$$\overline{W}_{n+2} = [I_{R(2R)^n} \overline{W}_{n+1} \quad W_{R(2R)^n}] \tag{41}$$

where

$$I_{R(2R)^n} = \begin{bmatrix} \underbrace{\begin{matrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & I \end{matrix}}_{R(2R)^n} \end{bmatrix}$$

and

$$W_{R(2R)^n} = \Bigg[\underbrace{W^T \cdots\cdots W^T}_{R(2R)^n}\Bigg]^T.$$

Similarly

$$\overline{W}_{n+1} = [I_{R(2R)^{n-1}} \overline{W}_n \quad W_{R(2R)^{n-1}}]. \tag{42}$$

Substituting (42) into (41) yields

$$\overline{W}_{n+2} = [I_{R(2R)^n} I_{R(2R)^{n-1}} \overline{W}_n \;\; I_{R(2R)^n} W_{R(2R)^{n-1}} \;\; W_{R(2R)^n}]. \tag{43}$$

In the matrix given by (43), the first matrix block takes the form

$$I_{R(2R)^n} I_{R(2R)^{n-1}} \overline{W}_n = \begin{bmatrix} \Bigg[\underbrace{\overline{W}_n(1, :)^T \cdots \overline{W}_n(1, :)^T}_{(2R)^2}\Bigg]^T \\ \vdots \\ \Bigg[\underbrace{\overline{W}_n(R(2R)^{n-1}, :)^T \cdots \overline{W}_n(R(2R)^{n-1}, :)^T}_{(2R)^2}\Bigg]^T \end{bmatrix} \tag{44}$$

the second matrix block takes the form

$$I_{R(2R)^n} W_{R(2R)^{n-1}} = \left[\left.\begin{matrix} \left[\underbrace{\omega_{-R}, \cdots, \omega_{-R}}_{2R}, \cdots, \underbrace{\omega_R, \cdots, \omega_R}_{2R}\right]^T \\ \vdots \\ \left[\underbrace{\omega_{-R}, \cdots, \omega_{-R}}_{2R}, \cdots, \underbrace{\omega_R, \cdots, \omega_R}_{2R}\right]^T \end{matrix}\right\} R(2R)^{n-1}\right] \tag{45}$$

and the third takes the form

$$W_{R(2R)^n} = \left[\left.\begin{matrix} \left[\underbrace{[\omega_{-R}, \cdots, \omega_R], \cdots, [\omega_{-R}, \cdots, \omega_R]}_{2R}\right]^T \\ \vdots \\ \left[\underbrace{[\omega_{-R}, \cdots, \omega_R], \cdots, [\omega_{-R}, \cdots, \omega_R]}_{2R}\right]^T \end{matrix}\right\} R(2R)^{n-1}\right]. \tag{46}$$

It is not difficult to observe from (43)–(46) that the elements of vector

$$W_{n+2} = \left[\left|\sum \overline{W}_{n+2}(1, :)\right|, \cdots, \left|\sum \overline{W}_{n+2}(R(2R)^{n+1}, :)\right|\right]^T$$

include all elements of vector

$$W_n = \left[\left|\sum \overline{W}_n(1, :)\right|, \cdots, \left|\sum \overline{W}_n(R(2R)^{n-1}, :)\right|\right]^T.$$

This implies that $\{\{W_n\}\} \in \{\{W_{n+2}\}\}$, that is

$$\Omega_n \in \Omega_{n+2} \tag{47}$$

or all frequencies in the nth-order nonlinear output are present in the $(n+2)$th-order nonlinear output.

This conclusion was proved before [9] under the assumption that $\omega_1, \cdots, \omega_R$ form a frequency base which means there does not exist a set of rational numbers $r_1, \cdots, r_R$ (not all zero) such that $r_1\omega_1 +, \cdots, +r_R\omega_R = 0$. Since no assumptions are made on $\omega_1, \cdots, \omega_R$ in the above derivation, the conclusion has now been established for arbitrary input frequencies.

It is straightforward from (47) that the frequencies in the system output

$$\Omega = \Omega_N \bigcup \Omega_{N-(2p^*-1)} \tag{48}$$

where the value to be taken by p^* could be $1, 2, \cdots, [N/2]$ where [.] denotes to take the integer part. The specific value of p^* depends on the system nonlinearities. If the system GFRF's $H_{N-(2i-1)}(.) = 0$ for $i = 1, \cdots, q-1$, and $H_{N-(2q-1)}(.) \neq 0$, then $p^* = q$.

