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Lie Algebras and the Stability of Nonlinear Systems

by

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Abstract

Lie algebras and the Cartan decomposition are used to study the stability of 'pseudo-linear' systems of differential equations.

Keywords:Lie Algebras, Nonlinear Systems.

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1 Introduction

In this paper we shall study the stability of the general nonlinear system

$$\dot{x} = f(x) \quad , \quad x \in \mathbf{R}^n \tag{1.1}$$

and, for simplicity, we shall assume that $x = 0$ is an isolated equilibirum point of (1.1), i.e:

$$f(0) = 0. \tag{1.2}$$

If f is analytic we may then write the equation (1.1) in the form

$$\dot{x} = A(x)x \quad , \quad x \in \mathbf{R}^n \tag{1.3}$$

where $A : \mathbf{R}^n \longrightarrow \mathbf{R}^{n^2}$ is a matrix-valued analytic function. We shall study systems of the form (1.3) by the use of Lie algebras. In fact we shall not generally assume that $A(x)$ is analytic and so we consider the system (1.3) in the case when A is merely continuous. As usual, $g\ell(\mathbf{R}^n)$ will denote the standard matrix Lie algebra consisting of all elements of $\mathbf{R}^{n^2}$ under the Lie bracket

$$[A, B] = AB - BA.$$

For any subset $S \subseteq \mathbf{R}^{n^2}$ we shall denote by $\mathcal{L}(S)$ the Lie subalgebra of $g\ell(\mathbf{R}^n)$ generated by S (i.e the Lie algebra given by the intersection

$$\mathcal{L}(S) = \cap_{S \subseteq \mathcal{L}} \mathcal{L}$$

of all Lie algebras containing S). Thus, if $A : \mathbf{R}^n \longrightarrow \mathbf{R}^{n^2}$ is a continuous matrix-valued function, as above, and the range of A is denoted by $\mathcal{R}(A)$, we define

$$\mathcal{L}_A = \mathcal{L}(\mathcal{R}(A)). \tag{1.4}$$

When considering the stability of (1.3) we may be tempted to conjecture that if the eigenvalues of $A(x)$ (as functions of x) are all strictly negative for all x, then the system is stable. This is, of course, false and we shall give a simple counterexample in the next section, which will lead, in the following section to a description of the case where the Lie algebra $\mathcal{L}_A$ is solvable. We shall then consider the case where $\mathcal{L}_A$ is semisimple, using the Cartan decomposition and give some examples.

The control theory of systems of the form (1.3) has been considered in ([1]) and the theory of Lie groups and Lie algebras has, of course, been applied to the symmetry properties of nonlinear systems ([2], [3]). However, it has not been used previously to study pseudo-linear systems of the form considered here.

2 A Counterexample

Consider the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -\lambda & \alpha(x_2) \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.1}$$

where

$$\alpha(x_2) = 1/x_2^2.$$

(Of course, this system has a singularity at $x_2 = 0$—we shall remedy this shortly).

For any initial condition (x_{10},x_{20}) with $x_{20} \neq 0$ this equation has the solution

$$\begin{aligned} x_2(t) &= e^{-\lambda t} x_{20} \\ x_1(t) &= e^{-\lambda t} x_{10} + \int_0^t e^{-\lambda(t-s)} \alpha(x_2(s)) x_2(s) ds \end{aligned}$$

Hence

$$\begin{aligned} x_1(t) &= e^{-\lambda t} x_{10} + \int_0^t e^{-\lambda(t-s)} \frac{e^{2\lambda s}}{x_{20}^2} e^{-\lambda s} x_{20} ds \\ &= e^{-\lambda t} x_{10} + \frac{e^{-\lambda t}}{x_{20}} \int_0^t e^{2\lambda s} ds \\ &= e^{-\lambda t} x_{10} + \frac{e^{-\lambda t}}{2\lambda x_{20}} (e^{2\lambda t} - 1) \\ &\longrightarrow \infty \end{aligned}$$

as $t \longrightarrow \infty$ (if $x_{20} > 0$,$\lambda > 0$). Suppose we take $x_{10} = 1$,$x_{20} = 1/2$,$\lambda = 1$.Then we have

$$\begin{aligned} x_1(t) &= e^t \\ x_2(t) &= \frac{1}{2} e^{-t} \end{aligned}$$

so that

$$x_1 x_2 = 1/2.$$

Let Ω be the set

$$\Omega = \{(x_1, x_2) : \ x_1 \ , \ x_2 > 0 \ , \ x_1 x_2 = 1/2\},$$

and define the function β by

$$\beta(x_1, x_2) = \begin{cases} \frac{1}{x_2^2} & , (x_1, x_2) \in \Omega \\ \gamma(x_1, x_2) & , (x_1, x_2) \notin \Omega \end{cases} \tag{2.2}$$

where γ is a C^∞ function which makes β continuous on the curve $x_1 x_2 = 1/2$ and for which

$$\gamma(x_1, x_2) = 0 \qquad \text{for } (x_1, x_2) \notin \Omega_\varepsilon$$

where Ω_ε is an ε-neighbourhood of Ω. It follows that the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & \beta(x_1, x_2) \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2.3}$$

is unstable since it is identically equal to (2.1) for the initial values $x_{10} = 1$, $x_{20} = 1/2$.

