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Abstract: Bounds on the gain for high order nonlinear transfer functions of wide class of nonlinear systems are derived. It is shown that the gain bounds can be expressed explicitly in terms of the coefficients of discrete time nonlinear models. The bounds can be used to check the convergence of the transfer function series and hence to determine the truncation order in practical applications.

1. Introduction

The conventional transfer function approach in linear systems analysis is well known and has been widely applied. When the system is nonlinear similar transfer functions can be defined based on multidimensional Laplace, or Fourier, transforms of the kernels in the Volterra series representation.[4,13,16,17] The nonlinear transfer function approach has received considerable attention from researchers in electrical and electronic engineering since the early 1960's, and more recently has been applied in structural engineering, physiology and control systems. It has been shown that a large class of nonlinear systems can be represented by the nonlinear transfer function approach and numerous applications have been reported in various disciplines.[8,11,14,15]

A large class of nonlinear systems can be represented by a *series* of transfer functions, $H_1, H_2, H_3, \cdots$ where $H_1 \neq 0$, $H_i = 0, i = 2,3,...$ represents the linear system case. The series may be infinite and the stronger the nonlinearities are, the more transfer functions are required. In practical applications the transfer function approach is most useful when the transfer function series converges rapidly. In this case the nonlinear system can be adequately described by just the first few terms. It is clearly desirable to have some knowledge on the bound of the gain of each transfer function, in a possibly infinite series, so that an informed decision regarding the truncation of the series and the approximating capabilities can be made. Chua and Liao (1991)[7] studied this problem and reported an experimental algorithm to measure the 'highest significant order' of the frequency response functions by applying a series of specially designed probing signals into the system. The method is straightforward providing special inputs can be applied but is less simple to implement for general nonlinear systems. In the present paper, a gain bound for each order of transfer function is derived in terms of the parameters of a discrete time model. This provides a unique insight into the relationship between the parameters in a time domain representation and the convergence of the nonlinear transfer functions in the frequency domain. Several examples are included to illustrate the properties and applications of the new bound.

2. Representations of Nonlinear Systems in the Time and Frequency Domain

By collecting input/output data from a nonlinear system and applying parameter estimation techniques a parametric model of the system can be identified. A general polynomial NARMAX (Nonlinear ARMAX) model [5,9,10] can be expressed in the form

$$y(t) = \sum_{m=1}^{M} y_m(t) \qquad (1)$$

where $y_m(t)$ is given by,

$$y_m(t) = \sum_{p=0}^{m} \sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1,\cdots,k_{p+q}) \prod_{i=1}^{p} y(t-k_i) \prod_{i=p+1}^{p+q} u(t-k_i) \qquad (2)$$

$c_{p,q}(\cdot)$ are the model parameters, M is the maximum degree of nonlinearity, and K is the maximum order or lag of the difference operation. Also notice that

$$p+q=m; \qquad k_i = 1,\cdots,K; \qquad \text{and} \quad \sum_{k_a,k_b=1}^{K} \equiv \sum_{k_a=1}^{K} \cdots \sum_{k_b=1}^{K} \qquad (3)$$

Equations (1) and (2) describe all the possible polynomial NARMAX models. After data sampling, structure detection, parameter estimation and model validation, all the unknown constants contained in eqn (2) will have been estimated.[1,2]

To illustrate the definitions in eqn.(1) and (2) a specific NARMAX model would be given for instance as

$$y(t) = 0.44u(t-1) + 0.5y(t-1) - 0.03u(t-1)u(t-1) - \qquad (4)$$

$$0.04u(t-2)u(t-1) - 0.06y(t-1)u(t-3) - 0.07y(t-2)y(t-3)$$

which may be obtained from the general form (1) and (2) with

$c_{0,1}(1) = 0.44$; $c_{1,0}(1) = 0.5$; $c_{0,2}(1,1) = -0.03$;

$c_{0,2}(2,1) = -0.04$; $c_{1,1}(1,3) = -0.06$; $c_{2,0}(2,3) = -0.07$; *else* $c_{p,q}(\cdot) = 0$;

For this specific example, the maximum degree of nonlinearity $M = 2$ and the maximum lag $K = 3$.

