On Wavelet Representations of Differential Equations

S.P.Banks and A.Moser

Department of Automatic Control and Systems Engineering

University of Sheffield

Mappin Street

SHEFFIELD S1 4DU

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Abstract

In this paper we consider the wavelet expansion of the solutions of nonlinear differential equations. We show that, using Lie series, an infinite dimensional linear system is obtained and we prove an existence result for periodic orbits. Moreover, the numerical computation of solutions is generalized from Euler's method to general wavelet expansions.

Keywords:Wavelets, Periodic orbits, Euler's method.

1 Introduction

Wavelet bases of Hilbert spaces have been studied extensively ([5],[8],[9],[3]) and have been applied largely to multiresolution analysis in signal processing ([6],[7]). The basic idea of multiresolution analysis is that one determines a linear operator A_{2^j} which approximates a signal $f(t) \in L^2(\mathbf{R})$ at resolution j. A_{2^j} is a projection operator and maps $L^2(\mathbf{R})$ onto a subspace V_{2^j}. We have

$$V_{2^j} \subseteq V_{2^{j+1}}$$

i.e. higher resolutions determine lower resolutions. Moreover,

$$f(t) \in V_{2^j} \quad \text{iff} \quad f(2t) \in V_{2^{j+1}}.$$

We also require that all the functions of all resolutions generate $L^2(\mathbf{R})$ i.e.

$$\cup_{j=-\infty}^{\infty} V_{2^j} \quad \text{is dense in } L^2(\mathbf{R})$$

and

$$\cap_{j=-\infty}^{\infty} V_{2^j} = 0$$

The simplest example of a resolution space is given by the usual discretization; i.e. V_1 is the space of all functions which are constant on the interval $(k, k+1)$ (and zero otherwise) so that V_{2^j} consists of functions constant on $(k2^{-j}, (k+1)2^{-j})$, $k \in \mathbf{Z}$.

In general we can find a scaling function $\phi(t) \in L^2(\mathbf{R})$ such that

$$\left(\sqrt{2^{-j}}\phi_{2^j}(t - 2^{-j}n)\right)_{n \in \mathbf{Z}}$$

is an orthonormal basis of V_{2^j} where $\phi_{2^j} = 2^j\phi(2^j t)$. Moreover, ϕ can be chosen to have compact support and be differentiable any (finite) number of times ([4]).

In this paper we study the application of wavelets to nonlinear dynamical systems and obtain an infinite dimensional linear wavelet equation equivalent to the original system. Moreover, we prove a limit cycle result in terms of wavelets. Finally, since Euler's method is essentially a wavelet expansion in terms of the simple rectangular wavelets described above, we can generalize it to give a multiresolution approximation to a nonlinear system.

2 Wavelet Representations of Dynamical Systems

Here we consider an analytic dynamical system of the form

$$\dot{x} = f(x) \tag{2.1}$$

for which no solution trajectory has finite escape time. (It is a simple matter to extend the results to systems with finite escape time by changing the time axis to τ such that

$$\tau = \tan\frac{\pi t}{2T}.)$$

Since (2.1) is analytic, the solution through any given point x is given by the Lie series ([1],[2]):

$$x(t) \quad = \quad e^{(tf\partial/\partial x)}x$$

$$= \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathcal{L}_f)^i x$$

where $\mathcal{L}_f$ denotes the Lie derivative with respect to f.

Let $(V_{2^j})_{j\in\mathbf{Z}}$ be a multiresolution approximation of $L^2(\mathbf{R})$ and consider the solution $x(t)$ at resolution j. Then we choose a scaling function $\phi(t) \in L^2(\mathbf{R})$ with compact support such that $\left(\sqrt{2^{-j}}\phi_{2^j}(t-2^{-j}n)\right)_{n\in\mathbf{Z}}$ is an orthonormal basis of V_{2^j} where $\phi_{2^j} = 2^j\phi(2^j t)$. Define

$$\psi_{jn}(t) = \sqrt{2^{-j}}\phi_{2^j}(t-2^{-j}n)\ ,\ j,n \in \mathbf{Z}. \tag{2.2}$$

