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The groups of automorphisms of the Lie algebras of triangular polynomial derivations

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Abstract

The group of automorphisms G_n of the Lie algebra $\mathfrak{u}_n$ of triangular polynomial derivations of the polynomial algebra $P_n = K[x_1, \dots, x_n]$ is found ($n \geq 2$), it is isomorphic to an iterated semi-direct product

$$\mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}'_n \times \mathbb{E}_n))$$

where $\mathbb{T}^n$ is an algebraic n-dimensional torus, $\mathrm{UAut}_K(P_n)_n$ is an explicit factor group of the group $\mathrm{UAut}_K(P_n)$ of triangular polynomial automorphisms, $\mathbb{F}'_n$ and $\mathbb{E}_n$ are explicit groups that are isomorphic respectively to the groups $\mathbb{I}$ and $\mathbb{J}^{n-2}$ where $\mathbb{I} := (1 + t^2K[[t]], \cdot) \simeq K^{\mathbb{N}}$ and $\mathbb{J} := (tK[[t]], +) \simeq K^{\mathbb{N}}$. It is shown that the adjoint group of automorphisms of the Lie algebra $\mathfrak{u}_n$ is equal to the group $\mathrm{UAut}_K(P_n)_n$.

Key Words: Group of automorphisms, Lie algebra, triangular polynomial derivations, automorphism, locally nilpotent derivation.

Mathematics subject classification 2010: 17B40, 17B66, 17B65, 17B30.

Contents

1 Introduction

Throughout, module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; K is a field of characteristic zero and K^* is its group of units; $P_n := K[x_1, \dots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^n} Kx^\alpha$ is a polynomial algebra over K where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n; $\mathrm{Aut}_K(P_n)$ is the group of automorphisms of the polynomial algebra P_n; $\mathrm{Der}_K(P_n) = \bigoplus_{i=1}^n P_n \partial_i$ is the Lie algebra of K-derivations of P_n; $A_n := K\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle = \bigoplus_{\alpha, \beta \in \mathbb{N}^n} Kx^\alpha \partial^\beta$ is the n'th *Weyl algebra*; for each natural number $n \geq 2$,

$$\mathfrak{u}_n := K\partial_1 + P_1\partial_2 + \cdots + P_{n-1}\partial_n$$

is the *Lie algebra of triangular polynomial derivations* (it is a Lie subalgebra of the Lie algebra $\mathrm{Der}_K(P_n)$) and $G_n := \mathrm{Aut}_K(\mathfrak{u}_n)$ is its group of automorphisms; $\delta_1 := \mathrm{ad}(\partial_1), \dots, \delta_n := \mathrm{ad}(\partial_n)$ are the inner derivations of the Lie algebra $\mathfrak{u}_n$ determined by the elements $\partial_1, \dots, \partial_n$ (where $\mathrm{ad}(a)(b) := [a, b]$).

The group of automorphisms G_n of the Lie algebra $\mathfrak{u}_n$. The aim of the paper is to find the group G_n (Theorem 5.3) and its explicit generators.

- (Theorem 5.3) *Let $\mathbb{I} := (1 + t^2K[[t]], \cdot)$ and $\mathbb{J} := (tK[[t]], +)$. Then for all $n \geq 2$,*

1. $G_n = \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}'_n \times \mathbb{E}_n))$.
2. $G_n \simeq \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{I} \times \mathbb{J}^{n-2})$.

The group $\mathbb{T}^n$ is an algebraic n-dimensional torus, $\mathrm{UAut}_K(P_n)_n := \mathrm{UAut}_K(P_n)/\mathrm{sh}_n$ is the factor group of the group of triangular polynomial automorphisms

$$\mathrm{UAut}_K(P_n) := \{\sigma \in \mathrm{Aut}_K(P_n) \,|\, \sigma(x_i) = x_i + a_i, \ a_i \in P_{i-1} \text{ for } i = 1, \ldots, n\}$$

modulo its normal subgroup

$$\mathrm{sh}_n := \{\sigma \in \mathrm{Aut}_K(P_n) \,|\, \sigma(x_i) = x_i, \ i = 1, \ldots, n-1; \ \sigma(x_n) = x_n + \lambda, \ \lambda \in K\},$$

$\mathbb{F}'_n \simeq \mathbb{I}$ and $\mathbb{E}_n \simeq \mathbb{J}^{n-2}$ are explicit subgroups of G_n (see below and Section 4). The group G_n is made up of two parts: the 'obvious' one, $\mathbb{T}^n \ltimes \mathrm{UAut}_K(P_n)_n$, and the 'non-obvious' one – $\mathbb{F}'_n \times \mathbb{E}_n \simeq \mathbb{I} \times \mathbb{J}^{n-2}$ – which is a much more massive group than the group $\mathbb{T}^n \ltimes \mathrm{UAut}_K(P_n)_n$.

The key ideas and the strategy of finding the group G_n. A group $G = G_1 \times_{ex} G_2$ is an *exact* product of its two subgroups G_1 and G_2 if every element g of the group G is a unique product g_1g_2 for some (unique) elements $g_1 \in G_1$ and $g_2 \in G_2$. The strategy of finding the group G_n is a (rather long) 'refining process' which is done in Sections 3–5. It consists of several steps. On each step the group G_n is presented as an exact or semi-direct product of several of its subgroups. Some of these subgroups are explicit groups and the other are defined in abstract terms (i.e., they satisfy certain properties, elements of which satisfy certain equations). Every successive step is a 'refinement' of its predecessor in the sense that 'abstract' subgroups are presented as explicit sets of automorphisms (i.e., the solutions are found to the defining equations of the subgroups).

In Section 3, the first step is done on the way of finding the group G_n. In Section 3, several important subgroups of the group G_n are introduced. These include the group $\mathrm{TAut}_K(P_n)_n$ and its subgroup $\mathcal{T}_n$ of triangular polynomial automorphisms with all constant terms being equal to zero,

$$\mathcal{T}_n := \{\sigma \in \mathrm{Aut}_K(P_n) \,|\, \sigma(x_1) = x_1, \sigma(x_i) = x_i + a_i \text{ where } \ a_i \in (x_1, \ldots, x_{i-1}), i = 2, \ldots, n\}$$

where $(x_1, \ldots, x_{i-1})$ is the maximal ideal of the polynomial algebra P_{i-1} generated by the elements $x_1, \ldots, x_{i-1}$; and the group

$$\mathrm{Sh}_{n-1} := \{\sigma \in \mathrm{Aut}_K(P_n) \,|\, \sigma(x_1) = x_1+\lambda_1, \ldots, \sigma(x_{n-1}) = x_{n-1}+\lambda_{n-1}, \sigma(x_n) = x_n \text{ where } \ \lambda_i \in K\}.$$

The most important subgroup of the group G_n is

$$\mathcal{F}_n := \{\sigma \in G_n \,|\, \sigma(\partial_1) = \partial_1, \ldots, \sigma(\partial_n) = \partial_n\},$$

as Theorem 3.8 demonstrates.

- (Theorem 3.8)
  1. $G_n = \mathrm{TAut}_K(P_n)_n \mathcal{F}_n = \mathcal{F}_n \mathrm{TAut}_K(P_n)_n$ *and* $\mathrm{TAut}_K(P_n)_n \cap \mathcal{F}_n = \mathrm{Sh}_{n-1}$.
  2. $G_n = \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} \mathcal{F}_n) = \mathbb{T}^n \ltimes (\mathcal{F}_n \times_{ex} \mathcal{T}_n)$.

As the groups $\mathbb{T}^n$ and $\mathcal{T}_n$ are explicit groups, the problem of finding the group G_n boils down to the problem of finding the group $\mathcal{F}_n$. This is done in Section 4.

- (Theorem 4.12) $\mathcal{F}_n = \mathrm{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n$, where

$$\begin{aligned}
\mathrm{Sh}_{n-2} &= \{\sigma \in \mathrm{Aut}_K(P_n)\,|\,\sigma(x_1)=x_1+\lambda_1,\ldots,\sigma(x_{n-2})=x_{n-2}+\lambda_{n-2},\\
&\quad \sigma(x_{n-1})=x_{n-1},\sigma(x_n)=x_n,\ \text{where } \lambda_i\in K\};\\
\mathbb{F}_n &= \{f\in 1+\partial_{n-1}K[[\partial_{n-1}]]\mid f(p_i\partial_i):=\begin{cases} p_i\partial_i & \text{if } i=1,\ldots,n-1,\\ f(p_n)\partial_n & \text{if } i=n,\end{cases}\\
&\quad \text{where } p_i\in P_{i-1}, i=1,\ldots,n\};\\
\mathbb{E}_n &= \begin{cases}\{e\} & \text{if } n=2,\\ \prod_{j=2}^{n-1}\mathbb{E}_{n,j} & \text{if } n\geq 3,\end{cases}\\
\mathbb{E}_{n,j} &= \{e'_j\in\partial_{j-1}K[[\partial_{j-1}]]\mid e'_j(p_i\partial_i):=\begin{cases} p_j\partial_j+e'_j(p_j)\partial_n & \text{if } i=j,\\ p_i\partial_i & \text{if } i\neq j.\end{cases}
\end{aligned}$$

As a corollary, the group G_n is presented as an exact product of its explicit subgroups.

- (Theorem 4.13) *Let* $\mathbb{I}=(1+t^2K[[t]],\cdot)$ *and* $\mathbb{J}=(tK[[t]],+)$. *Then for all* $n\geq 2$,
  1. $G_n=\mathbb{T}^n\ltimes(\mathcal{T}_n\times_{ex}(\mathrm{Sh}_{n-2}\times\mathbb{F}_n\times\mathbb{E}_n))=\mathrm{TAut}_K(P_n)_n\times_{ex}(\mathbb{F}'_n\times\mathbb{E}_n)$,
  2. $G_n\simeq\mathrm{TAut}_K(P_n)_n\times_{ex}(\mathbb{I}\times\mathbb{J}^{n-2})$.

In Section 5, the explicit form of the groups $\mathbb{T}^n$, $\mathrm{UAut}_K(P_n)_n$, $\mathbb{F}_n$ and $\mathbb{E}_n$ allows us to establish commutation relations between elements of these groups (Lemma 5.1 and Lemma 5.2). From which we deduce that the group $\mathrm{UAut}_K(P_n)_n$ is a *normal* subgroup of the group G_n. In combination with Theorem 4.13.(1), this fact yields the main result of the paper $G_n=\mathbb{T}^n\ltimes(\mathrm{UAut}_K(P_n)_n\rtimes(\mathbb{F}'_n\times\mathbb{E}_n))$ (Theorem 5.3.(1)), where $\mathbb{F}'_n=1+\partial^2_{n-1}K[[\partial_{n-1}]]\subset\mathbb{F}_n$.

At the end of Section 5, characterizations of the groups $\mathbb{F}_n$, $\mathbb{F}'_n$ and $\mathbb{E}_n$ are given in invariant terms (Proposition 5.6).

The canonical decomposition for an automorphism of the Lie algebra $\mathfrak{u}_n$.

By Theorem 4.13.(1), every automorphism $\sigma\in G_n=\mathbb{T}^n\ltimes(\mathcal{T}_n\times_{ex}(\mathrm{Sh}_{n-2}\times\mathbb{F}_n\times\mathbb{E}_n))$ is the unique product

$$\sigma=t\tau sfe' \ \text{ where } \ t\in\mathbb{T}^n,\ \tau\in\mathcal{T}_n,\ s\in\mathrm{Sh}_{n-2},\ f\in\mathbb{F}_n,\ e'\in\mathbb{E}_n.$$

This product is called the *canonical decomposition* of the automorphism $\sigma\in G_n$. In Section 6, for every automorphism $\sigma\in G_n$ explicit formulas are found (Theorem 6.1) for the automorphisms t, τ, s, f and e' via the elements $\{\sigma(s)\,|\,s\in S_n\}$ where the set $S_n:=\{\partial_1,x_1^j\partial_2,\ldots,x_{i-1}^j\partial_i,\ldots,x_{n-1}^j\partial_n\mid j\in\mathbb{N}\}$ is a set of generators for the Lie algebra $\mathfrak{u}_n$.

The adjoint group $\mathcal{A}(\mathfrak{u}_n)$ of the Lie algebra $\mathfrak{u}_n$. For a Lie algebra $\mathcal{G}$, the *adjoint group* $\mathcal{A}(\mathcal{G})$ is the subgroup of the group of automorphisms $\mathrm{Aut}_K(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ generated by the automorphisms $e^\delta:=\sum_{i\geq 0}\frac{\delta^i}{i!}$ where δ runs through the set of locally nilpotent inner derivations of the Lie algebra $\mathcal{G}$. All the inner derivations of the Lie algebra $\mathfrak{u}_n$ are locally nilpotent derivations [2]. In Section 7, we prove that the adjoint group $\mathcal{A}(\mathfrak{u}_n)$ of the Lie algebra $\mathfrak{u}_n$ is equal to the group $\mathrm{UAut}_K(P_n)_n$ (Theorem 7.1).

2 The Lie algebra $\mathfrak{u}_n$

In this section, for reader's convenience various results and properties of the Lie algebras $\mathfrak{u}_n$ are collected that are used in the rest of the paper. The details/proofs can be found in [2]. Since $\mathfrak{u}_n=\bigoplus_{i=1}^n\bigoplus_{\alpha\in\mathbb{N}^{i-1}}Kx^\alpha\partial_i$, the elements

$$X_{\alpha,i}:=x^\alpha\partial_i=x_1^{\alpha_1}\cdots x_{i-1}^{\alpha_{i-1}}\partial_i,\quad i=1,\ldots,n;\ \alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}^{i-1}, \tag{1}$$

form the K-basis $\mathcal{B}_n$ for the Lie algebra $\mathfrak{u}_n$. The basis $\mathcal{B}_n$ is called the *canonical basis* for $\mathfrak{u}_n$. For all $1 \leq i \leq j \leq n$, $\alpha \in \mathbb{N}^{i-1}$ and $\beta \in \mathbb{N}^{j-1}$,

$$[X_{\alpha,i}, X_{\beta,j}] = \begin{cases} 0 & \text{if } i = j, \\ \beta_i X_{\alpha+\beta-e_i,j} & \text{if } i < j, \end{cases} \tag{2}$$

where $e_1 := (1, 0, \ldots, 0), \ldots, e_n := (0, \ldots, 0, 1)$ is the canonical free $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $\mathbb{Z}^n$. The Lie algebra $\mathfrak{u}_n = \oplus_{i=1}^n P_{i-1}\partial_i$ is the direct sum of *abelian* (infinite dimensional when $i > 1$) Lie subalgebras $P_{i-1}\partial_i$ (i.e., $[P_{i-1}\partial_i, P_{i-1}\partial_i] = 0$) such that, for all $i < j$,

$$[P_{i-1}\partial_i, P_{j-1}\partial_j] = P_{j-1}\partial_j. \tag{3}$$

The Lie subalgebra $P_{i-1}\partial_i$ has the structure of the left P_{i-1}-module and ${}_{P_{i-1}}(P_{i-1}\partial_i) \simeq P_{i-1}$. By (3), the Lie algebra $\mathfrak{u}_n$ admits the finite strictly descending chain of ideals

$$\mathfrak{u}_{n,1} := \mathfrak{u}_n \supset \mathfrak{u}_{n,2} \supset \cdots \supset \mathfrak{u}_{n,i} \supset \cdots \supset \mathfrak{u}_{n,n} \supset \mathfrak{u}_{n,n+1} := 0 \tag{4}$$

where $\mathfrak{u}_{n,i} := \sum_{j=i}^n P_{j-1}\partial_j$ for $i = 1, \ldots, n$. By (3), for all $i < j$,

$$[\mathfrak{u}_{n,i}, \mathfrak{u}_{n,j}] \subseteq \begin{cases} \mathfrak{u}_{n,i+1} & \text{if } i = j, \\ \mathfrak{u}_{n,j} & \text{if } i < j. \end{cases} \tag{5}$$

For all $i = 1, \ldots, n$, there is the canonical isomorphism of Lie algebras

$$\mathfrak{u}_i \simeq \mathfrak{u}_n/\mathfrak{u}_{n,i+1}, \quad X_{\alpha,j} \mapsto X_{\alpha,j} + \mathfrak{u}_{n,i+1}. \tag{6}$$

In particular, $\mathfrak{u}_{n-1} \simeq \mathfrak{u}_n/P_{n-1}\partial_n$. The polynomial algebra P_n is an A_n-module: for all elements $p \in P_n$,

$$x_i * p = x_i p, \qquad \partial_i * p = \frac{\partial p}{\partial x_i}, \quad i = 1, \ldots, n.$$

Clearly, $P_n \simeq A_n/\sum_{i=1}^n A_n\partial_i$, $1 \mapsto 1 + \sum_{i=1}^n A_n\partial_i$. Since $\mathfrak{u}_n \subseteq A_n$, the polynomial algebra P_n is also a $\mathfrak{u}_n$-module.

Let V be a vector space over K. A K-linear map $\delta : V \to V$ is called a *locally nilpotent map* if $V = \cup_{i \geq 1} \ker(\delta^i)$ or, equivalently, for every $v \in V$, $\delta^i(v) = 0$ for all $i \gg 1$. When δ is a locally nilpotent map in V we also say that δ *acts locally nilpotently* on V. Every *nilpotent* linear map δ, that is $\delta^n = 0$ for some $n \geq 1$, is a locally nilpotent map but not vice versa, in general. Let $\mathcal{G}$ be a Lie algebra. Each element $a \in \mathcal{G}$ determines the derivation of the Lie algebra $\mathcal{G}$ by the rule $\mathrm{ad}(a) : \mathcal{G} \to \mathcal{G}$, $b \mapsto [a, b]$, which is called the *inner derivation* associated with a. The set $\mathrm{Inn}(\mathcal{G})$ of all the inner derivations of the Lie algebra $\mathcal{G}$ is a Lie subalgebra of the Lie algebra $(\mathrm{End}_K(\mathcal{G}), [\cdot, \cdot])$ where $[f, g] := fg - gf$. There is the short exact sequence of Lie algebras

$$0 \to Z(\mathcal{G}) \to \mathcal{G} \stackrel{\mathrm{ad}}{\to} \mathrm{Inn}(\mathcal{G}) \to 0,$$

that is $\mathrm{Inn}(\mathcal{G}) \simeq \mathcal{G}/Z(\mathcal{G})$ where $Z(\mathcal{G})$ is the *centre* of the Lie algebra $\mathcal{G}$ and $\mathrm{ad}([a, b]) = [\mathrm{ad}(a), \mathrm{ad}(b)]$ for all elements $a, b \in \mathcal{G}$. An element $a \in \mathcal{G}$ is called a *locally nilpotent element* (respectively, a *nilpotent element*) if so is the inner derivation $\mathrm{ad}(a)$ of the Lie algebra $\mathcal{G}$. Let J be a non-empty subset of $\mathcal{G}$ then $\mathrm{Cen}_{\mathcal{G}}(J) := \{a \in \mathcal{G} \mid [a, b] = 0 \text{ for all } b \in J\}$ is called the *centralizer* of J in $\mathcal{G}$. It is a Lie subalgebra of the Lie algebra $\mathcal{G}$.

Proposition 2.1 (Proposition 2.1, [2])

1. *The Lie algebra* $\mathfrak{u}_n$ *is a solvable but not nilpotent Lie algebra.*
2. *The finite chain of ideals (4) is the derived series for the Lie algebra* $\mathfrak{u}_n$, *that is* $(\mathfrak{u}_n)_{(i)} = \mathfrak{u}_{n,i+1}$ *for all* $i \geq 0$.

3. *The upper central series for the Lie algebra* $\mathfrak{u}_n$ *stabilizers at the first step, that is* $(\mathfrak{u}_n)^{(0)} = \mathfrak{u}_n$ *and* $(\mathfrak{u}_n)^{(i)} = \mathfrak{u}_{n,2}$ *for all* $i \geq 1$.

4. *Each element* $u \in \mathfrak{u}_n$ *acts locally nilpotently on the* $\mathfrak{u}_n$*-module* P_n.

5. *All the inner derivations of the Lie algebra* $\mathfrak{u}_n$ *are locally nilpotent derivations.*

6. *The centre* $Z(\mathfrak{u}_n)$ *of the Lie algebra* $\mathfrak{u}_n$ *is* $K\partial_n$.

7. *The Lie algebras* $\mathfrak{u}_n$ *where* $n \geq 2$ *are pairwise non-isomorphic.*

Proposition 2.1.(5) allows us to produces many automorphisms of the Lie algebra $\mathfrak{u}_n$. For every element $a \in \mathfrak{u}_n$, the inner derivation $\mathrm{ad}(a)$ is a locally nilpotent derivation, hence $e^{\mathrm{ad}(a)} := \sum_{i\geq 0} \frac{\mathrm{ad}(a)^i}{i!} \in G_n$. The adjoint group $\mathcal{A}(\mathfrak{u}_n) := \langle e^{\mathrm{ad}(a)} \,|\, a \in \mathfrak{u}_n \rangle$ coincides with the group $\mathrm{UAut}_K(P_n)_n$ (Theorem 7.1) which is a tiny part of the group G_n (Theorem 5.3).