In the example above, where a nonlinear system is subject to a two-tone input with $\omega_1 = 1$, $\omega_2 = 3$, assume that the maximum nonlinear order $N = 2$ and the first-order frequency-response function $H_1(.) \neq 0$. It is straightforward to show that in this case $p^* = 1$. Therefore, the output frequencies of the system can be determined using (48) as

$$\begin{aligned}\Omega &= \Omega_N \bigcup \Omega_{N-(2p^*-1)} = \Omega_2 \bigcup \Omega_1 \\ &= \{0, 2, 4, 6\} \bigcup \{1, 3\} = \{0, 1, 2, 3, 4, 6\}.\end{aligned}$$

Equations (39), (40), and (48) compose an algorithm for determining output frequencies of nonlinear systems under multiple inputs. This result actually extends the relationship between the input and output frequencies of linear systems to the nonlinear case when the systems are subject to multiple inputs and is therefore also of theoretical significance.

V. New Method for Evaluating Nonlinear Output Frequency Responses to Multiple Inputs

It has been shown in Section III that, based on (25) output components of nonlinear systems under multiple inputs at any frequencies of interest can be readily evaluated. Equations (39), (40), and (48) derived in Section IV provide an effective algorithm for determining possible output frequencies of nonlinear systems in this situation. Based on these two results, a new method is proposed below to evaluate nonlinear output frequency responses to multiple input excitations.

The basic idea of this new method is to determine all possible system output frequencies and the frequencies contributed by each order of system nonlinearities using the algorithm derived in Section IV. Thus, if the frequencies of interest are beyond the range of possible output frequencies, it is known immediately that the output responses at these frequencies are zero. If the frequencies of interest are within the range of possible output frequencies then the frequencies contributed by each order of system nonlinearities provide important information concerning which order of system nonlinearities could make a contribution to these frequencies of interest. Moreover, system output responses at the frequencies of interest are evaluated using (25) and the computation is implemented by first calculating the responses at these frequencies contributed by the nonlinear orders which really make contributions to these frequency components, and then simply making a summation of the results obtained for corresponding nonlinear orders.

A summary of the new method, which requires the frequency-domain model of the considered nonlinear system, i.e. the GFRF's, $H_n(j\omega_1, \cdots, j\omega_n)$, $n = 1, \cdots, N$, to be known *a priori*, is given below.

1) Calculate all possible output frequencies using (39), (40) and (48) to yield the set Ω.
2) For $n = 1, 2, \cdots, N$, calculate Ω_n to determine a set S_Ω which is composed of the numbers of the nonlinearity orders which have contributions to the output frequency $\omega_A \in \Omega$ at which the output component is to be evaluated.
3) Compute $\overline{Y}(j\omega_A)$ as below

$$\overline{Y}(j\omega_A) = \sum_{n \in S_\Omega} \overline{Y}_n(j\omega_A) \tag{49}$$

where

$$\begin{aligned}\overline{Y}_n(j\omega_A) = &\frac{1}{2^{n_1}} \sum_{i_1=-R,\, i_1\neq 0}^{R} \cdots \\ &\cdot \sum_{i_{n-1}=-R,\, i_{n-1}\neq 0}^{R} A(\omega_{i_1})\cdots A(\omega_{i_{n-1}}) \\ &\cdot A(\omega_A - \omega_{i_1} - \cdots - \omega_{i_{n-1}}) \\ &\cdot H_n[j\omega_{i_1}, \cdots, j\omega_{i_{n-1}}, \\ &\qquad j(\omega_A - \omega_{i_1} -, \cdots, -\omega_{i_{n-1}})]\end{aligned} \tag{50}$$

and

$$n_1 = \begin{cases} n, & \text{if } \omega_A = 0 \\ n-1, & \text{otherwise.}\end{cases}$$

4) Evaluate the output response at frequency ω_A as

$$y_{\omega_A}(t) = |\overline{Y}(j\omega_A)| \cos(\omega_A t + \angle \overline{Y}(j\omega_A)). \tag{51}$$

The example of the OTA-C circuit in Section I is considered again to illustrate the application of this method in the following.