Then the coefficients of the dolution $x(t)$ at resolution j are given by

$$\begin{aligned} x_{jn} \triangleq \langle x(t), \psi_{jn}(t)\rangle &= \int_{-\infty}^{\infty} \sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathcal{L}_f)^i x \cdot \psi_{jn}(t) dt \\ &= \sum_{i=0}^{\infty} \mu_{in}^j (\mathcal{L}_f)^i x \end{aligned} \tag{2.3}$$

where the 'moments' μ_{in}^j are given by

$$\mu_{in}^j = \langle \frac{t^i}{i!}, \psi_{jn}(t)\rangle.$$

Note that

$$\begin{aligned} \psi_{jn}(t) &= \sqrt{2^{-j}}\phi_{2^j}(t-2^{-j}n) \\ &= \sqrt{2^{-j}}\phi_{2^j}(t-2^{-j}(n-1)-2^{-j}) \\ &= \psi_{jn-1}(t-2^{-j}). \end{aligned}$$

Hence,

$$\mu_{in}^j = \int_{-\infty}^{\infty} \frac{t^i}{i!} \cdot \psi_{jn-1}(t-2^{-j}) dt$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} \frac{(t+2^{-j})^i}{i!} \cdot \psi_{jn-1}(t)dt \\
&= \sum_{k=0}^{i} \int_{-\infty}^{\infty} \frac{1}{i!} \begin{pmatrix} i \\ k \end{pmatrix} t^k 2^{-j(i-k)} \psi_{jn-1}(t)dt \\
&= \sum_{k=0}^{i} \int_{-\infty}^{\infty} \frac{2^{-j(i-k)}}{(i-k)!} \frac{t^k}{k!} \psi_{jn-1}(t)dt \\
&= \sum_{k=0}^{i} \alpha_{ik}^j \mu_{kn-1}^j
\end{aligned}
\tag{2.4}
$$

where

$$
\alpha_{ik}^j = \frac{2^{-j(i-k)}}{(i-k)!}.
$$

Next define the (infinite) vectors

$$
\begin{aligned}
m^j(n) &= (\mu_{0n}^j, \mu_{1n}^j, \cdots) \\
\Lambda(x) &= (I, ad_f, (ad_f)^2, (ad_f)^3, \cdots)x.
\end{aligned}
$$

We can then interpret (2.3) as saying that the n^{th} wavelet coefficient of the solution $x(t)$ of (2.1) at the resolution j is simply the inner product of the vectors $m^j(n)$ and $\Lambda(x)$ (where we regard Λ as an infinite vector of elements of $\mathbf{R}^n$). Moreover, the wavelet coefficients are linked in a very simple (linear) way by (2.4). Thus, if we define the infinite matrix

$$
{}^jA = (\alpha_{ik}^j) \tag{2.5}
$$

we have

$$
m^j(n+1) = {}^jAm^j(n). \tag{2.6}
$$

Remark In the following discussion we shall usually suppress the resolution level and simply write A for the matrix. The level should be clear from the context. Note that the structure of the matrix A is of the form

$$A = \begin{pmatrix} \alpha^j_{00} & 0 & 0 & 0 & 0 & \cdots & \cdots \\ \alpha^j_{10} & \alpha^j_{11} & 0 & 0 & 0 & \cdots & \cdots \\ \alpha^j_{20} & \alpha^j_{21} & \alpha^j_{22} & 0 & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Summarizing, we have

Theorem 2.1 The wavelet coefficients of the solution of (2.1) propagate according to the discrete dynamical system

$$m^j(n+1) = Am^j(n)$$

and are given by

$$\langle m^j(n), \Lambda(x) \rangle,$$

where we interpret the inner product in an obvious way. □

3 Limit Cycles

In this section we shall discuss the nature of limit cycles in terms of wavelet expansions. This may lead to some insight into the nature of other global (chaotic) structures where wavelet representations are natural tools. Consider first the Lie series approach.

Theorem 3.1 Suppose that the system

$$\dot{x} = f(x)$$

has a convergent Lie series through $x \in \mathbf{R}^n$ for all $t \geq 0$. Then it has a limit cycle through x with period τ if and onlt if $\Lambda(x) = (I, ad_f, (ad_f)^2, (ad_f)^3, \cdots)x$ is a characteristic vector of the infinite dimensional matrix

$$\begin{pmatrix} 1 & \tau & \frac{\tau^2}{2!} & \frac{\tau^3}{3!} & \cdots \\ & 1 & \tau & \frac{\tau^2}{2!} & \cdots \\ & & 1 & \tau & \cdots \\ & & & & \cdots \end{pmatrix} \tag{3.1}$$

with eigenvalue 1.

