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Observer Design For Nonlinear Systems

by

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Abstract

A method of designing an observer for a form of nonlinear systems is presented. The method gives one way of solving the comlexity in generalizing the well established theory of linear observers to the nonlinear systems in the form considered. The method is demonstrated on a practical model of the Ball and Beam system.

Keywords:Observers, Nonlinear Systems, Ball and Beam system.

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1 Introduction

In this paper we present a method for designing an observer for nonlinear systems of the form

$$\begin{aligned} \dot{x}(t) &= A(x)x + B(x)u \\ y(t) &= C(x)x \end{aligned} \quad (1)$$

The method gives one way of solving the complexity in generalizing the well established theory of linear observers (see; [4], [5], [6] and [7]) to the nonlinear systems in the form given in equation (1) above.

This problem has been addressed before by A.S. Hauksdottir and R.E. Fenton in 1988 and they have presented a method of design which requires transforming the nonlinear system in the form of (1) into what they called a nonlinear observer form (by analogy to the linear observer form) given by:

$$\begin{aligned} \dot{x}^{o} &= A^{o}(x^{o})x^{o} + B^{o}(x^{o})u \\ y(t) &= C^{o}x^{o} \end{aligned} \quad (2)$$

where the superscript 'o'denotes observer form, and

$$A^{o}(x^{o}) = \begin{bmatrix} a^{o}_{11}(x^{o}) & 1 & 0 & \dots & 0 \\ a^{o}_{21}(x^{o}) & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 1 \\ a^{o}_{n1}(x^{o}) & 0 & \dots & \dots & 0 \end{bmatrix}$$

$$B^{o}(x^{o}) = \begin{bmatrix} b^{o}_{1}(x^{o}) \\ b^{o}_{2}(x^{o}) \\ \cdot \\ \cdot \\ \cdot \\ b^{o}_{n}(x^{o}) \end{bmatrix}$$

$$C^{o} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

then they design an observer in analogous to the linear case. The major complexity with their method is the nonlinear transformation from system (1) to system (2) essentially required in the design. 'This involves solving a system of partial-differential equations and can get extremely involved, especially for higher-order systems ($n > 2$) '([3]).

2 Nonlinear Observer

In this section we develop an approach for designing an observer for systems in the form of (1). Consider the nonlinear system

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u \\ w &= H(x)x \end{aligned} \tag{3}$$

where x is the state vector, u is the control input, w is the output measurements of the system and they are of dimensions n, m and r respectively. $A(.) : R^n \longrightarrow R^{n^2}$, $B(.) : R^n \longrightarrow R^{nm}$ and $H(.) : R^n \longrightarrow R^{rn}$ are continuously differentiable matrix-valued functions with $H(x)$ being the observation on the states. A stabilizing control for the above nonlinear system has been buplished by the same authors (see [2]).

For the above system we define a nonlinear observer of the form

$$\dot{z} = F(x)z + G(x)x + T(x)B(x)u \tag{4}$$

where $T(x)$ is a matrix satisfying

$$F(x)T(x) - T(x)A(x) + G(x) = 0 \tag{5}$$

From equations 3, 4 and 5 we get

$$\begin{aligned}\frac{d}{dt}(z - T(x)x) &= \dot{z} - T(x)\dot{x} - \frac{\partial T(x)}{\partial x}\dot{x}x + x^T T(x)\dot{x} \\ &= F(x)(z - T(x)x) - \frac{\partial T(x)}{\partial x}(A(x)x + B(x)u)x \end{aligned} \quad (6)$$

where $F(x)$ can be assigned arbitrary eigenvalues.

If we define

$$\zeta = z - T(x)x \quad (7)$$

Then (6) will be

$$\dot{\zeta} = F(x)\zeta + h(x) \quad (8)$$

where

$$h(x) = -\frac{\partial T(x)}{\partial x}(A(x)x + B(x)u)x \quad (9)$$

Lemma 1. For the matrix $F(x)$ having assigned negative eigenvalues and a bounded $h(x)^*$, $\zeta(t)$ will converge to a ball of certain diameter.

