Monograph:
Banks, S.P. and Moser, A. (1992) On the Inversion of the n-Dimensional Laplace Transform. Research Report. ACSE Research Report 461 . Department of Automatic Control and Systems Engineering

On the Inversion of the n-Dimensional Laplace Transform

by

S.P.Banks and A.Moser

Department of Automatic Control and Systems Engineering

University of Sheffield

P.O.Box 600

Mappin Street

SHEFFIELD S1 4DU

Research Report No. 461

September 1992

Abstract

In this paper we study the inversion of the multi-dimensional Laplace transform by a combination of a general partial fraction expansion formula and the theory of residues. The ideas may be applied to nonlinear systems defined by Volterra series.

Keywords: Multi-dimensional Laplace transform, Multi-dimensional residue theory, Nonlinear systems.

1 Introduction

In this paper we shall consider the n-dimensional Laplace transform and obtain generalizations of the well-known properties of the one-dimensional transform used extensively in linear systems theory. In particular, we shall be interested in the computation of the inverse transform using the n-dimensional theory of residues ([5],[10],[8],[9],[1]). Most of our knowledge of the multi–dimensional inverse Laplace transform comes from tables obtained by evaluating the transform of given functions in the 'time domain' (defined by the coordinates $(t_1, \cdots, t_n) \in \mathbf{R}^n$). This can be found in a number of publications ([12],[6],[4]). The problem of inverting a general function of the Laplace variables is very difficult and its application in systems theory is therefore somewhat limited ([2]).

The main objective of this paper is to develop as complete a generalization of the ordinary one-dimensional Laplace transform theory (in the case of rational functions of s) to the n-dimensional case as possible. This will involve a generalized partial fraction expansion theory using the techniques of ideal theory. We shall then apply the theory of residues to compute the inverse transforms of the resulting quotient forms in which the denominators are irreducible polynomials in $\mathbf{C}^n[X]$, where $X = (X_1, \cdots, X_n)$ or $X = (s_1, \cdots, s_n)$ when we wish to emphasize the connection with the Laplace variables.

The application of these results to systems theory will be studied in a companion paper.

2 Mathematical Preliminaries

In this paper we shall need a number of mathematical ideas which are not commonly found in the systems literature. We shall discuss these briefly here, giving references to the background material.

Firstly, we shall need some elementary ring and ideal theory (see [7]). Let $\mathbf{C}[X] = \mathbf{C}[X_1, \cdots, X_n]$ denote the ring of polynomials in $X = (X_1, \cdots, X_n)$. An **ideal** I in $\mathbf{C}[X]$ (or indeed any ring) is a subset which is closed under subtraction and

$$I\mathbf{C}[X] \subseteq \mathbf{C}[X]I \subseteq I.$$

If $p \in \mathbf{C}[X]$, then the **principal ideal** generated by p, (p), is defined by

$$(p) = \{pq : q \in \mathbf{C}[X]\}.$$

More generally, if $S = \{p_i\}_{1 \leq i \leq v}$ is a set of polynomials, then the ideal generated by S is

$$(\{p_i\}) = \{\sum_{i=1}^{v} p_i q_i : q_i \in \mathbf{C}[X]\ \}.$$

For any ideal $I \subseteq \mathbf{C}[X]$, its **radical** $\sqrt{I}$ is defined by

$$\sqrt{I} = \{p \in \mathbf{C}[X]\ :\ p^k \in I \text{ for some } k > 0\}.$$

Given any two ideals I_1, I_2 we define their **sum** by

$$I_1 + I_2 = \{p_1 + p_2\ :\ p_1 \in I_1\ ,\ p_2 \in I_2\}.$$

Now, for any set of polynomials $S \subseteq \mathbf{C}[X]$, the **variety** defined by S, $V(S)$, is the zero set of all elements of S; i.e.

$$V(S) = \{X \in \mathbf{C}^n \ : \ p(X) = 0 \text{ for all } p \in S\}.$$

If S consists of a single polynomial p, we write $V(S) = V(p)$. Conversely, for any subset $W \subseteq \mathbf{C}^n$ we define the ideal

$$J(W) = \{p \in \mathbf{C}[X] : p(X) = 0 \text{ for each } X \in W\}.$$

(It is clear that J *is* an ideal.)

