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Multiple nonsmooth events in multi-degree-of-freedom vibro-impact systems

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Abstract.
The behaviour of a multi-degree-of-freedom vibro-impact system is studied using a two degree-of-freedom impact oscillator as a motivating example. A multi-modal model is used to simulate the behaviour of the system, and examine the complex dynamics which occurs when both degrees of freedom are subjected to a motion limiting constraint. In particular the chattering and sticking behaviour which occurs for low forcing frequencies is discussed. In this region a variety of nonsmooth events can occur, including newly studied phenomena such as sliding bifurcations. In this paper the multiple nonsmooth events which can occur in the two degree-of-freedom system are categorised, and demonstrated using numerical simulations.

Keywords: Vibro-impact, multiple constraint, chatter, sticking, sliding

1. Introduction

This paper deals with the dynamics of multi-degree-of-freedom impact oscillators subject to multiple motion limiting constraints. These systems consist of a set of coupled masses, where the motion of each of the masses is restricted by a series of motion limiting constraints. This paper will consider a two degree-of-freedom system with constraints placed a different distances from each mass [1]. Several authors have considered two constraints placed an equal distance either side of an oscillating mass, e.g. [2–4].

Two degree-of-freedom impact oscillator systems have been studied in relation to a range of applications [5–7]. More general multi-degree-of-freedom impact systems have also been considered by several other authors [8–11]. Of particular interest have been periodic impacting orbits which occur in multi-degree-of-freedom impact systems. Systems with a single impact stop have been studied [12, 13], and the method for finding period$(1, n)$ solutions developed for single degree-of-freedom impact oscillators by [14] has been extended to multi-degree-of-freedom impact oscillators [12]. Studies into bifurcations in this type of system have also been carried out, for example [15].

Chatter and sticking in single degree-of-freedom impact oscillators has been studied in detail [16, 17] and also noted in two degree-of-freedom systems [18]. The behaviour of periodic sticking motions in both single and multi-degree-of-freedom systems has also been studied [19]. The sticking phenomena observed here is analogous to the sliding in other systems [20], as discussed in [21]. In this paper we consider the same two degree-of-freedom example studied in [1], but examine the multi-dimensional nonsmooth events which can occur in the chatter and sticking region.

2. Mathematical Model

A schematic model of the generalized N degree-of-freedom coupled linear oscillator system with N lumped masses is shown schematically in Figure 1 (a). In general, the equations of motion for the coupled masses can be expressed as

$$m_i\ddot{x}_i+c_i(\dot{x}_i-\dot{x}_{i-1})+c_{i+1}(\dot{x}_i-\dot{x}_{i+1})+k_i(x_i-x_{i-1})+k_{i+1}(x_i-x_{i+1}) = f_i(t), \tag{1}$$

for $i = 1, 2 \ldots, N-1$ and

$$m_N\ddot{x}_N + c_N(\dot{x}_N - \dot{x}_{N-1}) + k_N(x_N - x_{N-1}) = f_N(t) \tag{2}$$

for $i = N$, where x_i and $f_i(t)$ represent the displacement and forcing of mass m_i, and an overdot is used to represent differentiation with respect to time t. These expressions govern the motion while all the displacements x_i are less than some fixed set of values s_i corresponding to the position of the motion constraints.

The equations of motion for the coupled masses can be expressed in matrix form as

$$[M]\ddot{\mathbf{x}} + [C]\dot{\mathbf{x}} + [K]\mathbf{x} = \mathbf{f}(t), \quad (x_i - s_i) \lesseqgtr 0 \quad \forall s_i \gtreqless 0 \tag{3}$$

where $[M], [C], [K]$ are the mass, damping and stiffness matrices respectively, $\mathbf{x} = \{x_1, x_2 \ldots, x_N\}^T$ is the displacement vector and $\mathbf{f}(t) = \{f_1, f_2 \ldots, f_N\}^T$ the external forcing vector. The coupling between masses occurs via the matrices $[C]$ and $[K]$, which are nondiagonal. The mass matrix $[M]$ is a diagonal matrix. Equation (3) has the dual condition for free flight that $(x_i - s_i) < 0$ for $s_i > 0$ and $(x_i - s_i) > 0$ for $s_i < 0$.

It is assumed that the damping matrix $[C]$ is linearly proportional to the stiffness matrix $[K]$, such that Equation (3) can be decoupled in the standard way [22]. In this work the case where $m_j = m$, $c_j = c$, $k_j = k$ for $j = 1, 2, \ldots, N$, is considered. Then Equation (3) can be written in the form

$$[I]\ddot{\mathbf{x}} + \frac{c}{m}[E]\dot{\mathbf{x}} + \frac{k}{m}[E]\mathbf{x} = \frac{1}{m}\mathbf{f}(t), \quad (x_i - s_i) \lesseqgtr 0 \quad \forall s_i \gtreqless 0 \tag{4}$$

where $[E]$ is the $N \times N$ coupling matrix

$$[E] = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & \dots & \ddots & \dots & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}, \tag{5}$$

and $[I]$ is the identity matrix.