Assume that $f[v_a(t)]$ can be approximated sufficiently well by a third-order polynomial $f[v_a(t)] = \sum_{n=1}^{3} g_n v_a^n(t)$, where $g_n \neq 0$, for $n = 1, 2, 3$, and the output response of the circuit to the multiple input $u(t) = \cos\, \omega_1 t + \cos\, \omega_2 t = \cos\, t + \cos\, 3t$ at the frequency of interest $\omega_A = 5$ is to be evaluated.

The GFRF's of the circuit system are given by (8). They are all zero in this case for $n > 3$, but not zero for $n = 1, 2, 3$ because $f[v_a(t)]$ can be approximated well by a third-order polynomial and $g_n \neq 0$, for $n = 1, 2, 3$. Obviously the maximum order of system nonlinearities in this case is $N = 3$. Because $H_2(.) \neq 0$, $p^* = 1$. Thus, using (39), (40) and (48) with $R = 2$, $\overline{W}_1 = [1, 3]^T$, $I = [1, 1, 1, 1]^T$ and $W = [-3, -1, 1, 3]^T$ yields $\Omega = \Omega_3 \bigcup \Omega_2 = \{0, 1, 2, 3, 4, 5, 6, 7, 9\}$ indicating that $\omega_A = 5$ belongs to the frequencies which possibly appear in the system output.

$\Omega_1, \Omega_2, \Omega_3$, obtained using (39) and (40) in this case, are $\Omega_1 = \{1, 3\}$, $\Omega_2 = \{0, 1, 2, 4, 6\}$, $\Omega_3 = \{1, 3, 5, 7, 9\}$. So $S_\Omega = \{3\}$ for the frequency of interest $\omega_A = 5$.

Using (50) and considering that in this specific case, $R = 2$, $A_1 = A_2 = 1$, $\omega_1 = 1$, $\omega_2 = 3$

$$A(\omega) = \begin{cases} A_i = 1, & \text{if } \omega \in \{\omega_i, i = \pm 1, \pm 2\} \\ & \qquad = \{-3, -1, 1, 3\} \\ 0, & \text{otherwise}\end{cases}$$

$$H_3(j\omega_1, j\omega_2, j\omega_3) = \frac{g_3 R_o}{j(\omega_1 + \omega_2 + \omega_3) R_o C + 1}$$

it follows that

$$\begin{aligned}\overline{Y}(j\omega_A) &= \sum_{n\in S_\Omega} \overline{Y}_n(j\omega_A) = \overline{Y}_3(j5) \\ &= \frac{1}{4}\frac{g_3 R_o}{j5R_oC + 1}\{3A(3) + 2A(1) + A(-1)\} \\ &= \frac{6}{4}\frac{g_3 R_o}{j5R_oC + 1}\end{aligned}$$

Thus, in this case, the output response component of the circuit at $\omega_A = 5$ is

$$y_{\omega_A=5}(t) = \frac{1.5|g_3|R_o}{\sqrt{(5R_oC)^2 + 1}} \cos(5t - \angle\, \tan^{-1}(5R_oC)).$$

The method developed and illustrated above provides an effective means for evaluating the output frequency responses of nonlinear systems under multiple inputs based on the system frequency-domain descriptions. Exact evaluation of system output frequency responses can only be achieved using both system models and exact knowledge of the corresponding input spectra. Multiple input signals can easily be generated with all parameters of the signals under control. Methods are currently available for estimating the GFRF's of nonlinear systems [13]–[15] and for systems such as some electronic circuits the GFRF's can even be derived directly from the system structure and parameters. Therefore, this method can, hopefully, be widely applied to analyze nonlinear behaviors of practical systems including electronic circuits at the system/circuit design and simulation stages. The application to nonlinear analysis of communication receivers will be discussed in the next section to demonstrate how to use this method in practical system analysis.

VI. Analysis on Nonlinear Interference and Distortion Effects in Communication Receivers

In communication systems, the modulated information signal from a transmitter is transmitted to a receiver where the signal is amplified and the information extracted. A simplified block diagram of a superheterodyne receiver is illustrated in Fig. 3. The block diagram of receivers in modern radio communication systems are essentially the same as this [16].