Note that

$$I_1 \subseteq I_2 \Rightarrow V(I_2) \subseteq V(I_1)$$

and

$$W_1 \subseteq W_2 \Rightarrow J(W_2) \subseteq J(W_1)$$

Moreover, we have

$$\begin{aligned} V(I_1 \cap I_2) &= V(I_1 \cdot I_2) = V(I_1) \cup V(I_2), \\ V(I_1 + I_2) &= V(I_1) \cap V(I_2) \end{aligned}$$

for any ideals $I_1, I_2 \in \mathbf{C}[X]$.

It can be shown that, for any ideal I,

$$J(V(I)) = \sqrt{(I)}.$$

If $I = \sqrt{(I)}$ we say that I is **closed**. An ideal I is **irreducible** if $I = I_1 \cap I_2$ implies that $I = I_1$ or $I = I_2$ and **prime** if $p_1 \cdot p_2 \in I$ implies that $p_1 \in I$ or $p_2 \in I$.

Similarly, a variety $V \subseteq \mathbf{C}^n$ is **irreducible** if $V = V_1 \cup V_2$ implies that $V = V_1$ or $V = V_2$. Any variety V can be written in the form

$$V = V_1 \cup \cdots \cup V_k$$

where V_i is irreducible and similarly any ideal I can be written

$$I = I_1 \cap \cdots \cap I_k$$

where I_i is irreducible and closed, i.e. $I_i = \sqrt{I_i}$.

If I is any ideal in $\mathbf{C}[X]$ we define the **quotient ring** $\mathbf{C}[X]/I$ to be the set of subsets

$$q + I \subseteq \mathbf{C}[X]$$

for all $q \in \mathbf{C}[X]$, with addition and multiplication defined in the obvious way.

Given a ring homomorphism

$$f : R_1 \longrightarrow R_2$$

for any two rings R_1, R_2, suppose that $f(I_1) \subseteq I_2$ for some given pair of ideals I_1, I_2. Then f induces a ring homomorphism

$$\overline{f} : R_1/I_1 \longrightarrow R_2/I_2.$$

If $\ker f \subseteq I_1$ then $\overline{f}$ is injective.

We shall also require some notions from algebraic topology and integration of forms. These will now be discussed briefly (see also [1] for a more extensive discussion). We shall use the singular homology and cohomology theories. Recall that a **smooth p-dimensional singular simplex** (or **p-simplex**) on a differentiable manifold X is a pair $\sigma_p = (\Delta_p, f)$ where

$$\Delta_p = \{t = (t_1, \cdots, t_p) \in \mathbf{R}^p : t_j \geq 0 \ , \ t_1 + \cdots + t_p \leq 1\}$$

and $f : \Delta_p \longrightarrow X$ is a smooth map. The simplex is **oriented** by the coordinates $t_1, \cdots, t_p$. (Different coordinates $\tau_1, \cdots, \tau_p$ induce the same orientation of σ_p if $\partial t/\partial \tau > 0$.) We denote by $-\sigma_p$ the simplex σ_p with the opposite orientation.

Let $\Sigma = \cup_{p=0}^{n} \Sigma_p$ be a dissection of a manifold X into singular simplexes. The group of p-chains $C_p(X)$ is the free abelian group generated by the p-simplexes in X, i.e. the set of formal sums

$$c_p = \sum_k m_k \sigma_p^k \ , \ \sigma_p^k \in \Sigma_p \ , \ m_i \in \mathbf{Z},$$

with obvious addition. The i^{th} **face** $\sigma_p^{(i)}$ of σ_p is a map $\Delta_{p-1} \longrightarrow \sigma_p$ such that

$$\sigma_p^{(i)} = \sigma_p \circ e_p^i$$

where e_p^i is the map $e_p^i : \Delta_{p-1} \longrightarrow \Delta_p$ which maps Δ_{p-1} to the face of Δ_p opposite the i^{th} vertex, with orientation induced from Δ_p. The **boundary** of a simplex σ_p is then defined as

$$\partial \sigma_p = \sum_{i=1}^{p} (-1)^i \sigma_{p-1}^{(i)},$$

and the boundary of the chain c_p by

$$\partial c_p = \sum_i m_k \partial(\sigma_p^k)^{(i)}.$$

Then we have

$$\partial\partial c_p = 0$$

for any chain c_p. A chain $c_p \in C_p(X)$ is a **cycle** if $\partial c_p = 0$ and a **boundary** if $c_p = \partial d_{p+1}$ for some chain d_{p+1}. Then we define

$$\begin{aligned} Z_p(X) &= \{c_p \in C_p(X) : c_p \text{ is a cycle}\} \\ B_p(X) &= \{c_p \in C_p(X) : c_p \text{ is a boundary}\} \end{aligned}$$

