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by

Q.M Zhu* & S.A Billings*

*Department of Control & Systems Engineering
The University of Sheffield
Mappin Street
Sheffield S1 4DT

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Approximation to Nonlinear Equations Using Nonlinear Rational Models

Q.M. Zhu, S.A. Billings

Department of Automatic Control and Systems Engineering,

University of Sheffield, Sheffield S1 4DU, UK

Abstract:

The feasibility of approximating several nonlinear dynamic equations which dominate several engineering disciplines is investigated using a nonlinear rational model formulation. The studies are based on identification from input and output data rather than analytical transformation from a nonlinear equation to a nonlinear rational model.

1 Introduction

It is common for just a few fundamental equations to dominate the theoretical study, experimental design, and interpretation of phenomenon in engineering and scientific systems. For example Maxwell's equations have found many important applications in the study of electromagnetic fields (Dearholt and Mcspadden 1973). The solution of the Boltzmann equation can be used to calculate the transport coefficients in solid state physics (Patterson 1971), and the famous Navier-Stockes equation describes the complex motion of a fluid in dynamic fluid fields (Batchelor 1967). In general these equations are fairly complicated so that the analytical solutions are difficult to achieve or even impossible. Often numerical methods such as the Runge-Kutta formulation can be applied to obtain numerical solutions using computers, but this approach may only be feasible for a specific range of data and it is difficult to achieve real time solutions for online applications. An alternative approach would be to simplify the complex equations by approximating them using system identification based on nonlinear polynomial, rational, or some other appropriate descriptions which represent the Fourier expansions of the partial differential equations to a finite number of components (Franceschini and Zanasi 1992).

Polynomial approximations are well known and can be justified by the Weierstrass theorem. Rational functions, which include the polynomial model as a subset, are often preferable to polynomials if they can be estimated because they tend to provide concise expressions with fast convergent approximation for functions with singular points. Both polynomial and rational functions have been widely applied to approximate many complicated nonlinear static functions and these formulations have been tabulated (for example, Luke 1975).

Applying polynomial or rational approximation to dynamic nonlinearities introduces two inherent problems. The first is the noise problem which arises from measurement errors, human error, stochastic disturbance, uncertainty and so on, the second is that there is often no exact analytic transformation into a polynomial or rational approximation or the analytic description is too complicated to transform mathematically. In order to deal with these problems the Nonlinear AutoRegressive and Moving Average with eXogenous inputs (NARMAX) model was introduced as a basis for the identification of complex nonlinear systems (Leontaritis and Billings 1985). NARMAX polynomial model identification was originally described a decade ago (Billings and Leontaritis 1981, Chen and Billings 1989) but NARMAX rational model identification was only recently considered because of the difficulties associated with the nonlinearity in the parameters (Billings and Chen 1989a, Billing and Zhu 1990, 1992, Zhu and Billings 1991, 1992).

Identification of polynomial model descriptions of a variety of nonlinear systems including a heat exchanger (Liu, Korenberg, Billings, and Fadzil 1987), a diesel generator (Billings and Fadzil 1988), an automotive diesel engine (Billings and Chen 1989b), nonlinear fluid loading systems (Worden, Stansby, Tomlinson, and Billings 1991) and several other systems have been reported. The application of the rational model to these types of problems is however only just beginning. The purpose of the present study is therefore to test the feasibility of using a rational model approximation to describe typical nonlinear stochastic dynamics in engineering systems. This will include the introduction of the stochastic rational model, an identification algorithm, and the application to the approximations of three nonlinear dynamic equations.

2 Rational model identification

Rational model identification is briefly described in this section. The the model description, parameter estimation, structure detection, and model validity tests are considered.

2.1 Model

The input-output description of the rational NARMAX model can be expressed as

$$y(k) = \frac{a(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))}{b(y(k-1), \cdots, y(k-r), u(k-1), \cdots, u(k-r), e(k-1), \cdots, e(k-r))} + e(k)$$

$$= \frac{a(k)}{b(k)} + e(k) = \frac{\sum_{j=1}^{num} p_{nj}(k)\theta_{nj}}{\sum_{j=1}^{den} p_{dj}(k)\theta_{dj}} + e(k) \tag{2.1}$$

This is a ratio of the polynomials numerator $a(k)$ and denominator $b(k)$ polynomials, where $u(k)$ and $y(k)$ represent the input and output at time k, and $e(k)$ is an unobservable independent noise sequence with zero mean and finite variance σ_e^2.

One of the distinctive features of the rational model eqn (2.1) is that the nonlinearity is a function of all the inputs, outputs, noises, and parameters. The nonlinear in the parameters gives rise to considerable difficulties in model structure detection and parameter estimation. To aid identification the rational model can be expanded as a linear in the parameter model but only at the expense of introducing highly correlated regressor terms and a series of rational model identification algorithms have been developed to overcome these problems (Billing and Zhu 1990, 1992, Zhu and Billings 1991, 1992).