The uniserial dimension. Let $(S, \leq)$ be a partially ordered set (a *poset*, for short), i.e., a set S admits a relation $\leq$ that satisfies three conditions: for all $a, b, c \in S$,

(i) $a \leq a$;

(ii) $a \leq b$ and $b \leq a$ imply $a = b$;

(iii) $a \leq b$ and $b \leq c$ imply $a \leq c$.

A poset $(S, \leq)$ is called an *Artinian* poset is every non-empty subset T of S has a *minimal element*, say $t \in T$, that is $t \leq t'$ for all $t' \in T$. A poset $(S, \leq)$ is a *well-ordered* if for all elements $a, b \in S$ either $a \leq b$ or $b \leq a$. A bijection $f : S \to S'$ between two posets $(S, \leq)$ and $(S', \leq)$ is an *isomorphism* if $a \leq b$ in S implies $f(a) \leq f(b)$ in S'. Recall that the *ordinal numbers* are the isomorphism classes of well-ordered Artinian sets. The ordinal number (the isomorphism class) of a well-ordered Artinian set $(S, \leq)$ is denoted by $\mathrm{ord}(S)$. The class of all ordinal numbers is denoted by W. The class W is well-ordered by 'inclusion' $\leq$ and Artinian. An associative addition '+' and an associative multiplication '$\cdot$' are defined in W that extend the addition and multiplication of the natural numbers. Every non-zero natural number n is identified with $\mathrm{ord}(1 < 2 < \cdots < n)$. Let $\omega := \mathrm{ord}(\mathbb{N}, \leq)$. More details on the ordinal numbers the reader can find in the book [5].

Definition, [2]. Let $(S, \leq)$ be a partially ordered set. The *uniserial dimension* $\mathrm{u.dim}(S)$ of S is the supremum of $\mathrm{ord}(\mathcal{I})$ where $\mathcal{I}$ runs through all the Artinian well-ordered subsets of S.

For a Lie algebra $\mathcal{G}$, let $\mathcal{J}_0(\mathcal{G})$ and $\mathcal{J}(\mathcal{G})$ be the sets of all and all non-zero ideals of the Lie algebra $\mathcal{G}$, respectively. So, $\mathcal{J}_0(\mathcal{G}) = \mathcal{J}(\mathcal{G}) \cup \{0\}$. The sets $\mathcal{J}_0(\mathcal{G})$ and $\mathcal{J}(\mathcal{G})$ are posets with respect to inclusion. A Lie algebra $\mathcal{G}$ is called *Artinian* (respectively, *Noetherian*) if the poset $\mathcal{J}(\mathcal{G})$ is Artinian (respectively, Noetherian). This means that every descending (respectively, ascending) chains of ideals stabilizers. A Lie algebra $\mathcal{G}$ is called a *uniserial* Lie algebra if the poset $\mathcal{J}(\mathcal{G})$ is a well-ordered set. This means that for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the Lie algebra $\mathcal{G}$ either $\mathfrak{a} \subseteq \mathfrak{b}$ or $\mathfrak{b} \subseteq \mathfrak{a}$.

Definition, [2]. Let $\mathcal{G}$ be an Artinian uniserial Lie algebra. The ordinal number $\mathrm{u.dim}(\mathcal{G}) := \mathrm{ord}(\mathcal{J}(\mathcal{G}))$ of the Artinian well-ordered set $\mathcal{J}(\mathcal{G})$ of nonzero ideals of $\mathcal{G}$ is called the *uniserial dimension* of the Lie algebra $\mathcal{G}$. For an arbitrary Lie algebra $\mathcal{G}$, the uniserial dimension $\mathrm{u.dim}(\mathcal{G})$ is the supremum of $\mathrm{ord}(\mathcal{I})$ where $\mathcal{I}$ runs through all the Artinian well-ordered sets of ideals.

If $\mathcal{G}$ is a Noetherian Lie algebra then $\mathrm{u.dim}(\mathcal{G}) \leq \omega$. So, the uniserial dimension is a measure of deviation from the Noetherian condition. The concept of the uniserial dimension makes sense for any algebras (associative, Jordan, etc.).

Let A be an algebra and M be its module, and let $\mathcal{J}_l(A)$ and $\mathcal{M}(M)$ be the sets of all the nonzero left ideals of A and of all the nonzero submodules of M, respectively. They are posets with respect to $\subseteq$. The *left uniserial dimension* of the algebra A is defined as $\mathrm{u.dim}(A) := \mathrm{u.dim}(\mathcal{J}_l(A))$ and the *uniserial dimension* of the A-module M is defined as $\mathrm{u.dim}(M) := \mathrm{u.dim}(\mathcal{M}(M))$, [2].

An Artinian well-ordering on the canonical basis $\mathcal{B}_n$ of $\mathfrak{u}_n$. Let us define an Artinian well-ordering $\leq$ on the canonical basis $\mathcal{B}_n$ for the Lie algebra $\mathfrak{u}_n$ by the rule: $X_{\alpha,i} > X_{\beta,j}$ iff $i < j$ or $i = j$ and $\alpha_{n-1} = \beta_{n-1}, \ldots, \alpha_{m+1} = \beta_{m+1}, \alpha_m > \beta_m$ for some m.

The next lemma is a straightforward consequence of the definition of the ordering $<$, we write $0 < X_{\alpha,i}$ for all $X_{\alpha,i} \in \mathcal{B}_n$.

Lemma 2.2 *If $X_{\alpha,i} > X_{\beta,j}$ then*

1. *$X_{\alpha+\gamma,i} > X_{\beta+\gamma,j}$ for all $\gamma \in \mathbb{N}^{i-1}$,*
2. *$X_{\alpha-\gamma,i} > X_{\beta-\gamma,j}$ for all $\gamma \in \mathbb{N}^{i-1}$ such that $\alpha - \gamma \in \mathbb{N}^{i-1}$ and $\beta - \gamma \in \mathbb{N}^{j-1}$,*
3. *$[\partial_k, X_{\alpha,i}] > [\partial_k, X_{\beta,j}]$ for all $k = 1, \ldots, i-1$ such that $\alpha_k \neq 0$, and*
4. *$[X_{\gamma,k}, X_{\alpha,i}] > [X_{\gamma,k}, X_{\beta,j}]$ for all $X_{\gamma,k} > X_{\alpha,i}$ such that $[X_{\gamma,k}, X_{\alpha,i}] \neq 0$, i.e., $\alpha_k \neq 0$.*

Let Ω_n be the set of indices $\{(\alpha, i)\}$ that parameterizes the canonical basis $\{X_{\alpha,i}\}$ of the Lie algebra $\mathfrak{u}_n$. The set $(\Omega_n, \leq)$ is an Artinian well-ordered set, where $(\alpha, i) \geq (\beta, j)$ iff $X_{\alpha,i} \geq X_{\beta,j}$, which is isomorphic to the Artinian well-ordered set $(\mathcal{B}_n, \leq)$ via $(\alpha, i) \mapsto X_{\alpha,i}$. We identify the posets $(\Omega_n, \leq)$ and $(\mathcal{B}_n, \leq)$ via this isomorphism. It is obvious that

$$\mathrm{ord}(\mathcal{B}_n) = \mathrm{ord}(\Omega_n) = \omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1, \tag{7}$$

$\Omega_2 \subset \Omega_3 \subset \cdots$ and $\mathcal{B}_2 \subset \mathcal{B}_3 \subset \cdots$. Let $[1, \mathrm{ord}(\Omega_n)] := \{\lambda \in \mathrm{W} \,|\, 1 \leq \lambda \leq \mathrm{ord}(\Omega_n)\}$. By (2), if $[X_{\alpha,i}, X_{\beta,j}] \neq 0$ then

$$[X_{\alpha,i}, X_{\beta,j}] < \min\{X_{\alpha,i}, X_{\beta,j}\}. \tag{8}$$

A classification of ideals of the Lie algebra $\mathfrak{u}_n$. By (8), the map

$$\rho_n : [1, \mathrm{ord}(\Omega_n)] \to \mathcal{J}(\mathfrak{u}_n), \;\; \lambda \mapsto I_\lambda := I_{\lambda,n} := \bigoplus_{(\alpha,i)\leq\lambda} KX_{\alpha,i}, \tag{9}$$

is a monomorphism of posets (ρ_n is an order-preserving injection).

Theorem 2.3 (Theorem 3.3, [2])

1. *The map (9) is a bijection.*
2. *The Lie algebra $\mathfrak{u}_n$ is a uniserial, Artinian but not Noetherian Lie algebra and its uniserial dimension is equal to* $\mathrm{u.dim}(\mathfrak{u}_n) = \mathrm{ord}(\Omega_n) = \omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1$.

An ideal $\mathfrak{a}$ of a Lie algebra $\mathcal{G}$ is called *proper* (respectively, *co-finite*) if $\mathfrak{a} \neq 0, \mathcal{G}$ (respectively, $\dim_K(\mathcal{G}/\mathfrak{a}) < \infty$). An ideal I of a Lie algebra $\mathcal{G}$ is called a *characteristic ideal* if it is invariant under all the automorphisms of the Lie algebra $\mathcal{G}$, that is $\sigma(I) = I$ for all $\sigma \in \mathrm{Aut}_K(\mathcal{G})$. It is obvious that an ideal I is a characteristic ideal iff $\sigma(I) \subseteq I$ for all $\sigma \in \mathrm{Aut}_K(\mathcal{G})$.

Corollary 2.4 (Corollary 3.7, [2]) *All the ideals of the Lie algebra $\mathfrak{u}_n$ are characteristic ideals.*

Each non-zero element u of $\mathfrak{u}_n$ is a finite linear combination

$$u = \lambda X_{\alpha,i} + \mu X_{\beta,j} + \cdots + \nu X_{\sigma,k} = \lambda X_{\alpha,i} + \cdots$$

where $\lambda, \mu, \ldots, \nu \in K^*$ and $X_{\alpha,i} > X_{\beta,j} > \cdots > X_{\sigma,k}$. The elements $\lambda X_{\alpha,i}$ and $\lambda \in K^*$ are called the *leading term* and the *leading coefficient* of u respectively, and the ordinal number $\mathrm{ord}(X_{\alpha,i}) = \mathrm{ord}(\alpha, i) \in [1, \mathrm{ord}(\Omega_n)]$, which is, by definition, the ordinal number that represents the Artinian well ordered set $\{(\beta, j) \in \Omega_n \,|\, (\beta, j) \leq (\alpha, i)\}$, is called the *ordinal degree* of u denoted by $\mathrm{ord}(u)$ (we hope that this notation will not lead to confusion). For all non-zero elements $u, v \in \mathfrak{u}_n$ and $\lambda \in K^*$,

(i) $\mathrm{ord}(u + v) \leq \max\{\mathrm{ord}(u), \mathrm{ord}(v)\}$ provided $u + v \neq 0$;
(ii) $\mathrm{ord}(\lambda u) = \mathrm{ord}(u)$;
(iii) $\mathrm{ord}([u, v]) < \min\{\mathrm{ord}(u), \mathrm{ord}(v)\}$ provided $[u, v] \neq 0$;

Corollary 2.5 (Corollary 3.8, [2]) *For all nonzero elements $u \in \mathfrak{u}_n$ and all automorphisms σ of the Lie algebra $\mathfrak{u}_n$,* $\mathrm{ord}(\sigma(u)) = \mathrm{ord}(u)$.

3 The structure of the group of automorphisms of the Lie algebra $\mathfrak{u}_n$

In this section, several important subgroups of the group G_n are introduced and studied. It is proved that the group G_n is an iterated semi-direct product and an exact product of some of them (Proposition 3.1, Theorem 3.8).

The groups $\mathbb{T}^n$ and $\mathcal{U}_n$. Let $G_n := \mathrm{Aut}_K(\mathfrak{u}_n)$ be the group of automorphisms of the Lie algebra $\mathfrak{u}_n$. The Lie algebra $\mathfrak{u}_n$ is uniserial, so

$$\sigma(I_\lambda) = I_\lambda \ \text{ for all } \ \lambda \in [1, \mathrm{ord}(\Omega_n)], \tag{10}$$

by Theorem 2.3. Moreover, by (9),

$$\sigma(X_{\alpha,i}) = \lambda_{\alpha,i} X_{\alpha,i} + \cdots \ \text{ for all } \ X_{\alpha,i} \in \mathcal{B}_n \tag{11}$$

for some scalar $\lambda_{\alpha,i} = \lambda_{\alpha,i}(\sigma) \in K^*$ where the three dots mean smaller terms, i.e., an element of $\sum_{(\beta,j)<(\alpha,i)} KX_{\beta,j}$. It follows that

$$\mathcal{U}_n := \{\sigma \in G_n \,|\, \sigma(X_{\alpha,i}) = X_{\alpha,i} + \cdots \ \text{ for all } \ X_{\alpha,i} \in \mathcal{B}_n\}$$

is a normal subgroup of the group G_n. The *algebraic n-dimensional torus* $\mathbb{T}^n$ is a subgroup of the group $\mathrm{Aut}_K(A_n)$ of automorphisms of the Weyl algebra A_n,

$$\mathbb{T}^n := \{t_\lambda : x_i \mapsto \lambda_i x_i, \ \partial_i \mapsto \lambda_i^{-1}\partial_i, 1 \leq i \leq n \,|\, \lambda = (\lambda_i) \in K^{*n}\} \simeq K^{*n},$$

that preserves the Lie algebra $\mathfrak{u}_n$. The group $\mathbb{T}^n$ can be seen as a subgroup of G_n,

$$\mathbb{T}^n := \{t_\lambda : X_{\alpha,i} \mapsto \lambda^\alpha \lambda_i^{-1} X_{\alpha,i} \text{ for all } X_{\alpha,i} \in \mathcal{B}_n \,|\, \lambda = (\lambda_i) \in K^{*n}\} \simeq K^{*n}$$

where $\lambda^\alpha := \prod \lambda_i^{\alpha_i}$.

Proposition 3.1 *1. The group $\mathcal{U}_n$ is a normal subgroup of G_n and $\mathcal{U}_n = \{\sigma \in G_n \,|\, \sigma(\partial_i) = \partial_i + \cdots$ for $i = 1, \ldots, n\}$.*

2. $G_n = \mathbb{T}^n \ltimes \mathcal{U}_n$ (the group G_n is the semidirect product of $\mathbb{T}_n$ and $\mathcal{U}_n$).

Proof. 1. We have already seen that $\mathcal{U}_n$ is a normal subgroup of G_n. It remains to show that the equality holds. Let R be the RHS of the equality. Then $\mathcal{U}_n \subseteq R$. It remains to show that $\mathcal{U}_n \supseteq R$, i.e., $\sigma \in R$ implies $\sigma \in \mathcal{U}_n$. We have to show that $\sigma(X_{\alpha,i}) = X_{\alpha,i} + \cdots$ for all $X_{\alpha,i} \in \mathcal{B}_n$. We use induction on $\lambda := \mathrm{ord}(X_{\alpha,i}) = \mathrm{ord}((\alpha, i)) \in [1, \mathrm{ord}(\Omega_n)]$. The initial case $\lambda = 1$ is obvious as $X_{\alpha,i} = \partial_n$ and $\sigma(\partial_n) = \partial_n$ since $\sigma \in R$.

Let $\lambda > 1$, and we assume that the result is true for all $\lambda' < \lambda$. If $X_{\alpha,i} = \partial_j$ for some j then it is nothing to prove. So, let $X_{\alpha,i} \in \mathcal{B}_n \backslash \{\partial_1, \ldots, \partial_n\}$, i.e., $\alpha \in \mathbb{N}^{i-1} \backslash \{0\}$. Let $j = \max\{k \,|\, \alpha_k \neq 0\}$. Then, by the very definition of the ordering $<$ on $\mathcal{B}_n$ (or use Lemma 2.2.(3)),

$$[\partial_j, X_{\alpha,i} + \cdots] = \alpha_j X_{\alpha-e_j,i} + \cdots \ \text{ and } \ [\oplus_{k=j+1}^i P_{k-1}\partial_k, X_{\alpha,i}] = 0.$$

Then applying the automorphism σ to the identity $[\partial_j, X_{\alpha,i}] = \alpha_j X_{\alpha-e_j,i}$ and using the fact that $\sigma(\partial_j) = \partial_j + u + v$ for some elements $u \in \oplus_{k=j+1}^i P_{k-1}\partial_k$ and $v \in \oplus_{k=i+1}^n P_{k-1}\partial_k$, we have

$$\begin{aligned} \alpha_j X_{\alpha-e_j,i} + \cdots &= \sigma(\alpha_j X_{\alpha-e_j,i}) = [\sigma(\partial_j), \sigma(X_{\alpha,i})] = [\partial_j + u + v, \lambda_{\alpha,i} X_{\alpha,i} + \cdots] \\ &= \lambda_{\alpha,i}\alpha_j X_{\alpha-e_j,i} + [v, \lambda_{\alpha,i} X_{\alpha,i}] + [\partial_j + u + v, \cdots] = \lambda_{\alpha,i}\alpha_j X_{\alpha-e_j,i} + \cdots. \end{aligned}$$

Hence, $\lambda_{\alpha,i} = 1$ since $\alpha_j \neq 0$.

2. By the very definition of the groups $\mathbb{T}^n$ and $\mathcal{U}_n$, $\mathbb{T}^n \cap \mathcal{U}_n = \{e\}$. As $\mathcal{U}_n$ is a normal subgroup of G_n it suffices to show that $G_n = \mathbb{T}^n \mathcal{U}_n$, i.e., every automorphism $\sigma \in G_n$ is a product $t\tau$ for some elements $t \in \mathbb{T}^n$ and $\tau \in \mathcal{U}_n$. By (11), $\sigma(\partial_i) = \lambda_i \partial_i + \cdots$ for all $i = 1, \ldots, n$ and for some

$\lambda = (\lambda_1, \dots, \lambda_n) \in K^{*n}$. Then $t_\lambda \sigma(\partial_i) = \partial_i + \cdots$ for all $i = 1, \dots, n$ where $t_\lambda \in \mathbb{T}^n$. By statement 1, $\tau := t_\lambda \sigma \in \mathcal{U}_n$, hence $\sigma = t_\lambda^{-1} \tau$, as required. $\square$

The group $\mathrm{TAut}_K(P_n)$ of triangular automorphisms of the polynomial algebra P_n. Let $\mathrm{Aut}_K(P_n)$ be the group of K-algebra automorphisms of the polynomial algebra P_n. Every automorphism $\sigma \in \mathrm{Aut}_K(P_n)$ is uniquely determined by the polynomials

$$x_1' := \sigma(x_1), \dots, x_n' := \sigma(x_n).$$

The inclusions of the polynomial algebras $P_1 \subset P_2 \subset \cdots$ yield the natural inclusions of their groups of automorphisms $\mathrm{Aut}_K(P_1) \subset \mathrm{Aut}_K(P_2) \subset \cdots$ where an automorphism $\sigma \in \mathrm{Aut}_K(P_n)$ is extended to the automorphism of the polynomial algebra P_{n+1} by the rule $\sigma(x_{n+1}) = x_{n+1}$.

The *group of triangular automorphisms* $\mathrm{TAut}_K(P_n)$ of the polynomial algebra P_n consists of all the automorphisms of P_n of the following type:

$$\sigma(x_i) = \lambda_i x_i + a_i, \;\; i = 1, \dots, n, \tag{12}$$

where $a_i \in P_{i-1}$ and $\lambda_i \in K^*$ for $i = 1, \dots, n$. The automorphism σ is uniquely determined by the elements a_i and λ_i, and we write $\sigma = [a_1, \dots, a_n; \lambda_1, \dots, \lambda_n]$. There are two distinct subgroups in $\mathrm{TAut}_K(P_n)$: the *algebraic n-dimensional torus* $\mathbb{T}^n$ (where $a_1 = \cdots = a_n = 0$) and the *group* $\mathrm{UAut}_K(P_n)$ *of triangular polynomial automorphisms* (where $\lambda_1 = \cdots = \lambda_n = 1$). Moreover, $\mathrm{UAut}_K(P_n)$ is a normal subgroup of $\mathrm{TAut}_K(P_n)$ and

$$\mathrm{TAut}_K(P_n) = \mathbb{T}^n \ltimes \mathrm{UAut}_K(P_n). \tag{13}$$

The group $\mathrm{UAut}_K(P_n)$ of triangular automorphisms of the polynomial algebra P_n. Every element $[a_1, \dots, a_n] := [a_1, \dots, a_n; 1, \dots, 1] \in \mathrm{UAut}_K(P_n)$ is uniquely determined by the polynomials $a_i \in P_{i-1}$, $i = 1, \dots, n$.