The natural frequencies are given by $\omega_{nj} = \sqrt{\lambda_j k/m}$ for $j = 1, 2, \dots, N$ where λ_j are the eigenvalues of matrix $[E]$, and the corresponding normalized eigenvectors ξ_j can be used to construct a orthogonal modal matrix $[\Psi] = [\{\xi_1\}, \{\xi_2\}, \dots, \{\xi_N\}]$. Equation (4) can then be transformed into a modal form by defining modal coordinates $\mathbf{x} = [\Psi]\mathbf{q}$ where $\mathbf{q} = \{q_1, q_2, \dots q_N\}^T$, such that

$$[I]\ddot{\mathbf{q}} + \frac{c}{m}[\Lambda]\dot{\mathbf{q}} + \frac{k}{m}[\Lambda]\mathbf{q} = \frac{1}{m}[\Psi]^T \mathbf{f}(t) \tag{6}$$

where $[\Lambda] = [\Psi]^T[E][\Psi]$ is the diagonal matrix of the eigenvalues, λ_j, $j = 1, 2, ...N$.

In this modal formulation the vector $\psi_i = \{\Psi_{i1}, \Psi_{i2}, \dots, \Psi_{iN}\}^T$, is defined such that an impact occurs when $\psi_i^T \mathbf{q} = x_i$. Hence equation 6 is valid only for $(\psi_i^T \mathbf{q} - s_i) \lesseqgtr 0 \quad \forall s_i \gtreqless 0$, which is equivalent to the condition that $(x_i - s_i) \lesseqgtr 0 \quad \forall s_i \gtreqless 0$ for the ith impacting mass.

The system is considered to be subject to harmonic forcing of the form $\mathbf{f}(t) = \mathbf{A}\cos(\Omega t)$, where $\mathbf{A} = \{A_1, A_2, \dots, A_N\}^T$, then equation 6 can be simplified such that for each mode

$$\ddot{q}_j + 2\zeta_j \omega_{nj} \dot{q}_j + \omega_{nj}^2 q = \frac{\hat{f}_j}{m}\cos(\Omega t), \quad j = 1, 2, \dots, N \tag{7}$$

where $\hat{\mathbf{f}} = [\Psi]^T \mathbf{A}$, $\hat{\mathbf{f}} = \{\hat{f}_1, \hat{f}_2, \dots, \hat{f}_N\}^T$ and $\zeta_j = (c/2)\sqrt{\lambda_j/km}$ is the modal damping coefficient. Equation (7) has the well known exact solution for under-damped oscillations $0 < \zeta_j < 1$ [22] such that for each mode an exact solution can be obtained, and from this the total displacements for $\mathbf{x}$ [1].

2.1. Modelling impacts for systems with multiple constraints

An impact occurs for the ith mass when $x_i = s_i$, while for $j \neq i$:$(x_j - s_j) \lesseqgtr 0 \quad \forall s_j \gtreqless 0$. To model the impact an instantaneous coefficient of restitution rule is used [23]

$$\dot{x}_i(t_+) = -r\dot{x}_i(t_-) \quad x_i = s_i \tag{8}$$

where, t_- is the time just before impact, t_+ is the time just after impact and r is the coefficient of restitution with a value in the range $r \in [0, 1]$.

For systems with multiple constraints the matrix form the coefficient of restitution rule is

$$\dot{\mathbf{x}}(t_+) = [R]\dot{\mathbf{x}}(t_-) \quad (x_i - s_i) = 0 \tag{9}$$

where $[R]$ is the $N \times N$ diagonal coefficient of restitution matrix. For a system with N impacting masses $[R]$ will have a different form depending on whether single, multiple or all the masses make contact during the impact process [1].

In modal form the coefficient of restitution rule, Equation 9, for a single impact becomes

$$[\Psi]\dot{\mathbf{q}}(\tau_+) = [R][\Psi]\dot{\mathbf{q}}(\tau_-), \qquad (\psi_i^T \mathbf{q} - s_i) = 0. \tag{10}$$

This leads to the relation for the modal velocities after impact

$$\dot{\mathbf{q}}(\tau_+) = [\hat{R}]\dot{\mathbf{q}}(\tau_-), \qquad (\psi_i^T \mathbf{q} - s_i) = 0, \tag{11}$$

where $[\hat{R}] = [\Psi]^{-1}[R][\Psi]$ is the set of matrices which represents a linear transform of modal velocities just before impact to modal velocities just after impact for all the possible impact cases.

2.2. Sticking motion

Sticking motions occur when one or more of the masses is held motionless against the stop for a finite period of time. Sticking motions can occur in multi-degree-of-freedom impact oscillators after a complete chatter sequence has occurred [16, 18]. A chatter sequence becomes complete when the time between two successive impacts, δt becomes small, while at the same time the forces on the mass hold it against the impact stop. So, in order to reach a sticking solution for a single mass (the pth say) chatter must be complete, i.e $\delta t \approx 0$ and the force acting on the sticking mass must hold it against the constraint, which is equivalent to the condition $F_p s_p > 0$. This is similar to conditions for a relay system referred to as the *reaching conditions* [20]. There is one possible exception to these conditions, that is if a mass comes into contact with the stop with zero velocity and acceleration and simultaneously $F_p s_p > 0$ becomes true. This non-generic case is an example of a grazing-sliding bifurcation discussed in section 3.3.