The nonlinear transfer function, which is also called the Generalised Frequency Response Function (GFRF), can then be obtained by mapping this time domain model into the frequency domain.[3,6,12] The nth order transfer function for the general NARMAX model eqn.(1) is given in terms of the model parameters by the following recursive relation:[12]

$$\left[1 - \sum_{k_1=1}^{K} c_{1,0}(k_1)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1}\right] H_n^{asym}(j\omega_1, \cdots, j\omega_n) =$$

$$+ \sum_{k_1,k_n=1}^{K} c_{0,n}(k_1, \cdots, k_n)\, e^{-j(\omega_1 k_1 + \cdots + \omega_n k_n)}$$

$$+ \sum_{p=2}^{n} \sum_{k_1,k_p=1}^{K} c_{p,0}(k_1, \ldots, k_p)\, H_{n,p}(j\omega_1, \cdots, j\omega_n) \qquad (5)$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1,k_{p+q}=1}^{K} c_{p,q}(k_1, \ldots, k_{p+q}) e^{-j(\omega_{n-q+1}k_{p+1} + \ldots + \omega_n k_{p+q})} H_{n-q,p}(j\omega_1, \ldots, j\omega_{n-q})$$

with the recursive relations

$$H_{n,p}^{asym}(\cdot) = \sum_{i=1}^{n-p+1} H_i^{asym}(j\omega_1, \ldots, j\omega_i) H_{n-i,p-1}(j\omega_{i+1}, \ldots, j\omega_n)\, e^{-j(\omega_1 + \ldots + \omega_i)k_p} \qquad (6)$$

Note that the recursion finishes with $p=1$, and that $H_{n,1}(j\omega_1, \cdots, j\omega_n)$ has the property

$$H_{n,1}(j\omega_1, \cdots, j\omega_n) = H_n(j\omega_1, \cdots, j\omega_n)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1} \tag{7}$$

3. Derivation of the Gain Bounds

Applying the triangle inequality and using the unity gain of complex exponentials, that is

$$\left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} |x_i| \quad \text{and} \quad |e^{j\theta}| = 1$$

to eqn (5) yields the following inequality

$$\left|\left[1 - \sum_{k_1=1}^{K} c_{1,0}(k_1)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1}\right] H_n^{asym}(j\omega_1, \cdots, j\omega_n)\right| \le$$

$$\sum_{k_1,k_n=1}^{K} |c_{0,n}(k_1, \cdots, k_n)|$$

$$+ \sum_{p=2}^{n} \sum_{k_1,k_p=1}^{K} |c_{p,0}(k_1, \cdots, k_p)| \cdot |H_{n,p}(j\omega_1, \cdots, j\omega_n)| \tag{8}$$

$$+ \sum_{q=1}^{n-1} \sum_{p=1}^{n-q} \sum_{k_1,k_{p+q}=1}^{K} |c_{p,q}(k_1, \ldots, k_{p+q})| \cdot |H_{n-q,p}(j\omega_1, \ldots, j\omega_{n-q})|$$

Because $H_{n,p}(\cdot) = 0$ for $n < p$ and

$$H_{n,0}(\cdot) = \begin{cases} 0 & \text{for } n \ne 0; \\ 1 & \text{for } n = 0; \end{cases}$$

the above equation can be re-written as

$$\left|\sum_{k_1=0}^{K} c_{1,0}(k_1)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1}\right| \cdot \left|H_n^{asym}(j\omega_1, \cdots, j\omega_n)\right| \le$$

$$\sum_{q=0}^{n} \sum_{p=0}^{n-q} \sum_{k_1,k_{p+q}=1}^{K} |c_{p,q}(k_1, \cdots, k_{p+q})|\, |H_{n-q,p}(j\omega_1, \cdots, j\omega_{n-q})| \tag{9}$$

where $p \ne 0$ when $q = 0$. Consider the recursive relation $H_{n,p}(\cdot)$ on the right hand side of the above inequality. Applying the triangle inequality again to eqn (6) yields

$$\left|H_{n,p}^{asym}(\cdot)\right| \le \sum_{i=1}^{n-p+1} |H_i^{asym}(j\omega_1, \cdots, j\omega_i)|\, |H_{n-i,p-1}(j\omega_{i+1}, \cdots, j\omega_n)| \tag{10}$$

There are some special cases which might be useful in deriving the gain bound, such as

$$H_{n,1}(j\omega_1, \cdots, j\omega_n) = H_n(j\omega_1, \cdots, j\omega_n)\, e^{-j(\omega_1 + \cdots + \omega_n)k_1} \tag{11}$$

so that

$$|H_{n,1}(j\omega_1, \cdots, j\omega_n)| = |H_n(j\omega_1, \cdots, j\omega_n)| \tag{12}$$

and

$$H_{n,n}(j\omega_1, \cdots, j\omega_n) = H_1(j\omega_1)\cdot H_1(j\omega_1) \cdots H_1(j\omega_n)\, e^{-j(\omega_1 k_n + \cdots + \omega_n k_1)} \tag{13}$$

hence

$$|H_{n,n}(j\omega_1, \cdots, j\omega_n)| = \prod_{i=1}^{n} |H_1(j\omega_i)| \tag{14}$$