Proof One can write equation (8) in the form:

$$\dot{\zeta} = F_0\zeta + (F(x) - F_0)\zeta + h(x) \quad (10)$$

---

*The boundness of $\frac{\partial T(x)}{\partial x}$ will be discussed later in this paper.

where F_0 is a constant (chosen) stability matrix. Then

$$\zeta = e^{F_0 t}\zeta_0 + \int_0^t e^{F_0(t-s)}[(F(x) - F_0)\zeta + h(x)]ds \tag{11}$$

With F_0 being a stability matrix we have that $\left\|e^{F_0 t}\right\| \leq Me^{-w_0 t}$ where M and w_0 are positive numbers, so that if we take norms to equation (11) we get:

$$\|\zeta\| \leq Me^{-w_0 t}\|\zeta_0\| + \int_0^t Me^{-w_0(t-s)}[\|(F(x) - F_0)\|\,\|\zeta\| + \|h(x)\|]ds \tag{12}$$

Let

$$y = e^{w_0 t}\|\zeta\| \tag{13}$$

then

$$y \leq C + M\int_0^t y(s)\,\|(F(x) - F_0)\|\,ds + M\int_0^t e^{w_0 s}\,\|h(x)\|\,ds \tag{14}$$

and

$$\dot{y} \leq My(t)\,\|(F(x) - F_0)\| + Me^{w_0 t}\,\|h(x)\| \tag{15}$$

where C in equation (14) is a positive number. If we chose the eigenvalues of the matrix $F(x)$ to be all equal to $-\lambda$ (this is possible as we have that both matrices $F(x)$ and F_0 can be assigned arbitrary eigenvalues), it follows

$$\|(F(x) - F_0)\| \leq d; \qquad d \geq 0 \tag{16}$$

then

$$y \leq Me^{dt}y_0 + M\int_0^t e^{d(t-s)}e^{w_0 s}\,\|h(x)\|\,ds \tag{17}$$

If we substitute back on y we get

$$\begin{aligned} \|\zeta\| &\leq Me^{dt}\,\|\zeta_0\| + M\int_0^t e^{(d-w_0)t}e^{(w_0-d)s}\,\|h(x)\|\,ds \\ &\leq Me^{dt}\,\|\zeta_0\| + \frac{M}{w_0-d}e^{(d-w_0)t}\left\{e^{(w_0-d)s}\right\}_0^t\|h(x)\| \\ &\leq Me^{dt}\,\|\zeta_0\| + \frac{M(1-e^{(d-w_0)t})}{w_0-d}\,\|h(x)\| \end{aligned} \tag{18}$$

and the result follows with (18) determining the diameter of the ball where $\|\zeta\|$ converges. □.

By the appropriate choice of the eigenvalues of the matrix $F(x)$ which will determine d and also noticing that w_0 is determined by the chosen stability matrix F_0, we can make sure that the diameter of the ball defined by equation (18) is small. Then we can write

$$\|\zeta\| \longrightarrow 0 \tag{19}$$

or

$$z - Tx \longrightarrow 0 \tag{20}$$

Suppose that the matrix

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix} \tag{21}$$

is invertible. Then

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix}^{-1} \begin{bmatrix} H(x) \\ T(x) \end{bmatrix} x = x$$

and therefore

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix}^{-1} \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} \simeq x \tag{22}$$

using (3) and (20).

3 Stability Analysis

Let us write

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix} = [M_1(x), M_2(x)] \tag{23}$$

then

$$\begin{aligned} u(t) &= K(x)M_1(x)w + K(x)M_2(x)z \\ &= K(x)M_1(x)H(x)x + K(x)M_2(x)z \end{aligned} \tag{24}$$

Define

$$\zeta = z - T(x)x \tag{25}$$

from equation (3) we get

$$\begin{aligned}\dot{x} &= (A(x) + B(x)K(x)M_1(x)H(x))x + B(x)K(x)M_2(x)(\zeta + T(x)x) \\ &= (A(x) + B(x)K(x))x + B(x)K(x)M_2(x)\zeta \end{aligned} \tag{26}$$

Also we have from (25)

$$\begin{aligned}\dot{\zeta} &= \dot{z} - T(x)\dot{x} - \frac{\partial T(x)}{\partial x}\dot{x}x \\ &= F(x)\zeta - \frac{\partial T(x)}{\partial x}[(A(x) + B(x)K(x))x + B(x)K(x)M_2(x)\zeta]x \end{aligned} \tag{27}$$

and the composite system (equations 26 and 27) is:

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A + BK & BKM_2 \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\partial T}{\partial x}[(A + BK)x + BKM_2\zeta]x \end{bmatrix} \tag{28}$$

where the eigenvalues of the matrecies $(A(x) + B(x)K(x))$ and F(x) can be assigned arbitrarily.