It follows that, given a system

$$\dot{x} = A(x)x \tag{2.4}$$

it is *not* sufficient for stability that the eigenvalues of $A(x)$ be all strictly negative (or even negative constants). Note that the function β given by (2.2) can easily be modified to be C^∞ everywhere, so that the system (2.4) is not necessarily globally stable even if the components of $A(x)$ are C^∞. In the next section we shall show that continuous triangular systems with negative eigenvalues functions are asymptotically stable at the origin.

3 Triangular Systems and Solvable Lie Algebras

In this section we shall consider the triangular system

$$\dot{x} = A(x)x \tag{3.1}$$

where

$$A(x) = \begin{pmatrix} -\lambda_1(x) & \alpha_{12}(x) & \cdots & \cdots & \cdots & \alpha_{1n}(x) \\ 0 & -\lambda_2(x) & \alpha_{23}(x) & \cdots & \cdots & \alpha_{2n}(x) \\ \vdots & & \ddots & & & \vdots \\ & & & & & \\ 0 & \cdots & \cdots & \cdots & \cdots & -\lambda_n(x) \end{pmatrix}.$$

We assume that all the elements of $A(x)$ are continuous in $x \in \mathbf{R}^n$ and that

$$\lambda_i(x) \geq \lambda > 0 \ , \ 1 \leq i \leq n \tag{3.2}$$

for all $x \in \mathbf{R}^n$ and some constant λ.

Theorem 3.1 Under the conditions just stated the origin is an asymptotically stable of equation (3.1) that 0 is an equilibrium point of (3.1) is trivial.

Proof That 0 is an equilibrium point of (3.1) is trivial. We shall show that the solution $x(t;x_0)$ of (3.1) through x_0 satisfies

$$| \, x_i(t, x_0) \, | \leq \delta e^{-\lambda t} p_i(t) \ ,1 \leq i \leq n \tag{3.3}$$

on some interval $t \in [0, \tau]$ where $\delta = \| x_0 \|$ and $p_i(t)$ is a polynomial in t, provided $\| x(t; x_0) \| < 1$. From this the result will follow since the functions

$$f_i(t) = e^{-\lambda t} p_i(t)$$

are all bounded for $t \geq 0$ and if we are put

$$K = \max_i \sup_k f_i(t)$$

we have by (3.3)

$$| x(t, x_0) | \leq \delta K \quad , \quad t > 0$$

and so if $\delta < 1/K$ we have

$$\| x(t, x_0) \| < 1 \ .$$

Hence (3.3) is valid for all $t \geq 0$ and we are done.

To prove (3.3) assume it is true for $n \geq i \geq j$.Then

$$\dot{x_j}(t) \leq -\lambda x_{j-1} + \sum_{\ell=j}^{n} \alpha_{j-1,\ell}(x) x_\ell(t) \ ,$$

i.e

$$\begin{aligned} | x_{j-1}(t) | &\leq e^{-\lambda t} | x_{j-1}(0) | + \sum_{\ell=j}^{n} \int_0^t e^{-\lambda(t-s)} M_{j-1,\ell} \delta e^{-\lambda s} p_\ell(s) ds \\ &\leq \delta e^{-\lambda t} \left(1 + \sum_{\ell=j}^{n} M_{j-1,\ell} \int_0^t p_\ell(s) ds \right) \end{aligned}$$

where $M_{i,j}$ is the maximum of the function $| \alpha_{ij}(x) |$ for $\| x \| \leq 1$, and the result follows by induction. □

Theorem 3.1 leads us to study equations of the form

$$\dot{x} = A(x) x \tag{3.4}$$

where $A(x)$ can be triangularized by a change of coordinates. Recall definition (1.4):

$$\mathcal{L}_A = \mathcal{L}(R(A)).$$

Theorem 3.2 If the Lie algebra $\mathcal{L}_A$ is solvable and the eigenvalues of $A(x)$ are strictly negative for all $x \in \mathbf{R}^n$, i.e.

$$\lambda_i(x) \geq \lambda > 0 \ ,1 \leq i \leq n \ ,$$

then the origin is an asymptotically stable equilibrium point of (3.4).

Proof. By Lie's theorem ([4]) if g is a solvable subalgebra of the Lie algebra $g\ell(\mathbf{R}^n)$, then there exist a basis of $\mathbf{R}^n$ such that all the elements of g in this basis are upper triangular. In these new coordinates the 'A' matrix in (3.5) will be of the same form as that in (3.1) and the result follows from theorem 3.1 □.