and the quotient group

$$H_p(X) = Z_p(X)/B_p(X)$$

is the **p-dimensional smooth singular homology group** of X. We write $c_p \sim 0$ if $c_p \in B_p(X)$ and $c_p \approx 0$ if $kc_p \sim 0$ for some integer k. We also define

$$H_p^w(X) = Z_p(X)/\{c_p \in C_p(X) : c_p \approx 0\}.$$

It is called the **weak homology group** of X and coincides with $H_p(X)$ in many cases and we shall assume this here. A set $\{c_p^i\} \subseteq Z_p(X)$ is called a **p-dimensional (homology) basis** if we can write any $[c] \in H_p(X)$ uniquely in the form

$$[c] = \sum_i m_i [c_p^i]$$

where $[c_p^i]$ denotes the image of c_p^i in $H_p(X)$ under the canonical projection.

If Y is another smooth manifold and $f : X \longrightarrow Y$ is a smooth map, then we define a map $\underline{f} : C_p(X) \longrightarrow C_p(Y)$ by

$$\underline{f}(\sigma_p) = (\Delta_p, f \circ g) \ , \ \sigma_p = (\Delta_p, g),$$

and since $\underline{f} \circ \partial = \partial \circ \underline{f}$, f generates a homomorphism

$$f_* : H_p(X) \longrightarrow H_p(Y).$$

Now let ω be a continuous p-form on X and let $\sigma_p = (\Delta_p, g)$ be a smooth p-simplex. Then we define the **integral** of ω over σ_p by

$$\int_{\sigma_p} \omega = \int_{\Delta_p} \overline{g}(\omega) = \int_{|\Delta_p|} A(t_1, \cdots, t_p) dt_1 \cdots dt_p$$

where $\overline{g}(\omega)$ is the pull-back form of ω under g. If ω is given locally by

$$\omega(x) = \sum_{i_1, \cdots, i_p} a_{i_1 \cdots i_p}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

then we can write

$$\overline{g}(\omega)(t) = \sum a_{i_1 \cdots i_p}(g(t)) dg_{i_1}(t) \wedge \cdots \wedge dg_{i_p}(t).$$

Also,

$$A(t) = \sum_{i_1, \cdots, i_p} a_{i_1 \cdots i_p}(g(t)) \frac{\partial(x_{i_1}, \cdots, x_{i_p}}{\partial(t_1, \cdots, t_p)}.$$

In general, for $c_p = \sum m_k \sigma_p^k \in C_p(X)$ we define

$$\int_{c_p} \omega = \sum_k m_k \int_{\sigma_p^k} \omega.$$

We also write

$$\int_{c_p} \omega = \langle \omega, c_p \rangle;$$

this specifies ω as a linear map $C_p \longrightarrow \mathbf{R}$ and this 'inner product' shows that the p-forms are in a one-to-one correspondence with the **cohomology group** $H^p(X)$ of X (indeed, we can take this as a definition of $H^p(X)$). We then have the following results:

Theorem 2.1 $\langle \overline{f}(\omega), c_p \rangle = \langle \omega, \underline{f}(c_p) \rangle$. □

Theorem 2.2 (Stokes' formula).

$$\int_{\partial c_p} \omega = \int_{c_p} d\omega$$

where $d\omega$ is the exterior derivative of ω. □

Corollary 2.3 If $c_p^1 \cong c_p^2$ and $\omega_1 = \omega_2 + d\phi$, then

$$\int_{c_p^1} \omega_1 = \int_{c_p^2} \omega_2.$$

□

Thus, if $[c_p] \in H_p(X)$ and $[c^p] = \{\omega\} \in H^p(X)$, then we can define

$$\int_{[c_p]} [c^p] = \int_{c_p} \omega.$$

Note that if $\{c_i\}$ is a basis of the p-dimensional homology of a manifold X, then for any cycle c, $\int_c \omega = \sum_i k_i \int_{c_i} \omega$ where $c = \sum k_i c_i$.