Proposition 3.2 *The exponential map $\mathfrak{u}_n \to \mathrm{UAut}_K(P_n)$, $\delta \mapsto e^\delta := \sum_{i\geq 0} \frac{\delta^i}{i!}$, is a bijection with the inverse map $\sigma \mapsto \ln(\sigma) := \ln(1-(1-\sigma)) := -\sum_{i\geq 1} \frac{(1-\sigma)^i}{i}$.*

Proof. By Proposition 2.1.(4), the exponential map is well-defined. Let us show that for every automorphism $\sigma = [a_1, \dots, a_n] \in \mathrm{UAut}_K(P_n)$ there is the unique derivation $\delta = \sum_{i=1}^n b_i \partial_i \in \mathfrak{u}_n$ (where $b_i \in P_{i-1}$ for $i = 1, \dots, n$) such that $\sigma = e^\delta$. Consider the system of equations where $\{b_i\}$ are unknown polynomials such that $\sigma(x_i) = e^\delta(x_i)$, $i = 1, \dots, n$, that is $x_i + a_i = x_i + (1-\partial)(b_i)$ where $\partial := -\sum_{i\geq 1} \frac{\delta^i}{(i+1)!}$ is a locally nilpotent map on P_n. Then

$$b_i = (1-\partial)^{-1}(a_i) = (\sum_{j\geq 0} \partial^j)(a_i) \in P_{i-1}.$$

For each $\sigma \in \mathrm{UAut}_K(P_n)$, the map $1 - \sigma : P_n \to P_n$ is a locally nilpotent map. So, the map $\ln(\sigma) = -\sum_{i\geq 1} \frac{(1-\sigma)^i}{i}$ makes sense. The rest is obvious. $\square$

For every element $[a_1, \dots, a_n] \in \mathrm{UAut}_K(P_n)$,

$$[a_1, \dots, a_n] = e^{a_n \partial_n} e^{a_{n-1}\partial_{n-1}} \cdots e^{a_1 \partial_1}. \tag{14}$$

For each natural number $i = 1, \dots, n$, the map $P_{i-1} \to e^{P_{i-1}\partial_i}$, $p_i \mapsto e^{p_i \partial_i}$, is an isomorphism of abelian groups. The group $\mathrm{UAut}_K(P_n)$ is an iterated semi-direct product of its subgroups $e^{P_{i-1}\partial_i}$, $i = 1, \dots, n$,

$$\mathrm{UAut}_K(P_n) = e^{P_{n-1}\partial_n} \rtimes e^{P_{n-2}\partial_{n-1}} \rtimes \cdots \rtimes e^{P_0 \partial_1}. \tag{15}$$

The set of all K-derivations $\mathrm{Der}_K(P_n) = \oplus_{i=1}^n P_n \partial_n$ of the polynomial algebra P_n is a Lie subalgebra of the Lie Weyl algebra $(A_n, [\cdot, \cdot])$, and $\mathfrak{u}_n$ is a Lie subalgebra of $\mathrm{Der}_K(P_n)$. Each automorphism $\sigma \in \mathrm{Aut}_K(P_n)$ induces an automorphism of the Lie algebra $\mathrm{Der}_K(P_n)$ (the change of variable) by the rule $\delta \mapsto \sigma\delta\sigma^{-1}$ where $\delta \in \mathrm{Der}_K(P_n)$. In particular, $\sigma(\frac{\partial}{\partial x_i})\sigma^{-1} = \frac{\partial}{\partial x_i'}$ where $x_i' := \sigma(x_i)$. Moreover,

$$\partial_i' := \frac{\partial}{\partial x_i'} = \sum_{j=1}^{n} \sigma(\frac{\partial \sigma^{-1}(x_j)}{\partial x_i})\partial_j. \qquad (16)$$

Let $\sigma \in \mathrm{TAut}_K(P_n)$ be as in (12). Then the automorphism $\sigma^{-1} \in \mathrm{TAut}_K(P_n)$ and so $\sigma^{-1} = (b_1, \ldots, b_n; \lambda_1^{-1}, \ldots, \lambda_n^{-1})$ for some $b_i \in P_{i-1}$, $i = 1, \ldots, n$. For the automorphism σ the equality (16) takes form

$$\partial_i' = \frac{\partial}{\partial x_i'} = \lambda_i^{-1}\partial_i + \sum_{j=i+1}^{n} \sigma(\frac{\partial \sigma^{-1}(b_j)}{\partial x_i})\partial_j \in K^*\partial_i + \sum_{j=i+1}^{n} P_{j-1}\partial_j. \qquad (17)$$

The groups Sh_n, $\mathcal{T}_n$ and $\mathrm{UAut}_K(P_n)_n$. Important subgroups of $\mathrm{TAut}_K(P_n)$ are the *shift group*

$$\mathrm{Sh}_n := \{\sigma : x_1 \mapsto x_1 + \mu_1, \ldots, x_n \mapsto x_n + \mu_n \,|\, \mu_i \in K, i = 1, \ldots, n\} \simeq K^n \qquad (18)$$

and the group

$$\mathcal{T}_n := \{\sigma = [0, a_2, \ldots, a_n; 1, \ldots, 1] \,|\, a_2(0) = a_3(0,0) = \cdots = a_n(0, \ldots, 0) = 0\}. \qquad (19)$$

So, an automorphism $\sigma = [a_1, a_2, \ldots, a_n; \lambda_1, \ldots, \lambda_n] \in \mathrm{TAut}_K(P_n)$ belongs to the group $\mathcal{T}_n$ iff $\lambda_1 = \cdots = \lambda_n = 1$ and all the constant terms of the polynomials a_i are equal to zero. Notice that $\mathcal{T}_n \subseteq \mathrm{UAut}_K(P_n) \subseteq \mathrm{TAut}_K(P_n)$. For each $i \geq 1$, let $\mathfrak{m}_i := \sum_{j=1}^{i} x_j P_i$, the maximal ideal of the polynomial algebra P_i generated by the elements $x_1, \ldots, x_i$. Set $\mathfrak{m}_0 := 0$. So, an automorphism $\sigma \in \mathrm{UAut}_K(P_n)$ belongs to the group $\mathcal{T}_n$ iff $\sigma(\mathfrak{m}_i) = \mathfrak{m}_i$ for all $i = 1, \ldots, n$. Let

$$\begin{aligned}
\mathrm{TAut}_K(P_n)_n &:= \{\sigma \in \mathrm{TAut}_K(P_n) \,|\, \sigma(x_n) = \lambda_n x_n + a_n \ \text{ where } \lambda_n \in K^*,\ a_n \in \mathfrak{m}_{n-1}\},\\
\mathrm{UAut}_K(P_n)_n &:= \{\sigma \in \mathrm{UAut}_K(P_n) \,|\, \sigma(x_n) = x_n + a_n \ \text{ where } a_n \in \mathfrak{m}_{n-1}\},\\
\mathrm{sh}_n &:= \mathrm{Sh}_n \cap \mathrm{Fix}_{\mathrm{TAut}_K(P_n)}(x_1, \ldots, x_{n-1}) = \{\sigma \in \mathrm{Sh}_n \,|\, \sigma(x_i) = x_i,\ i = 1, \ldots, n-1;\\
&\quad\ \sigma(x_n) = x_n + \lambda, \lambda \in K\} = e^{K\partial_n} := \{e^{\lambda\partial_n} \,|\, \lambda \in K\} \simeq (K, +).
\end{aligned}$$

The subsets $\mathrm{TAut}_K(P_n)_n$ and $\mathrm{UAut}_K(P_n)_n$ of the group $\mathrm{TAut}_K(P_n)$ are not subgroups, but sh_n is a subgroup. Clearly, $\mathrm{TAut}_K(P_n) = \mathrm{TAut}_K(P_n)_n\mathrm{sh}_n$, $\mathrm{TAut}_K(P_n)_n \cap \mathrm{sh}_n = \{e\}$ and the group sh_n is a normal subgroup of the group $\mathrm{TAut}_K(P_n)$. Therefore, the sets $\mathrm{TAut}_K(P_n)_n$ and $\mathrm{UAut}_K(P_n)_n$ can be identified with the factor groups $\mathrm{TAut}_K(P_n)/\mathrm{sh}_n$ and $\mathrm{UAut}_K(P_n)/\mathrm{sh}_n$, respectively, and as a result they have the group structure. Under these identifications we can write

$$\mathrm{TAut}_K(P_n)_n = \mathrm{TAut}_K(P_n)/\mathrm{sh}_n, \qquad (20)$$

$$\mathrm{UAut}_K(P_n)_n = \mathrm{UAut}_K(P_n)/\mathrm{sh}_n. \qquad (21)$$

Proposition 3.3 *1. $\sigma\mathfrak{u}_n\sigma^{-1} = \mathfrak{u}_n$ for all $\sigma \in \mathrm{TAut}_K(P_n)$.*

2. *The map $\omega : \mathrm{TAut}_K(P_n) \to G_n$, $\sigma \mapsto (\omega_\sigma : u \mapsto \sigma u \sigma^{-1})$, (where $u \in \mathfrak{u}_n$) is a group homomorphism with $\ker(\omega) = \mathrm{sh}_n$.*

3. *The map $\omega : \mathrm{TAut}_K(P_n)_n \to G_n$, $\sigma \mapsto \omega_\sigma$, is a group monomorphism.*

Proof. 1. Since $\mathrm{TAut}_K(P_n)$ is a group, to prove statement 1 it suffices to show that $\sigma\mathfrak{u}_n\sigma^{-1} \subseteq \mathfrak{u}_n$ for all elements $\sigma \in \mathrm{TAut}_K(P_n)$. Since $\mathfrak{u}_n = \sum_{i=1}^{n} P_{i-1}\partial_i$, it suffices to show that $\sigma P_{i-1}\partial_i\sigma^{-1} \subseteq \mathfrak{u}_n$ for all elements $\sigma \in \mathrm{TAut}_k(P_n)$ and $i = 1, \ldots, n$. This follows from (17) and the fact that $\sigma(P_{i-1}) = P_{i-1}$:

$$\sigma P_{i-1}\partial_i\sigma^{-1} = \sigma(P_{i-1})\sigma\partial_i\sigma^{-1} \subseteq P_{i-1}(K^*\partial_i + \sum_{j=i+1}^{n} P_{j-1}\partial_j) \subseteq \sum_{j=i}^{n} P_{j-1}\partial_j \subseteq \mathfrak{u}_n.$$

2. The map ω is a group homomorphism. By (16), $\mathrm{sh}_n \subseteq \ker(\omega)$. Let $\sigma \in \ker(\omega)$. It remains to show that $\sigma \in \mathrm{sh}_n$. By (17), $\partial_i' = \partial_i$ for all $i = 1, \dots, n$. Hence $\lambda_1 = \cdots = \lambda_n = 1$ and $\frac{\partial \sigma^{-1}(b_j)}{\partial x_i} = 0$ for all $i \leq j$. The second set of the conditions means that the elements $\sigma^{-1}(b_j) \in P_{j-1}$ are scalars. Summarizing, $\sigma^{-1}(x_i) = x_i + b_i$ where all $b_i \in K$. For all $i = 2, \dots, n$,

$$x_{i-1}\partial_i = \sigma^{-1}(x_{i-1}\partial_i)\sigma = \sigma^{-1}(x_i)\partial_i = x_{i-1}\partial_i + b_{i-1}\partial_i.$$

Hence, $b_1 = \cdots = b_{n-1} = 0$. Therefore, $\sigma \in \mathrm{sh}_n$, as required.

3. Statement 3 follows from statement 2 and (20). □

By Proposition 3.3.(3), we identify the group $\mathrm{TAut}_K(P_n)_n = \mathrm{TAut}_K(P_n)/\mathrm{sh}_n$ with its image in the group G_n, i.e., $\mathrm{TAut}_K(P_n)_n \subseteq G_n$. We identify the subgroup Sh_{n-1} of $\mathrm{Aut}_K(P_{n-1})$ with the following subgroup of $\mathrm{Aut}_K(P_n)$,

$$\mathrm{Sh}_{n-1} = \{\sigma \in \mathrm{Sh}_n \mid \sigma(x_n) = x_n\}. \tag{22}$$

$$\mathrm{Sh}_n = \mathrm{sh}_1 \times \cdots \times \mathrm{sh}_n = e^{K\partial_1} \times \cdots \times e^{K\partial_n} = e^{\sum_{i=1}^n K\partial_i}.$$

We say that a group G is the *exact product* of its two (or several) subgroups G_1 and G_2 and write $G = G_1 \times_{ex} G_2$ if every element $g \in G$ is a unique product $g = g_1 g_2$ where $g_1 \in G_1$ and $g_2 \in G_2$. Using the bijection $G \to G$, $g \mapsto g^{-1}$ and the fact that $(gh)^{-1} = h^{-1}g^{-1}$, we see that $G = G_1 \times_{ex} G_2$ iff $G = G_2 \times_{ex} G_1$. The next theorem describes the group G_n.

Proposition 3.4 *1.* $\mathrm{UAut}_K(P_n) = \mathcal{T}_n \times_{ex} \mathrm{Sh}_n$.

2. $\mathrm{TAut}_K(P_n) = \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} \mathrm{Sh}_n)$.

3. $\mathrm{TAut}_K(P_n)_n = \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} \mathrm{Sh}_{n-1}) = \mathbb{T}^n \ltimes \mathrm{UAut}_K(P_n)_n$.

4. $\mathrm{UAut}_K(P_n)_n = \mathcal{T}_n \times_{ex} \mathrm{Sh}_{n-1}$.

Proof. 1. It is obvious that $\mathrm{Sh}_n \cap \mathcal{T}_n = \{e\}$. To finish the proof of statement 1 it suffices to show that any automorphism $\sigma = [a_1, a_2, \dots, a_n; 1, \dots, 1] \in \mathrm{UAut}_K(P_n)$ is a product τs for some elements $\tau \in \mathcal{T}_n$ and $s \in \mathrm{Sh}_n$. Let $a_i = b_i + \mu_i$ where μ_i is the constant term of the polynomial a_i. Then $\tau = [0, b_2, \dots, b_n; 1, \dots, 1] \in \mathcal{T}_n$, $s = [\mu_1, \dots, \mu_n; 1, \dots, 1] \in \mathrm{Sh}_n$ and $\sigma = \tau s$.

2. Statement 2 follows from statement 1 and (13).

3. Notice that $\mathrm{Sh}_n = \mathrm{Sh}_{n-1} \times \mathrm{sh}_n$. Statement 3 follows from statement 1 and (20).

4. Statement 4 follows from statement 1 and (21). □

The following lemma gives a characterization of the shift group Sh_n via its action on the partial derivatives.

Lemma 3.5 $\mathrm{Fix}_{\mathrm{Aut}_K(P_n)}(\partial_1, \dots, \partial_n) = \mathrm{Sh}_n$ *and* $\mathrm{Fix}_{\mathrm{UAut}_K(P_n)_n}(\partial_1, \dots, \partial_n) = \mathrm{Sh}_{n-1}$.

Proof. For an automorphism $\sigma \in \mathrm{Aut}_K(P_n)$, let $\partial' := (\partial_1', \dots, \partial_n')^T$ where $\partial_i' := \sigma\partial_i\sigma^{-1}$ for $i = 1, \dots, n$, $\partial := (\partial_1, \dots, \partial_n)^T$ (where 'T' stands for the transposition) and $A = (A_{ij})$ be the $n \times n$ matrix where $A_{ij} := \sigma(\frac{\partial\sigma^{-1}(x_j)}{\partial x_i})$. The equalities (16) can be written in the matrix form as $\partial' = A\partial$. Then $\sigma \in \mathrm{Fix}_{\mathrm{Aut}_K(P_n)}(\partial_1, \dots, \partial_n)$ iff A is the identity matrix iff $\sigma \in \mathrm{Sh}_n$. The second equality follows from the first. □

The next theorem is a key result which is used in several proofs later.

Theorem 3.6 *Let* $\partial_1', \dots, \partial_n' \in \mathrm{Der}_K(P_n)$ *be commuting derivations such that* $\partial_i' := \mu_i\partial_i + \sum_{j=i+1}^n a_{ij}\partial_j$ *for* $i = 1, \dots, n$ *where* $\mu_i \in K^*$ *and* $a_{ij} \in P_{j-1}$. *Then*

1. *there exists an automorphism* $\sigma \in \mathrm{TAut}_K(P_n)$ *such that* $\partial_i' = \sigma\partial_i\sigma^{-1}$ *for* $i = 1, \dots, n$. *If* σ' *is another such an automorphism then* $\sigma' = \sigma s$ *for some* $\sigma \in \mathrm{Sh}_n$, *and vice versa.*

2. *There is the unique automorphism* $\sigma \in \mathbb{T}^n \ltimes \mathcal{T}_n$ *such that* $\partial_i' = \sigma\partial_i\sigma^{-1}$ *for* $i = 1, \ldots, n$. *The automorphism* σ *is defined as follows* $\sigma(x_i) = x_i'$ *for* $i = 1, \ldots, n$ *where the elements* x_i' *are defined recursively as follows:*

$$x_1' := \mu_1^{-1}x_1, \ \ x_i' := \phi_{i-1}\phi_{i-2}\cdots\phi_1(\mu_i^{-1}x_i), \ \ i = 2, \ldots, n, \tag{23}$$

$$\phi_i := \sum_{k\geq 0}(-1)^k\frac{x_i'^k}{k!}\partial_i'^k, \ \ i = 1, \ldots, n-1. \tag{24}$$

Proof. 1. The first part of statement 1 (concerning the existence of σ) follows from statement 2. If $\partial_i' = \sigma\partial_i\sigma^{-1} = \sigma'\partial_i\sigma'^{-1}$ for $i = 1, \ldots, n$ then $s := \sigma^{-1}\sigma' \in \mathrm{Sh}_n$, by Lemma 3.5. Hence, $\sigma' = \sigma s$, and vice versa (trivial).

2. We deduce statement 2 from two claims below. The uniqueness of σ follows from the second part of statement 1 (which has already been proved above) and the fact that $\mathbb{T}^n \ltimes \mathcal{T}_n \cap \mathrm{Sh}_n = \{e\}$.

Claim 1: $\partial_1', \ldots, \partial_n'$ *are commuting, locally nilpotent derivations of the polynomial algebra* P_n *such that* $\partial_i'(x_j') = \delta_{ij}$ *(the Kronecker delta) and* $\cap_{i=1}^n \mathrm{ker}_{P_n}(\partial_i') = K$.

It follows from Claim 1 and (Theorem 2.2, [1]) that the K-algebra homomorphism

$$\sigma : P_n \to P_n, \ \ x_i \mapsto x_i', \ \ i = 1, \ldots, n,$$

is an automorphism. Then (which is obvious) $\partial_i' = \sigma\partial_i\sigma^{-1}$ for $i = 1, \ldots, n$. We finish the proof of statement 2, by Claim 2.

Claim 2: $\sigma \in \mathbb{T}^n \ltimes \mathcal{T}_n$.

So, it remains to prove Claims 1 and 2. By the very definition of the derivations ∂_i', the following statements are obvious:

(i) $\partial_1', \ldots, \partial_n' \in \mathfrak{u}_n$;

(ii) $\partial_i'(P_j) \subseteq P_j$ for all $i, j = 1, \ldots, n$. Moreover, $\partial_i'(P_j) = 0$ for $i > j$;

(iii) $\partial_i'(\mu_i^{-1}x_i) = 1$ for $i = 1, \ldots, n$;

(iv) $\partial_1', \ldots, \partial_n'$ are locally nilpotent derivations of the polynomial algebra P_n;

(v) $\mathrm{Der}_K(P_n) = \oplus_{i=1}^n P_n\partial_i'$; and

(vi) $\cap_{i=1}^n \mathrm{ker}_{P_n}(\partial_i') = K$, by (v).

In view of (iv) and (vi), to finish the proof of Claim 1, we have to show that $\partial_i'(x_j') = \delta_{ij}$. To prove Claim 2, it suffices to show that $x_j' = \mu_j^{-1}x_j + a_j$ where $a_j \in \mathfrak{m}_{j-1} := \sum_{k=1}^{j-1} x_kP_{j-1}$ for $j = 1, \ldots, n$. To prove both statements we use induction on j. The initial case $j = 1$ follows from (ii) and (iii) and the fact that $x_1' = \mu_1^{-1}x_1$. Suppose that $j \geq 2$ and the result holds for all $j' < j$. By induction, $x_s' = \mu_s^{-1}x_s + a_s \in K^*x_s + \mathfrak{m}_{s-1} \subseteq \mathfrak{m}_s$ for $s = 1, \ldots, j-1$. Then, using repeatedly (ii) and (24), we see that $x_j' = \mu_j^{-1}x_j + a_j$ for some $a_j \in \mathfrak{m}_{j-1}$. Then $\partial_{j+1}'(x_j') = \cdots = \partial_n'(x_j') = 0$, by (ii). By (Theorem 2.2, [1]), $\partial_1'(x_j') = \cdots = \partial_{j-1}'(x_j') = 0$. For each $i = 1, \ldots, j-1$, $\partial_j'\phi_i = \phi_i\partial_j'$ since $\partial_j'(x_i') = 0$ for all $i = 1, \ldots, j-1$ (the last set of equalities follows from the set of equalities $\partial_j'(x_i) = 0$ for all $i = 1, \ldots, j-1$, and the fact that $x_s' = \mu_s^{-1}x_s + a_s$ for $s = 1, \ldots, j-1$). Therefore (by (iii)),

$$\partial_j'(x_j') = \partial_j'\phi_{j-1}\cdots\phi_1(\mu_j^{-1}x_j) = \phi_{j-1}\cdots\phi_1\partial_j'(\mu_j^{-1}x_j) = \phi_{j-1}\cdots\phi_1(1) = 1.$$

The proof of the inductive step is complete. $\square$

The next theorem states that the triangular polynomial automorphisms are *precisely* the polynomial automorphisms that respect the Lie algebra $\mathfrak{u}_n$.