To find the force F_p, we substitute $x_p = s_p$ and $\dot{x}_p = 0$ into the pth line of Equation (4). So for $1 \leq p < N$ from Equation (1) with all m, c and k values equal

$$F_p = c(\dot{x}_{p-1} + \dot{x}_{p+1}) + k(x_{p-1} + x_{p+1}) + f_p(t) + 2ks_p, \tag{12}$$

and for $p = N$ from Equation (2),

$$F_p = c\dot{x}_{p-1} + kx_{p-1} + f_p(t) - ks_p. \qquad (13)$$

The end of sticking is defined as when F_p changes sign.

As a result, Equations (12) and (13) set equal to zero, can be used to define the exit boundary of the sticking region. So, the region of sticking trajectories can be defined as S, which is bounded on one side by the exit boundary ∂S defined by $F_p = 0$.

3. A Two degree-of-freedom System Example

A two degree-of-freedom impact oscillator with multiple constraints is shown schematically in Figure 1 (b). The following parameter values have been selected: masses $m_1 = m_2 = 1$, stiffness $k_1 = k_2 = 1$, viscous damping $c_1 = c_2 = 0.1$, coefficient of restitution $r = 0.7$. From Equation (4), the equations of motion for two coupled masses can be expressed as

$$\ddot{x}_1 + \frac{c}{m}(2\dot{x}_1 - \dot{x}_2) + \frac{k}{m}(2x_1 - x_2) = \frac{A_1}{m}\cos(\Omega t), \qquad (14)$$

$$\ddot{x}_2 + \frac{c}{m}(\dot{x}_2 - \dot{x}_1) + \frac{k}{m}(x_2 - x_1) = \frac{A_2}{m}\cos(\Omega t). \qquad (15)$$

where x_1 represents the displacement of mass m_1 and x_2 the displacement of mass m_2. When $(x_i - s_i) = 0$ for $i = 1, 2$ an impact occurs and an instantaneous coefficient of restitution rule is applied via Equation (9). For this system the three $[R]$ matrices are

$$[R_1] = \begin{bmatrix} -r & 0 \\ 0 & 1 \end{bmatrix}, \quad [R_2] = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix}, \quad [R_3] = \begin{bmatrix} -r & 0 \\ 0 & -r \end{bmatrix}. \qquad (16)$$

The eigenvalues of the 2×2 coupling matrix $[E]$ are $\lambda_1 = 0.382$ and $\lambda_2 = 2.618$, and the corresponding normalised eigenvectors, $\xi_1 = [0.526, 0.851]^T$ and $\xi_2 = [-0.851, 0.526]^T$, which give the mode shapes for the non-impacting system. Using the modal transform described in Section 2, we can express the modal equations of motion for this example as

$$\ddot{q}_1 + 2\zeta_1\omega_{n1}\dot{q}_1 + \omega_{n1}q_1 = \frac{\hat{f}_1}{m}\cos(\Omega t), \qquad (17)$$

$$\ddot{q}_2 + 2\zeta_2\omega_{n2}\dot{q}_2 + \omega_{n2}q_2 = \frac{\hat{f}_2}{m}\cos(\Omega t). \qquad (18)$$

For this example there are two modal impact vectors, $\psi_1 = [\Psi_{11}, \Psi_{12}]$ and $\psi_2 = [\Psi_{21}, \Psi_{22}]$, such that $\psi_1\mathbf{q} = s_1$ and $\psi_2\mathbf{q} = s_2$, where $\mathbf{q} =$

$[q_1, q_2]^T$. For the numerical simulations in this paper we set the forcing amplitudes as $A_2 = 0$ and $A_1 = 0.5$ and take initial conditions $q_1(t_0) = q_2(t_0) = \dot{q}_1(t_0) = \dot{q}_1(t_0) = t_0 = 0$.

3.1. Solutions for sticking

In the case when $x_1 = s_1$ and $\dot{x}_1 = 0$ the reduced equation of motion with $A_2 = 0$, is

$$\ddot{x}_2 + \frac{c}{m}\dot{x}_2 + \frac{k}{m}(x_2 - s_1) = 0, \tag{19}$$

and the force which holds the mass against the stop during sticking is given by

$$F_2 = c\dot{x}_2 + k(x_2 - 2s_1) + A_1 \cos(\Omega t). \tag{20}$$

Equation (19) has the exact solution

$$x_2 = e^{-\hat{\zeta}\hat{\omega}_n(t-t_s)}(C_1 \cos(\hat{\omega}_d(t - t_s)) + C_2 \sin(\hat{\omega}_d(t - t_s))) + s_1, \tag{21}$$

where $\hat{\omega}_n = \sqrt{k/m}$, $\hat{\zeta} = c/2m\hat{\omega}_n$ and $\hat{\omega}_d = \hat{\omega}_n\sqrt{1 - \hat{\zeta}^2}$. At the start of the sticking period $t_s = t$ and the constants C_1 and C_2 can be found using the initial conditions $x_1(t_s) = s_1$ and $\dot{x}_1(t_s) = 0$.