Finally, we shall need Alexander-Pontryagin duality of a manifold (see [11]). This depends on the concept of **intersection index** which can be defined for two nondegenerate simplexes $\sigma_1 = \{\Delta_p, f_1\}, \sigma_2 = \{\Delta_q, f_2\}$, in an n-dimensional smooth orientable manifold X, where $p + q = n$, by

$$\chi(\sigma_1, \sigma_2) = \operatorname{sgn} \left. \frac{\partial(x_1, \cdots, x_n)}{\partial(t_1, \cdots, t_r, \tau_1, \cdots, \tau_q)} \right|_a$$

at a point a where σ_1 and σ_2 intersect transversally. Here, $(t_1, \cdots, t_p)$ and $(\tau_1, \cdots, \tau_q)$ are local coordinates at the point a determining the orientations of σ_1 and σ_2. If σ_1 and σ_2 do not intersect, we define

$$\chi(\sigma_1, \sigma_2) = 0.$$

For general chains $c_1 = \sum_i m_i \sigma_r^i$, $c_2 = \sum_j n_i \sigma_q^i$, with $r + q = n$, we define

$$\chi(c_1, c_2) = \sum_{i,j} m_i n_j \chi(\sigma_r^i, \sigma_q^j).$$

Now, if we have two boundaries $c_1 \in B_{r-1}(X)$ and $c_2 \in B_q(X)$ with $r + q = n$ and the carriers of c_1 and c_2 being disjoint, we define the **linking coefficient** of c_1 and c_2 by

$$v(c_1, c_2) = \chi(d_1, c_2)$$

where

$$\partial d_1 = c_1,$$

and d_1 is transversal to c_2. We then have

Theorem 2.4 Let S^n be an n-manifold homoemorphic to the n-dimensional sphere and T a polyhedral submanifold of S^n. Then, if $r + q = n$, the homology groups $H_{r-1}(T)$ and $H_q(S^n \backslash T)$ are isomorphic. Moreover, if $\{c_1, \cdots, c_p\}$ is an

$(r-1)$-dimensional homology basis of T, then there is a corresponding dual basis $\{d_1, \cdots, d_p\}$ of $S^n \backslash T$ such that

$$v(c_i, d_j) = \delta_{ij} \quad , \quad 1 \leq i, j \leq p.$$

□

By linearity cf the linking coefficients we obtain

Theorem 2.5 If F is analytic in $\mathbf{C}^n \backslash T$ and $\hat{T} = T \cup \{\infty\}$ is a subpolyhedron of $S^{2n} = \mathbf{C}^n \cup \{\infty\}$, then for any cycle $c \in Z_n(\mathbf{C}^n \backslash T)$ we have

$$\int_c F(z)dz = (2\pi i)^n \sum_{j=1}^{p} k_j R_j$$

where σ_j is an $(n-1)$-dimensional homology basis of $\hat{T}$ and

$$k_j = v(\sigma_j, c)$$

is the linking coefficient of c with the basis element σ_j and R_j is the 'residue' given by

$$R_j = \frac{1}{(2\pi i)^n} \int_{c_j} F(z)dz,$$

where c_j is the dual n-dimensional homology basis of $\mathbf{C}^n \backslash T$. □

3 Partial Fraction Expansion

In this section we shall generalize some results in [3].

Lemma 3.1 If

$$r(X) = \frac{p(X)}{q_1^{m_1}(X) \cdots q_k^{m_k}(X)}$$

where each $q_i(X) = q_i(X_1, \cdots, X_n)$ is irreducible and p and $q_1, \cdots, q_k$ are relatively prime, then we can write r in the form

$$r(X) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{p_{ij}(X)}{q_i^j} \tag{3.1}$$

for some $p_{ij}(X) \in \mathbf{C}[X_1, \cdots, X_n]$, $1 \leq i \leq k, 1 \leq j \leq m_i$, if and only if

$$(p) \subseteq (q_1') + \cdots + (q_k') \tag{3.2}$$

where

$$q_i' = \prod_{j \neq i} q_j^{m_j}.$$

Proof If (3.2) holds then there exist $v_i(X) \in bfC[x]$ such that

$$p(X) = \sum_{i=1}^{k} v_i(X) q_i'(X)$$

i.e.

$$r(X) = \sum_{i=1}^{k} \frac{v_i}{q_i^{m_i}(X)}$$

which is of the form (3.1) with

$$p_{im_i}(X) = v_i(X) \ , \ p_{ij}(X) = 0 \ \text{ if } 0 \leq j < m_i.$$

Conversely, if (3.1) holds for some $p_{ij}(X)$, then for any $t(X) \in (p)$ we have $t = pt_1$ for some $t_1 \in \mathbf{C}[X]$ and so

$$\begin{aligned} t &= \sum_{i=1}^{k} \sum_{j=1}^{m_i} t_1 p_{ij} (q_1^{m_1} \cdots q_j^{m_i - j} \cdots q_k^{m_k}) \\ &\in (q_1') + \cdots + (q_k'). \end{aligned} \tag{3.3}$$

$\square$

Corollary 3.1 In order that we may write $r(X)$ above in the form (3.1) it is necessary that

$$V(q_i) \cap V(q_j) \subseteq V(p).$$

Moreover, if the ideal $\sum(q_j')$ is minimal (i.e. $\sqrt{\sum(q_j')} = \sum(q_j')$) then this condition is also sufficient.