Theorem 3.7 *Let* $\sigma \in \mathrm{Aut}_K(P_n)$. *The following statements are equivalent.*

1. $\sigma \in \mathrm{TAut}_K(P_n)$.

2. $\sigma\mathfrak{u}_n\sigma^{-1} = \mathfrak{u}_n$

3. *For all* $i = 1, \ldots, n$, $\sigma\partial_i\sigma^{-1} = \mu_i\partial_i + \sum_{j=i+1}^n a_{ij}\partial_j$ *where* $\mu_i \in K^*$ *and* $a_{ij} \in P_{j-1}$.

Proof. 1. ($1 \Rightarrow 2$) Proposition 3.3.(1).

($2 \Rightarrow 3$) Suppose that $\sigma \mathfrak{u}_n \sigma^{-1} = \mathfrak{u}_n$. Then $\sigma^{-1}\mathfrak{u}_n\sigma = \mathfrak{u}_n$. The map $\omega_\sigma : \mathfrak{u}_n \to \mathfrak{u}_n$, $\delta \mapsto \sigma\delta\sigma^{-1}$, is a Lie algebra automorphism with $\omega_{\sigma^{-1}}$ as its inverse. Statement 3 follows from (10) and the definition of the ordering $<$ on $\mathfrak{u}_n$.

($3 \Rightarrow 1$) By Theorem 3.6, there is an automorphism $\tau \in \mathrm{TAut}_K(P_n)$ such that $\sigma\partial_i\sigma^{-1} = \tau\partial_i\tau^{-1}$ for all $i = 1, \dots, n$. Hence $\tau^{-1}\sigma \in \mathrm{Fix}_{\mathrm{Aut}_K(P_n)}(\partial_1, \dots, \partial_n) = \mathrm{Sh}_n \subseteq \mathrm{TAut}_K(P_n)$ (Lemma 3.5). Then $\sigma \in \mathrm{TAut}_K(P_n)$. $\square$

The group G_n is an exact product of its three subgroups. The group $\mathcal{U}_n$ contains the subgroup

$$\mathcal{F}_n := \mathrm{Fix}_{G_n}(\partial_1, \dots, \partial_n) := \{\sigma \in G_n \,|\, \sigma(\partial_1) = \partial_1, \dots, \sigma(\partial_n) = \partial_n\}. \tag{25}$$

This is the most important subgroup of G_n as Theorem 3.8.(2) shows. The next theorem represents the group G_n as an exact product of its three subgroups.

Theorem 3.8 *1. $G_n = \mathrm{TAut}_K(P_n)_n\mathcal{F}_n = \mathcal{F}_n\mathrm{TAut}_K(P_n)_n$ and $\mathrm{TAut}_K(P_n)_n \cap \mathcal{F}_n = \mathrm{Sh}_{n-1}$.*

2. $G_n = \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} \mathcal{F}_n) = \mathbb{T}^n \ltimes (\mathcal{F}_n \times_{ex} \mathcal{T}_n)$.

3. $\mathcal{U}_n = \mathcal{T}_n \times_{ex} \mathcal{F}_n$.

Proof. 2. Let $g \in G_n$. The elements $\partial_1, \dots, \partial_n$ of the Lie algebra $\mathfrak{u}_n$ commute then so do the elements $\partial_1' := g(\partial_1), \dots, \partial_n' := g(\partial_n)$ of $\mathfrak{u}_n$. By (11), the elements $\partial_1', \dots, \partial_n'$ satisfy the assumptions of Theorem 3.6, hence there is the *unique* automorphism $\sigma \in \mathbb{T}^n \ltimes \mathcal{T}_n$ such that $\partial_i' = \sigma\partial_i\sigma^{-1}$ for $i = 1, \dots, n$. Recall that we identified the group $\mathbb{T}^n \ltimes \mathcal{T}_n$ with its image in the group G_n (Proposition 3.3.(3)), i.e., the automorphism σ is identified with the automorphism $\omega_\sigma : \mathfrak{u}_n \to \mathfrak{u}_n$, $u \mapsto \sigma u \sigma^{-1}$. Then $\omega_{\sigma^{-1}}g(\partial_i) = \partial_i$ for all $i = 1, \dots, n$. Hence, $\omega_{\sigma^{-1}}g \in \mathcal{F}_n$, and statement 2 follows.

1. Proposition 3.3.(3), the inclusion $\mathbb{T}^n \ltimes \mathcal{T}_n \subseteq \mathrm{TAut}_K(P_n)_n$ (Proposition 3.4.(3)) and statement 2 imply the first two equalities in statement 1. Now, using Lemma 3.5,

$$\begin{aligned}\mathrm{TAut}_K(P_n)_n \cap \mathcal{F}_n &= \mathrm{Fix}_{\mathrm{TAut}_K(P_n)_n}(\partial_1, \dots, \partial_n) = \mathrm{TAut}_K(P_n)_n \cap \mathrm{Fix}_{\mathrm{Aut}_K(P_n)}(\partial_1, \dots, \partial_n) \\ &= \mathrm{TAut}_K(P_n)_n \cap \mathrm{Sh}_n = \mathrm{Sh}_{n-1}.\end{aligned}$$

3. Since $\mathcal{T}_n \times_{ex} \mathcal{F}_n \subseteq \mathcal{U}_n$ and $\mathcal{U}_n \cap \mathbb{T}^n = \{e\}$, statement 3 follows from statement 2:

$$\mathcal{U}_n = \mathcal{U}_n \cap G_n = \mathcal{U}_n \cap (\mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} \mathcal{F}_n)) = (\mathcal{U}_n \cap \mathbb{T}^n) \ltimes (\mathcal{T}_n \times_{ex} \mathcal{F}_n) = \mathcal{T}_n \times_{ex} \mathcal{F}_n. \quad \square$$

4 The group of automorphisms of the Lie algebra $\mathfrak{u}_n$

The aim of this section is to find the groups $\mathcal{F}_n$ (Theorem 4.12.(1)) and G_n (Theorem 4.13).

The $\mathfrak{u}_n$-module P_n. For each $n \geq 2$, $\mathfrak{u}_n$ is a Lie subalgebra of the Lie algebra $\mathfrak{u}_{n+1} = \mathfrak{u}_n \oplus P_n\partial_{n+1}$, $P_n\partial_{n+1}$ is an ideal of the Lie algebra $\mathfrak{u}_{n+1}$ and $[P_n\partial_{n+1}, P_n\partial_{n+1}] = 0$. In particular, $P_n\partial_{n+1}$ is a left $\mathfrak{u}_n$-module where the action of the Lie algebra $\mathfrak{u}_n$ on $P_n\partial_{n+1}$ is given by the rule (the adjoint action): $uv := [u, v]$ for all $u \in \mathfrak{u}_n$ and $v \in P_n\partial_{n+1}$. The polynomial algebra P_n is a left $\mathfrak{u}_n$-module.

Lemma 4.1 (Lemma 3.10, [2])

1. *The K-linear map $P_n \to P_n\partial_{n+1}$, $p \mapsto p\partial_{n+1}$, is a $\mathfrak{u}_n$-module isomorphism.*
2. *The $\mathfrak{u}_n$-module P_n is an indecomposable, uniserial $\mathfrak{u}_n$-module,* $\mathrm{u.dim}(P_n) = \omega^n$ *and* $\mathrm{ann}_{\mathfrak{u}_n}(P_n) = 0$.

The next proposition describes the algebra of all the $\mathfrak{u}_n$-homomorphisms (and its group of units) of the $\mathfrak{u}_n$-module P_n. The K-derivation $\frac{d}{dx_n}$ of the polynomial algebra $K[x_n]$ is also denoted by ∂_n.

Proposition 4.2 (Proposition 3.16, [2])

1. *The map* $\mathrm{End}_{\mathfrak{u}_n}(P_n) \to \mathrm{End}_{K[\partial_n]}(K[x_n]) = K[[\frac{d}{dx_n}]]$, $\varphi \mapsto \varphi|_{K[x_n]}$, *is a* K*-algebra isomorphism with the inverse map* $\varphi' \mapsto \varphi$ *where* $\varphi(x^\beta x_n^i) = X_{\beta,n}\varphi'(\frac{x_n^{i+1}}{i+1})$ *for all* $\beta \in \mathbb{N}^{n-1}$ *and* $i \in \mathbb{N}$.

2. *The map* $\mathrm{Aut}_{\mathfrak{u}_n}(P_n) \to \mathrm{Aut}_{K[\partial_n]}(K[x_n]) = K[[\frac{d}{dx_n}]]^*$, $\varphi \mapsto \varphi|_{K[x_n]}$, *is a group isomorphism with the inverse map as in the statement 1 (where* $K[[\frac{d}{dx_n}]]^*$ *is the group of units of the algebra* $K[[\frac{d}{dx_n}]]$*).*

The partial derivatives $\partial_1, \ldots, \partial_n$ are commuting locally nilpotent derivations of the polynomial algebra P_n. So, we can consider the skew power series algebra $P_n[[\partial_1, \ldots, \partial_n; \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]]$ which is also written as $P_n[[\partial_1, \ldots, \partial_n]]$, for short. Every element s of this algebra is a unique (formal) series $\sum_{\alpha\in\mathbb{N}^n} p_\alpha \partial^\alpha$ where $p_\alpha \in P_n$. The addition of two power series is defined in the obvious way $\sum_{\alpha\in\mathbb{N}^n} p_\alpha\partial^\alpha + \sum_{\alpha\in\mathbb{N}^n} q_\alpha\partial^\alpha = \sum_{\alpha\in\mathbb{N}^n}(p_\alpha + q_\alpha)\partial^\alpha$, and the multiplication satisfies the relations: $\partial_i p = p\partial_i + \partial_i(p)$ for all $i = 1, \ldots, n$ and $p \in P_n$. As the partial derivatives act locally nilpotently on the polynomial algebra P_n, the product of two power series can be written in the canonical form using the relations above, i.e., $(\sum_{\alpha\in\mathbb{N}^n} p_\alpha\partial^\alpha)(\sum_{\beta\in\mathbb{N}^n} q_\beta\partial^\beta) = \sum_{\gamma\in\mathbb{N}^n} r_\gamma\partial^\gamma$ for some polynomials $r_\gamma \in P_n$. For every series $s = \sum_{\alpha\in\mathbb{N}^n} p_\alpha\partial^\alpha \in P_n[[\partial_1, \ldots, \partial_n]]$ (where $p_\alpha \in P_n$), the action $s * p = \sum_{\alpha\in\mathbb{N}^n} p_\alpha\partial^\alpha * p$ is well-defined (the infinite sum is in fact a finite one). The algebra $P_n[[\partial_1, \ldots, \partial_n]]$ is the completion of the Weyl algebra $A_n = P_n[\partial_1, \ldots, \partial_n] = \oplus_{\alpha\in\mathbb{N}^n} P_n\partial^\alpha$. The polynomial algebra P_n is a left $P_n[[\partial_1, \ldots, \partial_n]]$-module. It is easy to show that the algebra homomorphism $P_n[[\partial_1, \ldots, \partial_n]] \to \mathrm{End}_K(P_n)$ is a monomorphism, and we identify the algebra $P_n[[\partial_1, \ldots, \partial_n]]$ with its image in $\mathrm{End}_K(P_n)$. The polynomial algebra $P_n = \cup_{i\geq 0} P_{n,\leq i}$ is a filtered algebra by the total degree of the variables where $P_{n,\leq i} := \sum\{Kx^\alpha \,|\, |\alpha| := \alpha_1 + \cdots + \alpha_n \leq i\}$ ($P_{n,\leq i}P_{n,\leq j} \subseteq P_{n,\leq i+j}$ for all $i, j \geq 0$). The vector space

$$\mathrm{End}_{\deg}(P_n) := \{f \in \mathrm{End}_K(P_n) \,|\, f(P_{n,\leq i}) \subseteq P_{n,\leq i}\}$$

is a subalgebra of $\mathrm{End}_K(P_n)$. Let $\mathrm{Aut}_{K[\partial_1,\ldots,\partial_n]}(P_n)$ be the group of invertible $K[\partial_1, \ldots, \partial_n]$-endomorphisms of the $K[\partial_1, \ldots, \partial_n]$-module P_n and $K[[\partial_1, \ldots, \partial_n]]^*$ be the group of units of the algebra $K[[\partial_1, \ldots, \partial_n]]$.

Proposition 4.3 *1.* $\mathrm{End}_K(P_n) = P_n[[\partial_1, \ldots, \partial_n]]$.

2. $\mathrm{End}_{\deg}(P_n) = P_n[[\partial_1, \ldots, \partial_n]]_{\deg} := \{\sum_{\alpha\in\mathbb{N}^n} p_\alpha\partial^\alpha \mid \deg(p_\alpha) \leq |\alpha|\}$.

3. $\mathrm{End}_{K[\partial_1,\ldots,\partial_n]}(P_n) = K[[\partial_1, \ldots, \partial_n]]$.

4. $\mathrm{Aut}_{K[\partial_1,\ldots,\partial_n]}(P_n) = K[[\partial_1, \ldots, \partial_n]]^*$.

Proof. 1. This is well-known and easy to prove.

2. Let R be the RHS of the equality in statement 2. The inclusion $\mathrm{End}_{\deg}(P_n) \supseteq R$ is obvious. We have to show that the reverse inclusion holds. Let $s = \sum p_\alpha\partial^\alpha \in \mathrm{End}_{\deg}(P_n)$. We have to prove that $\deg(p_\alpha) \leq |\alpha|$ for all α. We use induction on $d = |\alpha|$. The initial case $d = 0$ is obvious as $p_0 = s * 1 \in K$. Let $d > 0$ and suppose that the statement holds for all $d' < d$. Fix $\alpha \in \mathbb{N}^n$ such that $|\alpha| = d$. Then

$$P_{n,\leq d} \ni s * x^\alpha = \alpha! p_\alpha + \sum_{|\beta|<d} p_\beta\partial^\beta * x^\alpha,$$

and so $p_\alpha \in P_{n,\leq d}$ (since $\sum_{|\beta|<d} p_\beta\partial^\beta * x^\alpha \in P_{n,\leq d}$), as required.

3. An element $s = \sum_{\alpha\in\mathbb{N}^n} p_\alpha\partial^\alpha \in \mathrm{End}_K(P_n)$ belongs to $\mathrm{End}_{K[\partial_1,\ldots,\partial_n]}(P_n)$ iff $[\partial_i, s] = 0$ for all $i = 1, \ldots, n$ iff every $p_\alpha \in \cap_{i=1}^n \ker_{P_n}(\partial_i) = K$ for all $i = 1, \ldots, n$ (since $[\partial_i, s] = \sum \frac{\partial p_\alpha}{\partial x_i}\partial^\alpha$) iff $s \in K[[\partial_1, \ldots, \partial_n]]$.

4. Statement 4 follows from statement 3. $\square$

By Theorem 2.2, [1], the K-algebra endomorphisms $\phi_i := \sum_{k\geq 0}(-1)^k \frac{x_i^k}{k!}\partial_i^k : P_n \to P_n$ (where $i = 1, \dots, n$) commute and their composition

$$\phi := \prod_{i=1}^{n} \phi_i : P_n \to P_n, \quad \sum_{\alpha\in\mathbb{N}^n} \lambda_\alpha x^\alpha \mapsto \lambda_0, \tag{26}$$

is the projection onto the field K in the decomposition $P_n = \oplus_{\alpha\in\mathbb{N}^n} Kx^\alpha$. The next proposition is an easy corollary of this fact.

Proposition 4.4 *Let $s = \sum_{\alpha\in\mathbb{N}^n} \lambda_\alpha \partial^\alpha \in K[[\partial_1, \dots, \partial_n]] = \mathrm{End}_{K[\partial_1,\dots,\partial_n]}(P_n)$ where $\lambda_\alpha \in K$. Then $\lambda_\alpha = \phi s(\frac{x^\alpha}{\alpha!})$ for all $\alpha \in \mathbb{N}^n$ where $\alpha! := \prod_{i=1}^n \alpha_i!$.*

Proof. $\phi s(\frac{x^\alpha}{\alpha!}) = \phi(\lambda_\alpha + \cdots) = \lambda_\alpha$ where the three dots denote an element of the vector space $\oplus_{\alpha\in\mathbb{N}^n\backslash\{0\}} Kx^\alpha$. $\Box$

Lemma 4.5 *Let $\sigma \in \mathcal{F}_n$.*

1. *Then $\sigma|_{P_{n-1}\partial_n} : P_{n-1}\partial_n \to P_{n-1}\partial_n$ belongs to $I = 1 + \sum_{i=1}^{n-1} \delta_i K[[\delta_1, \dots, \delta_{n-1}]]$ where $\delta_i := \mathrm{ad}(\partial_i)$ for $i = 1, \dots, n-1$.*
2. *For all $\alpha \in \mathbb{N}^{n-1}\backslash\{0\}$, $\sigma(x^\alpha\partial_n) = p_\alpha\partial_n$ for some polynomial $p_\alpha \in P_{n-1}$ such that $p_\alpha = x^\alpha + \sum_{\beta\in\mathbb{N}^{n-1}, \beta\prec\alpha} \lambda_\beta x^\beta$ where $\beta = (\beta_i) \prec \alpha = (\alpha_i)$ iff $\beta_i \leq \alpha_i$ for all $i = 1, \dots, n-1$ and $\beta_j < \alpha_j$ for some j.*

Proof. 1. Recall that all ideals of the Lie algebra $\mathfrak{u}_n$ are characteristic ideals, and the vector space $P_{n-1}\partial_n$ is an ideal of the Lie algebra $\mathfrak{u}_n$. Therefore, the restriction map $\tau := \sigma|_{P_{n-1}\partial_n}$ is a well-defined map. Since $\sigma \in \mathcal{F}_n$, the map τ commute with the inner derivations δ_i where $i = 1, \dots, n-1$. The $\mathfrak{u}_{n-1}$-modules P_{n-1} and $P_{n-1}\partial_n$ are isomorphic (Corollary 4.1.(1)), hence $\tau \in \mathrm{Aut}_{K[\delta_1,\dots,\delta_{n-1}]}(P_{n-1}\partial_n) \simeq \mathrm{Aut}_{K[\partial_1,\dots,\partial_{n-1}]}(P_{n-1}) \simeq K[[\partial_1, \dots, \partial_{n-1}]]^*$. Then,

$$\tau \in \mathrm{Aut}_{K[\delta_1,\dots,\delta_{n-1}]}(P_{n-1}\partial_n) \simeq K[[\delta_1, \dots, \delta_{n-1}]]^*.$$

Since $\tau(\partial_n) = \sigma(\partial_n) = \partial_n$, we must have $\tau \in I$.

2. Statement 2 follows from statement 1. $\Box$

The subgroup $\mathbb{F}_n$ of $\mathcal{F}_n$. Let $\mathfrak{u}_1 := K\partial_1$, the abelian 1-dimensional Lie algebra. For $n \geq 2$, the set

$$\mathbb{F}_n := \mathrm{Fix}_{\mathcal{F}_n}(\mathfrak{u}_{n-1}) = \{\sigma \in \mathcal{F}_n \,|\, \sigma(u) = u \ \text{ for all } \ u \in \mathfrak{u}_{n-1}\} \tag{27}$$

is a subgroup of $\mathcal{F}_n$. Notice that $\mathbb{F}_2 = \mathcal{F}_2$. Recall that every ideal of the Lie algebra $\mathfrak{u}_n$ is a characteristic ideal (Corollary 2.4), $P_{n-1}\partial_n$ is an ideal of the Lie algebra $\mathfrak{u}_n = \mathfrak{u}_{n-1} \oplus P_{n-1}\partial_n$ such that $\mathfrak{u}_n/P_{n-1}\partial_n \simeq \mathfrak{u}_{n-1}$. In view of this Lie algebra isomorphism, for each natural number $n \geq 3$, there is the group homomorphism

$$\chi_n : \mathcal{F}_n \to \mathcal{F}_{n-1}, \quad \sigma \mapsto (u + P_{n-1}\partial_n \mapsto \sigma(u) + P_{n-1}\partial_n). \tag{28}$$

It is obvious that $\mathbb{F}_n \subseteq \ker(\chi_n)$. The ideal $P_{n-1}\partial_n$ of the Lie algebra $\mathfrak{u}_n$ is a (left) $\mathfrak{u}_n$-module and a $\mathfrak{u}_{n-1}$-module since $\mathfrak{u}_{n-1} \subseteq \mathfrak{u}_n$. Let $\mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}\partial_n)$ be the group of automorphisms of the $\mathfrak{u}_{n-1}$-module $P_{n-1}\partial_n$. The set $1 + \partial_{n-1}K[[\partial_{n-1}]]$ is a subgroup of the group $K[[\partial_{n-1}]]^*$ of units of the power series algebra $K[[\partial_{n-1}]]$.

The following proposition is an explicit description of the group $\mathbb{F}_n$.

