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Article:

A NOTE ON MODELLING MULTI-DEGREE OF FREEDOM VIBRO-IMPACT SYSTEMS USING COEFFICIENT OF RESTITUTION MODELS

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February 16, 2000

Letter to the Editor

Total number of pages 17. Number of Figures 5.

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1   INTRODUCTION

In this letter we consider the problem of modelling multi-degree of freedom impact oscillator systems using coefficient of restitution models. Impact oscillator systems have been studied extensively in recent years following the work of Shaw & Holmes [1]. Most of this work has been focused on models for single degree of freedom systems [2], although inevitably experimental systems studied were often multi-modal (see for example [3, 4, 5, 6]). In the majority of both mathematical and experimental impact oscillator studies the impact process was modelled using a coefficient of restitution rule. This rule was both effective and simple in capturing the key properties of an impacting body. However, in experimental studies using an impacting beam system, Thompson *et al* [7] noted that a low value of the coefficient of restitution must be used in numerical simulations of the system in order to match dynamical behaviour. Similar results were found by Weger *et al* [8]. Thompson *et al* [7] conjectured that this reduction in value for the coefficient of restitution was due to the transfer of energy into higher modes of vibration.

We consider here the dynamics of a multi-degree of freedom linear system, with proportional damping and a single point of impact. An energy balance approach for a periodic impacting motion is described. Then we derive a relationship between modal energy and the coefficient of restitution which exists for periodic impact orbits of multi-degree of freedom systems. For certain experimental systems this analysis can be used to estimate values for the coefficient of restitution [9]. Here we demonstrate the effects of increased modal behaviour by modelling a multi-modal beam system using models with one and four modes.

2   MATHEMATICAL MODEL

For a multi-modal system with N degrees of freedom such as the lumped mass model shown in figure 1, the equations of motion can be expressed in matrix form as

$$[M]\ddot{\mathbf{x}} + [C]\dot{\mathbf{x}} + [K]\mathbf{x} = \mathbf{f}(\tau) \quad x_N < x_s, \tag{1}$$

where $[M], [C], [K]$ are the mass, damping and stiffness matrices respectively, $\mathbf{x} = \{x_1, x_2 \ldots, x_N\}^T$ the displacement vector, x_s is the distance to the impact stop and $\mathbf{f}(\tau) = \{f_1, f_2 \ldots, f_N\}^T$ the external forcing vector. For a lumped mass system, the coupling between masses occurs via the matrices $[C]$ and $[K]$, which are nondiagonal. The mass matrix $[M]$ is a diagonal matrix. For this system, we assume that only mass N can impact. When an impact occurs, we apply a coefficient of restitution rule, which in matrix form can be written as

$$\dot{\mathbf{x}}(\tau_+) = [R]\dot{\mathbf{x}}(\tau_-) \quad x_N = x_s \tag{2}$$

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where $\tau-$ is the time just before impact, $\tau+$ is the time just after impact, $r \in [0, 1]$ is the coefficient of restitution and $[R] = \text{diag}\{1, 1, 1, ..., 1, 1, -r\}$. Thus the impacting mass, mass N, has it's velocity reversed and reduced by r at impact. This process is assumed to be *instantaneous*.

As we are primarily interested in systems with uniformly distributed parameters, we consider the case where $m_j = m$, $c_j = c$, $k_j = k$ for $j = 1, 2, \ldots, N$, such that c is proportional to k. Then we can write $[C] = c[E]$ and $[K] = k[E]$, where $[E]$ is the coupling matrix [10]. Then, by considering the undamped, unforced, (non-impacting) system the natural frequencies of the system are given by $\omega_{nj} = \sqrt{\lambda_j k/m}$ for $j = 1, 2, \ldots, N$ [11], where λ_j $j = 1, 2, \ldots, N$ are the eigenvalues of $[E]$. The eigenvectors ξ_j corresponding to each λ_j normalised such that $\parallel \xi_j \parallel = 1$ define the corresponding mode shapes of the system. Using these eigenvectors we can construct a modal matrix $[\Psi] = [\{\xi_1\}, \{\xi_2\}, \ldots, \{\xi_N\}]$. We can then define modal coordinates, using the linear transform $\mathbf{x} = [\Psi]\mathbf{q}$ where $\mathbf{q} = \{q_1, q_2, \ldots q_N\}^T$. Substituting this into equation 1 and premultiplying by $[\Psi]^T$ decouples the system ($[\Psi]$ is orthogonal such that $[\Psi]^T = [\Psi]^{-1}$) to give