System (28) can also be written in the form

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A + BK & BKM_2 \\ -\frac{\partial T}{\partial x}[(A + BK)x] & F - \frac{\partial T}{\partial x}BKM_2x \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \tag{29}$$

where the term $-\frac{\partial T}{\partial x} BKM_2 \zeta x$ has been put in the form $-\frac{\partial T}{\partial x} BKM_2 x \zeta$.

Let us now consider the general case of reduced order observer, where we have that

$$x = \begin{bmatrix} y \\ w \end{bmatrix}, \quad y \in E^r, \quad w \in E^{n-r}$$

then system (3) becomes

$$\begin{bmatrix} \dot{y} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A_{11}(x) & A_{12}(x) \\ A_{21}(x) & A_{22}(x) \end{bmatrix} \begin{bmatrix} y \\ w \end{bmatrix} + \begin{bmatrix} B_1(x) \\ B_2(x) \end{bmatrix} u \tag{30}$$

with the following dimensions: $A_{11}(x)$:rxr; $A_{12}(x)$:rx(n-r); $A_{21}(x)$:(n-r)xr; $A_{22}(x)$:(n-r)x(n-r); $B_1(x)$:rx1 and $B_2(x)$:(n-r)x1.

As in the linear case, the vector y is available and the control u is assumed known, then

$$\dot{y} = A_{11}(x)y + A_{12}(x)w + B_1(x)u$$

gives us

$$\dot{y} - A_{11}(x)y - B_1(x)u = A_{12}(x)w \tag{31}$$

with $A_{12}(x)$ as an observation on w.

From (30), we now have the following

$$\begin{aligned}\dot{w} &= A_{22}(x)w + A_{21}(x)y + B_2(x)u \\ \dot{y} &- A_{11}(x)y - B_1(x)u = A_{12}(x)w \end{aligned} \tag{32}$$

and we want to design an observer for the above system.

The proof of the following lemma (lemma 2) follows directly as in the linear case, for detailes see ([7]).

Lemma 2. The pair $(H(x), A(x))$ is observable if and only if the pair $(A_{12}(x), A_{22}(x))$ is observable.

In order to construct an observer, we recall that an identity observer for system (3) is given by

$$\dot{z} = (A(x) - L(x)H(x))z + L(x)H(x)x + B(x)u \tag{33}$$

Then an identity observer for system (32) can be defined in the form:

$$\begin{aligned}\dot{\hat{w}} &= (A_{22}(x) - L(x)A_{12}(x))\hat{w} + A_{21}(x)y \\ &\quad + L(x)A_{12}(x)w + B_2(x)u \\ &= (A_{22}(x) - L(x)A_{12}(x))\hat{w} + A_{21}(x)y + B_2(x)u \\ &\quad + L(x)(\dot{y} - A_{11}(x)y - B_1(x)u) \end{aligned} \tag{34}$$

selecting the (n-r)xr matrix $L(x)$ so that $(A_{22}(x) - L(x)A_{12}(x))$ is a stability matrix. We recall here that following the observability theorem we can chose the appropriate matrix $L(x)$ which assigne fixed eigenvalues to $(A_{22}(x) - L(x)A_{12}(x))$.

If we let

$$z = \hat{w} - L(x)y$$

then (34) becomes

$$\begin{aligned} \dot{z} &= (A_{22}(x) - L(x)A_{12}(x))z + (A_{22}(x) - L(x)A_{12}(x))L(x)y \\ &+ (A_{21}(x) - L(x)A_{11}(x))y + (B_2(x) - L(x)B_1(x))u \end{aligned} \tag{35}$$

or

$$\dot{z} = F(x)z + G(x)x + T(x)B(x)u \tag{36}$$

where

$$F(x) = A_{22}(x) - L(x)A_{12}(x) \tag{37}$$

$$G(x) = ((A_{22}(x) - L(x)A_{12}(x))L(x) + (A_{21}(x) - L(x)A_{11}(x), 0) \tag{38}$$