Ideally, when the receiver input consists of signals from many communication channels, tuning the receiver to the carrier frequency of a channel by changing the frequency of the local oscillator could allow only the information from the selected channel to be eventually recovered at the detector stage. However, this is correct only when the amplifier and intermediate frequency filter in the receiver are made up of ideal linear circuits, which is impossible in practice.

Fig. 4 illustrates the input and output frequency components of an amplifier which possesses nonlinearities up to the third-order [1]. If the amplifier and filter stages of a receiver have the frequency-response characteristics as shown in Fig. 4, where ω_1 and ω_2 represent the carrier frequencies of two different communication channels, it is not difficult to observe from the figure that owing to the intermodulation effects of the third-order nonlinearity, the signal from the second channel will definitely have an effect on the receiver output when the receiver is tuned to only select the signal from the first channel. This kind of interference between different communication channels and the resulting distortion on the transmitted information are clearly important problems which must be addressed in the design stage of communication receivers [16] and quantitative analysis of these nonlinear effects on the performance of communication receivers is necessary when dealing with these problems.

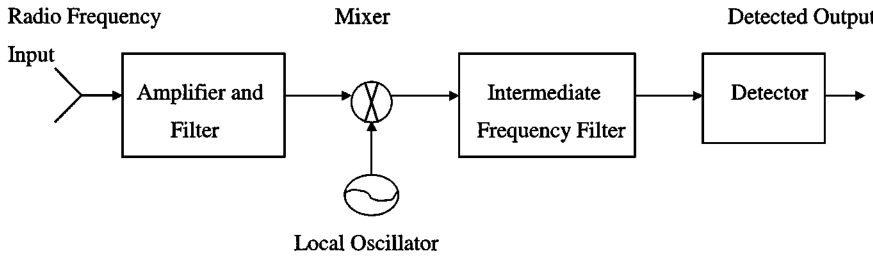

Fig. 3. A simplified block diagram of a superheterodyne receiver.

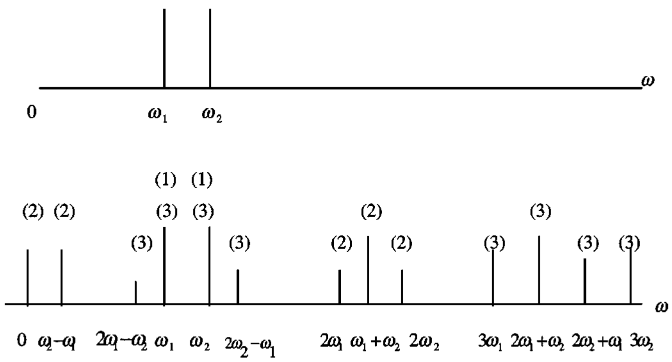

Fig. 4. The input and output frequency components of an amplifier possessing nonlinearities up to the third-order (the numbers in the curved brackets indicate the nonlinear orders which produce the corresponding frequency components).

Analysis of the effects of circuit nonlinearity on communication receivers was previously carried out based on theories of nonlinear circuit analysis and using the "frequency-mix vector" approach developed from the Volterra series theory of nonlinear systems [1], [8]. The nonlinear circuit analysis was applied to establish the nonlinear frequency-domain model of the receiver circuits to be analyzed and the "frequency-mix vector" approach was then used to evaluate and analyze the output frequency responses of the circuits to multiple inputs. There are two disadvantages to this approach. The first is that a complete and exact theoretical modeling for nonlinear circuits is usually impossible due to the fact that parameters of some devices may not be available and the second is associated with the problems of the "frequency-mix vector" approach which were discussed in Section II.

Based on the method in Section V and nonlinear system identification and frequency-domain analysis techniques developed by the authors, a different approach is proposed in the following to implement the analysis of frequency responses of nonlinear circuits in communication receivers. This approach can be applied at the design and testing stages to examine the effects of nonlinear interference and distortion on the receiver performance.