We next recall Cartan's criterion for solvability. Denote by

$$\mathcal{D}g = [g, g]$$

(the **derived algebra** of a Lie algebra g). Also we define the **Killing form** (X, Y) of g by

$$(X, Y) = \text{Tr ad } X \text{ ad } Y$$

where Tr denotes the trace of a linear operator and ad X is the linear operator defined by

$$(\text{ad } X)A = [X, A] \ , \ A \in g.$$

Cartan's criterion for solvable Lie algebra ([4]) states that g is solvable iff $(X, X) = 0$ for all $X \in \mathcal{D}g$.

In order to study Cartan's criterion in relation to the Lie algebra $\mathcal{L}_A$, we need to consider the latter in more detail. Let E_{ij} be the matrix which is zero apart from a 1 in the $(i,j)^{th}$ place. Then, if $A(x) = (a_{ij}(x))$ we can write

$$A(x) = \sum_{i,j=1}^{n} a_{ij}(x) E_{ij} \tag{3.5}$$

Let $\mathcal{M}_1$ denote the vector subspace of $\mathbf{R}^{n^2}$ generated by $\mathcal{R}(A)$.

Proposition3.3 $\mathcal{M}_1 = \mathbf{R}^{n^2}$ iff the functions $a_{ij}(x)$;$1 \leq i,j \leq n$ are linearly independent (as functions on $\mathbf{R}^{n^2}$).

Proof. if the functions a_{ij} are not linearly independent then we can write

$$\sum_{i,j} \alpha_{ij} a_{ij}(x) = 0 \ , \ \forall \ x \in \mathbf{R}^n$$

for some scalars α_{ij}, no all zero. We can assume that $\alpha_{11} \neq 0$; then

$$a_{11}(x) = \frac{1}{\alpha_{11}} \sum_{i,j \neq (1,1)} \alpha_{ij} a_{ij}(x)$$

and so

$$A(x) = \sum_{i,j \neq (1,1)} a_{ij}(x) \left(E_{ij} + \frac{\alpha_{ij}}{\alpha_{11}} E_{11} \right)$$

and the space $\mathcal{M}_1$ is generated by $\left\{ E_{ij} + \frac{\alpha_{ij}}{\alpha_{11}} E_{11} \right\}_{(i,j) \neq (1,1)}$.

To prove the converse, let $< \ , \ >$ be the standard inner product on $\mathbf{R}^{n^2}$, so that if $X = (x_{ij}), Y = (y_{ij}) \in \mathbf{R}^{n^2}$ then

$$< X, Y >= \sum_i \sum_j x_{ij} y_{ji} \ .$$

If $\mathcal{M}_1 \neq \mathbf{R}^{n^2}$ then there exists a nonzero matrix C orthogonal to $\mathcal{M}_1$, i.e.

$$< A(x), C >= \sum_{i,j=1}^{n} a_{ij}(x) < E_{ij}, C >= 0$$

for all $x \in \mathbf{R}^n$. However

$$C = \sum_{i,j=1}^{n} c_{i,j} E_{ij}$$

where not all the c_{ij}'s are zero. Hence

$$\begin{aligned} 0 &= \sum_{i,j=1}^{n} a_{ij}(x) < E_{ij}, \sum_{k,\ell=1}^{n} c_{k\ell} E_{k\ell} > \\ &= \sum_{i,j}^{n} c_{ij} a_{ij}(x) \end{aligned}$$

for all $x \in \mathbf{R}^n$ and the result follows. □

Corollary 3.4. If the functions $\{a_{ij}(x)\colon 1 \le i,j \le n\}$ are linearly independent then $\mathcal{L}_A$ cannot be solvable. □

In order to find the dimension and generators of $\mathcal{L}_A$ let

$$S = \{(i,j) : 1 \le i,j \le n\}$$

denote the set of ordered pairs of numbers from 1 to n. Suppose that the subset $S_1 \subseteq S$ is chosen so that the set

$$\{a_{ij}(x) : (i,j) \in S_1\}$$

is maximal linearly independent subset of the coefficient functions

$$\{a_{ij}(x) : (i,j) \in S\}.$$

Then any element $a_{k\ell}(x)$ for $(k,\ell) \in S \setminus S_1$ can be written

$$a_{k\ell}(x) = \sum_{(i,j)\in S_1} \alpha_{ij}^{k\ell} a_{ij}(x) \tag{3.6}$$

for some scalers $\alpha_{ij}^{k\ell}$. Hence by (3.5) we have

$$A(x) = \sum_{(i,j)\in S_1} a_{ij}(x) \left(E_{ij} + \sum_{(k,\ell)\in S\setminus S_1} \alpha_{ij}^{k\ell} E_{k\ell} \right)$$

Let

$$E_{ij}^{(1)} = E_{ij} + \sum_{(k,\ell)\in S\setminus S_1} \alpha_{ij}^{k\ell} E_{k\ell}$$

Then

$$A(x) = \sum_{(i,j)\in S_1} a_{ij}(x) E_{ij}^{(1)} \tag{3.7}$$

From (3.7) it follows that $\mathcal{M}_1$ is generated by the set $\{ E_{ij}^{(1)}; (i,j) \in S_1\}$.