Proof We have the expansion

$$x(t;x) = \sum_{n=0}^{\infty} \frac{t^n}{n!}(ad_f)^n x \tag{3.2}$$

for the solution of the equation through x. Similarly,

$$x(t+\tau;x) = \sum_{n=0}^{\infty} \frac{(t+\tau)^n}{n!}(ad_f)^n x$$

and so

$$x(t+\tau;x) \quad = \quad \sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \frac{(t)^k \tau^{n-k}}{n!}(ad_f)^n x$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{t^k}{k!(n-k)!}\tau^{n-k}(ad_f)^n x \\
&= \sum_{k=0}^{\infty}\frac{t^k}{k!}\sum_{n=k}^{\infty} \frac{\tau^{n-k}}{(n-k)!}(ad_f)^n x. \qquad (3.3)
\end{aligned}
$$

Since (3.1) and (3.2) are equal for all t, by periodicity, we have

$$(ad_f)^n x = \sum_{m=0}^{\infty} \frac{\tau^m}{m!}(ad_f)^{m+n} x$$

and the result follows. □

To consider the existence of limit cycles in the wavelet representation we shall assume, without loss of generality, that we are looking for a periodic orbit with period 1.

Lemma 3.1 If $x(t)$ is a periodic orbit with period 1, then the wavelet coefficients of x for a fixed resolution j ($j \geq 0$) are periodic with period $n = 2^j$, i.e.

$$x_{jn} = x_{j(n+2^j)}.$$

Proof

$$
\begin{aligned}
x_{j(n+2^j)} &= \int_{-\infty}^{\infty} x(t)\psi_{j(n+2^j)}(t)dt \\
&= \int_{-\infty}^{\infty} x(t)\sqrt{2^{-j}}\phi_{2^j}(t - 2^{-j}(n+2^j))dt \\
&= \int_{-\infty}^{\infty} x(t+1)\psi_{jn}(t)dt \\
&= x_{jn}
\end{aligned}
$$

since $x(t+1) = x(t)$. □

The next result follows directly from (2.6):

Theorem 3.2 In order that equation (3.1) has a limit cycle of period 1 it is necessary and sufficient that

$$\langle m^j(0), \Lambda(x)\rangle = \langle A^{2^j} m^j(0), \Lambda(x)\rangle$$

for all $j \geq 0$. □

4 Wavelets and Euler's Method

Most of our detailed knowledge of nonlinear differential equations comes from numerical computation. In this section we shall develop a generalization of Euler's well-known method to a numerical technique based on wavelet approximations. To begin, let $\{\psi_{jn}\}$ again be a wavelet basis as in (2.2) and consider the nonlinear equation

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

where $f(0) = 0$.

Then,

$$x(t) = x_0 + \int_0^t f(x(s))ds$$

and so

$$x(t) - x(t - 2^{-j}) = \int_{t-2^{-j}}^t f(x(s))ds. \tag{4.1}$$

Assuming $x \in L^2[0,\infty)$ (if not, we can always truncate x since we are only interested in the solution on a fixed time interval) then

$$x(t) \cong \sum_n x_{jn}\psi_{jn}(t)\ ,\ t \geq 0$$

is an approximation to x at resolution j, and so, from (4.1), we have

$$\sum_n x_{jn}\psi_{jn}(t) - \sum_n x_{jn}\psi_{jn}(t-2^{-j})$$
$$\cong \int_{t-2^{-j}}^{t} f\left(\sum_n x_{jn}\psi_{jn}(s)\right) ds$$

i.e.

$$\sum_n x_{jn}\psi_{jn}(t) - \sum_n x_{jn}\psi_{jn+1}(t)$$
$$\cong \int_{t-2^{-j}}^{t} f\left(\sum_n x_{jn}\psi_{jn}(s)\right) ds.$$