Application to Analysis of Nonlinear Effects in Communication Receivers

1) Establish a nonlinear difference model of the amplifier and filter circuit of the receiver to be analyzed using the Nonlinear AutoRegressive Moving Average model with exogenous inputs (NARMAX) methodology [18]–[19] and the input and output data from an experiment on the circuit in a prototype of the receiver. NARMAX methodology includes effective nonlinear system modeling techniques developed by Billings and coworkers which involve methods for model structure selection, parameter estimation, and model validation and can be used to produce a nonlinear difference model referred to as nonlinear autoregressive with exogenous input (NARX) model without using any prior knowledge of the identified system.

   The polynomial NARX model of a nonlinear system can be expressed as follows:

$$y(k) = \sum_{n=1}^{M} y_n(k)$$

   where $y_n(k)$ is a "nth-order output" given by

$$n(k) = \sum_{p=0}^{n} \sum_{l_1, l_{p+q}=1}^{K} c_{pq}(l_1, \cdots, l_{p+q}) \cdot \prod_{i=1}^{p} y(k-l_i) \prod_{i=p+1}^{p+q} u(k-l_i)$$

   with $p+q=n$, $l_i = 1, \cdots, K$, $i = 1, \cdots, p+q$ and $\sum_{l_1, l_{p+q}=1}^{K} \equiv \sum_{l_1=1}^{K} \cdots \sum_{l_{p+q}=1}^{K}$. K is the maximum lag and $y(.)$, $u(.)$, and $c_{pq}(.)$ are the output, input, and model coefficients, respectively.

   A specific NARX model such as, for example

$$\begin{aligned} y(k) = {} & 1.5593y(k-1) - 0.4582y(k-2) \\ & - 0.155\,85y(k-3) + 1.2829u(k-1) \\ & - 1.195u(k-3) + 4.8262u(k-3)u(k-3)u(k-3) \end{aligned}$$

   may be obtained from the above general form with $c_{01}(1) = 1.2829$, $c_{01}(3) = -1.195$, $c_{10}(1) = 1.5593$, $c_{10}(2) = -0.4582$, $c_{10}(3) = -0.155\,85$, $c_{03}(3, 3, 3) = 4.8262$, else $c_{pq}(.) = 0$.

2) Map the identified NARX model of the circuit into the frequency domain to yield the GFRF's of the circuit system $H_1(j\omega_1)$, $H_2(j\omega_1, j\omega_2)$, $\cdots$. The mapping from the polynomial NARX model to the frequency domain has been developed [14] and is given by

$$\left\{1 - \sum_{l_1=1}^{N} c_{10}(l_1) \exp\left[-j(\omega_1 +, \cdots, +\omega_n) l_1\right]\right\} \cdot H_n(j\omega_1, \cdots, j\omega_n) = \sum_{l_1, l_n=1}^{K} c_{0n}(l_1, \cdots, l_n) \exp\left[-j(\omega_1 l_1 +, \cdots, +\omega_n l_n)\right]$$

$$+\sum_{q=1}^{n-1}\sum_{p=1}^{n-q}\sum_{l_1,\,l_{p+q}=1}^{K} c_{pq}(l_1,\cdots,l_{p+q})$$
$$\cdot\exp\left[-j(\omega_{n-q+1}l_{p+1}+,\cdots,+\omega_n l_{p+q})\right]$$
$$\cdot H_{n-q,\,p}(j\omega_1,\cdots,j\omega_{n-q})$$
$$+\sum_{p=2}^{n}\sum_{l_1,\,l_p=1}^{K} c_{p0}(l_1,\cdots,l_p)H_{np}(j\omega_1,\cdots,j\omega_n)$$

where

$$H_{np}(j\omega_1,\cdots,j\omega_n)=\sum_{i=1}^{n-p+1}H_i(j\omega_1,\cdots,j\omega_i)\cdot H_{n-i,\,p-1}(j\omega_{i+1},\cdots,j\omega_n)\cdot\exp\left[-j(\omega_1+,\cdots,+\omega_i)l_p\right].$$