Clearly the recursive relation (10) will finally end at H_1 which can be expressed as

$$H_1(j\omega) = \frac{\sum_{k_1=1}^{K} c_{0,1}(k_1) e^{-j\omega k_1}}{\sum_{k_1=0}^{K} c_{1,0}(k_1) e^{-j\omega k_1}} \tag{15}$$

At this point it is necessary to define the lower bound of the denominator of H_1 because this will be dominant in all the higher order GFRF's. Denote

$$\mathbf{L} = \underset{\omega \in \mathbf{W}}{Inf} \left| \sum_{k_1=1}^{K} c_{1,0}(k_1) e^{-j\omega k_1} \right| \tag{16}$$

where $\mathbf{W}$ is the set containing all the possible input frequencies. Then from eqn.(15)

$$|H_1(j\omega)| \le \left| \frac{\sum_{k_1=1}^{K} c_{0,1}(k_1) e^{-j\omega k_1}}{\mathbf{L}} \right| \le \frac{1}{\mathbf{L}} \sum_{k_1=1}^{K} \left| c_{0,1}(k_1) \right| \tag{17}$$

Using the lower bound $\mathbf{L}$ the inequality of (9) can be written as

$$|H_n^{asym}(\cdot)| \le \frac{1}{\mathbf{L}} \sum_{q=0}^{n} \sum_{p=0}^{n-q} \sum_{k_1, k_{p+q}=1}^{K} |c_{p,q}(k_1, \cdots, k_{p+q})|\, |H_{n-q,p}(j\omega_1, \cdots, j\omega_{n-q})|$$

$$= \|H_n^{asym}\|_B \quad (p \ne 0 \text{ if } q=0) \tag{18}$$

where $\|\cdot\|_B$ means the gain bound of the given complex expression. With the bounded H_1 the right hand side of the above inequality is a constant independent of the frequency variables and can be computed recursively using inequality (10).

Notice that eqn.(5) only gives an asymmetric form of the nth order GFRF. Because the bound of H_n on the right hand side of (18) is independent of the frequency arguments so is the bound for all the asymmetric versions of the GFRF's. The symmetric GFRF which is defined by

$$H_n^{sym}(j\omega_1, \cdots, j\omega_n) = \frac{1}{n!} \sum_{\substack{all\ permutations \\ of\ \omega_1 \cdots \omega_n}} H_n^{asym}(j\omega_1, \cdots, j\omega_n) \tag{19}$$

will possess the same bound because

$$|H_n^{sym}(j\omega_1, \cdots, j\omega_n)| \leq \frac{1}{n!} \sum_{\substack{all\ permutations \\ of\ \omega_1 \cdots \omega_n}} |H_n^{asym}(j\omega_1, \cdots, j\omega_n)| \tag{20}$$

$$= \|H_n^{asym}\|_B .$$

4. Examples and Discussions

Consider for example a NARX model given by

$$y(t) = c_{0,1}u(t-1) + c_{0,2}u(t-1)u(t-3) + c_{2,1}u(t-2)y(t-2)y(t-3)$$
$$+ c_{1,1}y(t-1)u(t-1) \tag{21}$$

It is very simple to obtain the bounds of the GFRF's for this specific example

$$|H_1(\cdot)| \leq a_{0,1}$$

$$|H_2(\cdot)| \leq a_{0,2} + a_{1,1}a_{0,1}$$

$$|H_3(\cdot)| \leq a_{2,1}(a_{0,1})^2 + a_{1,1}a_{0,1}^3$$

where $a_{p,q} \equiv |c_{p,q}|$ is used to denote the absolute values of the model parameters. The bounds for all the recursively generated higher order GFRF's can be obtained in the same way in terms of the time domain model parameters. For more complex models the computation may be less simple. For example, if one more term is added to the location of the roots of the characteristic polynomial (the poles of the system), as well as the frequency set **W**. Assume that the characteristic polynomial $\sum_{k_1=1}^{K} c_{1,0}(k)\, p_{k_1}$ has K roots $p_1, p_2, \ldots, p_K$, then the denominator can be written as

$$\prod_{k=1}^{K} (e^{-j\omega} - p_k) = \prod_{k=1}^{K} S_k$$

where

$$S_k = e^{-j\omega} - p_k = U_k(\omega) e^{-j\phi_k(\omega)} \qquad k=1,2,\ldots,K$$

So that

$$\left| \sum_{k_1}^{K} c_{1,0}(k) e^{-j\omega k_1} \right| = \prod_{k=1}^{K} U_k(\omega)$$

In the complex plane, $e^{-j\omega}$ can be denoted by a point on the unit circle. Clearly a pole on the unit circle results in $|H_n(j\omega_1, \ldots, j\omega_n)| = \infty$ at $\omega = \angle p_k$. In this case **L**=0.