$$[I]\ddot{\mathbf{q}} + \frac{c}{m}[\Lambda]\dot{\mathbf{q}} + \frac{k}{m}[\Lambda]\mathbf{q} = \frac{1}{m}[\Psi]^T\mathbf{f}(\tau), \tag{3}$$

where $[\Lambda]$ is the diagonal eigenvalue matrix. We will consider only harmonic forcing of the form $\mathbf{f}(\tau) = \mathbf{A}\cos(\Omega\tau)$, $\mathbf{A} = \{A_1, A_2, \ldots, A_N\}^T$. Thus we can simplify equation 3 such that for each mode,

$$\ddot{q}_j + 2\zeta_j\omega_{nj}\dot{q}_j + \omega_{nj}^2 q = \frac{F_j}{m}\cos(\Omega\tau), \quad j = 1, 2, \ldots, N \tag{4}$$

where $\mathbf{F} = [\Psi]^T\mathbf{A}$, $\mathbf{F} = \{F_1, F_2, \ldots, F_N\}^T$ and $\zeta_j = (c/2)\sqrt{\lambda_j/km}$ is the modal damping coefficient.

However, the motion of the system is constrained such that $x_N < x_s$ during excitation. If we define the vector $\psi = \{\Psi_{N1}, \Psi_{N2}, \ldots, \Psi_{NN}\}^T$, then in terms of modal coordinates an impact occurs when $\psi^T\mathbf{q} = x_s$. Hence equation 3 is valid only for $\psi^T\mathbf{q} < x_s$ which is equivalent to the condition that $x_N < x_s$.

We are considering a linear coupled system in physical coordinates, $\mathbf{x}$, with a plane of discontinuity representing an impact law; the coefficient of restitution. This plane of discontinuity crosses only a single coordinate axis x_N, in phase space. Transforming this system into modal coordinates, using the linear transform defined by the modal matrix $[\Psi]$ decouples the equations of motion. However, transforming the discontinuity into this modal space results in all modal coordinates becoming discontinuous. So now the system is coupled via the impact event.

3   MODAL ENERGY ANALYSIS

For a multi-degree of freedom system, energy loss occurs at impact via the coefficient of restitution rule, and during free flight due to viscous damping. Now we consider a modal energy analysis

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of a steady state vibro-impact motion of a multi-degree of freedom impact oscillator. From this analysis we derive a relation between energy lost during impact and free flight (modal energy).

In modal space the coefficient of restitution rule, equation 2, becomes

$$[\Psi]\dot{\mathbf{q}}(\tau_+) = [R][\Psi]\dot{\mathbf{q}}(\tau_-), \qquad \psi^T\mathbf{q} = x_s. \tag{5}$$

This leads to the relation for the modal velocities after impact

$$\dot{\mathbf{q}}(\tau_+) = [\hat{R}]\dot{\mathbf{q}}(\tau_-), \qquad \psi^T\mathbf{q} = x_s, \tag{6}$$

where $[\hat{R}] = [\Psi]^{-1}[R][\Psi]$ is the matrix which represents a linear transform of modal velocities just before impact to modal velocities just after impact. This transformation also represents the discontinuous jump in velocities at impact.

Premultiplying the (modal) equation of motion for an N degree of freedom system, equation 3, by $m\dot{\mathbf{q}}^T$ and integrating with respect to τ gives an expression for the energy at time τ^*

$$\frac{m}{2}(\dot{\mathbf{q}}^T\dot{\mathbf{q}}(\tau^*) - \dot{\mathbf{q}}^T\dot{\mathbf{q}}(\tau_i)) + \frac{k}{2}(\mathbf{q}^T[\Lambda]\mathbf{q}(\tau^*) - \mathbf{q}^T[\Lambda]\mathbf{q}(\tau_i))$$

$$= \int_{\tau_i}^{\tau^*} \dot{\mathbf{q}}^T\mathbf{F}\cos(\Omega\tau)\mathrm{d}\tau - c\int_{\tau_i}^{\tau^*} \dot{\mathbf{q}}^T[\Lambda]\dot{\mathbf{q}}\mathrm{d}\tau, \tag{7}$$

where $\tau_i < \tau^* < \tau_{i+1}$ is a time between two consecutive impacts, τ_i and τ_{i+1}. The terms on the right hand side of this expression represent the modal forcing energy FE and modal damping energy DE respectively. For each mode, we refer to $FE_j - DE_j$ at the end of a period as the *residual* modal energy, where

$$FE_j = \int_{\tau_i}^{\tau^*} \dot{q}_j F_j \cos(\Omega\tau)\mathrm{d}\tau,$$

$$DE_j = c\int_{\tau_i}^{\tau^*} \dot{q}_j\lambda_j\dot{q}_j\mathrm{d}\tau. \tag{8}$$