$$T(x) = (-L(x), I_{n-r}) \tag{39}$$

Its clear from equations (37-39) that all the matrices which construct the observer (36) are expressed in terms of the "chosen" matrix $L(x)$. This raises the requirement that the matrix $L(x)$ should be bounded. To get this bound we consider equation (37) and let us assume that the matrix $(A_{12}(x)A_{12}^T(x))$ is invertible. Define the "generalised inverse" of the rx(n-r) matrix to be

$$A_{12}^{r}(x) = A_{12}^T(x)(A_{12}(x)A_{12}^T(x))^{-1} \tag{40}$$

Using (40) with (37), we get

$$L(x) = A_{22}(x)A_{12}^{r}(x) - F(x)A_{12}^{r}(x) \tag{41}$$

Recall that for A an mxn matrix, the notation $\|A\|$ denotes the nonnegative square root of the sum of squares of the moduli of the elements of A; $\|A\|^2 = trA^{r}A$ and $\|A\| > 0$ unless A=0, then $\|A\| = 0$. Then from equation (41) we can get

$$\|L(x)\| \leq \|A_{22}(x)\| \, \|A_{12}^{r}(x)\| + \|F(x)\| \, \|A_{12}^{r}(x)\| \tag{42}$$

noticing that the matrix $F(x)$ is bounded by the moduli of the largest of its chosen eigenvalues, (42) gives the required bound on the matrix $L(x)$ which in turn gives bound on the matricies $G(x)$ and $T(x)$ (see eqns.(38) and (39)).

Also from (42) we can get an estimate of $\frac{\partial L}{\partial x}$, this can be done by taking the derivative of (41) with respect to x, i.e.

$$\frac{\partial L(x)}{\partial x} = A_{22}(x)\frac{\partial A_{12}^{\tau}(x)}{\partial x} + \frac{\partial A_{22}(x)}{\partial x}A_{12}^{\tau}(x) - F(x)\frac{\partial A_{12}^{\tau}(x)}{\partial x} - \frac{\partial F(x)}{\partial x}A_{12}^{\tau}(x)$$

which provide us with the inequality

$$\begin{aligned} \left\|\frac{\partial L(x)}{\partial x}\right\| &\leq \|A_{22}(x)\|\left\|\frac{\partial A_{12}^{\tau}(x)}{\partial x}\right\| + \left\|\frac{\partial A_{22}(x)}{\partial x}\right\|\|A_{12}^{\tau}(x)\| \\ &+ \|F(x)\|\left\|\frac{\partial A_{12}^{\tau}(x)}{\partial x}\right\| + \left\|\frac{\partial F(x)}{\partial x}\right\|\|A_{12}^{\tau}(x)\| \end{aligned} \tag{43}$$

and this bounds $\frac{\partial L}{\partial x}$. Similarly from (38 and 39) equation 43 provides a bound on $\frac{\partial G}{\partial x}$ and $\frac{\partial T}{\partial x}$.

We recall that the composite system is in the form

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A+BK & BKM_2 \\ 0 & F \end{bmatrix}\begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{\partial T}{\partial x}[(A+BK)x + BKM_2\zeta]x \end{bmatrix} \tag{44}$$

the eigenvalues of the matrices $(A(x)+B(x)K(x))$ and $F(x)$ are to be assigned arbitrary and $\frac{\partial T}{\partial x}$ is bounded. This system can also be written in the form

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A+BK & BKM_2 \\ 0 & F \end{bmatrix}\begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{\partial T}{\partial x}(A+BK) & BKM_2x \end{bmatrix}\begin{bmatrix} x \\ \zeta \end{bmatrix} \tag{45}$$

Let$\psi = (x \quad \zeta)^T$, then we have

$$\dot{\psi} = \Gamma(x)\psi + \Pi(x)\psi \tag{46}$$

where $\Gamma(x)$ and $\Pi(x)$ are as defined in (45) also $\Gamma(x)$ is a stability matrix (with chosen eigenvalues).

One can write (46) as

$$\dot{\psi} = \Gamma_0\psi + (\Gamma(x) - \Gamma_0)\psi + \Pi(x)\psi \tag{47}$$

where Γ_0 is a constant stability matrix then, following the same steps as in the proof of lemma 1, we get

$$\|\psi\| \leq Ne^{-wt}\|\psi_0\| + \int_0^t Ne^{-wt}[\|\Gamma(x) - \Gamma_0\| + \|\Pi(x)\|]\,\|\psi(x)\|\,ds \tag{48}$$

where $\left\|e^{\Gamma_0 t}\right\| \leq Ne^{-wt}$ with N and w positive numbers.