Proposition 4.6 *For $n \geq 2$, the map $\eta_n : 1 + \partial_{n-1}K[[\partial_{n-1}]] \to \mathbb{F}_n$, $s = \sum \lambda_i \partial_{n-1}^i \mapsto \eta_n(s)$, is a group isomorphism where, for $p \in P_{n-1}$, $\eta_n(s) : p\partial_n \mapsto (\sum \lambda_i \frac{\partial^i p}{\partial x_{n-1}^i})\partial_n$. So, $\eta_n(s)$ acts on the elements of the algebra $\mathfrak{u}_n$ as $\sum \lambda_i \mathrm{ad}(\partial_{n-1})^i$. In particular, the group $\mathbb{F}_n = 1 + \delta_{n-1}K[[\delta_{n-1}]]$ is abelian where $\delta_{n-1} = \mathrm{ad}(\partial_{n-1})$ (equally, we can write $\mathbb{F}_n = 1 + \partial_{n-1}K[[\partial_{n-1}]]$). Moreover, $\eta_n = (\xi_{n-1}\pi_{n-1}\rho_n)^{-1}$ where the maps ρ_n, π_{n-1} and ξ_{n-1} are defined in the proof (see (29), (31) and (32) respectively).*

Proof. Since $\mathbb{F}_n = \mathrm{Fix}_{\mathcal{F}_n}(\mathfrak{u}_{n-1})$ and $P_{n-1}\partial_n$ is an abelian characteristic ideal of the Lie algebra $\mathfrak{u}_n$, the restriction map

$$\rho_n : \mathbb{F}_n \to \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}\partial_n)_n, \quad \sigma \mapsto \sigma|_{P_{n-1}\partial_n}, \tag{29}$$

is a group isomorphism where $\mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}\partial_n)_n := \{\varphi \in \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}\partial_n) \,|\, \varphi(\partial_n) = \partial_n\}$. The map

$$P_{n-1}\partial_n \to P_{n-1}, \quad p\partial_n \mapsto p, \tag{30}$$

is a $\mathfrak{u}_{n-1}$-module isomorphism, and so $\mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}\partial_n) \simeq \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1})$ and

$$\pi_{n-1} : \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}\partial_n)_n \simeq \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1})_1 := \{\sigma \in \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1}) \,|\, \sigma(1) = 1\}. \tag{31}$$

By Proposition 4.2.(1,2), the restriction map

$$\xi_{n-1} : \mathrm{Aut}_{\mathfrak{u}_{n-1}}(P_{n-1})_1 \to \mathrm{Aut}_{K[\partial_{n-1}]}(K[x_{n-1}])_1 = 1 + \partial_{n-1}K[[\partial_{n-1}]], \quad \varphi \mapsto \varphi|_{K[x_{n-1}]}, \tag{32}$$

is a group isomorphism where $\mathrm{Aut}_{K[\partial_{n-1}]}(K[x_{n-1}])_1 := \{\sigma \in \mathrm{Aut}_{K[\partial_{n-1}]}(K[x_{n-1}]) \,|\, \sigma(1) = 1\}$. Then the automorphism η_n is equal to $(\xi_{n-1}\pi_{n-1}\rho_n)^{-1}$. $\square$

Remark. It is useful to identify the groups $\mathbb{F}_n = 1 + \delta_{n-1}K[[\delta_{n-1}]]$ and $1 + \partial_{n-1}K[[\partial_{n-1}]]$ via the isomorphism η_n, i.e.,

$$\mathbb{F}_n = 1 + \delta_{n-1}K[[\delta_{n-1}]] = 1 + \partial_{n-1}K[[\partial_{n-1}]]. \tag{33}$$

The first equality is used when the action of automorphisms of the group $\mathbb{F}_n$ on elements of the Lie algebra $\mathfrak{u}_n$ is considered. The second equality is used when we want to stress how the polynomial coefficient of ∂_n is changed under this action on the ideal $P_{n-1}\partial_n$ of the Lie algebra $\mathfrak{u}_n$.

The subgroup $\mathbb{E}_n$ of $\mathcal{F}_n$. For $n \geq 3$, let

$$\begin{aligned} \mathbb{E}_n &:= \mathrm{Fix}_{\ker(\chi_n)}(P_{n-1}\partial_n) = \{\sigma \in \ker(\chi_n) \,|\, \sigma(u) = u \text{ for all } u \in P_{n-1}\partial_n\} & (34)\\ &= \{\sigma \in \mathcal{F}_n \,|\, \sigma(u) = u \text{ for all } u \in P_{n-1}\partial_n; \sigma(v) - v \in P_{n-1}\partial_n \text{ for all } v \in \mathfrak{u}_{n-1}\}. & (35) \end{aligned}$$

Clearly, $\mathbb{F}_n \cap \mathbb{E}_n = \{e\}$ since $\mathfrak{u}_n = \mathfrak{u}_{n-1} \oplus P_{n-1}\partial_n$. Consider the vector space (of certain 1-cocycles) $Z^1_{n-1} := \{c \in \mathrm{Hom}_K(\mathfrak{u}_{n-1}, P_{n-1}\partial_n) \,|\, c(\partial_1) = \cdots = c(\partial_{n-1}) = 0$ and

$$c([u,v]) = [c(u), v] + [u, c(v)] \tag{36}$$

for all $u, v \in \mathfrak{u}_{n-1}\}$. In particular, Z^1_{n-1} is an additive (abelian) group.

Lemma 4.7 *For $n \geq 3$, the map $\Delta_n : \mathbb{E}_n \to Z^1_{n-1}, \sigma \mapsto \sigma - 1 : u \mapsto \sigma(u) - u$ (where $u \in \mathfrak{u}_{n-1}$), is a group isomorphism with the inverse map $c \mapsto \sigma_c$ where $\sigma_c(u) = \begin{cases} u + c(u) & \text{if } u \in \mathfrak{u}_{n-1}, \\ u & \text{if } u \in P_{n-1}\partial_n. \end{cases}$ In particular, $\mathbb{E}_n$ is an abelian group (and a vector space over the field K).*

Proof. Let us show that the map Δ_n is well-defined. Let $\sigma \in \mathbb{E}_n$. Then $c = \sigma - 1 \in \mathrm{Hom}_K(\mathfrak{u}_{n-1}, P_{n-1}\partial_n)$ (since $\sigma \in \ker(\chi_n)$), and $c(\partial_1) = \cdots = c(\partial_n) = 0$ (since $\sigma \in \mathcal{F}_n$). The condition (36) follows from the facts that σ is a Lie algebra homomorphism, $\mathrm{im}(c) \subseteq P_{n-1}\partial_n$ and $[P_{n-1}\partial_n, P_{n-1}\partial_n] = 0$. In more details, for all elements $u, v \in \mathfrak{u}_{n-1}$,

$$\begin{aligned} 0 &= \sigma([u,v]) - [\sigma(u), \sigma(v)] = [u,v] + c([u,v]) - [u,v] - [c(u), v] - [u, c(v)] - [c(u), c(v)] \\ &= c([u,v]) - [c(u), v] - [u, c(v)]. \end{aligned}$$

The map Δ_n is a group homomorphism: for all elements $\sigma_1, \sigma_2 \in \mathbb{E}_n$,

$$\Delta_n(\sigma_1\sigma_2) = \sigma_1\sigma_2 - 1 = \sigma_1 - 1 + \sigma_1(\sigma_2 - 1) = \sigma_1 - 1 + \sigma_2 - 1 = \Delta_n(\sigma_1) + \Delta_n(\sigma_2),$$

we used the fact that $\mathrm{im}(\sigma_2 - 1) \subseteq P_{n-1}\partial_n$ and $\sigma_1(u) = u$ for all elements $u \in P_{n-1}\partial_n$ (since $\sigma_1 \in \mathbb{E}_n$). Since $\ker(\Delta_n) = \{\mathrm{id}\}$, the map Δ_n is a monomorphism.

To finish the proof of the lemma it suffices to show that the map $Z^1_{n-1} \to \mathbb{E}_n$, $c \mapsto \sigma_c$, is well-defined (indeed, this claim guarantees the surjectivity of the map Δ_n, i.e., Δ_n is a group isomorphism; then it is obvious that the map $c \mapsto \sigma_c$ is the inverse of Δ_n). By the very definition, the K-linear map $\sigma_c : \mathfrak{u}_n \to \mathfrak{u}_n$ is a bijection. To finish the proof of the claim it suffices to show that σ_c is a Lie algebra homomorphism (since then $\sigma_c \in \ker(\chi_n)$ and $\sigma_c \in \mathrm{Fix}_{\ker(\chi_n)}(P_{n-1}\partial_n) = \mathbb{E}_n$). Since $\mathfrak{u}_n = \mathfrak{u}_{n-1} \oplus P_{n-1}\partial_n$ and $[P_{n-1}\partial_n, P_{n-1}\partial_n] = 0$, it suffices to verify that, for all elements $u, v \in \mathfrak{u}_{n-1}$, $\sigma_c([u,v]) = [\sigma_c(u), \sigma_c(v)]$. This follows from (36) and $[c(u), c(v)] = 0$:

$$\sigma_c([u,v]) - [\sigma_c(u), \sigma_c(v)] = [u,v] + c([u,v]) - [u,v] - [c(u), v] - [u, c(v)] - [c(u), c(v)] = 0. \ \square$$

The subgroup $\ker(\chi_n)$ **of** $\mathcal{F}_n$. The next corollary describes the groups $\ker(\chi_n)$ when $n \geq 3$.

Corollary 4.8 *Let $n \geq 3$. Then*

1. $\ker(\chi_n) = \mathbb{F}_n \ltimes \mathbb{E}_n$.
2. *Each element $\sigma \in \ker(\chi_n)$ is the unique product ef where $e \in \mathbb{E}_n$, $e(u) = u + c(u)$ and $c(u) := \sigma(u) - u$ for all $u \in \mathfrak{u}_{n-1}$, and $f := e^{-1}\sigma \in \mathbb{F}_n$.*

Proof. It is obvious that $\mathbb{F}_n \cap \mathbb{E}_n = \{e\}$ and $\tau\mathbb{E}_n\tau^{-1} \subseteq \mathbb{E}_n$ for all $\tau \in \mathbb{F}_n$. Therefore, $\mathbb{F}_n \ltimes \mathbb{E}_n \subseteq \ker(\chi_n)$. Now, to finish the proof of both statements it suffices to show that each element $\sigma \in \ker(\chi_n)$ is the product ef where e and f are as in statement 2. It is easy to check that $c \in Z^1_{n-1}$ (where $c(u) = \sigma(u) - u$ for $u \in \mathfrak{u}_{n-1}$): for all elements $u, v \in \mathfrak{u}_{n-1}$,

$$\begin{aligned} c([u,v]) &= \sigma([u,v]) - [u,v] = [\sigma(u), \sigma(v)] - [u,v] = [u + c(u), v + c(v)] - [u,v] \\ &= [c(u), v] + [u, c(v)]. \end{aligned}$$

By Lemma 4.7, $\Delta_n^{-1}(c) \in \mathbb{E}_n$. Notice that $\Delta_n^{-1}(c) = e$. Now, $e^{-1}\sigma \in \mathbb{F}_n$ since, for all elements $u \in \mathfrak{u}_{n-1}$,

$$e^{-1}\sigma(u) = e^{-1}(u + c(u)) = e^{-1}(u) + c(u) = u - c(u) + c(u) = u. \quad \square$$

We will see that $\ker(\chi_n) = \mathbb{F}_n \times \mathbb{E}_n$ (Theorem 4.12.(3)).

Let $n \geq 3$. For each $i = 1, \ldots, n-2$, $\mathfrak{u}_i + \mathcal{D}_{n-1}$ is a Lie subalgebra of the Lie algebra $\mathfrak{u}_n$ where $\mathfrak{u}_1 := K\partial_1$ and $\mathcal{D}_{n-1} := \oplus_{i=1}^{n-1} K\partial_i$. Notice that the Lie algebra $\mathfrak{u}_n$ is the (adjoint) $\mathfrak{u}_i + \mathcal{D}_{n-1}$-module, and the vector spaces $P_i\partial_{i+1}$, $P_i\partial_n$ and $P_{n-1}\partial_n$ are $\mathfrak{u}_i + \mathcal{D}_{n-1}$-submodules of $\mathfrak{u}_n$. The $\mathfrak{u}_i + \mathcal{D}_{n-1}$-modules $P_i\partial_{i+1}$ and $P_i\partial_n$ are annihilated by the elements $\partial_{i+1}, \ldots, \partial_{n-1}$. Our goal is to give an explicit description of the group Z^1_{n-1} (Corollary 4.10). The group Z^1_{n-1} turns out to be the direct product of certain subgroups described in Proposition 4.9. Let

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n)_0 &:= \{\varphi \in \mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n) \,|\, \varphi(\partial_{i+1}) = 0\}, \\ \mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_i\partial_n)_0 &:= \{\varphi \in \mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_i\partial_n) \,|\, \varphi(\partial_{i+1}) = 0\}. \end{aligned}$$

Proposition 4.9 *Let $n \geq 3$ and $\mathfrak{u}_1 = K\partial_1$. Then*

1. *For all $i = 1, \ldots, n-2$, $\mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n) = \mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_i\partial_n) \simeq \mathrm{End}_{\mathfrak{u}_i}(P_i) \simeq K[[t]]$. Moreover, the map*

$$\alpha_n : K[[t]] \to \mathrm{Hom}_{\mathfrak{u}_i + \mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n), \ \sum_{j \geq 0} \lambda_j t^j \mapsto \varphi,$$

(where $\lambda_j \in K$) is an isomorphism of vector spaces where, for all elements $\beta \in \mathbb{N}^{i-1}$ and $k \in \mathbb{N}$,

$$\varphi(x^\beta x_i^k \partial_{i+1}) := [X_{\beta,i}, \sum_{j \geq 0} \lambda_j (\mathrm{ad}\,\partial_i)^j ((k+1)^{-1} x_i^{k+1} \partial_n)] = \sum_{j \geq 0} \lambda_j (\mathrm{ad}\,\partial_i)^j (x^\beta x_i^k \partial_n).$$

2. For all $i = 1, \dots, n-2$, $\mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n)_0 = \mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_i\partial_n)_0 \simeq tK[[t]]$. Moreover, the map

$$\alpha_n : tK[[t]] \to \mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n)_0, \ \sum_{j\geq 1}\lambda_j t^j \mapsto \varphi,$$

(where $\lambda_j \in K$) is an isomorphism of vector spaces where, for all elements $\beta \in \mathbb{N}^{i-1}$ and $k \in \mathbb{N}$,

$$\varphi(x^\beta x_i^k \partial_{i+1}) := [X_{\beta,i}, \sum_{j\geq 1}\lambda_j (\mathrm{ad}\,\partial_i)^j((k+1)^{-1}x_i^{k+1}\partial_n)] = \sum_{j\geq 1}\lambda_j(\mathrm{ad}\,\partial_i)^j(x^\beta x_i^k\partial_n).$$

3. For every $i = 1, \dots, n-2$, the K-linear map

$$\beta_{n,i} : \mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n)_0 \to Z^1_{n-1}, \ \ \psi \mapsto c_\psi,$$

is an injection where, for $u \in P_{j-1}\partial_j$, $j = 1, \dots, n-1$, $c_\psi(u) := \begin{cases} \psi(u) & \text{if } j = i+1, \\ 0 & \text{if } j \neq i+1. \end{cases}$

Proof. 1. Let $\delta_i := \mathrm{ad}(\partial_i)$ for $i = 1, \dots, n-1$ and $\psi \in H := \mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n)$. For all $j = i+1, \dots, n-1$, $[\partial_j, \psi(P_i\partial_{i+1})] = \psi([\partial_j, P_i\partial_{i+1}]) = \psi(0) = 0$, hence

$$\mathrm{im}(\psi) \subseteq \bigcap_{j=i+1}^{n-1} \ker_{P_{n-1}\partial_n}(\delta_j) = (\bigcap_{j=i+1}^{n-1} \ker_{P_{n-1}}(\partial_j))\partial_n = P_i\partial_n.$$

Therefore, $H = \mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_i\partial_n)$. The maps

$$P_i\partial_{i+1} \to P_i\partial_n \to P_i, \ \ p\partial_{i+1} \to p\partial_n \to p$$

(where $p \in P_i$) are $\mathfrak{u}_i + \mathcal{D}_{n-1}$-module isomorphisms. The $\mathfrak{u}_i + \mathcal{D}_{n-1}$-modules $P_i\partial_{i+1}$ and $P_i\partial_n$ are annihilated by the elements $\partial_{i+1}, \dots, \partial_{n-1}$. So,

$$H = \mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_i\partial_n) \simeq \mathrm{End}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i) \simeq \mathrm{End}_{\mathfrak{u}_i}(P_i) \simeq K[[t]],$$

by Proposition 4.2.(1). The map $\alpha_n : K[[t]] \to H$ is the inverse of the above isomorphism $H \simeq K[[t]]$ (see Proposition 4.2.(1)).

2. Statement 2 follows from statement 1.

3. Let $c = c_\psi$. By the very definition of c, $c(\partial_1) = \cdots = c(\partial_{n-1}) = 0$. We have to show that $c([u, v]) = [c(u), v] + [u, c(v)]$ for all elements $u \in P_{s-1}\partial_s$ and $v \in P_{t-1}\partial_t$ where $s, t = 1, \dots, n-1$. Without loss of generality we may assume that $s \leq t$.

If $s \neq i+1$ and $t \neq i+1$ then $[u, v] \notin P_i\partial_{i+1}$, and the equality above trivially holds ($0 = 0+0$).

If $s < i+1$ and $t = i+1$ then $c(u) = 0$, and the equality that we have to check reduces to the equality $\psi([u, v]) = [u, \psi(v)]$ which is obviously true as the map ψ is a $\mathfrak{u}_i + \mathcal{D}_{n-1}$-homomorphism.

If $s = t = i+1$ then the equality that we have to check reduces to the equality $\psi([u, v]) = [\psi(u), v] + [u, \psi(v)]$ which is obviously true as $[u, v] \in [P_i\partial_{i+1}, P_i\partial_{i+1}] = 0$, $[\psi(u), v] \in [P_i\partial_n, P_i\partial_{i+1}] = 0$, and $[u, \psi(v)] \in [P_i\partial_{i+1}, P_i\partial_n] = 0$ (since $\mathrm{im}(\psi) \subseteq P_i\partial_n$, see the proof of statement 1). $\square$

In combination with Proposition 4.9, the following corollary gives an explicit description of the vector space Z^1_{n-1}, and, as a result, using Lemma 4.7 we have an explicit description of the group $\mathbb{E}_n$ and its generators.

Corollary 4.10 *Let $n \geq 3$. Then the K-linear map*

$$\beta_n := \bigoplus_{i=1}^{n-2}\beta_{n,i} : \bigoplus_{i=1}^{n-2}\mathrm{Hom}_{\mathfrak{u}_i+\mathcal{D}_{n-1}}(P_i\partial_{i+1}, P_{n-1}\partial_n)_0 \to Z^1_{n-1}, \ \ (\psi_1, \dots, \psi_{n-2}) \mapsto c_{\psi_1} + \cdots + c_{\psi_{n-2}},$$

is a bijection where $c_{\psi_i} = \beta_{n,i}(\psi_i)$. In particular, $Z^1_{n-1} \simeq (tK[[t]])^{n-2}$, the direct sum of $n-2$ copies of the vector space $tK[[t]]$.