In the case when $x_2 = s_2$ and $\dot{x}_2 = 0$, the reduced equation of motion is given by

$$\ddot{x}_1 + 2\frac{c}{m}\dot{x}_1 + \frac{k}{m}(2x_1 - s_2) = \frac{A_1}{m}\cos(\Omega t). \tag{22}$$

The force which holds the mass against the stop during sticking is given by

$$F_1 = c\dot{x}_1 + k(x_1 - s_2). \tag{23}$$

Equation (22) has the exact solution

$$\begin{aligned} x_1 = e^{-2\hat{\zeta}\hat{\omega}_n(t-t_s)}(C_1 \cos(2\omega_d^*(t - t_s)) + C_2 \sin(2\omega_d^*(t - t_s))) \\ +C_3 \cos(\Omega t - \phi^*) - s_2/2, \end{aligned} \tag{24}$$

where $\hat{\omega}_n = \sqrt{k/m}$, $\hat{\zeta} = c/2m\hat{\omega}_n$ and $\omega_d^* = \hat{\omega}_n\sqrt{0.5 - \hat{\zeta}^2}$ and t_0 is taken at the start of the sticking period. Full details of the derivation of these sticking solutions can be found in [1].

3.2. Periodic sticking motion

A numerically computed example of sticking motion is shown in Figure 2 (a) with stop distance values $s_1 = -0.3$ and $s_2 = 0.1$, forcing amplitude $A_1 = 0.5$, and forcing frequency $\Omega = 0.2$. The figure shows the displacement of both masses for this set of parameter values. The motion is period infinity steady state motion and each mass has a complete chatter sequence and sticking period during one excitation period. Complete chatter motions with sticking are referred to as period infinity periodic motions because an infinite number of instantaneous impacts occur in one period [16]. In Figure 2 (b) we show a close up of the chatter sequence computed for mass 2. The chatter peaks diminish approximately exponentionally and can be studied via mappings [16, 17].

There is a special class of orbits which have a complete chatter sequence, but then lift off before a finite sticking time occurs — referred to as border orbits [17]. In vibro-impact systems these border orbits represent the boundary between the sticking and non-sticking regimes as a system parameter is varied. This can be seen in Figure 3 where examples of bifurcation diagrams for the two degree-of-freedom system with unequal constraints of different sign, computed for a forcing amplitude $A_1 = 0.5$ are shown. In Figure 3 (a) the impact velocity, $v_1 = \dot{x}_1$ of mass 1 is shown against forcing frequency, Ω in the range 0.25–0.7, and in (b) the impact velocity, $v_2 = \dot{x}_2$ is shown against the same frequency range. We note that because $s_1 < 0$ the impact velocities for mass 1 are all less than zero, and likewise as $s_2 > 0$ the impact velocities for mass 2 are all greater than zero.

The region of sticking motions all exist at forcing frequencies, $\Omega < 0.5$, which can be seen from the chatter impact velocities successively decreasing toward zero. At low frequency, $\Omega < 0.5$ periodic sticking motions preceded by complete chatter exist. Then as Ω is increased past the sticking region, chatter becomes incomplete – and the system passes through a border orbit. In fact careful observation shows that for mass 1, chatter remains incomplete for $0.3 \lessapprox \Omega \lessapprox 0.35$ and $0.37 \lessapprox \Omega \lessapprox 0.4$, such that additional border orbits exist close to these parameter values.

In Section 2.2 the relationship $F_p = 0$ was used to define the boundary in phase space where sticking ends. For example, for the system shown in Figure 2, the case when $x_2 = s_2$ and $\dot{x}_2 = 0$, the system trajectories during sticking will be restricted to the $x_1, \dot{x}_1$ space. By setting Equation (23) to zero we define the relationship for the end of sticking as $\dot{x}_1 = -(k/c)x_1 + (k/c)s_2 = \dot{x}_1 = -10x_1 + 1$, which defines the exit boundary of the sticking region S which is denoted ∂S. For sticking to exist the condition $F_p s_p > 0$ must apply, which in this case is the region on the positive side of the ∂S. Note also that ∂S includes the point $(0.1, 0)$ which corresponds to the $(x_2, \dot{x}_2)$ values during sticking. However, when mass 2 sticks, setting Equation (20) to zero results in a relationship for the end of sticking given by $\dot{x}_2 = -(k/c)(x_2 - 2s_1) - A_1 \cos(\Omega t) = \dot{x}_2 = -10(x_2 + 0.6) - 0.5 \cos(\Omega t)$.

This exit boundary for the sticking region is now dependent of the forcing frequency Ω.

In the work on sliding orbits by [20], S was defined using Utkin's equivalent control method. However, in this case only a subset of the system states are restricted to S, with the result that we cannot define S simply in terms of the system parameters alone, we must include some of the system states.