Let

$$\|\Gamma(x) - \Gamma_0\| + \|\Pi(x)\| \leq \epsilon$$

by Gronwall's inequality, we have

$$\|\psi\| \leq e^{(-w+N\epsilon)t}\,\|\psi_0\| \tag{49}$$

which proof the asymptotic stability of (46) provided that $w > N\epsilon$.

Remark. From (37) we have

$$F(x) = A_{22}(x) - L(x)A_{12}(x) \tag{50}$$

Then in reference to the proof of lemma 1, we have

$$\begin{aligned}
\|F(x) - F_0\| &= \|(A_{22}(x) - L(x)A_{12}(x)) - (A_{22}(0) - L(0)A_{12}(0))\| \\
&= \|(A_{22}(x) - A_{22}(0)) - (L(x)A_{12}(x) - L(0)A_{12}(0))\| \\
&= \|A_{22}(x) - A_{22}(0) - L(x)A_{12}(x) + L(0)A_{12}(0) \\
&- L(x)A_{12}(0) + L(x)A_{12}(0)\| \\
&= \|(A_{22}(x) - A_{22}(0)) - (L(x)A_{12}(x) - L(x)A_{12}(0)) \\
&+ (L(x)A_{12}(0) - L(0)A_{12}(0))\| \\
&\leq \|(A_{22}(x) - A_{22}(0))\| + \|L(x)\| \, \|(A_{12}(x) - A_{12}(0))\| \\
&+ \|(L(x) - L(0))\| \, \|A_{12}(0)\| \qquad (51)
\end{aligned}$$

which states the condition of equation (16) in terms of the chosen matrix $L(x)$. In summary we have proved the following theorem:

Theorem 1. For nonlinear systems in the form

$$\begin{aligned} \dot{x}(t) &= A(x)x + B(x)u \\ y(t) &= C(x)x \end{aligned} \tag{52}$$

where x is the state vector, u is the control input, w is the output measurements of the system and they are of dimensions n, m and r respectively. $A(.) : R^n \longrightarrow R^{n^2}$, $B(.) : R^n \longrightarrow R^{nm}$ and $H(.) : R^n \longrightarrow R^{rn}$ are continuously differentiable matrix-valued functions with $H(x)$ being the observation on the states.

A nonlinear observer can be defined for system (52) as

$$\dot{z} = F(x)z + G(x)x + T(x)B(x)u \tag{53}$$

where $T(x)$ is a matrix satisfying

$$F(x)T(x) - T(x)A(x) + G(x) = 0 \tag{54}$$

For this observer, if we define ζ to be the difference between observer states z and $T(x)x$, i.e.

$$\zeta = z - T(x)x \tag{55}$$

Then with matrix $F(x)$ assigned negative eigenvalues and a bounded $h(x)$

[defined by equation(9)], ζ will converge to a ball the diameter which is determined by equation (18).

Considering the general case of a reduced order observer, the elements of nonlinear observer (53) are given by

$$F(x) = A_{22}(x) - L(x)A_{12}(x) \tag{56}$$

$$G(x) = ((A_{22}(x) - L(x)A_{12}(x))L(x) + (A_{21}(x) - L(x)A_{11}(x), 0) \tag{57}$$

and

$$T(x) = (-L(x), I_{n-r}) \tag{58}$$

where system (52) has been written in the form of (30), and $L(x)$ is a chosen matrix which satisfy the following inqualities

$$\|L(x)\| \leq \|A_{22}(x)\| \, \|A_{12}^{\tau}(x)\| + \|F(x)\| \, \|A_{12}^{\tau}(x)\| \tag{59}$$

and

$$\begin{aligned} \left\|\frac{\partial L(x)}{\partial x}\right\| &\leq \|A_{22}(x)\| \left\|\frac{\partial A_{12}^{\tau}(x)}{\partial x}\right\| + \left\|\frac{\partial A_{22}(x)}{\partial x}\right\| \|A_{12}^{\tau}(x)\| \\ &+ \|F(x)\| \left\|\frac{\partial A_{12}^{\tau}(x)}{\partial x}\right\| + \left\|\frac{\partial F(x)}{\partial x}\right\| \|A_{12}^{\tau}(x)\| \end{aligned} \tag{60}$$

and as all the matrices which construct the observer are defined in terms of $L(x)$, the above inqualities provide bounds on matrices $F(x)$, $G(x)$ and $T(x)$

as will as bounds on their rate of change with respect to the state vector $x(t)$.