Proof By lemma 1, we have the necessary condition

$$(p) \subseteq (q_1') + \cdots + (q_k')$$

and so

$$V\left(\sum_{j=1}^{k}(q_j')\right) \subseteq V(p).$$

However,

$$V\left(\sum_{j=1}^{k}(q_j')\right) = \cup_{i \neq j}\left(V(q_i) \cap V(q_j)\right)$$

and the necessity is true.

For sufficiency, we have

$$V\left(\sum_{j=1}^{k}(q_j')\right) \subseteq V(p)$$

as above and so

$$\sqrt{(p)} \subseteq \sqrt{\left(\sum_{j=1}^{k}(q_j')\right)}$$

and if $\sqrt{\sum(q_j')} = \sum(q_j')$ then

$$(p) \subseteq \sqrt{(p)} \subseteq \sum_{j=1}^{k}(q_j')$$

and the sufficiency follows from lemma 3.1. □

Corollary 3.2 We can write

$$\frac{1}{q_1^{m_1}(X)\cdots q_k^{m_k}(X)} = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{p_{ij}(X)}{q_i^j(X)}$$

for some $p_{ij}(X) \in \mathbf{C}[X]$ if and only if

$$(1) = \sum_{i=1}^{k}(q_i')$$

i.e. if and only if the ideals (q_i') are **comaximal**. □

Consider next the evaluation of the polynomials $p_{ij}(X)$ in the expansion (3.1) when they exist. It is easy to see that such p_{ij}'s are not unique, but if

$$r(X) = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{p_{ij}(X)}{q_i^j(X)} = \sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\overline{p}_{ij}(X)}{q_i^j(X)} \tag{3.4}$$

then

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{(p_{ij} - \overline{p}_{ij})}{q_i^j} = 0$$

i.e.

$$\sum_{i=1}^{k}\sum_{j=1}^{m_i}(p_{ij} - \overline{p}_{ij})q_1^{m_1}\cdots q_i^{m_i-j}\cdots q_k^{m_k} = 0. \tag{3.5}$$

Consider the polynomials $p_{11}, \cdots, p_{1m_1}$ and the sequence of (generalized) coordinate rings

$$\mathbf{C}[X]/(q_1) \subseteq \mathbf{C}[X]/(q_1^2) \subseteq \cdots \subseteq \mathbf{C}[X]/(q_1^{m_1}). \tag{3.6}$$

By (3.3), since q_1 is a factor of each term $q_1^{m_1} \cdots q_i^{m_i - j} \cdots q_k^{m_k}$, apart from when $i = 1$ and $j = m_1$, it follows that q_1 must divide $p_{1m_1} - \overline{p}_{1m_1}$ so that

$$p_{1m_1} - \overline{p}_{1m_1} \in (q_1).$$

Hence, p_{1m_1} is uniquely determined modulo the ideal (q_1), i.e.

$$p_{1m_1} \in \mathbf{C}[X]/(q_1).$$

Now consider the ring homomorphism

$$\gamma_1 : \mathbf{C}[X] \longrightarrow \mathbf{C}[X]$$

given by

$$\gamma_1(p) = p \cdot q_1 \ , \ p \in \mathbf{C}[X]$$

i.e. multiplication by q_1. Then γ_1 induces an injection

$$\overline{\gamma}_1 : \mathbf{C}[X]/(q_1) \longrightarrow \mathbf{C}[X]/(q_1^2),$$

which is the inclusion map in (3.4). Since $p_{1m_1} = p_{2m_2}$ in $\mathbf{C}[X]/(q_1)$, we have

$$p_{1m_1} = p_{2m_2} \ \text{ in } \mathbf{C}[X]/(q_1^2).$$

Hence, in $\mathbf{C}[X]/(q_1)$, (3.3) reduces to

$$\begin{aligned} &\sum_{i=2}^{k} \sum_{j=1}^{m_i} (p_{ij} - \overline{p}_{ij}) q_1^{m_1} \cdots q_i^{m_i - j} \cdots q_k^{m_k} + \\ &\quad + \sum_{j=1}^{m_i - 1} (p_{ij} - \overline{p}_{ij}) q_1^{m_1 - j} q_2^{m_2} \cdots q_k^{m_k} = 0. \end{aligned} \tag{3.7}$$

