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1.0 Introduction

The nonlinear polynomial model has been studied in detail and the model characteristics, identification and properties are now well known. In contrast the rational model, defined as the ratio of two polynomial models, is much more difficult to estimate and has only recently been considered. But both models are closely related and the present study introduces a unified least squares estimation algorithm for these two representations. This algorithm provides unbiased parameter estimates even when current noise terms, involving $e(t)$, are present in the regressed model terms. Current noise terms which are not present in traditional linear and polynomial models (Ljung 1987, Soderstrom and Stoica 1989) are induced when the rational model is expanded and are related to the concepts of noise in variables or noisy input measurements (Tugnait 1992).

Two real data sets which relate to the wave forces acting on structures in fluid loading systems for both unidirectional and directional sea states are analyzed to illustrate the identification of nonlinear rational models using the unified algorithm.

2.0 Generalized NARMAX model

In a practical environment some uncertain behaviors or stochastic phenomena are often encounted and model fitting based on stochastic data will be necessary. The NARMAX model

$$y(t) = F(y^{t-1}, u^{t-1}, e^{t-1}) + e(t) \tag{2.1}$$

can be used to describe the stochastic behaviors of both linear and nonlinear systems, where t (t=1, 2, ...) is the time index and

$$y = [y(t-1), \ldots y(t-r)]$$
$$u = [u(t-1), \ldots u(t-r)]$$
$$e = [e(t-1), \ldots e(t-r)]$$

(2.2)

are output, input and noise vectors respectively and $F(.)$ can be a linear or nonlinear function.

The generalization or extension of the stochastic NARMAX model can be considered in three stages. The first and fundamental sub-model set is the polynomial formulation (Billings and Leontarittis 1981), the second extension is the extended model set including exponential, absolute value, logarithmic, and trigonometrical terms (Billings and Chen 1989a), and the third extension is the introduction of the rational model (Billings and Chen 1989b, Billings and Zhu 1991). The first two sub-model sets are naturally characterized as linear in the parameters.

Extending the model in eqn (2.1) to accommodate current noise terms gives

$$y(t) = F(y^{t-1}, u^{t-1}, e^{t-1}, e_1(t), \ldots, e_{ni}(t)) + e(t)$$

(2.3)

where $e_j(t)$ are independent current noises, with zero mean and finite variance σ_{ej}^2, which are internally addictive on the inputs and/or regression terms. The current noise terms (eg $e_j(t)$) may be induced by different noise effects on different variables or regressors. The model of eqn (2.1) shown in Fig. 1(a) assumes that all the current noise sources can be lumped together at the output but the model of eqn (2.3) shown in Fig. 1(b) allows for different noise sequences on different variables. Eqn (2.3) can be expressed as

$$y(t) = \sum_{j=1}^{ni} \phi_j(t) \theta_j + e(t)$$

(2.4)

where

$$\phi_j(t) = p_j(t) (v_j(t) + e_j(t))$$

(2.5)

and the current noise free terms $p_j(t) = p_j(y^{t-1}, u^{t-1}, e^{t-1})$ and $v_j(t) = v_j(y^{t-1}, u^{t-1}, e^{t-1})$ may be in the form of polynomial, exponential, absolute value, logarithmic, trigonometrical, orother functions. Substituting eqn (2.5) into eqn (2.4) yields

$$y(t) = \sum_{j=1}^{m} (p_j(t) v_j(t) + p_j(t) e_j(t)) \theta_j + e(t)$$

(2.6)

Eqn (2.6) provides a model representation when the measurements of different variables and the output are noise contaminated. Consider the rational model as an example to show the generality of this model structure

$$y(t) = \frac{a(t)}{b(t)} + e(t) = \frac{\sum_{j=1}^{num} p_{nj}(t)\theta_{nj}}{\sum_{j=1}^{den} p_{dj}(t)\theta_{dj}} + e(t)$$