This is a measure of modal energy gain during free flight, over one period of motion for each mode, and the sum over all modes, with $FE = \sum_{j=1}^{N} FE_j$ and $DE = \sum_{j=1}^{N} DE_j$, represents the energy gain for the whole system.

We evaluate this expression for a period one, one impact motion, denoted $\mathbf{P}(1,1)$. A numerical simulation of such a $\mathbf{P}(1,1)$ orbit is shown in figure 2 for a two degree of freedom system. The phase portrait is plotted in both modal (2 (a)) and physical (2 (b)) coordinates.

We examine the whole period of motion between two impacts τ_i and τ_{i+1} for such an orbit by setting $\tau^* = \tau_{i+1}$. As we are considering the system from one impact to the next, the modal displacement will be the same at τ_i and τ_{i+1}, and hence the potential energy term (second term in equation 7) is zero. For each impact $\tau_i \equiv \tau_+$ and $\tau_{i+1} \equiv \tau_-$. The kinetic energy term (first

term in equation 7) can be evaluated using the relations $\dot{\mathbf{q}}(\tau_-) = [\Psi]^{-1}\dot{\mathbf{x}}(\tau_-)$, $\dot{\mathbf{q}}^T(\tau_-) = \dot{\mathbf{x}}^T(\tau_-)[\Psi]$, $\dot{\mathbf{q}}(\tau_+) = [\Psi]^{-1}[R]\dot{\mathbf{x}}(\tau_-)$ and $\dot{\mathbf{q}}^T(\tau_+) = \dot{\mathbf{x}}^T(\tau_-)[R][\Psi]$, to give

$$\frac{m}{2}(\dot{\mathbf{x}}^T(\tau_-)\dot{\mathbf{x}}(\tau_-) - \dot{\mathbf{x}}^T(\tau_-)[R][R]\dot{\mathbf{x}}(\tau_-)), \tag{9}$$

which reduces to

$$\frac{m}{2}v_N^2(1 - r^2) \tag{10}$$

where v_N denotes the velocity of mass N at impact (τ_-). Thus we can reduce equation 7 to an energy balance expression for a $\mathbf{P}(1,1)$ orbit

$$\frac{m}{2}v_N^2(1 - r^2) = \sum_{j=1}^{N}(FE_j - DE_j). \tag{11}$$

The energy lost at impact (left hand side of equation 11) is equal to the sum of the residual energy of all the modes in the system. In other words, the energy lost at impact, IE say, is equal to the energy input, FE, less the energy dissipated due to viscous damping, DE. So $IE = FE - DE$ or $FE = IE + DE$ for a $\mathbf{P}(1,1)$ orbit.

By rearranging equation 11 we can obtain an expression for the coefficient of restitution

$$r = \sqrt{1 - \frac{2}{mv_N^2}\{\sum_{j=1}^{N}(FE_j - DE_j)\}}. \tag{12}$$

Then we can reduce equation 12 to

$$r = \sqrt{1 - \frac{RE}{KE_i}} \tag{13}$$

where $KE_i = mv_N^2/2$ is the kinetic energy at impact, and $RE = \sum_{j=1}^{N}(FE_j - DE_j)$ is the residual energy for all modes.

We know by definition that $r \in [0, 1]$, is a real positive quantity. The kinetic energy at impact $KE_i > 0$ is a strictly positive quantity (excluding zero velocity impacts, which do not occur in stable periodic motion—the case considered here). Thus we can see that for real values of r, $0 < RE < KE_i$. These bounds apply to $\mathbf{P}(1,1)$ orbits in systems with an arbitrary N degrees of freedom. Then from equation 13, we can see that by increasing residual energy or decreasing kinetic energy at impact, the coefficient of restitution is reduced. This is shown in figure 3 (a) for $KE_i = 1.0$, and in figure 3 (b) for $KE_i > RE$. Thus from figure 3, we see that for a multi-modal system, it is possible to have a complete range of r values $0 \leq r \leq 1$, dependent on the energy balance of the system.