Once the value of **L** is known the recursive computation can be easily implemented by direct programme code. It is always wise to inspect the bound for each order of GFRF's before computing the entire function.

The bound expression might also be useful for checking if the underlying system is suitable to be studied using the nonlinear transfer function approach. For example if the coefficients of an identified model have values such that the computed bounds of the GFRF's are quickly decreasing as the order increases, then the GFRF can be confidently used, for example to design a controller or analyse the system behaviour. If however the computed bounds are increasing with the order, or converge slowly, then from a practical point of view, GFRF should be used with caution.

5. Conclusions

An expression for the gain bounds of the terms in the nonlinear transfer functions series has been derived. The gain bound is given as a function of the parameters in a discrete time model and provides valuable insight into approximation capability and truncation of the series.

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References

1. BILLINGS, S. A., "An overview of nonlinear systems identification," *7th IFAC symp. ident. and syst. par. est.*, pp. 725-729, 1985.
2. BILLINGS, S.A. AND TSANG, K.M., "Spectral Analysis for Nonlinear Systems. Part I - Parametric Nonlinear Spectral Analysis," *J. Mechanical Systems and Signal Processing*, vol. 3, no. 4, pp. 319-339, 1989a.
3. BILLINGS, S.A. AND PEYTON JONES, J.C, "Mapping Nonlinear Integro-Differential Equations Into the Frequency Domain," *Int J. Control*, vol. 52(4), pp. 863-879, 1990.
4. BUSSGANG, J.J., EHRMAN, L., AND GRAHAM, J.W., "Analysis of Nonlinear Systems with Multiple Inputs," *Proc. IEEE*, vol. 62, pp. 1088-1119, 1974.
5. CHEN, S. AND BILLINGS, S.A., "Representations of Nonlinear Systems: The NARMAX Model," *Int J. Control*, vol. 49, pp. 1013-1032, 1989.
6. CHUA, L.O. AND NG, C.Y., "Frequency Domain Analysis of Nonlinear Systems: General Theory," *IEE Journal Electronic Circuits and Systems*, vol. 3, no. 4, pp. 165-185, 1979.
7. CHUA, L.O. AND LIAO, Y., "Measuring Volterra kernels III: how to estimate the highest significant order," *Int. J. Circuits Theory Appl. (UK)*, vol. 19(2), pp. 189-206, 1991.
8. HONG, J.Y., KIM, Y.C., AND POWERS, E.J., "On modelling the nonlinear relationship between fluctuations with nonlinear transfer functions," *Proc. of IEEE*, vol. 64(8), pp. 1026-1027, 1980.
9. LEONTARITIS, I.J AND BILLINGS, S.A., "Input-output Parametric Models for Non-Linear Systems Part I: Deterministic Non-Linear Systems," *Int J. Control*, vol. 41, no. 2, pp. 303-328, 1985.
10. LEONTARITIS, I.J AND BILLINGS, S.A., "Input-output Parametric Models for Non-Linear Systems Part II: Stochastic Non-Linear Systems," *Int J. Control*, vol. 41, no. 2, pp. 329-344, 1985.
11. MITZEL, G., CLANCY, S., AND RUGH, W., "On Transfer Function Representations for Homogeneous Nonlinear Systems," *IEEE Trans. on Automatic Control*, vol. AC-24, pp. 242-249, 1979.
12. PEYTON JONES, J.C. AND BILLINGS, S.A., "A Recursive Algorithm for Computing the Frequency Response of a Class of Nonlinear Difference Equation Models," *Int J. Control*, vol. 50, no. 5, pp. 1925-1940, 1989.
13. RUGH, W.J., *Nonlinear System Theory - The Volterra/Wiener Approach*, Johns Hopkins Univ. Press, Baltimore and London, 1981.

14. SAIN, P.M., SAIN, M.K., AND MICHEL, A.N., "Higher order methods for nonlinear feedback design," *Proc. of the 33rd Midwest Symp. On Circuits and Systems*, vol. 1, pp. 68-71, 1991.

15. VINH, T., CHOUYCHAI, T., LIU, H., AND DJOUDER, M., *Second Order Transfer Function: Computation and Physical Interpretation*, 5th IMAC, London, 1987.

16. WEINER, D.D. AND SPINA, J.F., *Sinusoidal Analysis and Modelling of Weakly Nonlinear Circuits*, Van Nostrand, New York, 1980.

17. ZHANG, H. AND BILLINGS, S.A., "Analysing the transfer function of nonlinear systems in the frequency domain," *J. Mech. Systems and Signal Processing*, vol. 7, no. 6, pp. 531-550, 1993.