(2.7)

where $a(t)$ with terms $p_{nj}(t)$ and the associated parameters θ_{nj} and $b(t)$ with terms $p_{dj}(t)$ and the associated parameters θ_{dj} are the numerator and denominator polynomials respectively. In order to facilitate identification eqn (2.7) is expanded into a linear in the parameters expression to give

$$Y(t) = \sum_{j=1}^{num} p_{nj}(t)\theta_{nj} + \sum_{j=2}^{den} p_{dj}(t) y(t)\theta_{dj} + b(t) e(t)$$
$$= \sum_{j=1}^{num} p_{nj}(t)\theta_{nj} + \sum_{j=2}^{den} p_{dj}(t) \left(\frac{a(t)}{b(t)} + e(t)\right)\theta_{dj} + b(t) e(t)$$

(2.8)

which is obtained by multiplying $b(t)$ on both sides of eqn (2.7) and then moving all the terms except $Y(t)=y(t)p_{d1}(t)$ (letting $\theta_{d1}=1$ without loss of generality) to the right hand side.

Comparing eqn (2.8) with eqn (2.6) shows that

$$p_j(t) = p_{dj}(t) \qquad v_j(t) = \frac{a(t)}{b(t)} \qquad e_j(t) = e(t)$$

(2.9)

in the denominator terms and

$$p_j(t) = p_{nj}(t) \qquad v_j(t) = 1 \qquad e_j(t) = 0$$

(2.10)

in the numerator terms. The expanded rational model in eqn (2.8) therefore gives a class of models with current noise sequences in regressors.

3.0 A unified least squares algorithm

NARMAX model identification usually consists of following steps

(i) Model based term selection.

(ii) Parameter estimation.

(iii) Model validation.

The first two steps can be based on a least squares type algorithm. The model term selection can be obtained using an extension of the err (error reduction ratio) test of Billings and Chen (1989b) which selects significant terms according to the contribution that each makes to the reduction of the estimated noise variance (Zhu and Billings 1993). Parameter estimation is normally achieved by either a conventional least squares estimator or the orthogonal estimator but correlated noise must be accommodated if bias is to be avoided. Model validation tests the results obtained from the algorithms.

By considering the model of eqn (2.6) a unified least squares algorithm can be derived which gives a basis for structure detection and parameter estimation for all the varieties of NARMAX model sets. Writing eqn (2.6) in vector notation

$$Y = (PV + PE)\Theta + e \tag{3.1}$$

where

$$Y = [y(1), \ldots, y(N)]^T$$

$$PV = \begin{bmatrix} p_1(1)v_1(1) & \ldots & p_{nl}(1)v_{nl}(1) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ p_1(N)v_1(N) & \ldots & p_{nl}(N)v_{nl}(N) \end{bmatrix}$$

$$PE = \begin{bmatrix} p_1(1)e_1(1) & \dots & p_m(1)e_m(1) \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ p_1(N)e_1(N) & \dots & p_m(N)e_m(N) \end{bmatrix}$$

$$\Theta = [\theta_1, \dots, \theta_m]^T$$
$$e = [e(1), \dots, e(N)]^T$$

(3.2)

and N is the data length. Let

$$\Phi = PV + PE$$

(3.3)

Then the formal least squares parameter estimator is

$$\hat{\Theta} = [\Phi^T\Phi]^{-1}\Phi^T Y$$
$$= [[PV]^T PV + [PE]^T PE]^{-1}[[PV]^T Y + [PE]^T Y]$$

(3.4)

where, by the probability limit property with large N (Wilks 1962),

$$\frac{1}{N}[PV]^T PE \approx Plim\left[\frac{1}{N}[PV]^T PE\right] = 0$$

(3.5)

because $e_j(t)$ in PE is an independent zero mean noise and $Plim\ [f]$ denotes the probability limit of $[f]$. Define

$$Bias_1 = Plim\left[\frac{1}{N}[PE]^T PE\right] \approx \frac{1}{N}[PE]^T PE$$
$$Bias_2 = Plim\left[\frac{1}{N}[PE]^T Y\right] \approx \frac{1}{N}[PE]^T Y$$

(3.6)

which can be thought of as a form of auto-correlation of the error terms and the cross-correlation between the output and the error terms in regressors respectively.