3.3. Multiple nonsmooth events

Sliding bifurcations have been studied in detail for relay systems [20] and friction oscillators [24]. The occurrence of the multi-sliding bifurcation in vibro-impact systems was first highlighted by [19] (referred to as a rising bifurcation) and has been studied for two degree-of-freedom systems by [21]. Multi-sliding occurs when a sliding orbit in the region S becomes tangent (in a similar way to grazing) to the exit boundary ∂S as a system parameter is varied. In the system studied here ∂S is defined by the condition $F_p = 0$, and so we can characterize a multi-sliding event using F_p. An example is shown in Figure 4, where the force F_2 is shown just before, Figure 4 (a) and just after a multi-sliding bifurcation, Figure 4 (b). In both Figure 4 (a) and (b), the solid line represents the force on the mass during the sticking phase. In this example a negative force is required to keep mass 1 stuck to the motion constraint. So when the force signal intersects the zero axis, the mass lifts off from the constraint. In the special case when the force becomes tangent to the zero axis, a multi-sliding bifurcation occurs — Figure 4 (a). In Figure 4 (b), the situation just after multi-sliding is shown, where the sticking orbit is now divided into two segments. In between the segments a chatter sequence exists [21].

In addition to multi-sliding, other sliding bifurcations can occur in vibro-impact systems. For example, for the special case of $r = 0$, the system will have a grazing-sliding bifurcation each time a grazing event occurs — exceptions to this are discussed in [25]. This is similar to the grazing-sliding in the friction oscillator example studied by [24]. The non-generic example mentioned in section 2.2 of a mass coming into contact with the constraint with zero velocity and as the force changed sign, is also an example of grazing-sliding.

In general sliding bifurcations occur to sliding orbits from an individual mass in the MDOF system — including all the examples we have shown thus far. However, because there are two masses in the system, there is also the possibility of nonsmooth events occurring simultaneously to both masses. The three nonsmooth events we are considering in these examples are impact, grazing and sticking. It is therefore possible to list the possible combinations of multiple nonsmooth events in a two mass system.

Table I.: Possible multiple nonsmooth events.

Event	
Dual Grazing	DG
Dual Sticking	DS
Double Impact	DI
Grazing-Sticking	GS
Grazing-Impact	GI
Sticking-Impact	SI

An example is shown in Figure 5 (a) where mass 1 has a complete chatter and sticking sequence, and mass 2 has two impacts during the sticking phase of mass 1 (SI), where both motion constraints are set at $s_1 = s_2 = 0.3$. This is distinct from a double impact (DI) which is the case where $x_1 = s_1$ and $x_2 = s_2$ simultaneously without chatter or sticking — a rare event. However, because our model of the impact assumes that an infinite number of impacts occur during sticking, the points in Figure 5 (a) could possibly be viewed as double impacts (DI) — i.e. both mass 1 and 2 are in contact with the motion constraints simultaneously. Despite this, it is preferable if a distinction between these two cases is made.

A second example is shown in Figure 5 (b), where the motion constraint for mass 2 has been moved to $s_2 = 0.45$, and the coefficient of restitution is close to zero ($r = 0.001$). In this case, mass 1 has a very short chatter sequence, and a long sticking phase. At $t \approx 920$, mass 2 grazes with $s_2 = 0.45$ while mass 1 is stuck (GS) — not the same as the grazing-sliding bifurcation — subsequent dynamics involve dual sticking of the masses (DS).

Finally in Figure 5 (c), an example is shown which is close to dual sticking (DS), at very low forcing frequency of $\Omega = 0.02$, with $r = 0.7$. In this case mass 1 is stuck to s_1, while mass 2 has a very long chatter sequence, and also appears to be stuck (at this scale), but in fact is still chattering at the end of the sticking phase of mass 1 — a reduction of Ω and/or r leads to the dual sticking case (DS).

4. Conclusions

In this paper we have considered the dynamics of multi-degree-of-freedom impact oscillators with multiple constraints using a two degree-of-freedom example to illustrate the dynamical complexities of these systems. We have considered the mathematical modelling of these multiple constraint systems using a modal formulation, including a modal form of the coefficient of restitution rule to model single, multiple and simultaneous impact events. The focus of this paper has been the chatter and sticking region for a two degree-of-freedom example. In particular we have discussed two types of sliding bifurcation — multi-sliding and grazing-sliding — which can occur in these systems.

The fact that the system considered has two masses means that nonsmooth events such as impact, grazing and sticking can occur simultaneously for certain ranges of parameter values. The possible combination of these events has been discussed, and numerical simulations of three examples; impact and sticking, sticking and grazing and the dual sticking case have been presented.