4   MODELLING PHYSICAL VIBRO-IMPACT PROBLEMS

The problem of estimating a value for the coefficient of restitution arises when modelling a physical system such as an impacting beam [7, 12, 8]. In [7] a value for r was selected using

the material type during impact, steel on steel, as the criteria. Using reference sources such as [13] one can estimate that for steel on steel impact a coefficient of restitution value in the range $0.85 - 0.95$ should be chosen. As an alternative direct experimental measurements can be made of the velocity before and after impact to find the coefficient [14]. However, in multi-modal systems the influence of higher modes after impact may impede this process. In addition it is clear from [13] and equation 12 that the coefficient of restitution is a nonlinear function of impact velocity as well as material property. For a particular periodic impact motion the impact velocity is constant, so we can consider a value for the coefficient of restitution to be constant in this case only.

As an example, we consider the vibro-impact cantilever beam system described in detail by Thompson *et al* [7], see also [12, 15]. A time series taken from a typical $\mathbf{P}(1,1)$ motion is shown in figure 4. When modelling the dynamics of this system Thompson *et al* [7] found that a coefficient of restitution value of 0.2 was required in order to simulate the motion of the beam for a range of frequency values close to the first natural frequency of the beam. This simulation was carried out using a single degree of freedom model. A simulated time series from this model is shown in figure 5 (a). We can see that this simulation captures the essential dynamics of the system, in terms of periodicity and maximum (positive) displacement amplitude.

In order to obtain a multi modal simulation, we have used a model based on a Galerkin reduction of the Euler-Bernoulli equation for the vibro-impacting beam system [15]. This method incorporates an instantaneous impact rule, and assumes that the impact occurs at the free end of the beam. The result of simulating the motion of the free end of the beam using four modal coordinates and a coefficient of restitution value of 0.75 is shown in figure 5 (b). This value of the coefficient of restitution is much closer to the steel on steel material range $0.85 - 0.95$, which suggests that the four degree of freedom model is a closer representation of the physical system, in terms of both dynamical behaviour and energy transfer. The fact that the four degree of freedom system models the dynamics of the beam more closely than the single degree of freedom model is perhaps not surprising. However, the energy balance is distinctly different between the two models, and demonstrates the difficulties encountered when trying to model the behaviour of these complex systems.

As for many systems, the use of a low dimensional model may be sufficient to model the key dynamics, the problem is that an appropriate reduced coefficient value needs to be estimated. This can be achieved by using equation 12 in conjunction with experimentally recorded data [15, 9].

5   CONCLUSIONS

When modelling physical phenomena such as an impacting beam, the main focus of attention is on simulating the dynamics of the physical system. Thompson *et al* [7] show that for an impacting beam (forced close to it's first natural frequency) this can be achieved with a simple (single degree of freedom) numerical model. However, although the simple model captures the qualitative dynamics of the system, it fails to simulate the energy loss characteristics. These must be accounted for in the model by reducing the coefficient of restitution value to an appropriate level to account for the the energy dissipated due to the excitation of higher modes at impact.

Here we have shown the relationship which exists between the coefficient of restitution and modal energy during periodic impacting motion of an arbitrary N-degree of freedom system. Using simulations of experimental data from impacting beam experiments, we have demonstrated the effect of using multi-modal models on the value of the coefficient of restitution.

References

[1] S. W. Shaw and P. J. Holmes. 1983 *Journal of Sound and Vibration* **90**(1) 129–155. A periodically forced piecewise linear oscillator.

[2] A. B. Nordmark. 1991 *Journal Of Sound and Vibration* **145**(2) 275–297. Non-periodic motion caused by grazing incidence in an impact oscillator.

[3] F. C. Moon and S. W. Shaw. 1983 *International Journal of Non-Linear Mechanics* **18**(6) 465–477. Chaotic vibrations of a beam with non-linear boundary conditions.

[4] S. W. Shaw. 1985 *Journal of Sound and Vibration* **99**(2) 199–212. Forced vibrations of a beam with one-sided amplitude constraint: Theory and experiment.