From the above analysis the unbiased least squares estimate of the parameters for the model of eqn (2.6) is

$$\hat{\Theta} = [\Phi^T\Phi - [PE]^T PE]^{-1}[\Phi^T Y - [PE]^T Y]$$
$$= [\Phi^T\Phi - bias_1]^{-1}[\Phi^T Y - bias_2]$$

(3.7)

The well known orthogonal least squares algorithms can also be applied to such an expression by transforming $\Phi^T\Phi$ into an orthogonal normal matrix with appropriate corrections to the normal matrix and correlation vector (Zhu and Billings 1993).

The following analysis is used to show that the algorithms to identify the polynomial and rational models are particular implementations of the unified algorithm given in eqn (3.7). For the polynomial model the unified algorithm reduces to

$$\hat{\Theta} = [\Phi^T\Phi - [PE]^T PE]^{-1}[\Phi^T Y - [PE]^T Y]$$
$$= [\Phi^T\Phi]^{-1}[\Phi^T Y]$$

(3.8)

which is a straightforward unbiased estimator. This follows because the matrix $PE = 0$ since the normal matrix does not depend on the current noise. The covariance of the estimator is given by

$$Cov\hat{\Theta} = \sigma_e^2[\Phi^T\Phi]^{-1}$$

(3.9)

This type of algorithm and various alternatives to it have been extensively studied (Billings and Chen 1989a).

For the rational model the unified algorithm reduces to

$$\hat{\Theta} = [\Phi^T\Phi - [PE]^T PE]^{-1}[\Phi^T Y - [PE]^T Y]$$
$$= [\Phi^T\Phi - \sigma_e^2\Psi]^{-1}[\Phi^T Y - \sigma_e^2\psi]$$

(3.10)

where $[PE]^T PE=\sigma_e^2\Psi$ and $[PE]^T Y=\sigma_e^2\psi$, and σ_e^2 is the noise variance. This algorithm was introduced as a new method of identification for the rational model (Billings and Zhu

1991, 1993, Zhu and Billings 1991, 1993). The covariance matrix of the algorithm was derived by Zhu and Billings (1991)

$$Cov\hat{\Theta} = \sigma_e^2\sigma_b^2[\Phi^T\Phi - \sigma_e^2\Psi]^{-1}, \sigma_e^2 < 1 \tag{3.11}$$

where σ_b^2 is the denominator variance of the rational model. Consider a simple rational model to illustrate the algorithm

$$y(t) = \frac{a(t)}{b(t)} = \frac{a_1 u(t-1)e(t-1)}{1 + b_1 y^2(t-1)} + e(t) \tag{3.12}$$

The linear in the parameters expression is given by multiplying $b(t)$ on both sides of eqn (3.12) and then moving all the terms except $y(t)$ to the right hand side

$$y(t) = a_1 u(t-1)e(t-1) - b_1 y^2(t-1)y(t) + b(t)e(t) \tag{3.13}$$

where

$$\hat{\Theta} = [\hat{a}_1 \quad \hat{b}_1]^T$$

$$\Phi = \begin{bmatrix} u(0)e(0) & y^2(0)y(1) \\ & \cdots \\ & \cdots \\ & \cdots \\ u(N-1)e(N-1) & y^2(N-1)y(N) \end{bmatrix}$$

$$\Psi = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{t=1}^{N} y^4(t-1) \end{bmatrix}$$

$$\psi = \begin{bmatrix} 0 \\ -\sum_{t=1}^{N} y^2(t-1) \end{bmatrix}$$

$$Y = [y(1)\ldots y(N)]^T$$

(3.14)

Unbiased parameter estimates can be obtained by substituting eqn (3.14) into eqn (3.10). Inspection of eqn (3.8) and eqn (3.10) shows that the critical difference between the algorithms for the polynomial and rational models is the current noise terms which are included in the later model.