Proposition 3.5 The set $\{ E_{ij}^{(1)} :(i,j) \in S_1\}$ is linearly independent in $\mathbf{R}^{n^2}$.

Proof. Suppose, on the contrary, that

$$\sum_{(i,j)\in S_1} \beta_{ij} E_{ij}^{(1)} = 0$$

for some scalers β_{ij} not all zero. Then, if $(p,q) \in S_1$

$$\sum_{(i,j)\in S_1} \beta_{ij} \langle E_{ij}^{(1)}, E_{pq}\rangle = 0 \tag{3.8}$$

But,

$$< E_{ij}^{(1)}, E_{pq} >= \delta_{ij}^{pq}.$$

Hence, by (3.8), $\beta_{pq} = 0$ for all $(p,q) \in S_1$, and the result follows □

We must next determine a basis for $[\mathcal{M}_1, \mathcal{M}_1] \ominus \mathcal{M}_1$.

To do this consider the bracket

$$\begin{aligned} [A(x), A(y)] &= [\sum_{(i,j)\in S_1} a_{ij}(x) E_{ij}^{(1)}, \sum_{(k,\ell)\in S_1} a_{k\ell}(y) E_{k\ell}^{(1)}] \\ &= \sum_{(i,j),(k\ell)\in S_1} a_{ij}(x) a_{k\ell}(y) [E_{ij}^{(1)}, E_{k\ell}^{(1)}] \end{aligned}$$

Let $P_{\mathcal{M}_1}[E_{ij}^{(1)}, E_{k\ell}^{(1)}]$ be the projection of $[E_{ij}^{(1)}, E_{k\ell}^{(1)}]$ on $\mathcal{M}_1$ and define

$$F_{ijk\ell} = (I - P_{\mathcal{M}_1})[E_{ij}^{(1)}, E_{k\ell}^{(1)}]$$

i.e. the part of $[E_{ij}^{(1)}, E_{k\ell}^{(1)}]$ off $\mathcal{M}$. Write

$$F_{ijk\ell} = \sum_{v,w} f_{vw}^{ijk\ell} E_{vw} \ ;$$

then

$$\begin{aligned}(I - P_{\mathcal{M}_1})[A(x), A(y)] &= \sum_{(i,j),(k,\ell)\in S_1} \sum_{v,w} a_{ij}(x) a_{k\ell}(y) f_{vw}^{ijk\ell} E_{vw} \\ &= \sum_{v,w} \sum_{(i,j),(k,\ell)\in S_1} \{a_{ij}(x) a_{k\ell}(y) f_{vw}^{ijk\ell}\} E_{vw}\end{aligned}$$

(Note that $P_{\mathcal{M}_1}$ is given explicitly as follows. We can apply Gram-Schmit's procedure to orthogonalize the $E_{ij}^{(1)}$. Call the orthogonolized matrices $\overline{E}_{ij}^{(1)}$.

$$P_{\mathcal{M}_1}([E_{vw}^{(1)}, E_{k\ell}^{(1)}]) = \sum_{(i,j)\in S_1} \alpha_{ij}^{vwk\ell} \overline{E}_{ij}^{(1)}$$

for some scalars α_{ij}. Then

$$\begin{aligned}< P_{\mathcal{M}_1}([E_{vw}^{(1)}, E_{k\ell}^{(1)}]), \overline{E}_{pq}^{(1)} > &= \sum_{(i,j)\in S_1} \alpha_{ij}^{vwk\ell} < \overline{E}_{ij}^{(1)}, \overline{E}_{pq}^{(1)} > \\ &= \alpha_{pq}^{vwk\ell}.\end{aligned}$$

Let $S_2 \subseteq S$ be chosen so that

$$\{b_{vw}(x,y) : \ (v,w) \in S_2\}$$

is a maximal linearly independent set of functions over $\mathbf{R}^n \times \mathbf{R}^n$ where

$$b_{vw}(x,y) = \sum_{(i,j),(k,\ell)\in S_1} a_{ij}(x) a_{k\ell}(y) f_{vw}^{ijk\ell} \tag{3.9}$$

Then,

$$(I - P_{\mathcal{M}_1})[A(x), A(y)] = \sum_{(i,j)\in S_2} b_{ij}(x,y)\left(E_{ij} + \sum_{(k,\ell)\in S\setminus S_2} \beta_{ij}^{k\ell} E_{k\ell}\right)$$

where we have written

$$b_{k\ell}(xy) = \sum_{(i,j)\in S_2} \beta_{ij}^{k\ell} b_{ij}(x,y) \; .$$

Let $E_{ij}^{(2)} = E_{ij} + \sum_{(k,\ell)\in S\setminus S_2} \beta_{ij}^{k\ell} E_{k\ell}$, $(i,j) \in S_2$. As before, the set $\{E_{ij}^{(2)}\}_{(i,j)\in S_2}$ is linearly independent in $\mathbf{R}^{n^2}$ and generates $[\mathcal{M}_1, \mathcal{M}_1] \ominus \mathcal{M}_1$.