With the information about the unobservable states of system (52) provided by observer (53), and writting the assumed invertable matrix

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix} \tag{61}$$

in the form

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix} = [M_1(x), M_2(x)] \tag{62}$$

a stabilizing feedback control for nonlinear system (52) is given by

$$u(t) = K(x)M_1(x)H(x)x + K(x)M_2(x)z \tag{63}$$

and the composite system is

$$\begin{bmatrix} \dot{x} \\ \dot{\zeta} \end{bmatrix} = \begin{bmatrix} A+BK & BKM_2 \\ 0 & F \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{\partial T}{\partial x}(A+BK) & BKM_2x \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix} \tag{64}$$

Let$\psi = (x \quad \zeta)^T$, then the above system can be written in the form

$$\dot{\psi} = \Gamma(x)\psi + \Pi(x)\psi \tag{65}$$

If we define Γ_0 as a constant stability matrix then we have that $\left\| e^{\Gamma_0 t} \right\| \leq N e^{-wt}$ where N and w are positive numbers,then system (65) is asymptotically stable provided that $w > N\epsilon$. $\square$.

4 Example

In this section we shall consider the application of the above theory to the Ball and Beam system shown in Fig.1.

In this system the beam is symmetric and is made to rotate in a vertical plane by applying a torque at the point of rotation (the centre). The ball is restricted to frictionless sliding along the beam (as a bead along a wire). This allows for complete rotations and arbitrary angular accelerations of the beam without the ball losing contact with the beam. We shall be interested in controlling the position of the ball along the beam i.e. we would like the ball to track an arbitrary trajectory.

Let the moment of inertia of the beam be J, the mass of the ball be M, and the acceleration of the gravity be G. If we choose the angle ϕ of the beam and the position r of the ball as a generalised coordinates for this system, then the Lagrangian equations of the motion are given by

$$
\begin{aligned}
0 &= \dot{r} + G sin\phi - r\dot{\phi}^2 \\
t_0 &= (Mr^2 + J)\dot{\phi} + 2Mr\dot{r}\dot{\phi} + MGrcos\phi
\end{aligned}
\tag{66}
$$

where t_0 is the torque applied to the beam and there is no force applied to the ball. Using the invertable transformation

$$
t_0 = 2Mr\dot{r}\dot{\phi} + MGrcos\phi + (Mr^2 + J)u \tag{67}
$$

to define a new input u the system can be written in state space form as

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - G sin x_3 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{68}
$$

$$
y = x_1
$$

where $x = (x_1, x_2, x_3, x_4)^T =: (r, \dot{r}, \phi, \dot{\phi})^T$ is the state and $y = h(x) := r$ is the output of the system (i.e. the variable that we want to control).

System (54) is in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 & 0 & f(x_3) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (69)$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

where

$$f(x_3) = -G(1 - \frac{x_3^2}{3!} + \frac{x_3^4}{5!} - +...)$$

For this system we notice that n=4 and r=1 (i.e. only one state, x_1, is available at the output).

From the above we see that

$$\begin{aligned} A_{12}A_{12}^T &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \end{bmatrix} \end{aligned}$$

so that the assumption of (30) is satisfied.

4.1 Determination of the matrix $F(x)$.

We recall that

$$F(x) = A_{22}(x) - L(x)A_{12}(x)$$

Then

$$F(x) = \begin{bmatrix} -l_1(x) & f(x_3) & 0 \\ -l_2(x) & 0 & 1 \\ -l_3(x) & 0 & 0 \end{bmatrix}$$