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Figure Captions

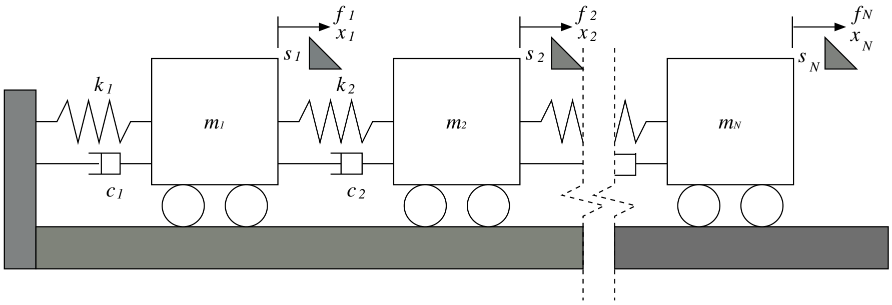
- Figure 1. Schematic representation of an impact oscillator with multiple motion limiting constraints (a) N degree-of-freedom (b) a 2 degree-of-freedom.

- Figure 2. Numerically computed displacement time series of a two degree-of-freedom impact oscillator with constraints $s_1 = -0.3$ and $s_2 = 0.1$; Solid line mass 1; dashed line mass 2. (a) showing chatter and sticking motion. (b) close up of the chatter region for mass 2.

- Figure 3. Numerically computed two degree-of-freedom impact oscillator bifurcation diagram for case (b) with impact stops $s_1 = -0.3, s_2 = 0.1$. Parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 0.1$, $r = 0.7$, forcing $A_2 = 0.0$, $A_1 = 0.5$. (a) Impact velocity v_1 *vs* forcing frequency Ω. (b) Impact velocity v_2 *vs* forcing frequency Ω.

- Figure 4 Multi-sliding bifurcation. Force F_2 during a sticking phase of mass one. (a) Just before multi-sliding $\Omega = 0.256$, (b) just after multi-sliding $\Omega = 0.2561$.

- Figure 5. Numerically computed displacement-time series of a two degree-of-freedom impact oscillator showing with constraints $s_1 = 0.3$ solid line mass 1 dashed line mass 2 (a) additional impacts during sticking $\Omega = 0.2561$, $s_2 = 0.3$ (b) Grazing on mass 2, while sticking on mass 1 for $r = 0.001$, $\Omega = 0.2995$ $s_2 = 0.45$ (c) Just before dual sticking $r = 0.7$, $\Omega = 0.02$, $s_2 = 0.3$.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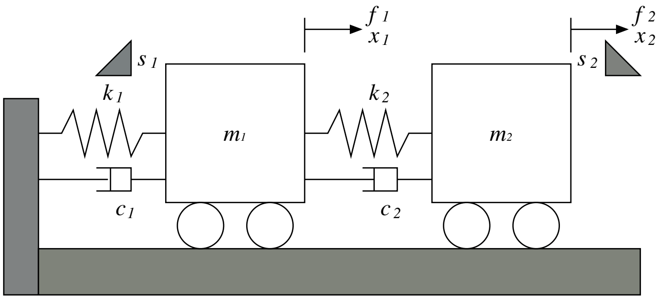
x 1
s 1
k 1
m1
c 1
f 2
x 2
s 2
k 2
m2
c 2
fN
x N
s N
mN

(a)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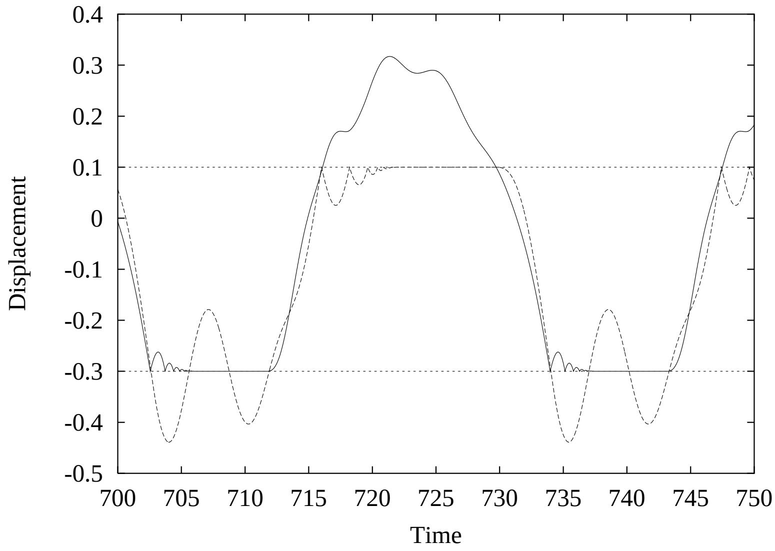
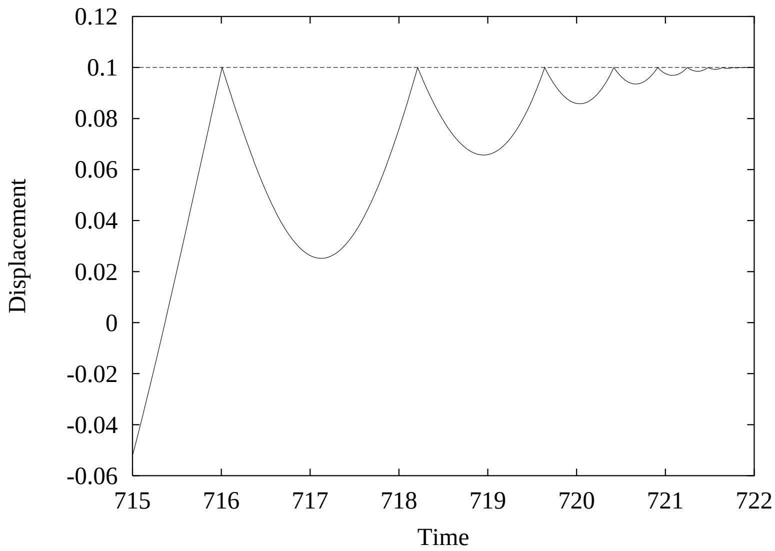
x 1
s 1
k 1
m1
c 1
f 2
x 2
s 2
k 2
m2
c 2