4.0 Identification of fluid loading systems

Rational NARMAX models of two nonlinear fluid loading systems have been estimated using the unified algorithm in section 3.0 combined with orthogonal term selection (Zhu and Billings 1993) and model validation test procedures (Billings and Voon 1986).

The first identification S_1 involved fitting a model to relate wave force to flow velocity for unidirectional wave profiles acting on a fixed smooth cylinder from the Delta flume facility. The estimation algorithm was applied with an initial model consisting of 90 terms with specifications *numerator degree* = *denominator degree* = 2, *input lag* = *output lag* = 2 and *noise lag* = 4. The input and output sequences are shown in Fig. 2 and the identified rational model took the form

$$y(t) = \frac{a(t)}{b(t)} + e(t)$$

(3.15)

where

$$a(t) = 1.03y(t-1) - 0.14y(t-2) + 402.3u(t-1) - 407.1u(t-2) - 7.09u^2(t-1) + 0.55e(t-2) - 0.92u(t-2)e(t-3) + 0.67$$

$$b(t) = 1 - 0.04u^2(t-1) - 0.02u(t-1)e(t-1)$$

(3.16)

The one step ahead predictions and residuals are illustrated in Fig.3 and model validity tests are shown in Fig. 4. All these results suggest that an adequate model of the wave flume system has been obtained.

The second identification S_2 involved fitting a model relating wave force to flow velocity in directional sea states with a prominent current for the Christchurch Bay Tower. The input and output sequences for this system are shown in Fig. 5 and the identified rational model took the form

$$y(t) = \frac{a(t)}{b(t)} + e(t) \tag{3.17}$$

where

$$a(t) = 1.74y(t-1) - 0.82y(t-2) + 0.001y^2(t-1) - 26.9u^2(t-2) + 25.9u(t-1)u(t-2)$$

$$b(t) = 1 + 0.00001y(t-1) + 0.001y(t-2) - 1.70u^2(t-1) - 1.53u^2(t-2) \quad 3.19u(t-1)u(t-2) \tag{3.18}$$

The one step ahead predictions and residuals are illustrated in Fig. 6 and model validity tests are shown in Fig. 7. All these results suggest that an adequate model of the system has been obtained.

5.0 Conclusions

A unified least squares algorithm has been applied to identify rational models of two real fluid loading nonlinear systems.

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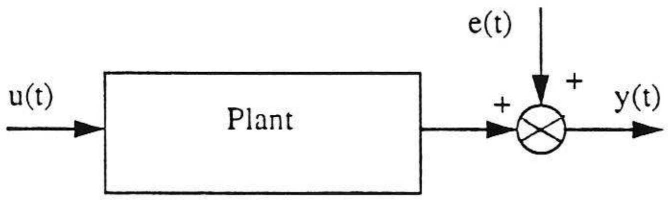
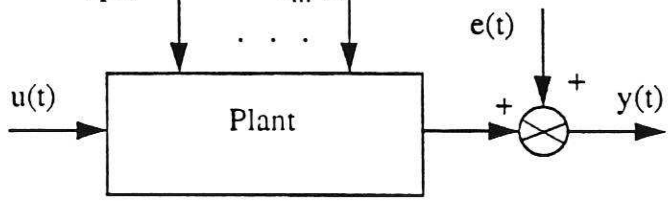