[5] J. P. Cusumano, M. T. Sharkady, and B. W. Kimble. 1994 *Philosophical Transactions of the Royal Society of London A* **347** 421–438. Experimental measurements of dimensionality and spatial coherence in the dynamics of a flexible-beam impact oscillator.

[6] R. H. B. Fey, E. L. B. van de Vorst, D. H. van Campden, A. de Kraker, G. J. Meijer, and F. H. Assinck. Chaos and bifurcations in a multi-dof beam system with nonlinear support. In *Nonlinearity and chaos in engineering dynamics*, J. M. T. Thompson and S. R. Bishop, Eds. Chichester: John Wiley 1994 ch. 9, pp. 125–139.

[7] M. G. Thompson, S. R. Bishop, and S. Foale. 1994 Machine Vibration **3** 10–17. An experimental study of low velocity impacts.

[8] J. Weger, D. Binks, J Molenaar, and W. van de Water. 1996 *Physical Review Letters* **76**(21) 3951–3954. Generic behaviour of grazing impact oscillators.

[9] D. J. Wagg and S. R. Bishop. 1998 A multi-degree of freedom approach to coefficient of restitution models for impact oscillators. In *IUTAM Symposium on: Unilateral Multibody Dynamics* pp. 145–154.

[10] G. M. L. Gladwell. 1986 *Inverse problems in vibration.* Dordrecht: Kluwer.

[11] S. P. Timoshenko, D.H. Young, and W. Weaver Jr. 1974 *Vibration problems in engineering.* Van Nostrand USA.

[12] S. R. Bishop, M. G. Thompson, and S. Foale. 1996 *Proceedings of the Royal Society of London A* **452** 2579–2592. Prediction of period-1 impacts in a driven beam.

[13] W. Goldsmith. 1960 *Impact.* London: Edward Arnold.

[14] N. HINRICHS, M. OESTREICH, AND K. POPP. 1997 *Chaos, Solitons and Fractals* **8**(4) 535–558. Dynamics of oscillators with impact and friction.

[15] D. J. WAGG. *Vibro-impact dynamics of engineering systems.* PhD thesis University of London 1998.

FIGURE CAPTIONS

- Figure 1: Schematic representation of an N degree of freedom impact oscillator

- Figure 2: Two degree of freedom impact oscillator numerically generated phase portraits. Parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 0.1$, $x_s = 0.1$, $A_2 = 0.0$, $A_1 = 0.5$, $\Omega = 0.9$ and $r = 0.7$. (a) Modal coordinates: Solid line q_1, broken line q_2. (b) Physical coordinates: Solid line x_1, broken line x_2.

- Figure 3: Coefficient of restitution as a function of energy for multi-degree of freedom impact oscillators. (a) r vs RE with $KE_i = 1.0$. (b) r as a function of RE and KE_i, $KE_i > RE$.

- Figure 4: Experimental $\mathbf{P}(1,1)$ beam data. Parameter values $A = 2.5$ volts, $x_s = -0.2$volts and $\Omega = 138.42$Hz. Sampling rate: 4000 samples/sec.

- Figure 5: Numerical simulations of the vibro-impact beam data shown in figure 4 (a) Single degree of freedom model $r = 0.2$. (b) Four degree of freedom model $r = 0.75$.
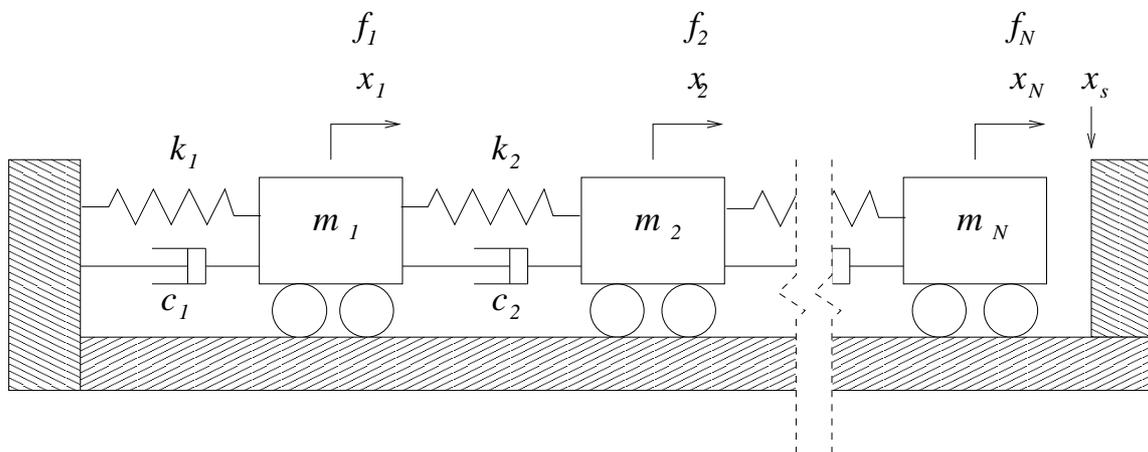
Figure 1:

(a)

(b)

Figure 2:

(a)

(b)
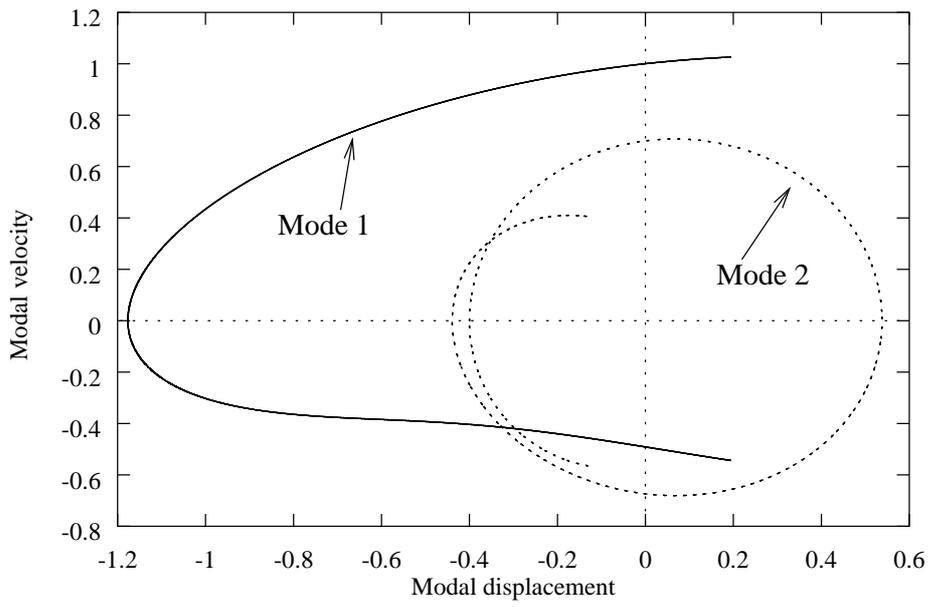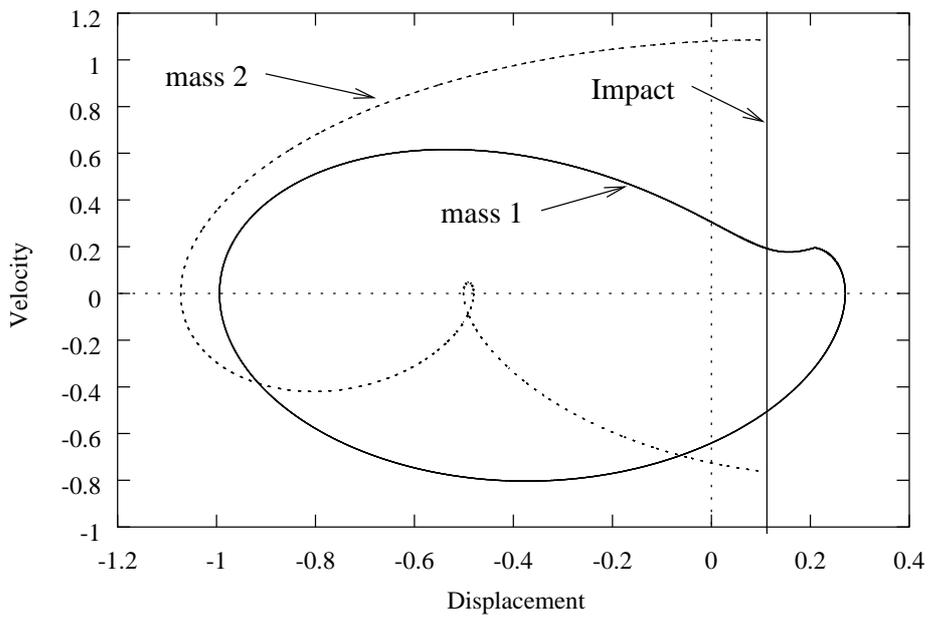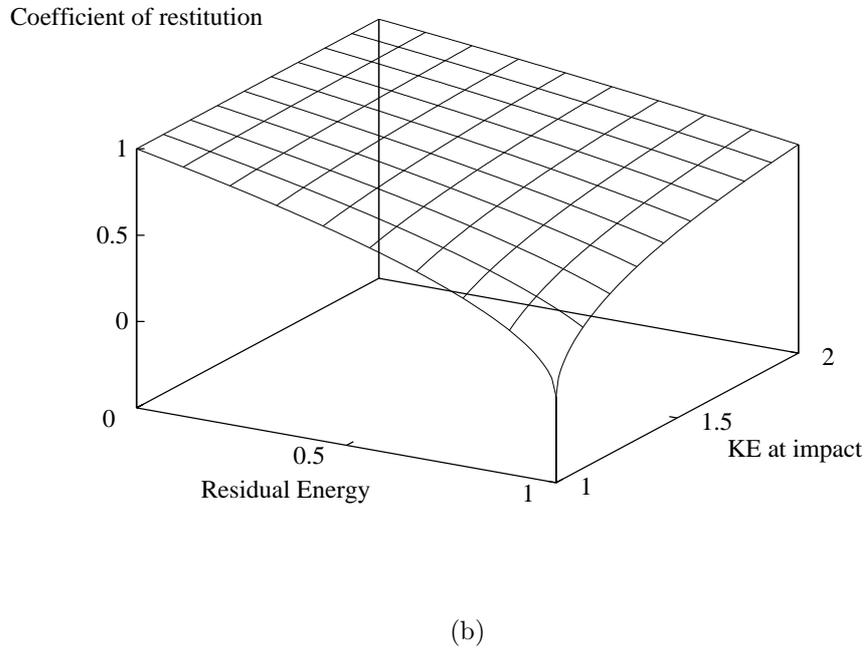