We now have found a basis for $\mathcal{M}_2 \triangleq \mathcal{M}_1 \oplus ([\mathcal{M}_1, \mathcal{M}_1] \ominus \mathcal{M}_1)$, i.e. $\{E_{ij}^{(1)}\}_{(i,j)\in S_1}$, $\{E_{ij}^{(2)}\}_{(i,j)\in S_2}$. Continuing in this way we can repeat the above procedure with $\mathcal{M}_1$ replaced by $\mathcal{M}_2$ and consider a typical term of $[\mathcal{M}_1, \mathcal{M}_1]$ of the form

$$B = \left[\sum_{(i,j)\in S_1} a_{ij}(x) E_{ij}^{(1)} + \sum_{(i,j)\in S_2} b_{ij}(y,z) E_{ij}^{(2)}, \sum_{(i,j)\in S_1} a_{ij}(p) E_{ij}^{(1)} + \sum_{(i,j)\in S_2} b_{ij}(q,r) E_{ij}^{(2)}\right]$$

Then $(I - P_{\mathcal{M}_2})B$ is of the form

$$\sum_{v,w} C_{vw}(x,y,z,p,q,r) E_{vw}$$

for some set of functions c_{vw} over $\mathbf{R}^{6n}$ and we search for a linearly independent subset of them. Hence we have proved.

Theorem 3.6 Given a system

$$\begin{aligned} \dot{x} &= A(x)x \\ &= (a_{ij}(x))x \\ &= \sum_{i,j=1}^{n} a_{ij}(x) E_{ij} \end{aligned}$$

we can find a basis for $\mathcal{L}_A$ of the form $\{E_{ij}^{(1)}\}_{(i,j)\in S_1}, \{E_{ij}^{(2)}\}_{(i,j)\in S_2}, \dots \{E_{ij}^{(K)}\}_{(i,j)\in S_K}$, where the matrices $E_{ij}^{(1)}$ form a basis of $\mathcal{M}_1$, the linear subspace of $\mathbf{R}^{n^2}$ generated by $\mathcal{R}(A)$ and $E_{ij}^{(p)}$ is a basis of

$$[\mathcal{M}_{p-1}, \mathcal{M}_{p-1}] \ominus \mathcal{M}_{p-1} \quad .$$

With each set S_p there is associated a set of functions $a_{ij}^{p}(\overbrace{x, y, \dots, t}^{m_p})$. The elements $E_{ij}^{(1)}$ are found by choosing a basis $\{a_{ij}(x)\}_{(i,j)\in S_1}$ where the number of variables m_p is given by the relation

$$m_1 = n \quad , \quad m_k = 2(\sum_{i=1}^{k-1} m_i)$$

($a_{ij}^{1}(x)$ is just $a_{ij}(x)$ and $a_{ij}^{2}(x, y) = b_{ij}(x, y)$, given in (3.9)), of the linear space spanned by the functions $a_{ij}(x)$ and setting

$$E_{ij}^{(1)} = E_{ij} + \sum_{(k,\ell)\in S\setminus S_1} \alpha_{ij}^{k\ell} E_{k\ell}$$

where $\alpha_{ij}^{k\ell}$ are given by (3.6). The remainder are given inductively as follows. If $\{E_{ij}^{(p-1)}\}_{(i,j)\in S_{p-1}}$ and $\mathcal{M}_{p-1}$ have been found we can write a typical term of $[\mathcal{M}_{p-1}, \mathcal{M}_{p-1}]$ in the form

$$B = \left[\sum_{k=1}^{p-1} \sum_{(i,j)\in S_k} a_{ij}^{k} \, \overbrace{(x^{k,1}, y^{k,1}, \cdots, t^{k,1})}^{m_k} \, E_{ij}^{(k)} \, , \sum_{k=1}^{p-1} \sum_{(i,j)\in S_k} a_{ij}^{k}(x^{k,2}, \cdots, t^{k,2}) E_{ij}^{(k)} \right]$$

Then $(I - P_{\mathcal{M}_{p-1}})B$ can be written in the form

$$\sum_{i,j} a_{ij}^{p} \, \overbrace{(x, y, \cdots, t)}^{m_p} \, E_{ij}$$

and S_p is given as the set of suffices of a linearly independent set of the a_{ij}^{p}'s. □

To simplify the notation we shall write any element of $\mathcal{L}_A$ in the form

$$\sum_{i\in I} c_i(x^i)F_i$$

where the index set I is equal to $S_1 \vee \cdots \vee S_K$ (disjoint union), x^k is an m_k-vector variable and c_i is an approprate function of a^k_{ij} of the vector variable $x^i = (x, y, z, \cdots, t)$ (of appropriate size) and F_i is the corresponding basis element $E^{(k)}_{ij}$.