(a) Current noise at the output



(b) Current noise at the input and output

Figure 1 Block diagrams for NARMAX models

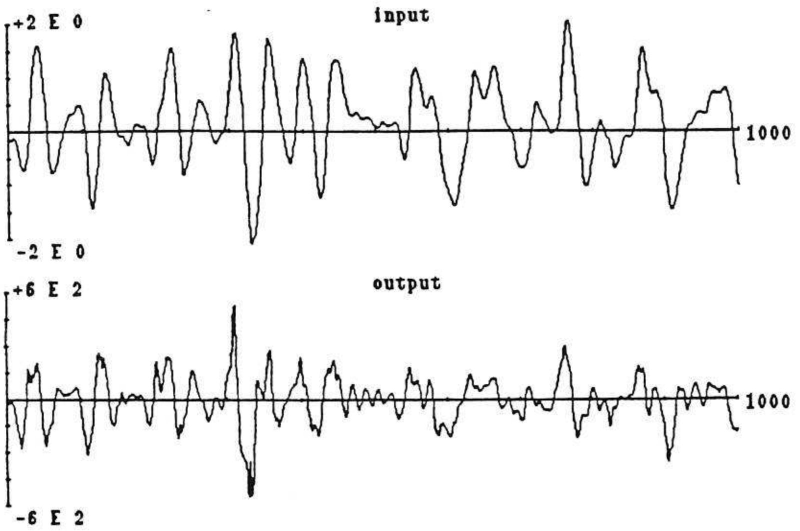


Figure 2 Input and output for S_1

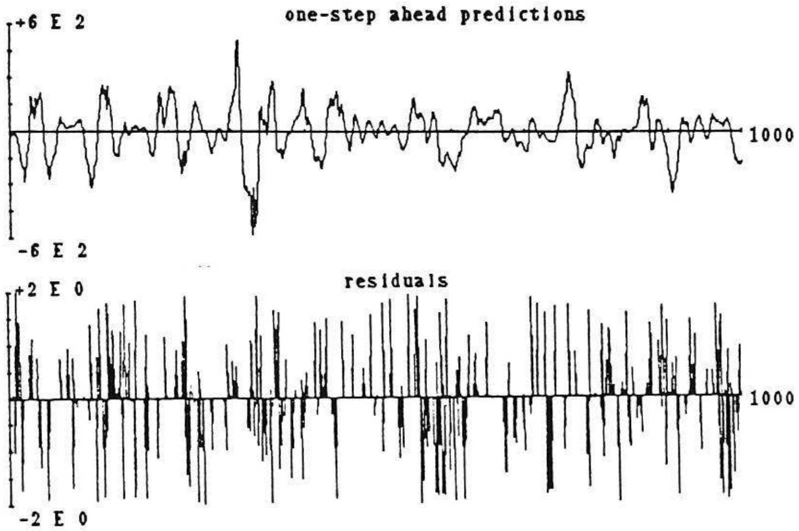


Figure 3 One step ahead predictions and residuals for S_1