(b)

Figure 1.

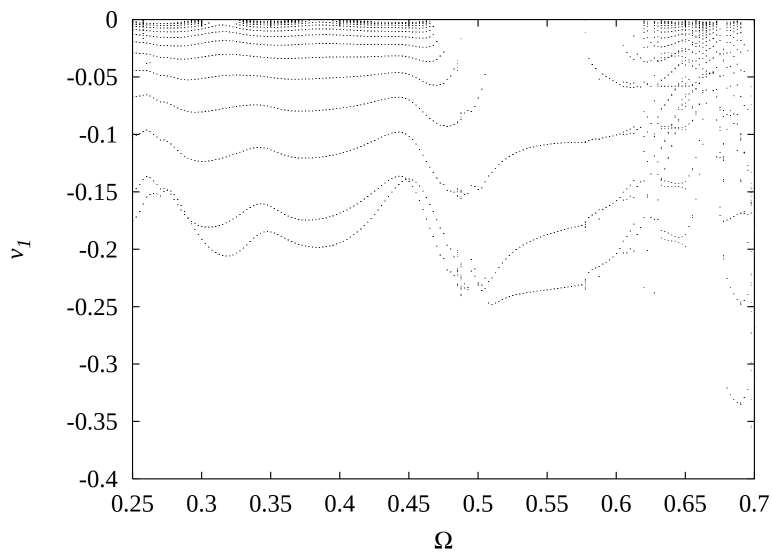

0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
Displacement
700
705
710
715
720
725
730
735
740
745
750
Time

(a)


0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
Displacement
715
716
717
718
719
720
721
722
Time

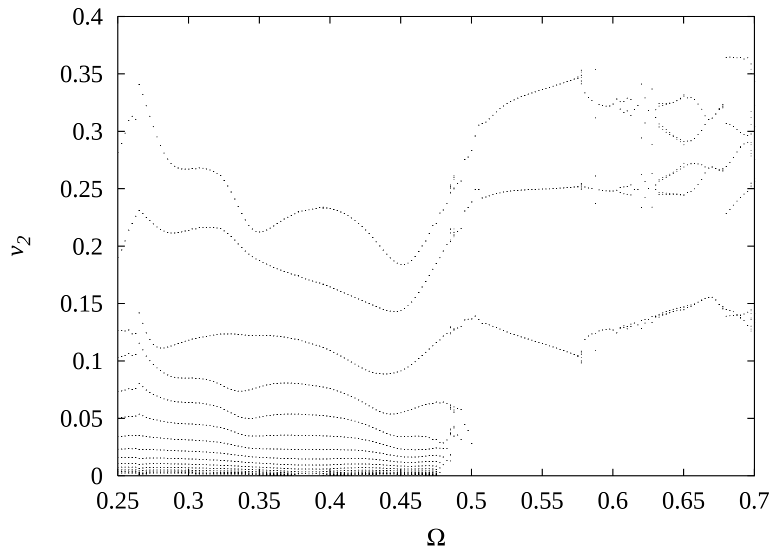
(b)

Figure 2.



0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.3
-0.35
-0.4
0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6 0.65 0.7
v_1
Ω

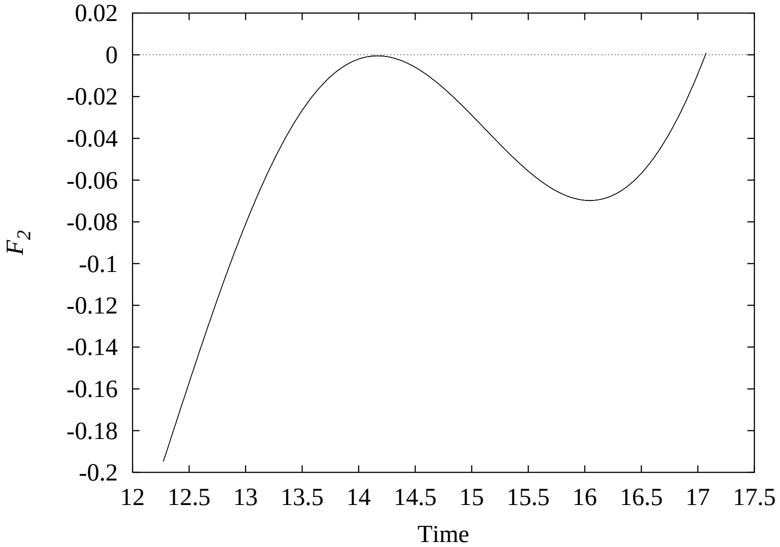
(a)