Arguing as before we see that, since q_1^2 divides every term in (3.5) apart from the term where $i = 1, j = m_1 - 1$ (provided $m_1 > 1$), we have

$$p_{1m_1-1} - \overline{p}_{1m_1-1} \in (q_1^2)$$

i.e. p_{1m_1-1} is uniquely determined modulo the ideal (q_1^2), i.e.

$$p_{1m_1-1} \in \mathbf{C}[X]/(q_1^2).$$

Continuing in this way, we obtain

Lemma 3.2 In the expansion (3.1), p_{ij} is uniquely determined modulo the ideal $(q_i^{m_i-j+1})$ so that p_{ij} is well-defined in $\mathbf{C}[X]/(q_i^{m_i-j+1})$. □

In order to determine the polynomials p_{ij} in the expansion (3.1) we need to define operators corresponding to 'differentiation by q_i' for $1 \leq i \leq n$. We shall do this in the case of two variables X_1, X_2; the general case is an easy extension. Hence we require to find the polynomials p_{1i}, p_{2j} in the expansion

$$\frac{p(X_1, X_2)}{q_1^{m_1}(X_1, X_2)q_2^{m_2}(X_1, X_2)} = \frac{p_{11}}{q_1} + \frac{p_{12}}{q_1^2} + \cdots + \frac{p_{1m_1}}{q_1^{m_1}} + \frac{p_{21}}{q_2} + \cdots + \frac{p_{2m_2}}{q_2^{m_2}} \quad (3.8)$$

where we have omitted the variables X_1, X_2 on the right for convenience. Note first that if an expression of the form (3.8) exists then we can write

$$p = vq_2^{m_2} + wq_1^{m_1}$$

for some polynomials v and w. Note that v is uniquely defined in $\mathbf{C}[X]/(q_1^{m_1})$ and w is uniquely defined in $\mathbf{C}[X]/(q_2^{m_2})$. We can find suitable polynomials v and w from the division algorithm. We then have

$$v = p_{11}q_1^{m_1-1} + p_{12}q_1^{m_1-2} + \cdots + p_{1m_1}. \quad (3.9)$$

Suitable values of $p_{11}, \cdots, p_{1m_1}$ can again be found from v by successive application of the division algorithm:

$$\begin{aligned} v &= \alpha_1 q_1 + \beta_1 \\ &= (\alpha_2 q_1 + \beta_2) q_1 + \beta_1, \end{aligned}$$

where $\alpha_1 = \alpha_2 q_1 + \beta_2$. Similarly,

$$\begin{aligned} v &= ((\alpha_3 q_1 + \beta_3) q_1 + \beta_2) q_1 + \beta_1 \\ &= \cdots \\ &= \alpha_{m_1 - 1} q_1^{m_1 - 1} + \beta_{m_1 - 1} q_1^{m_1 - 2} + \cdots + \beta_1. \end{aligned} \tag{3.10}$$

Identifying (3.7) with (3.8) gives appropriate values for $p_{11}, \cdots, p_{1m_1}$ (and similarly for $p_{21}, \cdots, p_{2m_2}$). As we know, of course, the polynomials p_{ij} are not uniquely defined in $\mathbf{C}[X]$, only in $\mathbf{C}[X]/(q_i^{m_i - j + 1})$.

We can, however, choose 'canonical forms' for p_{ij} in certain cases. For this we need the well-known division lemma:

Lemma 3.3 If R is a ring with identity and p, q are polynomials in $R[X]$ whose highest coefficients are units in R, then there is a unique representation

$$p = \alpha q + \beta$$

where deg β <deg q. □

Considering the case of two variables again as above, we can repeat the procedure in (3.8) and if we write the polynomials as functions of X_1 with coefficients in $\mathbf{C}[X_2]$ (say), then provided each polynomial has highest order coefficient independent of X_2 (i.e. a unit in $\mathbf{C}[X_2]$) the polynomials $\alpha_{m_1-1}, \beta_{m_1-1}, \cdots, \beta_1$ will be uniquely defined.