We are now in a position to study the condition for solvability of $\mathcal{L}_A$ in more detail. We require

$$(X, X) = \text{Tr ad X ad X} = 0$$

for each $X \in [\mathcal{L}_A, \mathcal{L}_A] = \mathcal{D}_{\mathcal{L}_A}$.

Proposition 3.7 If $\{e_i\}$ is a basis of a matrix Lie algebra L then, for any element $X \in L$ we have

$$(X, X) = \sum_{i=1}^{n} \alpha_{ii}$$

where

$$[X, [X, e_i]] = \sum_{j=1}^{n} \alpha_{ij} e_j \quad .$$

□

Now write

$$[F_i, F_j] = \sum_{k\in I} \gamma^k_{ij} F_k$$

i.e. γ^k_{ij} are the structure constants of $\mathcal{L}_A$ with respect to the basis $\{F_i\}_{i\in I}$. If

$[X, Y] \in \mathcal{D}_{\mathcal{L}_A}$ we have

$$X = \sum_{i \in I} c_i(x^i) F_i \ , Y = \sum_{j \in I} c_j(y^j) F_j$$

and so

$$\begin{aligned}
[[X, Y], [[X, Y], F]] &= \sum_{i,j,k,\ell \in I} c_i(x^i) c_j(y^j) c_k(x^k) c_\ell(y^\ell) [[E_i, E_j], [[F_k, F_\ell], F_p]] \\
&= \sum_{i,j,k,\ell \in I} \sum_{i',j',k',\ell' \in I} c_i(x^i) c_j(y^j) c_k(x^k) c_\ell(y^\ell) \gamma_{ij}^{k'} \gamma_{k'j'}^{\ell'} \gamma_{i'p}^{j'} \gamma_{k\ell}^{i'} F_{\ell'}
\end{aligned}$$

By proposition 3.7 we have

Theorem 3.8 $\mathcal{L}_A$ is solvable if

$$\sum_{i,j,k,\ell \in I} \sum_{i',j',k',\ell' \in I} c_i(x^i) c_j(y^j) c_k(x^k) c_\ell(y^\ell) \gamma_{ij}^{k'} \gamma_{k'j'}^{\ell'} \gamma_{i'l'}^{j'} \gamma_{kl}^{i'} = 0$$

for all vectors x^i, y^j. □

3.9 Example We shall give a very simple example to illustrate the Lie algebra $\mathcal{L}_A$ and to show how the conditions for solvability often simplify considerably. Thus, consider the system

$$\begin{aligned}
\dot{x} &= \frac{1}{2} \begin{pmatrix} -2 - x_1^2 & 2x_1 + x_1^2 & x_1^2 \\ -x_1 & -2 + x_1 & x_1 \\ x_1 - x_1^2 & x + x_1^2 & -2 - x_1 + x_1^2 \end{pmatrix} x \\
&= \frac{1}{2}(-2 - x_1^2) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{pmatrix} x
\end{aligned}$$

$$-\frac{x_1}{2}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -2 & 0 & 3 \end{pmatrix} x$$

$$+(-2+x_1)\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{pmatrix} x$$

It is easy to check that $\mathcal{L}_A$ is a three-dimensional Lie algebra which is therefore solvable.

4 Semisimple Lie Algebras

In this section we shall consider the case when $\mathcal{L}_A$ is a semisimple Lie algebra. The standard structure theory of such algebras will be used without further comment; the reader can find the details in various well known references, e.g. [5]. Thus again consider the nonlinear system

$$\dot{x} = A(x)x \tag{4.1}$$

and let $\mathcal{L}_{Ac}$ denote the complexified Lie algebra [1] generated by all the matrices $A(x)$, $x \in \mathbf{R}_n$. We shall assume that $\mathcal{L}_{Ac}$ is semisimple. Let h be a Cartan (i.e. maximal Abelian) subalgebra of $\mathcal{L}_{Ac}$ of dimension N and let

$$\mathcal{L}_{Ac} = h \oplus \sum_{\alpha\in\Sigma} \mathcal{L}_{Ac}^{\alpha} \tag{4.2}$$

[1] i.e. $\mathcal{L}_A \otimes \mathbf{C}$ where $\mathcal{L}_A$ is defined as before

be a root space decomposition of $\mathcal{L}_{Ac}$. We shall again denote by (.,.) the Killing form on $\mathcal{L}_{Ac}$. Recall the following properties of the decomposition (4.2):

(i). (.,.) is nondegenerate on h.

(ii). $\mathcal{L}_{Ac}$ has N linearly independent roots and each root space $\mathcal{L}^{\alpha}_{Ac}$ is one dimensional.

(iii). If $E_\alpha \in \mathcal{L}^{\alpha}_{Ac}$, then (by definition)

$$(\text{ad H})E_\alpha = \alpha(H)E_\alpha$$

for all $H \in h$.