Proof. In view of the explicit nature of the maps $\beta_{n,i}$, the map β_n is an injection. It remains to show that the map β_n is a surjection. Let $c \in Z^1_{n-1}$. Then, by (36),

$$\psi_1 := c|_{P_1\partial_2} \in \mathrm{Hom}_{\mathcal{D}_{n-1}}(P_1\partial_2, P_{n-1}\partial_n)_0 = \mathrm{Hom}_{\mathfrak{u}_1+\mathcal{D}_{n-1}}(P_1\partial_2, P_{n-1}\partial_n)_0,$$

and so $c_{\psi_1} := \beta_{n,1}(\psi_1) \in Z^1_{n-1}$, by Proposition 4.9.(3). By (36),

$$c_2' := c - c_{\psi_1} \in \mathrm{Hom}_{\mathfrak{u}_2+\mathcal{D}_{n-1}}(\mathfrak{u}_{n-1}, P_{n-1}\partial_n)_0$$

since $c_2'(\mathfrak{u}_2 + \mathcal{D}_{n-1}) = 0$. Then $\psi_2 := c_2'|_{P_2\partial_3} \in \mathrm{Hom}_{\mathfrak{u}_2+\mathcal{D}_{n-1}}(P_2\partial_3, P_{n-1}\partial_n)_0$, and so $c_{\psi_2} := \beta_{n,2}(\psi_2) \in Z^1_{n-1}$, by Proposition 4.9.(3). Then $c_3' := c - c_{\psi_1} - c_{\psi_2} \in \mathrm{Hom}_{\mathfrak{u}_3+\mathcal{D}_{n-1}}(\mathfrak{u}_{n-1}, P_{n-1}\partial_n)_0$, by (36) and the fact that $c_3'(\mathfrak{u}_3 + \mathcal{D}_{n-1}) = 0$. Continue this process (or use induction) we obtain the decomposition $c = c_{\psi_1} + \cdots + c_{\psi_{n-2}}$ where $c_{\psi_i} \in \mathrm{im}(\beta_{n,i})$ for all $i = 1, \ldots, n-2$. This mean that the map β_n is a surjective map, as required. $\square$

The structure of the group $\mathbb{E}_n$. For each $i = 2, \ldots, n-1$, let $\mathbb{E}_{n,i} := \mathrm{im}(\Delta_n^{-1}\beta_{n,i-1})$ (notice the shift by 1 of the indices when comparing them with the indices in Corollary 4.10). Then $\mathbb{E}_{n,i} = \{e_i'(s_i) \,|\, s_i \in \partial_{i-1}K[[\partial_{i-1}]]\} \simeq (\partial_{i-1}K[[\partial_{i-1}]], +)$ (via $e_i'(s_i) \mapsto s_i$) where, for all $j = 1, \ldots, n$ and $\alpha \in \mathbb{N}^{j-1}$,

$$e_i'(s_i)(x^\alpha\partial_j) = \begin{cases} x^\alpha\partial_j + s_i(x^\alpha)\partial_n & \text{if } j = i, \\ x^\alpha\partial_j & \text{if } j \neq i, \end{cases} \tag{37}$$

$$\mathbb{E}_n = \prod_{i=2}^{n-1} \mathbb{E}_{n,i}. \tag{38}$$

So, each element $e' \in \mathbb{E}_n$ is the unique product $e' = e_2' \cdots e_{n-1}'$ with $e_i' = e_i'(s_i) \in \mathbb{E}_{n,i}$ and each automorphism e_i' is uniquely determined by a series $s_i = \sum_{j\geq 1} \lambda_{ij}\partial_{i-1}^j$ with $\lambda_{ij} \in K$. By (37), for all elements $x^\alpha\partial_j \in \mathfrak{u}_n$ where $\alpha \in \mathbb{N}^{j-1}$,

$$e'(x^\alpha\partial_j) = \begin{cases} e_j'(x^\alpha\partial_j) & \text{if } 2 \leq j \leq n-1, \\ x^\alpha\partial_j & \text{if } j = 1, n, \end{cases} = \begin{cases} x^\alpha\partial_j + s_j(x^\alpha)\partial_n & \text{if } 2 \leq j \leq n-1, \\ x^\alpha\partial_j & \text{if } j = 1, n. \end{cases} \tag{39}$$

Equivalently, for all elements $u = \sum_{i=1}^n p_i\partial_i \in \mathfrak{u}_n$ where $p_i \in P_{i-1}$ for all $i = 1, \ldots, n$,

$$e'(u) = u + \sum_{i=2}^{n-1} s_i(p_i)\partial_n. \tag{40}$$

Lemma 4.11 *For $n \geq 4$, $\mathrm{im}(\chi_n) \cap \mathbb{E}_{n-1} = \{e\}$.*

Proof. Let $\sigma \in \mathbb{E}_{n-1}$. Then for all polynomials $p_i \in P_i$, $i = 1, \ldots, n-2$, by (40),

$$\sigma(\sum_{i=1}^{n-2} p_i\partial_i) = \sum_{i=1}^{n-2} p_i\partial_i + (\sum_{i=2}^{n-2} s_i(p_i))\partial_{n-1}$$

where $s_i = \sum_{j\geq 1} \lambda_{ij}(\mathrm{ad}\,\partial_{i-1})^j$ for some scalars $\lambda_{ij} \in K$. Suppose that the automorphism σ belongs to $\mathrm{im}(\chi_n)$, i.e., $\sigma = \chi_n(\sigma')$ for some $\sigma' \in \mathcal{F}_n$. By Lemma 4.5, $\sigma'(x_{n-1}\partial_n) = x_{n-1}\partial_n + \lambda\partial_n$ for some $\lambda \in K$. Then applying the automorphism σ' to the equality $[\sum_{i=1}^{n-2} p_i\partial_i, x_{n-1}\partial_n] = 0$ in the algebra $\mathfrak{u}_n$ yields the equality $(\sum_{i=2}^{n-2} s_i(p_i))\partial_n = 0$ for all polynomials $p_i \in P_{i-1}$, $i = 2, \ldots, n-2$. We used the fact that ∂_n is a central element of the Lie algebra $\mathfrak{u}_n$ and $[P_{n-1}\partial_n, P_{n-1}\partial_n] = 0$. Hence, all $s_i = 0$, i.e., $\sigma = e$. $\square$

By Lemma 4.5.(1), there is the short exact sequence of group homomorphisms

$$1 \to \ker(\mathrm{res}_n) \to \mathcal{F}_n \stackrel{\mathrm{res}_n}{\to} 1 + \delta_{n-1}K[[\delta_{n-1}]] \to 1, \tag{41}$$

where $\text{res}_n(\sigma) := \sigma|_{K[x_{n-1}]\partial_n} : K[x_{n-1}]\partial_n \to K[x_{n-1}]\partial_n$ with $\text{res}_n(\mathbb{F}_n) = 1 + \delta_{n-1}K[[\delta_{n-1}]]$ (Proposition 4.6), and the natural inclusion $\mathbb{F}_n = 1 + \delta_{n-1}K[[\delta_{n-1}]] \subseteq \mathcal{F}_n$ is a splitting for the epimorphism res_n. Therefore,

$$\mathcal{F}_n = \mathbb{F}_n \ltimes \ker(\text{res}_n). \tag{42}$$

The group $\mathcal{F}_n$ contains the groups Sh_{n-2}, $\mathbb{F}_n$ and $\mathbb{E}_n$. Let us show that *the elements of the groups* Sh_{n-2}, $\mathbb{F}_n$ *and* $\mathbb{E}_n$ *pairwise commute.* Let $u = \sum_{i=1}^n p_i\partial_i = u_{n-1} + u_n$ where $p_i \in P_{i-1}$ for all $i = 1, \dots, n$, $u_{n-1} = \sum_{i=1}^{n-1} p_i\partial_i$, $u_n = p_n\partial_n$, $e' \in \mathbb{E}_n$, $f = 1 + \sum_{i\geq 1} \lambda_i\partial_{n-1}^i \in \mathbb{F}_n$ where $\lambda_i \in K$ and $s \in \text{Sh}_{n-1}$. We assume that (40) holds for the element e'. Then

$$\begin{aligned} e'f(u) &= e'(u_{n-1} + f(u_n)) = u_{n-1} + \sum_{i=2}^{n-1} s_i(p_i)\partial_n + f(u_n), \\ fe'(u) &= f(u + \sum_{i=2}^{n-1} s_i(p_i)\partial_n) = u_{n-1} + f(u_n) + \sum_{i=2}^{n-1} s_i(p_i)\partial_n. \end{aligned}$$

The last equality holds since $f(\sum_{i=2}^{n-1} s_i(p_i)\partial_n) = \sum_{i=2}^{n-1} s_i(p_i)\partial_n$. This follows from the inclusions and $s_i(p_i) \in P_{i-1}$ for all $i = 2, \dots, n-1$ ($\partial_{n-1}(s_i(p_i)) = 0$ for all $i = 2, \dots, n-1$). Therefore, $e'f = fe'$. Every element $s \in \text{Sh}_{n-2}$ can be uniquely written as $s = e^{\sum_{i=1}^{n-2}\lambda_i\partial_i}$ where $\lambda_i \in K$. Then it is obvious that $sf = fs$. Finally, $se = es$ since, for all elements $u \in \mathfrak{u}_n$,

$$se(u) = s(u + \sum_{i=2}^{n-1} s_i(p_i)\partial_n) = s(u) + \sum_{i=2}^{n-1} ss_i(p_i)\partial_n = s(u) + \sum_{i=2}^{n-1} s_is(p_i)\partial_n = es(u),$$

as $ss_i = s_is$ for all elements $i = 1, \dots, n-1$, and $s(\partial_j) = \partial_j$ for all elements $j = 1, \dots, n$.

By Corollary 4.8.(1), $\ker(\chi_n) = \mathbb{F}_n \times \mathbb{E}_n$. For $n = 2$, $\text{Sh}_0 := \{e\}$ and $\mathbb{E}_2 := \{e\}$. The subgroup of the group $\mathcal{F}_n$ that these three groups generate is an abelian group. It is easy to see that $\text{Sh}_{n-2} \cap (\mathbb{F}_n \times \mathbb{E}_n) = \{e\}$. Hence, $\text{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n \subseteq \mathcal{F}_n$. The next theorem shows that, in fact, the equality holds.

Theorem 4.12 *1.* $\mathcal{F}_n = \text{Sh}_{n-2} \times \ker(\chi_n) = \text{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n$.

2. $\text{im}(\chi_n) = \text{Sh}_{n-2}$.

3. The short exact sequence of group homomorphisms $1 \to \ker(\chi_n) \to \mathcal{F}_n \stackrel{\chi_n}{\to} \text{Sh}_{n-2} \to 1$ *is a split short exact sequence and the natural inclusion* $\text{Sh}_{n-2} \subseteq \mathcal{F}_n$ *is a splitting of the epimorphism* χ_n, $\ker(\chi_n) = \mathbb{F}_n \times \mathbb{E}_n$.

4. $\ker(\text{res}_n) = \text{Sh}_{n-2} \times \mathbb{E}_n$ *and* $\mathcal{F}_n = \mathbb{F}_n \times \ker(\text{res}_n)$.

Proof. 1-3. We use induction on $n \geq 2$ to prove all three statements simultaneously. The initial step $n = 2$ is trivial as $\mathcal{F}_2 = \mathbb{F}_2$, $\mathcal{F}_1 = \{e\}$, $\text{Sh}_0 := \{e\}$ and $\mathbb{E}_1 = \{e\}$. So, let $n > 2$ and we assume that all three statements hold for all $n' < n$. By induction, $\mathcal{F}_{n-1} = \text{Sh}_{n-3} \times \mathbb{F}_{n-1} \times \mathbb{E}_{n-1}$, and as a result

$$\chi_n : \mathcal{F}_n \to \mathcal{F}_{n-1} = \text{Sh}_{n-3} \times \mathbb{F}_{n-1} \times \mathbb{E}_{n-1}.$$

To show that statements 1-3 hold it suffices only to prove statement 2, that is $\text{im}(\chi_n) = \text{Sh}_{n-2}$ (since then statement 3 follows as $\ker(\chi_n) = \mathbb{F}_n \times \mathbb{E}_n$; statement 3 and Corollary 4.8.(1) imply statement 1). The subgroup Sh_{n-3} of $\mathcal{F}_n$ is mapped isomorphically/identically onto its image $\chi_n(\text{Sh}_{n-3}) = \text{Sh}_{n-3}$. Then $\text{im}(\chi_n) = \text{Sh}_{n-3} \times I$ where $I := \text{im}(\chi_n) \cap (\mathbb{F}_{n-1} \times \mathbb{E}_{n-1})$. Notice that $\text{Sh}_{n-2} = \text{Sh}_{n-3} \times \text{sh}_{n-2}$, $\text{sh}_{n-2} = e^{K\partial_{n-2}} \subseteq \mathbb{F}_{n-1} = 1 + \partial_{n-2}K[[\partial_{n-2}]]$ (recall the identification (33)). To finish the proof it remains to show that $I = \text{sh}_{n-2}$.

Let $\sigma \in \mathcal{F}_n$ be such that $\sigma' := \chi_n(\sigma) \in \mathbb{F}_{n-1} \times \mathbb{E}_{n-1}$, $\sigma' = fe'$ for some automorphisms $f \in \mathbb{F}_{n-1}$ and $e' \in \mathbb{E}_{n-1}$. We have to show that $\sigma' \in \text{sh}_{n-2}$. Since $\mathbb{F}_n \subseteq \ker(\chi_n)$, in view of (42), without loss of generality we may assume that $\sigma \in \ker(\text{res}_n)$, that is $\sigma(x_{n-1}^i\partial_n) = x_{n-1}^i\partial_n$

for all $i \geq 0$. Recall that $\mathfrak{u}_n = \mathfrak{u}_{n-2} \oplus P_{n-2}\partial_{n-1} \oplus P_{n-1}\partial_n = \mathfrak{u}_{n-1} \oplus P_{n-1}\partial_n$. For all elements $u \in P_{n-2}\partial_{n-1}$,

$$\sigma(u) = fe'(u) + c(u) = f(u) + c(u)$$

for some map $c \in \mathrm{Hom}_K(P_{n-2}\partial_{n-1}, P_{n-1}\partial_n)$ ($e'(u) = u$ for all elements $u \in P_{n-2}\partial_{n-1}$). The automorphism $f = 1 + \sum_{i\geq 1} \lambda_i \partial_{n-2}^i \in \mathbb{F}_{n-1} = 1 + \partial_{n-2}K[[\partial_{n-2}]]$ (where $\lambda_i \in K$) acts on the elements $x^\alpha \partial_i \in \mathfrak{u}_{n-1}$ as $f(x^\alpha\partial_i) = f(x^\alpha)\partial_i$ where

$$f(x^\alpha) = (1 + \sum_{i\geq 1} \lambda_i \partial_{n-2}^i)(x^\alpha).$$

When we apply the automorphism σ to the following identities in the Lie algebra $\mathfrak{u}_n$,

$$[x_{n-2}^i\partial_{n-1}, (j+1)^{-1}x_{n-1}^{j+1}\partial_n] = x_{n-2}^i x_{n-1}^j \partial_n, \quad i, j \in \mathbb{N},$$

it yields the identities

$$\begin{aligned} \sigma(x_{n-2}^i x_{n-1}^j\partial_n) &= [\sigma(x_{n-2}^i\partial_{n-1}), \sigma((j+1)^{-1}x_{n-1}^{j+1}\partial_n)] \\ &= [f(x_{n-2}^i)\partial_{n-1} + c(x_{n-2}^i\partial_{n-1}), (j+1)^{-1}x_{n-1}^{j+1}\partial_n] = f(x_{n-2}^i)x_{n-1}^j\partial_n \end{aligned}$$

since $[c(x_{n-2}^i\partial_{n-1}), (j+1)^{-1}x_{n-1}^{j+1}\partial_n] \subseteq [P_{n-1}\partial_n, P_{n-1}\partial_n] = 0$. Similarly, when we apply the automorphism σ to the following identities in the Lie algebra $\mathfrak{u}_n$,

$$[x_{n-2}^i\partial_{n-1}, x_{n-2}^j x_{n-1}\partial_n] = x_{n-2}^{i+j}\partial_n, \quad i, j \in \mathbb{N},$$

we deduce the identities

$$f(x_{n-2}^i)f(x_{n-2}^j)\partial_n = f(x_{n-2}^{i+j})\partial_n, \quad i, j \in \mathbb{N}.$$

In more detail,

$$\begin{aligned} f(x_{n-2}^{i+j})\partial_n &= \sigma(x_{n-2}^{i+j}\partial_n) = \sigma([x_{n-2}^i\partial_{n-1}, x_{n-2}^j x_{n-1}\partial_n]) = [\sigma(x_{n-2}^i\partial_{n-1}), \sigma(x_{n-2}^j x_{n-1}\partial_n)] \\ &= [f(x_{n-2}^i\partial_{n-1}) + c(x_{n-2}^i\partial_{n-1}), f(x_{n-2}^j)x_{n-1}\partial_n] \\ &= [f(x_{n-2}^i)\partial_{n-1}, f(x_{n-2}^j)x_{n-1}\partial_n] = f(x_{n-2}^i)f(x_{n-2}^j)\partial_n \end{aligned}$$

since $[c(x_{n-2}^i\partial_{n-1}), f(x_{n-2}^j)x_{n-1}\partial_n] \in [P_{n-1}\partial_n, P_{n-1}\partial_n] = 0$. Therefore, $f(x_{n-2}^{i+j}) = f(x_{n-2}^i)f(x_{n-2}^j)$ for all $i, j \in \mathbb{N}$. This means that the map $f = 1 + \sum_{i\geq 1}\lambda_i\partial_{n-2}^i : K[x_{n-2}] \to K[x_{n-2}]$ is an automorphism of the polynomial algebra $K[x_{n-2}]$. Since $f(x_{n-2}) = x_{n-2} + \lambda_1$, we must have $f = e^{\lambda_1\partial_{n-2}} \in \mathrm{sh}_{n-2}$. Replacing the automorphism σ by the automorphism $f^{-1}\sigma$, we may assume that $f = e$, and so $\sigma' = e' \in \mathbb{E}_{n-1}$. By Lemma 4.11, $\sigma' \in \mathrm{im}(\chi_n) \cap \mathbb{E}_{n-1} = \{e\}$. Therefore, $I = \mathrm{sh}_{n-2}$.

4. Notice that $\mathrm{Sh}_{n-2} \times \mathbb{E}_n \subseteq \ker(\mathrm{res}_n)$. By (42), $\mathcal{F}_n = \mathbb{F}_n \ltimes \ker(\mathrm{res}_n)$. By statement 1, $\mathcal{F}_n = \mathbb{F}_n \times \mathrm{Sh}_{n-2} \times \mathbb{E}_n$. Therefore, $\ker(\mathrm{res}_n) = \mathrm{Sh}_{n-2} \times \mathbb{E}_n$ and $\mathcal{F}_n = \mathbb{F}_n \times \ker(\mathrm{res}_n)$. $\square$

It follows from the inclusion $\mathrm{sh}_{n-1} = e^{K\partial_{n-1}} \subseteq \mathbb{F}_n = 1 + \partial_{n-1}K[[\partial_{n-1}]]$ (see (33)) that

$$\mathbb{F}_n = \mathrm{sh}_{n-1} \times \mathbb{F}'_n \tag{43}$$

where $\mathbb{F}'_n = 1 + \partial_{n-1}^2 K[[\partial_{n-1}]] = 1 + \delta_{n-1}^2 K[[\delta_{n-1}]]$ (see (33)). So,

$$\mathbb{F}'_n = \{f \in 1+\partial_{n-1}^2K[[\partial_{n-1}]] \mid f(p_i\partial_i) := \begin{cases} p_i\partial_i & \text{if } i = 1, \ldots, n-1, \\ f(p_n)\partial_n & \text{if } i = n, \end{cases} \text{where } p_i \in P_{i-1}, i = 1, \ldots, n\}. \tag{44}$$

Moreover, $\mathbb{F}_n = \prod_{i\geq 1} e^{K\delta_{n-1}^i} \simeq K^{\mathbb{N}}$ and $\mathbb{F}'_n = \prod_{i\geq 2} e^{K\delta_{n-1}^i} \simeq K^{\mathbb{N}}$.

The next theorem describes the group G_n as an exact product of its explicit subgroups.

Theorem 4.13 *Let $\mathbb{I} := (1 + t^2K[[t]], \cdot)$ and $\mathbb{J} := (tK[[t]], +)$. Then for all $n \geq 2$,* *1.* $G_n = \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} (\mathrm{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n)) = \mathrm{TAut}_K(P_n)_n \times_{ex} (\mathbb{F}'_n \times \mathbb{E}_n)$,

2. $G_n \simeq \mathrm{TAut}_K(P_n)_n \times_{ex} (\mathbb{I} \times \mathbb{J}^{n-2})$.

Proof. 1. The first equality follows from Theorem 3.8.(2) and Theorem 4.12.(1). The second equality follows from the first one, the equality $\mathbb{F}_n = \mathrm{sh}_{n-1} \times \mathbb{F}'_n$ (see (43)) and Proposition 3.4.(3).

2. Statement 2 follows from statement 1 and the facts that $\mathbb{F}'_n \simeq \mathbb{I}$ and $\mathbb{E}_n \simeq \mathbb{J}^{n-2}$ (Corollary 4.10). $\Box$

In Section 5, Theorem 4.13 will be strengthen (Theorem 5.3). Roughly speaking, the exact product will be replaced by a semi-direct product.

5 The group of automorphism of the Lie algebra $\mathfrak{u}_n$ is an iterated semi-direct product

The aim of this section is to show that the group G_n is an iterated semi-direct product $\mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}'_n \times \mathbb{E}_n))$ (Theorem 5.3), that none of its subgroups $\mathrm{TAut}_K(P_n)_n$, $\mathbb{F}'_n \times \mathbb{E}_n$, $\mathbb{F}_n \times \mathbb{E}_n$, $\mathbb{F}'_n$, $\mathbb{F}_n$, $\mathbb{E}_n$ and $\mathbb{E}_{n,i}$ is a normal subgroup (Corollary 5.5), and to give characterizations of the groups $\mathbb{F}_n$, $\mathbb{F}'_n$ and $\mathbb{E}_n$ of G_n in invariant terms (Proposition 5.6). The proof of the results are based on the following two lemmas.

Lemma 5.1 *Let $t_\lambda \in \mathbb{T}^n$ where $\lambda = (\lambda_1, \ldots, \lambda_n) \in K^{*n}$; $e'_i(s_i) \in \mathbb{E}_{n,i}$ where $i = 2, \ldots, n-1$, $s_i = \sum_{j \geq 1} \lambda_{ij} \partial_{i-1}^j$ and $\lambda_{ij} \in K$; and $f = 1 + \sum_{k \geq 1} \mu_k \delta_{n-1}^k \in \mathbb{F}_n$ where $\delta_{n-1} = \mathrm{ad}(\partial_{n-1})$. Then*

1. $t_\lambda e'_i(s_i) t_\lambda^{-1} = e'_i(\lambda_i \lambda_n^{-1} t_\lambda s_i t_\lambda^{-1}) \in \mathbb{E}_{n,i}$ *where* $t_\lambda s_i t_\lambda^{-1} = \sum_{j \geq 1} \lambda_{ij} \lambda_{i-1}^{-j} \partial_{i-1}^j$.