Example Consider the rational function

$$\frac{X_1^2}{(X_1 - X_2^2)^2(X_1 + 2X_2^2)} = \frac{p_1}{(X_1 - X_2^2)} + \frac{p_2}{(X_1 - X_2^2)^2} + \frac{p_3}{(X_1 + 2X_2^2)}$$

Note that such a decomposition is possible by corollary 3.1, since

$$V(X_1 - X_2^2) \cap V(X_1 + 2X_2^2) = \{0\} \subseteq V(X_1^2)$$

and

$$\sqrt{(X_1^2)} = (X_1) \subseteq \sqrt{(X_1 - X_2^2)^2 + (X_1 + 2X_2^2)}$$

To find p_3 we merely multiply the left hand side by $(X_1 + 2X_2^2)$ and set $X_1 + 2X_2^2 = 0$; i.e.

$$\begin{aligned} p_3 &= \frac{X_1^2}{(X_1 - X_2^2)^2}\Big|_{X_1+2X_2^2=0} \\ &= \frac{4}{9}. \end{aligned}$$

Now we must write X_1^2 in the form

$$X_1^2 = r(X_1 + 2X_2^2) + \frac{4}{9}(X_1 - X_2^2)^2.$$

A simple calculation shows that

$$r = \frac{5}{9}X_1 - \frac{2}{9}X_2^2.$$

Thus,

$$r = \frac{5}{9}X_1 - \frac{2}{9}X_2^2 = p_1(X_1 - X_2^2) + p_2.$$

Regarding the polynomials as elements of $(\mathbf{C}[X_2])[X_1]$ we have

$$p_1 = \frac{5}{9}\ ,\ p_2 = \frac{1}{3}X_2^2.$$

Hence,

$$\frac{X_1^2}{(X_1 - X_2^2)^2(X_1 + 2X_2^2)} = \frac{5}{9(X_1 - X_2^2)} + \frac{X_2^2}{3(X_1 - X_2^2)^2} + \frac{4}{9(X_1 + 2X_2^2)}.$$

4 The Inverse Laplace Transform

We shall next apply the theory of residues to the problem of inverting the n-dimensional Laplace transform. We start from the general expression for the inverse transform, namely

$$f(t_1, \cdots, t_n) = \frac{1}{(2\pi i)^n}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \cdots \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} F(s_1, \cdots, s_n)e^{(s_1 t_1 + \cdots + s_n t_n)}ds_1 \cdots ds_n . \quad (4.1)$$

The following result specifies the conditions under which (4.1) holds and its proof can be found in ([6]):

Theorem 4.1 Suppose that the function $f(t_1, \cdots, t_n)$ has partial derivatives of all orders up to n, i.e. $\frac{\partial^{|\mathbf{i}|}}{\partial t^{\mathbf{i}}}f$ exists for $|\mathbf{i}| \leq n$, where $\mathbf{i} = (i_1, \cdots, i_n), t^{\mathbf{i}} = t_1^{i_1} \cdots t_n^{i_n}$ and $|\mathbf{i}| = \sum_{j=1}^{n} i_j$. Moreover, assume that

$$|f(t_1, \cdots, t_n)| < Qe^{(k_1 t_1 + \cdots + k_n t_n)}$$

for some constants $Q, k_1, \cdots, k_n$. Then, if

$$F(s_1, \cdots, s_n) = \int_0^\infty \cdots \int_0^\infty e^{-(s_1 t_1 + \cdots + s_n t_n)} dt_1 \cdots dt_n$$

we have

$$f(t_1, \cdots, t_n) = \frac{1}{(2\pi i)^n} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \cdots \int_{\sigma_n - i\infty}^{\sigma_n + i\infty} F(s_1, \cdots, s_n) e^{\sum_{i=1}^n s_i t_i} ds_1 \cdots ds_n \tag{4.2}$$

for $\sigma_i > k_i$. □

Consider the space $\mathbf{C}^n$ with the one point $\{\infty\}$ added, using the standard one-point compactification topology. Then $\mathbf{C}^n \cup \{\infty\}$ is topologically a real $2n$-dimensional sphere S^{2n}. Let T be the singular set of F, i.e.

$$T = \{(s_1, \cdots, s_n) \in \mathbf{C}^n : F \text{ is not analytic at } (s_1, \cdots, s_n)\},$$

and let $\hat{T} = T \cup \{\infty\}$.