(iv).For any root $\alpha \in \Sigma$ there exists a unique $H_\alpha \in h$ such that

$$(H, H_\alpha) = \alpha(H)$$

for all $H \in h$.

Corresponding to the decomposition (4.2) we can write (4.1) in the form

$$\dot{x} = H(x)x + \sum_{\alpha\in\Sigma} e_\alpha(x)E_\alpha x \tag{4.3}$$

where

$$H(x) \in h \quad , \quad x \in \mathbf{R}^n \quad , \quad E_\alpha \in \mathcal{L}^{\alpha}_{Ac} \; .$$

Using condition (iv) we can write

$$H(x) = \sum_{\alpha\in\Sigma'} h_\alpha(x)H_\alpha$$

for some functions $h_\alpha(x)$ where Σ' is a set of N linearly independent roots.

Define the function

$$F(x) = 2\sum_{\alpha\in\Sigma'}\sum_{\beta\in\Sigma'}\left(\frac{\partial h_\alpha}{\partial x}A(x)x\right)h_\beta(x)\beta(H_\alpha) \tag{4.4}$$

In the following we shall fix a linearly independent set of roots Σ'. We then have

Theorem 4.1 Suppose that $\mathcal{L}_{Ac}$ is a semisimple Lie algebra and let (4.2) be a root space decomposition. Then the system (4.1) is asymptotically stable in the large under either of the following set of conditions:

(I). (a). All the roots are real-valued on (the algebra generated by) $H(x)$.

(b). If $x \neq 0$, then for at least one root $\alpha \in \Sigma$ we have $\alpha(H(x)) \neq 0$. For $x = 0$, $H(0) = 0$.

(c). If $\| x \| \to \infty$ then for at least one root $\beta \in \Sigma$ we have $| \beta(H(x)) | \to \infty$

(d). $F(x) < 0$, $x \neq 0$.

(II).(a). All the roots are pure imaginary on $H(x)$.

(b),(c) as in (I).

(d). $F(x) > 0$, $x \neq 0$.

Proof. In case (I) consider the function

$$V(x) = (H(x), H(x)).$$

We have

$$V(x) = \mathrm{Tr}((\text{ ad } H(x))(\text{ ad } H(x))$$

Since $\{ H_\alpha , \alpha \in \Sigma' \}$, $\{ E_\beta , \beta \in \Sigma \}$ is a basis of the vector space $\mathcal{L}_{Ac}$ and

$$(\text{ ad } H(x))(\text{ ad } H(x))H_\alpha = 0 \quad , \quad \alpha \in \Sigma'$$

(since H is Abelian)

$$(\text{ ad } H(x))(\text{ ad } H(x))E_\beta = \beta^2(H(x))E_\beta$$

(by property (iii)), we have

$$V(x) = \sum_{\alpha \in \Sigma} \alpha^2(H(x)) \tag{4.5}$$

By I(a) and I(b) we have

$$V(x) > 0 , \quad x \neq 0 \tag{4.6}$$

$$V(0) = 0. \tag{4.7}$$

Now,

$$\begin{aligned} \dot{V}(x) &= \frac{d}{dt}(H(x), H(x)) \\ &= \frac{d}{dt}\Big(\sum_{\alpha \in \Sigma'} h_\alpha(x) H_\alpha, \sum_{\beta \in \Sigma'} h_\beta(x) H_\beta\Big) \\ &= \frac{d}{dt} \sum_{\alpha \in \Sigma'} \sum_{\beta \in \Sigma'} h_\alpha(x) h_\beta(x) \beta(H_\alpha) \end{aligned}$$

(by property (iv)), so

$$\dot{V}(x) = 2 \sum_{\alpha \in \Sigma'} \sum_{\beta \in \Sigma'} \left(\frac{\partial h_\alpha}{\partial x} A(x) x \right) h_\beta(x) \beta(H_\alpha) = F(x)$$

where we have used the symmetry of (.,.) so that

$$\begin{aligned} \beta(H_\alpha) &= (H_\alpha, H_\beta) \\ &= \alpha(H_\beta). \end{aligned}$$

By I(d) we have

$$\dot{V}(x) < 0.$$

The result now follows from I(c) and Lyapunov's main stability theorem.

Case II follows in much the same way except that

$$V(x) < 0$$

since $\alpha^2(H(x)) < 0$ because the roots have pure imaginary values. □.