2. $t_\lambda f t_\lambda^{-1} = 1 + \sum_{k \geq 1} \mu_k \lambda_{n-1}^{-k} \delta_{n-1}^k \in \mathbb{F}_n$.

3. If, in addition, $f \in \mathbb{F}'_n$ (i.e., $\mu_1 = 0$) then $t_\lambda f t_\lambda^{-1} = 1 + \sum_{k \geq 2} \mu_k \lambda_{n-1}^{-k} \delta_{n-1}^k \in \mathbb{F}'_n$.

Proof. 1. Let $\sigma = t_\lambda e'_i(s_i) t_\lambda^{-1}$ and $\sigma' := e'_i(\lambda_i \lambda_n^{-1} t_\lambda s_i t_\lambda^{-1})$. Then $\sigma(x^\alpha \partial_j) = x^\alpha \partial_j = \sigma'(x^\alpha \partial_j)$ for all $j \neq i$ and $\alpha \in \mathbb{N}^{j-1}$. For all elements $\alpha \in \mathbb{N}^{i-1}$,

$$\begin{aligned} \sigma: \quad x^\alpha \partial_i &\overset{t_\lambda^{-1}}{\mapsto} \lambda^{-\alpha} \lambda_i x^\alpha \partial_i \overset{e'_i(s_i)}{\mapsto} t_\lambda^{-1}(x^\alpha \partial_i) + \lambda^{-\alpha} \lambda_i s_i(x^\alpha) \partial_n \\ &\overset{t_\lambda}{\mapsto} \quad x^\alpha \partial_i + \lambda^{-\alpha} \lambda_i t_\lambda s_i t_\lambda^{-1}(t_\lambda(x^\alpha)) \cdot \lambda_n^{-1} \partial_n = x^\alpha \partial_i + \lambda_i \lambda_n^{-1} t_\lambda s_i t_\lambda^{-1}(x^\alpha) \partial_n = \sigma'(x^\alpha \partial_i). \end{aligned}$$

Therefore, $\sigma = \sigma'$.

2 and 3. Straightforward. $\Box$

Let G be a group and $a, b \in G$. Then $[a, b] := aba^{-1}b^{-1}$ is the (group) *commutator* of elements a and b of the group G.

Lemma 5.2 *Let $\tau = e^{a_s \partial_s} \in \mathrm{UAut}_K(P_n)_n$ where $a_s \in P_{s-1}$ and $1 \leq s \leq n$; $e'_i(s_i) \in \mathbb{E}_{n,i}$ where $2 \leq i \leq n-1$, $s_i = \sum_{j \geq 1} \lambda_{ij} \partial_{i-1}^j$ and $\lambda_{ij} \in K$; and $f = 1 + \sum_{i \geq 1} \mu_i \delta_{n-1}^i \in \mathbb{F}_n$ where $\mu_i \in K$ and $\delta_{n-1} = \mathrm{ad}(\partial_{n-1})$. Then*

1. $[e'_i(s_i), \tau] = \begin{cases} e^{s_i(a_i)\partial_n} & \text{if } s = i, \\ e & \text{if } s \neq i. \end{cases}$

2. $[\tau, f] = \begin{cases} e & \text{if } 1 \leq s < n, \\ e^{-f'(a_n)\partial_n} & \text{if } s = n, \end{cases}$ *where* $f'(a_n) := \sum_{i \geq 1} \mu_i \partial_{n-1}^i(a_n)$.

Proof. Recall that $\tau : x_s \mapsto x_s + a_s$, $x_i \mapsto x_i$ for $i \neq s$, $\partial_j \mapsto \begin{cases} \partial_j - \frac{\partial a_s}{\partial x_j}\partial_s & \text{if } j < s, \\ \partial_j & \text{if } j \geq s. \end{cases}$ In the arguments below, the decomposition $\mathfrak{u}_n = \oplus_{i=1}^n P_{i-1}\partial_i$ is often used.

1. Since $e'(-s_i) = e'(s_i)^{-1}$, the equality in statement 1 is equivalent to the equality

$$[e'(s_i)^{-1}, \tau] = \begin{cases} e^{-s_i(a_i)\partial_n} & \text{if } s = i, \\ e & \text{if } s \neq i. \end{cases}$$

Notice that $[e'(s_i)^{-1}, \tau] = e'(s_i)^{-1} \cdot \tau e'(s_i)\tau^{-1}$. Let $e'_i = e'_i(s_i)$ and $\sigma = \tau e'_i \tau^{-1}$. We consider three cases separately: $s < i$, $s = i$, and $s > i$.

Case 1: $s < i$. We have to show that $\sigma = e'_i$. The automorphisms $\tau^{\pm 1}$ and e'_i respect the vector spaces $V_- := \oplus_{1 \leq j < i} P_{j-1}\partial_j$ and $V_+ := \oplus_{i < j \leq n} P_{j-1}\partial_j$ (i.e., the vector spaces are invariant under the action of the automorphisms). Moreover, the automorphism e'_i acts as the identity map on both of them, hence so does the automorphism σ. In particular, $\sigma|_{V_- \oplus V_+} = e'_i|_{V_- \oplus V_+}$. For all elements $\alpha \in \mathbb{N}^{i-1}$,

$$\begin{aligned} \sigma : x^\alpha \partial_i &\overset{\tau^{-1}}{\mapsto} \tau^{-1}(x^\alpha)\partial_i \overset{e'_i}{\mapsto} \tau^{-1}(x^\alpha\partial_i) + s_i\tau^{-1}(x^\alpha)\partial_n \\ &\overset{\tau}{\mapsto} x^\alpha\partial_i + \tau s_i \tau^{-1}(x^\alpha)\partial_n = x^\alpha\partial_i + s_i(x^\alpha)\partial_n = e'_i(x^\alpha\partial_i) \end{aligned}$$

since $\tau s_i(x^\alpha) = s_i \tau(x^\alpha)$ as $s < i$. Therefore, $\sigma = e'_i$.

Case 2: $s = i$. We have to show that $c := [e'^{-1}_i(s_i), \tau] = e^{-s_i(a_i)\partial_n}$.

The automorphisms τ, e'_i and $\xi := e^{-s_i(a_i)\partial_n}$ respect the vector space V_+, hence so do the automorphisms σ and c. Moreover, the automorphisms e'_i and ξ act on V_+ as the identity map. In particular, $c|_{V_+} = \xi|_{V_+}$. Let $1 \leq j < i$ and $\alpha \in \mathbb{N}^{j-1}$. Then

$$\begin{aligned} c : x^\alpha\partial_j &\overset{\tau^{-1}}{\mapsto} x^\alpha(\partial_j + \frac{\partial a_i}{\partial x_j}\partial_i) \overset{e'_i}{\mapsto} \tau^{-1}(x^\alpha\partial_j) + s_i(x^\alpha \frac{\partial a_i}{\partial x_j})\partial_n \\ &= \tau^{-1}(x^\alpha\partial_j) + x^\alpha\partial_j s_i(a_i)\partial_n \overset{\tau}{\mapsto} x^\alpha\partial_j + x^\alpha\partial_j s_i(a_i)\partial_n = x^\alpha(\partial_j + \partial_j s_i(a_i)\partial_n) \\ &\overset{e'^{-1}_i}{\mapsto} x^\alpha(\partial_j + \partial_j s_i(a_i)\partial_n) = \xi(x^\alpha\partial_j), \end{aligned}$$

since $\tau(x^\alpha\partial_j s_i(a_i)) = x^\alpha\partial_j s_i(a_i)$ as $x^\alpha\partial_j s_i(a_i) \in P_{i-1}$. Finally, for all elements $\alpha \in \mathbb{N}^{i-1}$,

$$\begin{aligned} c : x^\alpha\partial_i &\overset{\tau^{-1}}{\mapsto} x^\alpha\partial_i \overset{e'_i}{\mapsto} x^\alpha\partial_i + s_i(x^\alpha)\partial_n \overset{\tau}{\mapsto} x^\alpha\partial_i + \tau s_i(x^\alpha)\partial_n \\ &= x^\alpha\partial_i + s_i(x^\alpha)\partial_n = e'_i(x^\alpha\partial_i) \overset{e'^{-1}_i}{\mapsto} x^\alpha\partial_i = \xi(x^\alpha\partial_i). \end{aligned}$$

Therefore, $c = \xi$.

Case 3: $s > i$. We have to show that $\sigma = e'_i$. The automorphisms $\tau^{\pm 1}$ and e'_i respect the subspace $V = \oplus_{j \neq i} P_{j-1}\partial_j$ of $\mathfrak{u}_n$. Moreover, $e'_i|_V = \mathrm{id}_V$. Therefore, $\sigma|_V = \mathrm{id}_V = e'_i|_V$. For all elements $\alpha \in \mathbb{N}^{i-1}$,

$$\begin{aligned} \sigma : x^\alpha\partial_i &\overset{\tau^{-1}}{\mapsto} \tau^{-1}(x^\alpha)\partial_i + \tau^{-1}(x^\alpha)\frac{\partial a_s}{\partial x_i}\partial_s = x^\alpha\partial_i + x^\alpha\frac{\partial a_s}{\partial x_i}\partial_s \overset{e'_i}{\mapsto} \tau^{-1}(x^\alpha\partial_i) + s_i(x^\alpha)\partial_n \\ &\overset{\tau}{\mapsto} x^\alpha\partial_i + s_i(x^\alpha)\partial_n = e'_i(x^\alpha\partial_i) \end{aligned}$$

since $\tau\partial_i\tau^{-1} = \tau(\partial_i) = \partial_i - \frac{\partial a_s}{\partial x_i}\partial_s$ and $\tau^{\pm 1}(x^\alpha) = x^\alpha$ (since $s > i$ and $\alpha \in \mathbb{N}^{i-1}$). Therefore, $\sigma = e'_i$.

2. Case 1: $1 \leq s < n$. In this case, $\tau\partial_{n-1}\tau^{-1} = \partial_{n-1}$, and so $\tau\delta_{n-1} = \delta_{n-1}\tau$. This implies that the maps τ and $f = 1 + \sum_{i \geq 1} \mu_i \delta^i_{n-1}$ commute.

Case 2: $s = n$. Let $c := [\tau, f]$. In this case, both automorphisms τ and f respect the vector space $P_{n-1}\partial_n$. Moreover, the automorphism τ acts as the identity map on it, hence so does the automorphism c. Clearly, the automorphism $e^{-f'(a_n)\partial_n}$ acts as the identity map on

$P_{n-1}\partial_n$. Therefore, $c|_{P_{n-1}\partial_n} = e^{-f'(a_n)\partial_n}|_{P_{n-1}\partial_n}$. Consider the action of the automorphism c on the elements $x^\alpha\partial_i$ where $\alpha \in \mathbb{N}^{i-1}$ and $1 \leq i < n$,

$$\begin{aligned} c: x^\alpha\partial_i &\overset{f^{-1}}{\mapsto} x^\alpha\partial_i \overset{\tau^{-1}}{\mapsto} x^\alpha(\partial_i + \frac{\partial a_n}{\partial x_i}\partial_n) \overset{f}{\mapsto} \tau^{-1}(x^\alpha\partial_i) + f'(x^\alpha\frac{\partial a_n}{\partial x_i})\partial_n = \tau^{-1}(x^\alpha\partial_i) + x^\alpha f'(\frac{\partial a_n}{\partial x_i})\partial_n \\ &\overset{\tau}{\mapsto} x^\alpha(\partial_i + f'(\frac{\partial a_n}{\partial x_i})\partial_n) = x^\alpha(\partial_i + \frac{\partial f'(a_n)}{\partial x_i}\partial_n) = e^{-f'(a_n)\partial_n}(x^\alpha\partial_i). \end{aligned}$$

Then $c = e^{-f'(a_n)\partial_n}$. $\square$

Statements 1 and 2 of Lemma 5.2 can be rewritten as follows (where $2 \leq i \leq n-1$)

$$e_i'(s_i)e^{a_s\partial_s}e_i'(s_i)^{-1} = \begin{cases} e^{s_i(a_s)\partial_n}e^{a_s\partial_s} & \text{if } s = i, \\ e^{a_s\partial_s} & \text{if } s \neq i, \end{cases} \tag{45}$$

$$fe^{a_s\partial_s}f^{-1} = \begin{cases} e^{a_s\partial_s} & \text{if } 1 \leq s < n, \\ e^{(a_n+f'(a_n))\partial_n} = e^{f(a_n\partial_n)} = e^{f(a_n)\partial_n} & \text{if } s = n. \end{cases} \tag{46}$$

Recall that the map $P_{n-1}\partial_n \to P_{n-1}$, $p\partial_n \mapsto p$, is a $\mathfrak{u}_n$-module isomorphism. Under this isomorphism the action of the element ∂_{n-1} on the ideal $P_{n-1}\partial_n$ of the Lie algebra $\mathfrak{u}_n$, which is $\delta_{n-1} = \mathrm{ad}(\partial_{n-1})$, becomes the partial derivative ∂_{n-1} on the polynomial algebra P_{n-1}. So, the expression $f(a_n)$ in (46) makes sense, it simply means $(1+f')(a_n)$.

The next theorem is one of the main results of the paper.

Theorem 5.3 *Let* $\mathbb{I} := (1 + t^2K[[t]], \cdot)$ *and* $\mathbb{J} := (tK[[t]], +)$*. Then for all* $n \geq 2$,

1. $G_n = \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}_n' \times \mathbb{E}_n))$.
2. $G_n \simeq \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{I} \times \mathbb{J}^{n-2})$.
3. $\mathrm{UAut}_K(P_n)_n$ *is a normal subgroup of the group* G_n.
4. $\mathcal{U}_n = \mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}_n' \times \mathbb{E}_n)$.

Proof. 3. Statement 3 follows from statement 1.

1. By Proposition 3.4.(4), $\mathrm{UAut}_K(P_n)_n = \mathcal{T}_n \times_{ex} \mathrm{Sh}_{n-1} = \mathcal{T}_n \times_{ex} (\mathrm{Sh}_{n-2} \times \mathrm{sh}_{n-1})$. By (43), $\mathbb{F}_n = \mathrm{sh}_{n-1} \times \mathbb{F}_n'$. Then, by Theorem 4.13.(1),

$$\begin{aligned} G_n &= \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} (\mathrm{Sh}_{n-2} \times \mathrm{sh}_{n-1} \times \mathbb{F}_n' \times \mathbb{E}_n)) = \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \times_{ex} (\mathbb{F}_n' \times \mathbb{E}_n)) \\ &= \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}_n' \times \mathbb{E}_n)), \text{ by (45) and (46).} \end{aligned}$$

2. Statement 2 follows from statement 1 (see Theorem 4.13.(2)).

4. Statement 4 follows from the obvious inclusion $\mathrm{UAut}_K(P_n)_n \rtimes (\mathbb{F}_n' \times \mathbb{E}_n) \subseteq \mathcal{U}_n$, statement 1 and Proposition 3.1.(2). $\square$

Corollary 5.4 1. *The group* $\mathcal{T}_n' := \{[0, \ldots, 0, a_n] \mid a_n \in \mathfrak{m}_{n-1}\} = e^{\mathfrak{m}_{n-1}\partial_n}$ *is a normal subgroup of the groups* G_n *and* $\mathrm{UAut}_K(P_n)_n$.

2. $\mathrm{UAut}_K(P_n)_n = \mathrm{UAut}_K(P_{n-1}) \ltimes \mathcal{T}_n' \subseteq G_n$ *(this is the equality of subgroups of* G_n*).*
3. $G_n/\mathcal{T}_n' = \mathbb{T}^n \ltimes (\mathrm{UAut}_K(P_{n-1}) \times \mathbb{F}_n' \times \mathbb{E}_n)$.

Proof. 2. Statement 2 is obvious.

1. Statement 1 follows from statement 2, the equalities (45) and (46), and Theorem 5.3.(1).

3. Statement 3 follows from Theorem 5.3.(1) and statement 2. $\square$

We say that subgroups E and F of a group G *commute* if $ef = fe$ for all elements $e \in E$ and $f \in F$. By (45), the subgroups $\mathcal{T}_n'$ and $\mathbb{E}_n$ commute.

Corollary 5.5 1. *The group* $\mathrm{TAut}_K(P_n)_n$ *is not a normal subgroup of the group* G_n.

2. *None of the groups* $\mathbb{F}'_n \times \mathbb{E}_n$, $\mathbb{F}_n \times \mathbb{E}_n$, $\mathbb{F}'_n$, $\mathbb{F}_n$; $\mathbb{E}_n$ *and* $\mathbb{E}_{n,i}$ *where* $2 \leq i \leq n-1$ *and* $n \geq 3$ *is a normal subgroup of the group* G_n.

Proof. The corollary follows from Theorem 5.3.(1), Lemma 5.1 and Lemma 5.2. $\square$

A formula for multiplication of two elements of the group G_n. Since the group G_n is the iterated semi-direct product of four groups and one of them is $\mathrm{UAut}_K(P_n)_n$ which is also a semi-direct product of n of its subgroups, the multiplication in the group G_n looks messy. Surprisingly, it looks less messy than one might expect when we change the order in the presentation of an element of the group G_n as a product of four automorphisms. By Theorem 5.3.(1), every element σ of G_n can be written as the product

$$\sigma = \tau t e' f' \tag{47}$$

where $\tau = [a_1, \ldots, a_n] = e^{a_n\partial_n} \cdots e^{a_1\partial_1} \in \mathrm{UAut}_k(P_n)_n$ and $a_i \in P_{i-1}$ for $i = 1, \ldots, n-1$ and $a_n \in \mathfrak{m}_{n-1}$; $t = t_{(\lambda_1,\ldots,\lambda_n)} \in \mathbb{T}^n$; $e' = e'_2(s_2)\cdots e'_{n-1}(s_{n-1}) \in \mathbb{E}_n$ where $s_i = \sum_{j\geq 1} \nu_{ij}\partial_{i-1}^j \in \partial_{i-1}K[[\partial_{i-1}]]$ for $i = 2, \ldots, n-1$ and $\nu_{ij} \in K$; and $f' = \prod_{i\geq 2} e^{\mu_i \partial_{n-1}^i} = 1 + \sum_{i\geq 2} \mu'_i \partial_{n-1}^i \in \mathbb{F}'_n = 1 + \partial_{n-1}^2 K[[\partial_{n-1}]]$. To make notations simpler and computations more transparent we write $[a_i]$ for $e^{a_i\partial_i}$ sometimes.

$$e'f'[a_1, \ldots, a_n](e'f')^{-1} = e^{(f'(a_n)+\sum_{i=2}^{n-1} s_i(a_i))\partial_n}[a_1, \ldots, a_{n-1}] = e^{((-1+f')(a_n)+\sum_{i=2}^{n-1} s_i(a_i))\partial_n}[a_1, \ldots, a_n]. \tag{48}$$

In more detail, the second equality is obvious. Using (45) and (46) we obtain the first one:

$$\begin{aligned} e'f'[a_1, \ldots, a_n](e'f')^{-1} &= e'f'[a_n][a_1, \ldots, a_{n-1}](e'f')^{-1} = f'[a_n]f'^{-1} \cdot e'[a_1, \ldots, a_{n-1}]e'^{-1} \\ &= e^{f'(a_n)\partial_n} \cdot e'[a_{n-1}]e'^{-1} \cdots e'[a_i]e'^{-1} \cdots e'[a_2]e'^{-1} \cdot a_1 \\ &= e^{f'(a_n)\partial_n} \cdot e^{s_{n-1}(a_{n-1})\partial_n}[a_{n-1}] \cdots e^{s_i(a_i)\partial_n}[a_i] \cdots e^{s_2(a_2)\partial_n}[a_2] \cdot a_1 \\ &= e^{(f'(a_n)+\sum_{i=2}^{n-1} s_i(a_i))\partial_n}[a_1, \ldots, a_{n-1}]. \end{aligned}$$

Let $\sigma_1 = \tau_1 t_1 e'_1 f'_1$ be another element of the group G_n where $\tau_1 = [b_1, \ldots, b_n]$. Then using (48) we obtain the formula for multiplication in the group G_n:

$$\sigma\sigma_1 = \tau e^{t(((-1+f')(b_n)+\sum_{i=2}^{n-1} s_i(b_i))\partial_n)}\omega_t(\tau_1) \cdot tt_1 \cdot \omega_{t_1^{-1}}(e')e'_1 \cdot \omega_{t_1^{-1}}(f')f'_1 \tag{49}$$

where $\omega_{t_1^{-1}}(g) = t_1^{-1}gt_1$.