Hence, by orthogonality of the wavelets, we get

$$x_{jm} - x_{jm-1} \cong \langle \int_{\cdot-2^{-j}}^{\cdot} f\left(\sum_n x_{jn}\psi_{jn}(s)\right) ds, \psi_{jm}\rangle. \qquad (4.2)$$

The bracket on the right hand side is an integral over the support of ψ_{jm}. The main drawback with this expression is that it depends on values of x_{jn} for $n > m$ since the wavelets overlap, in general. In fact, if $[-\sigma_j, \sigma_j]$ is the support of ψ_{0j} and $L = [\sigma_j/2^j] + 1$, where $[x]$ denotes the integer part of x, then (4.2) leads to the difference equation

$$x_{jm} - x_{jm-1} = F(x_{jm-L}, \cdots, x_{jm+L}) \qquad (4.3)$$

for some function F. This is clearly not particularly convenient for numerical computation since F depends on 'future' values of x_{jn}. We can overcome this difficulty with the following simple result, which essentially says that a nonlinear differential equation may be approximated by a delay equation.

Lemma 4.1 Consider the differential equation

$$\dot{x}(t) = f(x(t)) \quad , \quad x(0) = x_0 \tag{4.4}$$

(where f is continuously differentiable) which is assumed to have a unique solution and associate with it the simple delay equation

$$\dot{y}(t) = f(y(t-\delta)) \quad , \quad y(t) = x_0 \ , \ -\delta \le t \le 0. \tag{4.5}$$

Then the solution of (4.5) converges to that of (4.4) as $\delta \longrightarrow 0$, on any compact interval $[0, T]$ on which the solutions exist.

Proof We shall give a quick outline of this simple result. From (4.4) and (4.5) we have

$$\begin{aligned} x(t) &= x_0 + \int_0^t f(x(s))ds \\ y(t) &= x_0 + \int_0^t f(y(s-\delta))ds \end{aligned}$$

and so

$$\epsilon(t) = \int_0^t (f(x(s)) - f(y(s-\delta)))ds$$

where

$$\epsilon(t) = x(t) - y(t).$$

Hence,

$$||\epsilon(t)|| \le \int_0^t ||f(x(s)) - f(y(s))|| + \int_0^t ||f(y(s)) - f(y(s-\delta))||ds$$

$$\leq \int_0^t \left\| \frac{\partial f}{\partial x}(\overline{x}(s)) \right\| ||e(s)||ds + \int_0^t \left\| \frac{\partial f}{\partial x}(\overline{y}(s)) \right\| ||y(s) - y(s-\delta)||ds$$

by the mean-value theorem, where $\overline{x}$ is between $x(s)$ and $y(s)$ and $\overline{y}(s)$ is between $y(s)$ and $y(s-\delta)$. It is easy to see that we may assume that the matrix norms $\left\| \frac{\partial f}{\partial x}(\overline{x}(s)) \right\|$ and $\left\| \frac{\partial f}{\partial x}(\overline{y}(s)) \right\|$ are bounded independently of δ (by M, say) and so a simple application of Gronwall's lemma shows that

$$||e(t)|| \leq e^{Mt} M \sup_{[0,T]} ||y(s) - y(s-\delta)||.$$

By uniform continuity of the solution of (4.5) over $[0, T]$, the result follows. □

Lemma 4.1 and the reasoning leading to (4.3) now give the following generalization of Euler's method:

Theorem 4.2 Consider the differential equation

$$\dot{x}(t) = f(x(t))$$

and let x_{jn} be the wavelet coefficients of $x(t)$ at resolution j. Then the coefficients may be approximated by

$$x_{jm} - x_{jm-1} = F(x_{jm-2L-1}, \cdots, x_{jm-1}) \tag{4.6}$$

where, as before,

$$L = [\sigma_j / 2^j] + 1. \tag{4.7}$$

Proof Indeed, the coefficients satisfying (4.6) are the approximate Fourier coefficients of the solution of the delay equation

$$\dot{x}(t) = f(x(t - L - 1)). \tag{4.8}$$

□

Remark The accuracy of the approximation increases with j, i.e. as $j \longrightarrow \infty$, $L \longrightarrow \infty$ and lemma 4.1 then applies. This, of course, directly generalizes the basic idea of Euler's method–as the step length decreases, the accuracy improves and discretization is equivalent to using a rectangular wavelet.

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