4.2 Example Consider the simple Lie algebra of all skew-symmetric 3×3 complex matrices with basis

$$M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and structural formulas

$$[M_1, M_2] = M_3 \ ,[M_2, M_3] = M_1 \ ,[M_3, M_1] = M_2$$

If

$$H = iM_3 \ , \ E_1 = i(M_1 + iM_2) \ , \ E_{-1} = i(M_1 - iM_2)$$

then

$$[H, E_1] = E_1 \ , \ [H, E_{-1}] = -E_{-1} \ , \ [E_1, E_{-1}] = 2H.$$

The one-dimentional supspace $h = \{\lambda H : \lambda \in \mathbf{C}\}$ is a Cartan subalgebra and the two roots are λ and $-\lambda$ with root vectors E_1 and E_{-1}. For H_λ (as in (iv)) we can take

$$H_\lambda = \frac{1}{2}H$$

Now consider the 'skew-symmetric' differential equation

$$\dot{x}_1 = -f_1(x)x_2 + f_2(x)x_3$$

$$
\begin{aligned}
\dot{x}_2 &= f_1(x)x_1 - f_3(x)x_3 \\
\dot{x}_3 &= -f_2(x)x_1 + f_3(x)x_2
\end{aligned}
$$

This can be written in the form

$$
\begin{aligned}
\dot{x} &= (f_1(x)M_3 + f_2(x)M_2 + f_3(x)M_1) \\
&= \left(-if_1(x)H - \frac{1}{2}(f_2(x) + if_3(x))E_1 + \frac{1}{2}(f_2(x) - if_3(x))E_2\right) x \\
&\triangleq A(x)x.
\end{aligned}
$$

In this case,

$$
H(x) = -if_1(x)H
$$

and so the roots are $\pm if_1(x)$ which are pure imaginary. There is only one linearly independent root and so (4.4) becomes

$$
F(x) = \left(-i\frac{\partial f_1}{\partial x}A(x)x\right)(-if_1(x))
$$

since

$$
\lambda(H_\lambda) = \frac{1}{2}.
$$

Hence

$$
F(x) = -\frac{1}{2}\frac{\partial(f_1^2)}{\partial x}A(x)x
$$

and so for stability we require

$$
\begin{aligned}
&\frac{\partial f_1^2}{\partial x_1}(-f_1(x)x_2 + f_2(x)x_3) + \frac{\partial f_1^2}{\partial x_2}(-f_1(x)x_1 - f_3(x)x_3) \\
&\qquad \frac{\partial f_1^2}{\partial x_3}(-f_2(x)x_1 + f_3(x)x_2) < 0
\end{aligned}
$$

for each $x \neq 0$ and $f_1(x) \neq 0$ for $x \neq 0$. Thus, for example, if

$$f_1 = x_1^2 + x_2^2 + x_3^4 \ , \ f_2 = -x_1 x_3 \ , \ f_3 = x_2 x_3$$

we have

$$\begin{aligned} F(x) &= 2x_1 x_3 f_2 \left(1 - 2x_3^2\right) + 2x_3 x_2 f_3 \left(2x_3^2 - 1\right) \\ &= -2x_1^2 x_3^2 \left(1 - 2x_3^2\right) - 2x_2^2 x_3^2 \left(1 - 2x_3^2\right) \end{aligned}$$

and we have stability on the region where

$$x_3^2 < 1/2 \ .$$

4.3 Example Consider a system of the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Then $\mathcal{L}_{Ac}$ is the Lie algebra of 2×2 trace zero matrices. The Cartan subalgebra h is one-dimentional and spanned by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The root is given by $2a(x)$ and so

$$F(x) = 4(\frac{\partial a}{\partial x})A(x)xa(x)$$

and for stability we require

$$F(x) < 0$$

i.e.

$$a(x)\left(\frac{\partial a}{\partial x_1}(a(x)x_1 + b(x)x_2) + \frac{\partial a}{\partial x_2}(c(x)x_1 - a(x)x_2)\right) < 0 \ .$$

Similar stability conditions can be obtained for systems which generates the other clasical Lie algebras.

5 Conclusions

In this paper we have proved stability results for nonlinear systems of the form

$$\dot{x} = A(x)x$$

by considering the Lie algebra $\mathcal{L}_A$ (or $\mathcal{L}_{A^c}$) generated by the matrices $A(x)$, for all $x \in \mathbf{R}^n$. We have seen, in particular, that solvable and semisimple Lie algebras can play a significant role here.

Another possible approach in the case when $A(x)$ is an analytic matrix-valued function is to define the graded Lie algebra of functions in the following way. Let

$$A(x) = \sum_{\mathbf{i}=0}^{\infty} x^{\mathbf{i}} A_{\mathbf{i}}$$

where

$$x^{\mathbf{i}} = x_1^{i_1} \cdots x_n^{i_n}$$

and let g be the Lie subalgebra of the Lie algebra of all matrices generated by

$$\{A_{\mathbf{i}} : \ |\mathbf{i}| \geq 0\}.$$

If

$$T_{\mathbf{i}} = A_{\mathbf{i}} x^{\mathbf{i}}$$

then

$$[T_{\mathbf{i}}, T_{\mathbf{j}}] = [A_{\mathbf{i}}, A_{\mathbf{j}}] x^{\mathbf{i}+\mathbf{j}}. \tag{5.1}$$

This defines a kind of 'n-dimensional' Kac-Moody algebra graded by the index $\mathbf{i}$. Such algebras have been used in the physics of superstrings (see [6]) and we shall consider their application to nonlinear systems in a future paper.

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