Characterizations of the subgroups $\mathbb{F}_n$, $\mathbb{F}'_n$ and $\mathbb{E}_n$. By (Corollary 3.12, [2]), the ideal $I_{\omega^{n-2}+1} = Kx_{n-1}\partial_n + \sum_{\alpha\in\mathbb{N}^{n-2}} Kx^\alpha\partial_n$ is the least ideal of the Lie algebra $\mathfrak{u}_n$ which is a faithful $\mathfrak{u}_{n-1}$-module (with respect to the adjoint action). Hence, its predecessor $I_{\omega^{n-2}} = \sum_{\alpha\in\mathbb{N}^{n-2}} Kx^\alpha\partial_n$ is the largest ideal of the Lie algebra $\mathfrak{u}_n$ which is *not* a faithful $\mathfrak{u}_{n-1}$-module. By (Corollary 5.4, [2]), the ideal $P_{n-1}\partial_n$ is the least ideal I of the Lie algebra $\mathfrak{u}_n$ such that the Lie factor algebra $\mathfrak{u}_n/I$ is isomorphic to the Lie algebra $\mathfrak{u}_{n-1}$. The next proposition gives characterizations of the groups $\mathbb{F}_n$, $\mathbb{F}'_n$ and $\mathbb{E}_n$.

Proposition 5.6 *Let* $n \geq 2$. *Then*

1. $\mathbb{F}'_n = \mathrm{Fix}_{G_n}(\mathfrak{u}_{n-1} + I_{\omega^{n-2}+1})$ *where* $I_{\omega^{n-2}+1} = Kx_{n-1}\partial_n + \sum_{\alpha\in\mathbb{N}^{n-2}} Kx^\alpha\partial_n$.
2. $\mathbb{F}_n = \mathrm{Fix}_{G_n}(\mathfrak{u}_{n-1} + I_{\omega^{n-2}})$ *where* $I_{\omega^{n-2}} = \sum_{\alpha\in\mathbb{N}^{n-2}} Kx^\alpha\partial_n$.
3. $\mathbb{E}_n = \mathrm{Fix}_{G_n}(\mathcal{D}_n + P_{n-1}\partial_n)$ *where* $\mathcal{D}_n = \sum_{i=1}^n K\partial_i$.

Proof. 1. Let R be the RHS of the equality in statement 1. The inclusion $\mathbb{F}'_n \subseteq R$ is obvious. Since $\partial_1, \ldots, \partial_n \in \mathfrak{u}_{n-1} + I_{\omega^{n-2}+1}$, we have the inclusion $R \subseteq \mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \mathcal{F}_n$. By Theorem 4.12.(1),

$$\mathcal{F}_n = \mathrm{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n = \mathrm{Sh}_{n-2} \times (\mathrm{sh}_{n-1} \times \mathbb{F}'_n) \times \mathbb{E}_n = \mathrm{Sh}_{n-1} \times \mathbb{F}'_n \times \mathbb{E}_n.$$

Now, $R = \mathbb{F}'_n \times (R \cap (\mathrm{Sh}_{n-1} \times \mathbb{E}_n))$. By looking at the action of the elements of the group $\mathrm{Sh}_{n-1} \times \mathbb{E}_n$ on the elements $x_1\partial_2, x_2\partial_3, \ldots, x_{n-1}\partial_n$ of the Lie algebra $\mathfrak{u}_{n-1} + I_{\omega^{n-2}+1}$, we conclude that $R \cap (\mathrm{Sh}_{n-1} \times \mathbb{E}_n) = R \cap \mathbb{E}_n$. Every element of the group $\mathbb{E}_n$ is uniquely determined by its action on $\mathfrak{u}_{n-1}$. Therefore, $R \cap \mathbb{E}_n = \{e\}$, i.e., $R = \mathbb{F}'_n$.

2. Let R be the RHS of the equality in statement 2. Since $\mathfrak{u}_{n-1} + I_{\omega^{n-2}} \subseteq \mathfrak{u}_{n-1} + I_{\omega^{n-2}+1}$, $\mathbb{F}'_n = \mathrm{Fix}_{G_n}(\mathfrak{u}_{n-1} + I_{\omega^{n-2}+1}) \subseteq \mathrm{Fix}_{G_n}(\mathfrak{u}_{n-1} + I_{\omega^{n-2}}) = R$, by statement 1. Clearly, $\mathrm{sh}_{n-1} \subseteq R$. Therefore, $\mathbb{F}_n = \mathrm{sh}_{n-1} \times \mathbb{F}'_n \subseteq R$. Since $\partial_1, \ldots, \partial_n \in \mathfrak{u}_{n-1} + I_{\omega^{n-2}}$, we have the inclusion $R \subseteq \mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \mathcal{F}_n$. By Theorem 4.12.(1), $\mathcal{F}_n = \mathrm{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n$. Now, $R = \mathbb{F}_n \times (R \cap (\mathrm{Sh}_{n-2} \times \mathbb{E}_n))$. By looking at the action of the elements of the group $\mathrm{Sh}_{n-2} \times \mathbb{E}_n$ on the elements $x_1\partial_2, x_2\partial_3, \ldots, x_{n-2}\partial_{n-1}$ of the Lie algebra $\mathfrak{u}_{n-1} + I_{\omega^{n-2}}$, we conclude that $R \cap (\mathrm{Sh}_{n-2} \times \mathbb{E}_n) = R \cap \mathbb{E}_n$. Every element of the group $\mathbb{E}_n$ is uniquely determined by its action on $\mathfrak{u}_{n-1}$. Therefore, $R \cap \mathbb{E}_n = \{e\}$, i.e., $R = \mathbb{F}_n$.

3. Let R be the RHS of the equality in statement 3. Then $\mathbb{E}_n \subseteq R$. Clearly, $R \subseteq \mathrm{Fix}_{G_n}(\partial_1, \ldots, \partial_n) = \mathcal{F}_n = \mathrm{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n$. Now, $R = \mathbb{E}_n \times (R \cap (\mathrm{Sh}_{n-2} \times \mathbb{F}_n)) = \mathbb{E}_n$ since $R \cap (\mathrm{Sh}_{n-2} \times \mathbb{F}_n) = \{e\}$, by looking at the action of the group $\mathrm{Sh}_{n-2} \times \mathbb{F}_n$ on the ideal $P_{n-1}\partial_n$. $\square$

6 The canonical decomposition for an automorphism of the Lie algebra $\mathfrak{u}_n$

By Theorem 4.13.(1), every automorphism $\sigma \in G_n = \mathbb{T}^n \ltimes (\mathcal{T}_n \times_{ex} (\mathrm{Sh}_{n-2} \times \mathbb{F}_n \times \mathbb{E}_n))$ is the unique product $\sigma = t\tau s f e'$ where $t \in \mathbb{T}^n$, $\tau \in \mathcal{T}_n$, $s \in \mathrm{Sh}_{n-2}$, $f \in \mathbb{F}_n$ and $e' \in \mathbb{E}_n$. This product is called the *canonical decomposition* for the automorphism $\sigma \in G_n$. It is a trivial observation that every automorphism of a Lie algebra is uniquely determined by its action on any generating set for the Lie algebra. Our goal is to find explicit formulas for the automorphisms t, τ, s, f and e' via the elements $\{\sigma(s) \,|\, s \in S_n\}$ where S_n is a certain set of generators for the Lie algebra $\mathfrak{u}_n$ (Theorem 6.1).

Theorem 6.1 *Let $n \geq 2$.*

1. *The set $S_n := \{\partial_1, x_1^j\partial_2, \ldots, x_{i-1}^j\partial_i, \ldots, x_{n-1}^j\partial_n \,|\, j \in \mathbb{N}\}$ is a set of generators for the Lie algebra $\mathfrak{u}_n$.*
2. *Let $\sigma \in G_n$ and $\sigma = t\tau s f e'$ be its canonical decomposition. Below, explicit formulas are given for the automorphisms t, τ, s, f and e' via the elements $\{\sigma(s) \,|\, s \in S_n\}$.*
   - (a) *$t = t_{(\lambda_1,\ldots,\lambda_n)}$ where $\sigma(\partial_i) = \lambda_i^{-1}\partial_i + \cdots$ for $i = 1, \ldots, n$ where the three dots mean smaller terms with respect to the ordering (i.e., an element of $\oplus_{j>i}P_{j-1}\partial_j$);*
   - (b) *$\tau : P_n \to P_n$, $x_i \mapsto x'_i$, where $x'_1 = x_1$ and $x'_i := \phi_{i-1}\phi_{i-2}\cdots\phi_1(x_1)$ for $i = 2, \ldots, n$; $\phi_i := \sum_{k\geq 0}(-1)^k \frac{x_i'^k}{k!}\partial_i'^k$ and $\partial'_i := t^{-1}\sigma(\partial_i)$ for $i = 1, \ldots, n-1$;*
   - (c) *$f = 1 + \sum_{i\geq 1} f_i\delta_{n-1}^i$ where $\delta_{n-1} = \mathrm{ad}(\partial_{n-1})$, $f_i \in K$ and $f_i\partial_n = \Phi_{n-1}(t\tau)^{-1}\sigma(\frac{x_{n-1}^i}{i!}\partial_n)$ where $\Phi_{n-1} := \sum_{k\geq 0}(-1)^k\frac{x_{n-1}^k}{k!}\delta_{n-1}^k$ and $\delta_{n-1} = \mathrm{ad}(\partial_{n-1})$.*
   - (d) *$s(x_i) = x_i + \mu_i$ for $i = 1, \ldots, n-2$ where $(t\tau f)^{-1}\sigma(x_i\partial_{i+1}) = \mu_i\partial_{i+1} + x_i\partial_{i+1} + \cdots$ (the three dots denote an element of $\oplus_{j>i}P_{j-1}\partial_j$);*
   - (e) *by (37) and (38), $e' = e'_2 \cdots e'_{n-1}$ is the unique product where $e'_i \in \mathbb{E}_{n,i}$ for $i = 2, \ldots, n-1$, and, for all $j = 1, \ldots, n$ and $\alpha \in \mathbb{N}^{j-1}$,*

$$e'_i(x^\alpha\partial_j) = \begin{cases} x^\alpha\partial_i + s_i(x^\alpha)\partial_n & \text{if } j = i, \\ x^\alpha\partial_j & \text{if } j \neq i, \end{cases}$$

where $s_i = \sum_{j\geq 1}\nu_{ij}\partial_{i-1}^j$, $\nu_{ij} \in K$, $\nu_{ij}\partial_n = \Phi_{i-1}((t\tau s f)^{-1}\sigma - 1)(\frac{x_{i-1}^j}{j!}\partial_i)$, $\Phi_{i-1} := \sum_{k\geq 0}(-1)^k\frac{x_{i-1}^k}{k!}\delta_{i-1}^k$ and $\delta_{i-1} = \mathrm{ad}(\partial_{i-1})$.

Proof. 1. We use induction on $n \geq 2$. The initial case $n=2$ is obvious as the set S_2 is a K-basis for the Lie algebra $\mathfrak{u}_2$. Suppose that $n > 2$ and the result holds for all $n' < n$. By induction, S_{n-1} is a set of generators for the Lie algebra $\mathfrak{u}_{n-1}$. Notice that $\mathfrak{u}_n = \mathfrak{u}_{n-1} \oplus P_{n-1}\partial_n$ and, for all elements $\alpha \in \mathbb{N}^{n-2}$ and $j \in \mathbb{N}$, $x^\alpha x_{n-1}^j \partial_n = [x^\alpha \partial_{n-1}, (j+1)^{-1} x_{n-1}^{j+1}\partial_n]$. To finish the proof of statement 1 notice that $S_n = S_{n-1} \cup \{x_{n-1}^j \partial_n \,|\, j \in \mathbb{N}\}$ and the set of elements $\{x^\alpha x_{n-1}^j \partial_n\}$ is a K-basis for the vector space $P_{n-1}\partial_n$.

2. Statement (a) is obvious (Proposition 3.1). For all elements $i = 1, \ldots, n$, $sfe(\partial_i) = \partial_i$ (since $sfe \in \mathcal{F}_n$, Theorem 4.12.(1)). Then $\tau(\partial_i) = \tau sfe(\partial_i) = t^{-1}\sigma(\partial_i)$ and statement (b) follows from Theorem 3.6.(2). The automorphisms s and e' act as the identity map on the vector space $V := K[x_{n-1}]\partial_n$. Therefore, $f|_V = fse'|_V = sfe'|_V = (t\tau)^{-1}\sigma|_V : V \to V$ and $f = 1 + \sum_{i\geq 1} f_i \delta_{n-1}^i$ for some scalars $f_i \in K$. By Proposition 4.4, $f_i \partial_n = \Phi_{n-1}(t\tau)^{-1}\sigma(\frac{x_{n-1}^i}{i!}\partial_n)$. This finishes the proof of statement (c).

For all elements $i = 1, \ldots, n-2$,

$$(t\tau f)^{-1}\sigma(x_i\partial_{i+1}) = se(x_i\partial_{i+1}) = s(x_i\partial_{i+1} + \cdots) = \mu_i\partial_{i+1} + x_i\partial_{i+1} + \cdots ,$$

and statement (d) follows.

By (37) and (38), $e' = e_2' \cdots e_{n-1}'$ is the unique product where $e_i' \in \mathbb{E}_{n,i}$ for $i = 2, \ldots, n-1$, and, for all $j = 1, \ldots, n$ and $\alpha \in \mathbb{N}^{j-1}$,

$$e_i'(x^\alpha \partial_j) = \begin{cases} x^\alpha \partial_i + s_i(x^\alpha)\partial_n & \text{if } j = i, \\ x^\alpha \partial_j & \text{if } j \neq i, \end{cases}$$

where $s_i = \sum_{j\geq 1} \nu_{ij}\partial_{i-1}^j$ and $\nu_{ij} \in K$. For each $i = 0, \ldots, n-1$,

$$(e-1)|_{K[x_{i-1}]\partial_i} = ((t\tau sf)^{-1}\sigma - 1)|_{K[x_{i-1}]\partial_i} : K[x_{i-1}]\partial_i \to K[x_{i-1}]\partial_n, \;\; p\partial_i \mapsto s_i(p)\partial_n$$

where $p \in K[x_{i-1}]$. By Proposition 4.4, $\nu_{ij}\partial_n = \Phi_{i-1}((t\tau sf)^{-1}\sigma - 1)(\frac{x_{i-1}^j}{j!}\partial_i)$. $\square$

7 The adjoint group of automorphisms of the Lie algebra $\mathfrak{u}_n$

The aim of this section is to show that the adjoint group $\mathcal{A}(\mathfrak{u}_n)$ of automorphisms of the Lie algebra $\mathfrak{u}_n$ is equal to the group $\mathrm{UAut}_K(P_n)_n$ (Theorem 7.1).

Let $\mathcal{G}$ be a Lie algebra over the field K and $\mathrm{LN}(\mathcal{G})$ be the set of locally nilpotent elements of the Lie algebra $\mathcal{G}$. Recall that an element $g \in \mathcal{G}$ is called a *locally nilpotent element* if the inner derivation $\mathrm{ad}(g)$ of the Lie algebra $\mathcal{G}$ is a locally nilpotent derivation. The set $\mathrm{LN}(\mathcal{G})$ is an $\mathrm{Aut}_K(\mathcal{G})$-invariant set. Each locally nilpotent element g yields the automorphism $e^{\mathrm{ad}(g)} := \sum_{i\geq 0} \frac{\mathrm{ad}(g)^i}{i!}$ of the Lie algebra $\mathcal{G}$ which is called an *inner automorphism* of the Lie algebra $\mathcal{G}$. The subgroup of $\mathrm{Aut}_K(\mathcal{G})$, $\mathcal{A}(\mathcal{G}) := \langle e^{\mathrm{ad}(g)} \,|\, a \in \mathrm{LN}(\mathcal{G})\rangle$, is called the *adjoint group* (of automorphisms) of the Lie algebra $\mathcal{G}$. The adjoint group $\mathcal{A}(\mathcal{G})$ is a *normal* subgroup of the group $\mathrm{Aut}_K(\mathcal{G})$ since $\sigma e^{\mathrm{ad}(g)}\sigma^{-1} = e^{\mathrm{ad}(\sigma(g))}$ for all automorphisms $\sigma \in \mathrm{Aut}_K(\mathcal{G})$.

The aim of this section is to prove the next theorem.

Theorem 7.1 *1.* $\mathcal{A}(\mathcal{G}) = \mathrm{UAut}_K(P_n)_n$.

2. The map $\mathrm{UAut}_K(P_n)_n \to \mathcal{A}(\mathcal{G})$, $e^a \mapsto e^{\mathrm{ad}(a)}$, *is the identity map where* $a \in \mathfrak{u}_n'$ *(recall that* $\mathrm{UAut}_K(P_n)_n \subset G_n$*), i.e., for all elements* $u \in \mathfrak{u}_n$, $e^a u e^{-a} = e^{\mathrm{ad}(a)}(u)$.

The proof of Theorem 7.1, which is given at the end of the section, is based on the following proposition that is interesting on its own.

The Lie algebra $\mathrm{Der}_K(P_n)$ is a left P_n-module, and so $P_n\mathfrak{u}_n \subseteq \mathrm{Der}_K(P_n)$. The polynomial algebra P_n is a left $\mathrm{Der}_K(P_n)$-module and a left $\mathfrak{u}_n$-module. The action of an element $\delta \in \mathrm{Der}_K(P_n)$ on the polynomial algebra P_n is denoted either by $\delta * p$ or $\delta(p)$. Every element $u \in \mathfrak{u}_n$ is a locally nilpotent derivation of the polynomial algebra P_n (Proposition 2.1.(4)). Then $e^u \in \mathrm{Aut}_K(P_n)$.

Proposition 7.2 *Let* $u, v \in \mathfrak{u}_n$ *and* $p \in P_n$. *Then*

1. $e^u(v * p) = e^{\mathrm{ad}(u)}(v) * e^u(p)$ *(where* $e^u \in \mathrm{Aut}_K(P_n)$*).*
2. $e^{\mathrm{ad}(u)}(v) = e^u v e^{-u}$ *(where* $e^{\mathrm{ad}(u)} \in G_n$*).*
3. $e^{\mathrm{ad}(u)}(pv) = e^u(p) e^{\mathrm{ad}(u)}(v)$.

Proof. 1.

$$\begin{aligned} e^u(v * p) &= (\sum_{i\geq 0} \frac{u^i}{i!})v(p) = \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}\mathrm{ad}(u)^j(v)u^{i-j}(p) \\ &= \sum_{i\geq 0}\sum_{j+k=i}\frac{\mathrm{ad}(u)^j}{j!}(v)\frac{u^k}{k!}(p) = e^{\mathrm{ad}(u)}(v) * e^u(p). \end{aligned}$$

2. Recall that $e^{-u} \in \mathrm{UAut}_K(P_n)_n$. In statement 1, replacing the polynomial p by the polynomial $e^{-u}(p)$, we have the equality $e^u v e^{-u} * p = e^{\mathrm{ad}(u)}(v) * p$, for all polynomials $p \in P_n$. Therefore, $e^u v e^{-u} = e^{\mathrm{ad}(u)}(v)$.

3. For all natural numbers $s \gg 0$, $\mathrm{ad}(u)^s(pv) = 0$, $u^s(p) = 0$ and $\mathrm{ad}(u)^s(v) = 0$. So, the infinite sums below are finite sums:

$$\begin{aligned} e^{\mathrm{ad}(u)}(pv) &= (\sum_{i\geq 0}\frac{\mathrm{ad}(u)^i}{i!})(pv) = \sum_{i\geq 0}\frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}u^j(p)\mathrm{ad}(u)^{i-j}(v) \\ &= \sum_{i\geq 0}\sum_{j+k=i}\frac{u^j}{j!}(p)\frac{\mathrm{ad}(u)^k}{k!}(v) = e^u(p)e^{\mathrm{ad}(u)}(v). \ \square \end{aligned}$$

Proof of Theorem 7.1. 1. By Proposition 3.2, the map $\mathfrak{u}_n \to \mathrm{UAut}_K(P_n)$, $u \mapsto e^u$, is a bijection. In particular, $\mathrm{UAut}_K(P_n) = \{e^u \,|\, u \in \mathfrak{u}_n\}$. By Proposition 3.3.(2), the map (where $v \in \mathfrak{u}_n$)

$$\exp : \mathrm{UAut}_K(P_n) \to G_n, \quad e^u \mapsto (v \mapsto e^u v e^{-u}),$$

is a group homomorphism such that $\ker(\exp) = \mathrm{sh}_n$ and $\mathrm{im}(\exp) \simeq \mathrm{UAut}_K(P_n)/\mathrm{sh}_n = \mathrm{UAut}_K(P_n)_n$. By Proposition 7.2.(2), $e^u v e^{-u} = e^{\mathrm{ad}(u)}(v)$ for all elements $u, v \in \mathfrak{u}_n$. It follows from this fact that, for all elements $u_1, \ldots, u_s \in \mathfrak{u}_n$,

$$e^{\mathrm{ad}(u_1)}\cdots e^{\mathrm{ad}(u_s)}(v) = e^{u_1}\cdots e^{u_s} v (e^{u_1}\cdots e^{u_s})^{-1}.$$

By Proposition 3.2.(2), $e^{u_1}\cdots e^{u_s} = e^u$ for some element $u \in \mathfrak{u}_n$. Then, $e^{\mathrm{ad}(u_1)}\cdots e^{\mathrm{ad}(u_s)}(v) = e^u v e^{-u}$, i.e., $e^{\mathrm{ad}(u_1)}\cdots e^{\mathrm{ad}(u_s)} \in \mathrm{UAut}_K(P_n)_n$.

2. Statement 2 follows from Proposition 7.2.(2). $\square$

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