Theorem 4.2 Let γ be any cycle in $C_n(S^{2n} \backslash T)$ which is weakly homologous to the cycle $S^n \backslash \{\infty\} \in C_n(S^{2n} \backslash \{\infty\})$ defined by the set

$$S^n \backslash \{\infty\} = \{(s_1, \cdots, s_n) \in \mathbf{C}^n : s_k = \sigma_k + iy, -\infty < y < \infty\}.$$

Then,

$$\begin{aligned} f(t_1, \cdots, t_n) &= \mathcal{L}^{-1}(F(s_1, \cdots, s_n)) \\ &= \frac{1}{(2\pi i)^n} \int_\gamma F(s_1, \cdots, s_n) e^{\sum_{i=1}^n s_i t_i} ds_1 \wedge \cdots \wedge ds_n. \end{aligned}$$

Proof This follows directly from corollary 2.3 and theorem 4.1. □

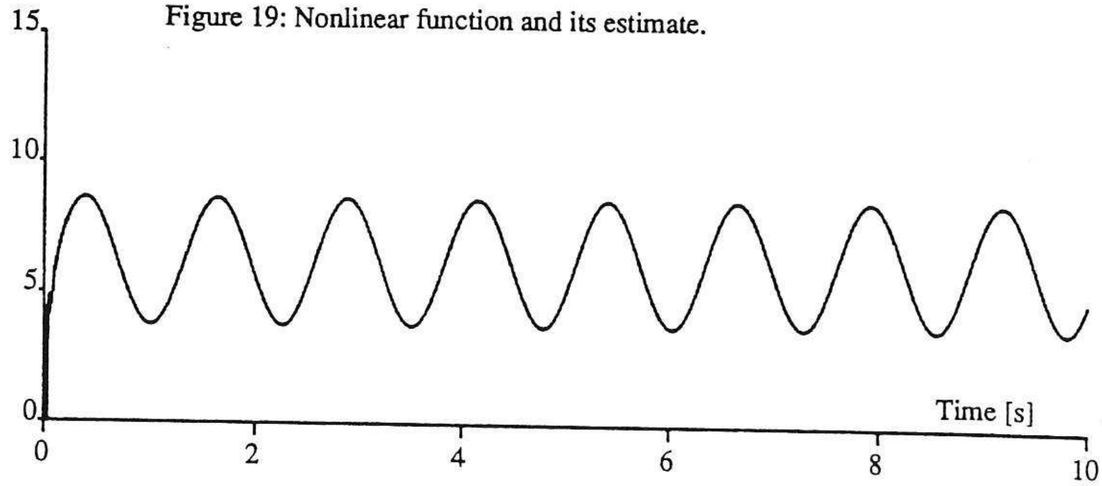

Figure 19: Nonlinear function and its estimate.

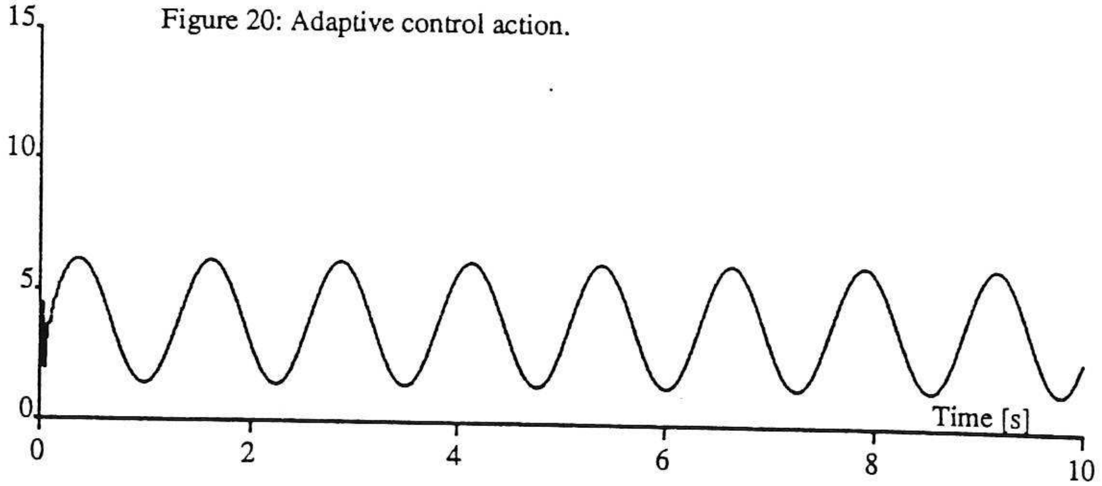

Figure 20: Adaptive control action.