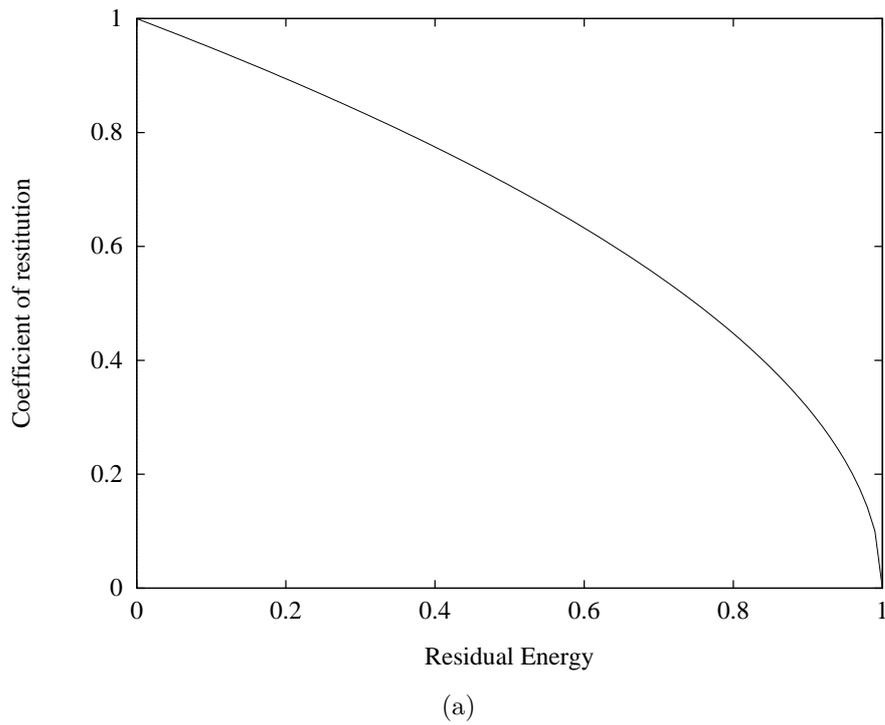
Figure 3:

Figure 4:

(a)

(b)

Figure 5:

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