We want to chose l_i above so that $F(x)$ has the desired eigenvalues. Let these desired eigenvalues for this system be λ_1, λ_2 and λ_3 (real and negative eigenvalues). Then the elements of the vector $L(x)$ will be

$$\begin{aligned} l_1 &= -(\lambda_1 + \lambda_2 + \lambda_3) \\ l_2 &= \frac{(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)}{f(x_3)} \\ l_3 &= -\frac{(\lambda_1\lambda_2\lambda_3)}{f(x_3)} \end{aligned}$$

so that

$$F(x) = \begin{bmatrix} k_1 & f(x_3) & 0 \\ \frac{-k_2}{f(x_3)} & 0 & 1 \\ \frac{k_3}{f(x_3)} & 0 & 0 \end{bmatrix}$$

where $k_1 = (\lambda_1 + \lambda_2 + \lambda_3)$, $k_2 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)$ and $k_3 = (\lambda_1\lambda_2\lambda_3)$. Let us chose $\lambda_1 = \lambda_2 = \lambda_3 = -2$, then from the above we have that $k_1 = -6$, $k_2 = 12$ and $k_3 = -8$ and so

$$F(x) = \begin{bmatrix} -6 & f(x_3) & 0 \\ \frac{-12}{f(x_3)} & 0 & 1 \\ \frac{-8}{f(x_3)} & 0 & 0 \end{bmatrix}$$

and

$$L(x) = \begin{bmatrix} 6 \\ \frac{12}{f(x_3)} \\ \frac{8}{f(x_3)} \end{bmatrix}$$

it follows that

$$G(x) = \begin{bmatrix} x_4^2 - 24 & 0 & 0 & 0 \\ \frac{-64}{f(x_3)} & 0 & 0 & 0 \\ \frac{-48}{f(x_3)} & 0 & 0 & 0 \end{bmatrix}$$

and

$$T(x) = \begin{bmatrix} -6 & 1 & 0 & 0 \\ \frac{-12}{f(x_3)} & 0 & 1 & 0 \\ \frac{-8}{f(x_3)} & 0 & 0 & 1 \end{bmatrix}$$

It easily seen that (5) is satisfied.

It follows that

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 \\ \frac{-12}{f(x_3)} & 0 & 1 & 0 \\ \frac{-8}{f(x_3)} & 0 & 0 & 1 \end{bmatrix} \tag{70}$$

and

$$\begin{bmatrix} H(x) \\ T(x) \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ \frac{12}{f(x_3)} & 0 & 1 & 0 \\ \frac{8}{f(x_3)} & 0 & 0 & 1 \end{bmatrix} \tag{71}$$

For a tracking problem we recall (see [1]) that the control is given by

$$u = kM_1Hx + kM_2z - R^{-1}B^Ts_l \tag{72}$$

where

$$k = -R^{-1}B^TPx \tag{73}$$

and s_l is the vector $lim_{t \longrightarrow \infty} s$ where s is the solution of

$$\dot{s} = -(A - BR^{-1}B^T P)^T s + Qr \tag{74}$$

and r is the set point. With the calculations carried out so far we are now able to form the composite system below

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{z_1} \\ \dot{z_2} \\ \dot{z_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_4^2 & 0 & f(x_3) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ (x_4^2 - 24) & 0 & 0 & 0 & -6 & f(x_3) & 0 \\ \frac{-64}{f(x_3)} & 0 & 0 & 0 & \frac{-12}{f(x_3)} & 0 & 1 \\ \frac{-48}{f(x_3)} & 0 & 0 & 0 & \frac{-8}{f(x_3)} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

The simulation results of the above system in Fig.2 show that the closed loop system with the control given in equation (72) provides good tracking for the trajectory $5 * cos(pi * t/30)$.

5 Conclusion.

In this paper we have studied a method for designing an observer for nonlinear systems of the form:

$$\dot{x}(t) = A(x)x + B(x)u$$
$$y(t) = C(x)x$$

In the method we have generalized the theory of linear observer to nonlinear systems in the form given above. Similar to the linear observer theory, we have seen that the convergence of the observer states to the exact system states is determined by the appropriate choice of the eigenvalues of the 'observer matrix '$F(x)$, which gives us in the nonlinear system a ball of convergence defined by equation (18).

We have shown also that all the elements of the observer i.e. $F(x)$, $G(x)$ and $T(x)$ are defined in terms of a chosen matrix $L(x)$ (eqns. 37-39). Then it has been shown that $L(x)$ is bounded with respect to the change in the states (eqn.36) which in turn bounds the matrices $F(x)$, $G(x)$ and $T(x)$. The condition of equation (16) has also been reduced to a condition in terms of the matrix $L(x)$ (eqn.51). The result of this paper have been demonstrated on a practical Ball-Beam system.