For the specific NARX model above, for example, the mapping can be readily obtained from the general relationship as

$$H_1(j\omega_1)=\frac{1.2819\exp(-j\omega_1)-1.195\exp(-3j\omega_1)}{\left\{\begin{matrix}1-1.5593\exp(-j\omega_1)+0.4582\exp(-2j\omega_1)\\+0.155\,85\exp(-3j\omega_1)\end{matrix}\right\}}$$
$$H_2(j\omega_1,j\omega_2)=0$$
$$H_3(j\omega_1,j\omega_2,j\omega_3)=\frac{4.8262\exp\left[-j3(\omega_1+\omega_2+\omega_3)\right]}{\left\{\begin{matrix}1-1.5593\exp\left[-j(\omega_1+\omega_2+\omega_3)\right]\\+0.4582\exp\left[-j2(\omega_1+\omega_2+\omega_3)\right]\\+0.155\,85\exp\left[-j3(\omega_1+\omega_2+\omega_3)\right]\end{matrix}\right\}}$$
$$H_n(j\omega_1,\cdots,j\omega_n)=0,\qquad n\geq 4.$$

Notice that the above procedure shows how to obtain the GFRF's of the discrete time model of a nonlinear circuit. If the frequency-domain description of the continuous time model of the circuit is required, the results are essentially the same as the results obtained for the discrete time model.

3) Determine the maximum order N of nonlinearities in the circuit using the method in [20] concerning truncation of the Volterra series expansion of nonlinear systems. But for the above simple specific NARX model, it is clear that $N=3$.

4) Evaluate the output frequency response of the circuit to a multiple input using the method in Section V to examine the nonlinear effects on the performance of the communication receiver. This involves the following steps.

   a) Select the frequencies $\omega_1,\cdots,\omega_R$ and corresponding magnitude and phase for the multiple input to be applied. These frequencies could, for example, be the carrier frequencies of the different communication channels associated with the receiver which is to be analyzed.

   b) For $n=2,\cdots,N$, determine the frequency set Ω_n, which contains output frequencies contributed by the nth-order circuit nonlinearity, using the algorithm given by (39) and (40), and then determine a set $\overline{S}_\Omega$ which is composed of the numbers of the nonlinearity orders that have contributions to the output frequency ω_A with $\omega_A\in\{\omega_1,\cdots,\omega_R\}$. The specific value of ω_A here depends on which communication channel is required to be analyzed for nonlinear interference and distortion.

   c) Evaluate

$$\sum_{n\in\overline{S}_\Omega}\overline{Y}_n(j\omega_A)$$

where $\overline{Y}_n(j\omega_A)$ is evaluated using (50) to examine the nonlinear interference effect on the communication channel associated with the carrier frequency ω_A. Notice that the complete output component at frequency ω_A is given by

$$\overline{Y}(j\omega_A)=\overline{Y}_1(j\omega_A)+\sum_{n\in\overline{S}_\Omega}\overline{Y}_n(j\omega_A)$$

where the first term represents the linear output response of the circuit to the multiple input at frequency ω_A, which, without nonlinear interference and distortion reflected by the second term, should be the output frequency response of the circuit to the signal channel associated with the carrier frequency ω_A.

These procedures can be readily coded and implemented such that the new approach can be directly applied for the interference and distortion analysis of communication receivers at the design and testing stages.

In engineering, two-tone ($R=2$) tests are commonly used to experimentally quantify the degree of nonlinearity of a nonlinear communication system or device. It is obvious that the new approach can be readily applied to perform the same analysis. In addition to this, the new approach also allows the analysis to be easily implemented when the system is subject to a arbitrary R ($R>2$) tone sinusoidal excitation so as to be able to accommodate complicated but more practical situations, which is impossible to be analyzed based on simple experimental studies.

VII. Conclusion

The behavior of practical systems, including electronic circuits, usually exhibit nonlinear characteristics although measures are often taken to try to compensate for undesirable nonlinear effects. It is therefore important to evaluate system output responses so as to estimate how the nonlinearities affect the system performance. Multiple inputs are typical signals which are used to excite systems when the system performance in the frequency domain is to be investigated. The existing methods for this investigation are almost all based on a concept known as the "frequency-mix vector." This concept is useful for explaining how the output frequencies of nonlinear systems are generated but it is difficult to use to evaluate the output response at frequencies of interest. In order to overcome this problem, a new method is developed in the present study to evaluate the frequency responses of nonlinear systems under multiple inputs. This method circumvents difficulties associated with the existing "frequency-mix vector" based approaches and provides an effective means to investigate nonlinear behaviors of practical systems including electronic circuits at the system design and simulation stages. The application of the method to nonlinear analysis of communication receivers has been studied and specific procedures are proposed which can be directly used in practice for this application.

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