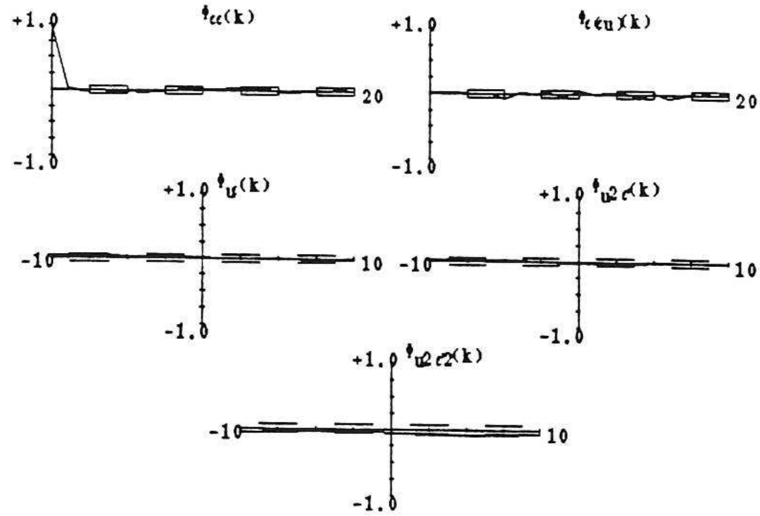


Figure 4 Model validation for S_1

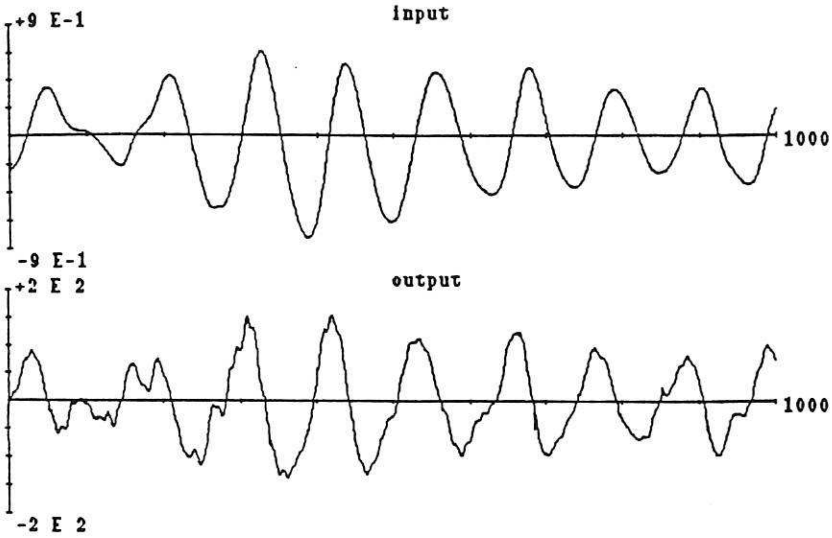


Figure 5 Input and output for S_2

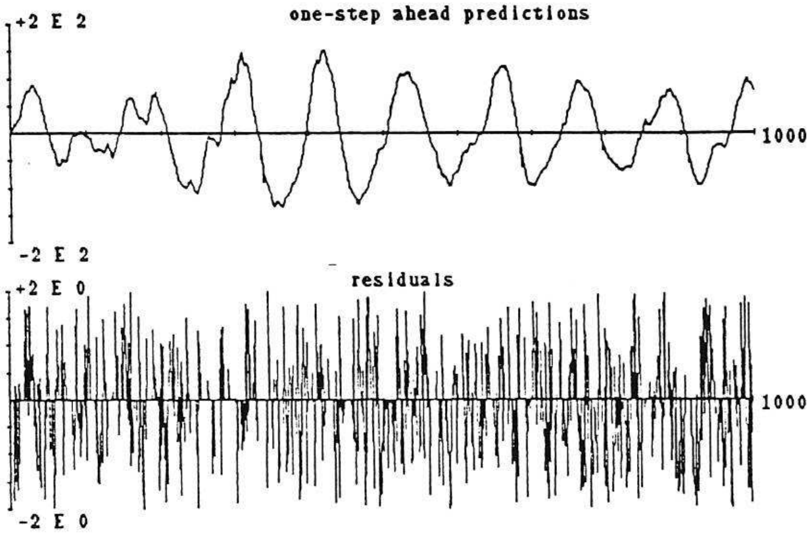


Figure 6 One step ahead predictions and residuals for S_2

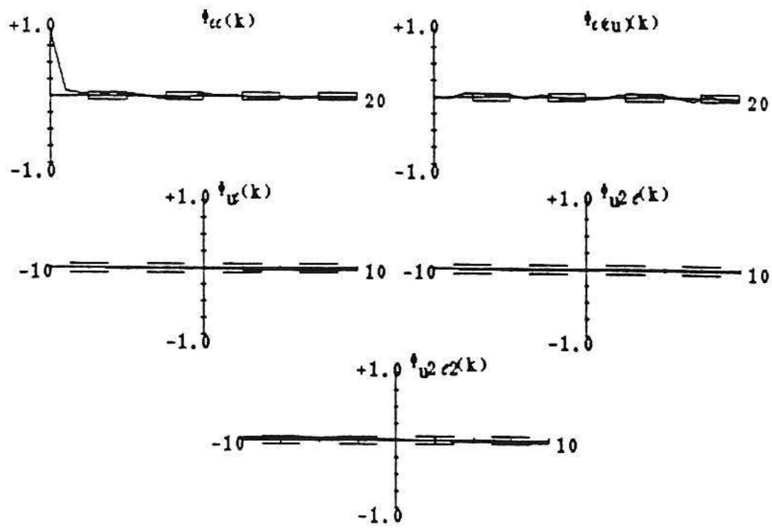


Figure 7 Model validation for S_2