0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0.25 0.3 0.35 0.4 0.45 0.5 0.55 0.6 0.65 0.7
v_2
Ω

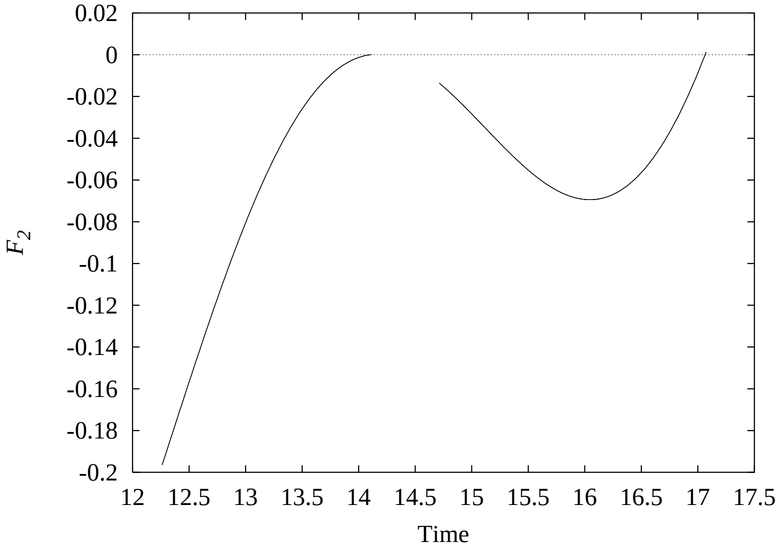
(b)

Figure 3.



0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14
-0.16
-0.18
-0.2
F_2
12 12.5 13 13.5 14 14.5 15 15.5 16 16.5 17 17.5
Time

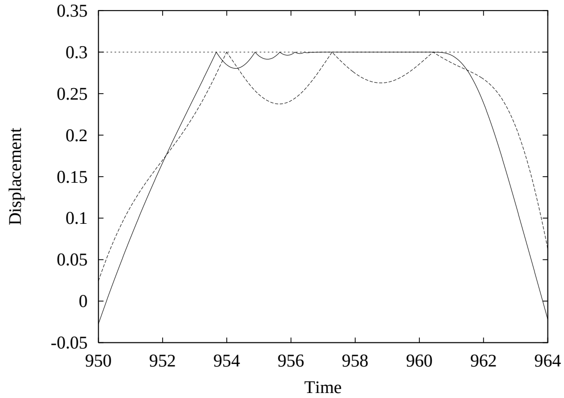
(a)



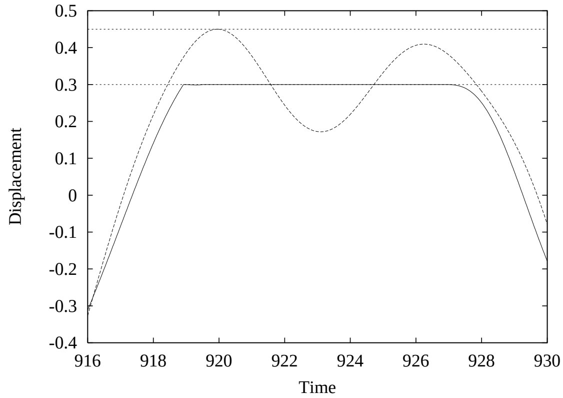
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
-0.12
-0.14
-0.16
-0.18
-0.2
F_2
12 12.5 13 13.5 14 14.5 15 15.5 16 16.5 17 17.5
Time

(b)

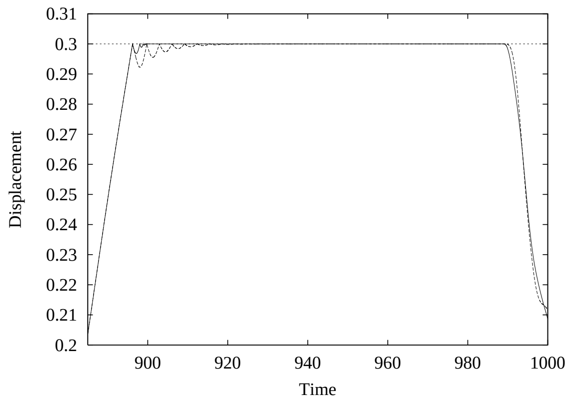
Figure 4.


Displacement
Time
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
950
952
954
956
958
960
962
964

(a)


Displacement
Time
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
916
918
920
922
924
926
928
930

(b)


Displacement
Time
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0.3
0.29
0.28
0.27
0.26
0.25
0.24
0.23
0.22
0.21
0.2
900
920
940
960
980
1000

(c)

Figure 5.