The linear in the parameters expression of eqn (2.1) may be obtained by multiplying $b(k)$ on both sides of eqn (2.1) and then moving all the terms except $y(k)p_{d1}(k)\theta_{d1}$ to the right hand side to give

$$Y(k) = a(k) - y(k) \sum_{j=2}^{den} p_{dj}(k)\theta_{dj} + b(k)e(k)$$

$$= \sum_{j=1}^{num} p_{nj}(k)\theta_{nj} - \sum_{j=2}^{den} y(k)p_{dj}(k)\theta_{dj} + \zeta(k) \tag{2.2}$$

where

$$Y(k) = y(k)p_{d1}(k)|_{\theta_{d1}=1}$$

$$= p_{d1}(k)\frac{a(k)}{b(k)} + p_{d1}(k)e(k) \tag{2.3}$$

Alternatively divide all the right hand side terms by θ_{d1} and redefine symbols to give essentially $\theta_{d1} = 1$. The third term on the right hand side in eqn (2.2) is given by

$$\zeta(k) = b(k)e(k)$$

$$= (\sum_{j=1}^{den} p_{dj}(k)\theta_{dj})e(k)$$

$$= p_{d1}(k)e(k) + (\sum_{j=2}^{den} p_{dj}(k)\theta_{dj})e(k) \tag{2.4}$$

where

$$E[\zeta(k)] = E[b(k)]E[e(k)] = 0 \tag{2.5}$$

providing $e(k)$ has been reduced to an uncorrelated sequence.

2.2 Parameter estimation

The objective of parameter estimation is to obtain unbiased estimates of the rational model parameters from noisy input and output data. A new orthogonal rational model estimator (ORME) algorithm (Zhu and Billings 1992) is briefly described below.

Eqn (2.2) may be written in vector notation as

$$\begin{aligned} Y(k) &= \phi(k)\Theta + \zeta(k) \\ &= \hat{\phi}(k)\Theta + p_{d1}(k)e(k) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \phi(k) &= [\phi_n(k) \;\; \phi_d(k)] \\ &= [p_{n1}(k) \cdots p_{nnum}(k) \; -p_{d2}(k)y(k) \cdots -p_{dden}(k)y(k)] \\ &= [p_{n1}(k) \cdots p_{nnum}(k) \; -p_{d2}(k)(\frac{a(k)}{b(k)} + e(k)) \cdots -p_{dden}(k)(\frac{a(k)}{b(k)} + e(k))] \end{aligned} \tag{2.7}$$

$$\begin{aligned} \Theta^T &= [\Theta_n \; \Theta_d] \\ &= [\theta_{n1} \cdots \theta_{nnum} \; \theta_{d2} \cdots \theta_{dden}] \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \hat{\phi}(k) &= [\phi_n(k) \;\; \hat{\phi}_d(k)] \\ &= [p_{n1}(k) \cdots p_{nnum}(k) \; -p_{d2}(k)\frac{a(k)}{b(k)} \cdots -p_{dden}(k)\frac{a(k)}{b(k)}] \end{aligned} \tag{2.9}$$

and the subscriptions n and d refer to numerator and denominator terms respectively. Notice that the matrix $\hat{\phi}(k)$ cannot be obtained directly because $\frac{a(k)}{b(k)}$ cannot be measured.

Consider an orthogonal transformation of the original model eqn (2.6)

$$\begin{aligned} Y(k) &= w(k)G + \zeta(k) \\ &= w(k)G + b(k)e(k) \\ &= \sum_{j=1}^{num} w_{nj}(k)g_{nj} + \sum_{j=2}^{den} w_{dj}(k)g_{dj} + b(k)e(k) \end{aligned} \tag{2.10}$$

where

$$G = [G_n \;\; G_d]$$
$$= [g_{n1}, \cdots, g_{nnum}, g_{d2}, \cdots, g_{dden}] \tag{2.11}$$

and

$$w(k) = [w_n(k) \;\; w_d(k)]$$
$$= [ww_n(k) \;\; ww_d(k)] + [e_n(k) \;\; e_d(k)]e(k)$$
$$= [ww_{n1}(k), \cdots, ww_{nnum}(k), ww_{d2}(k), \cdots, ww_{dden}(k)]$$
$$+ [e_{n1}(k), \cdots, e_{nnum}(k), e_{d2}(k), \cdots, e_{dden}(k)]e(k) \tag{2.12}$$

where $[e_n(k) \;\; e_d(k)]e(k)$, the inherent error in the orthogonal transformation, represents all terms which include the factor $e(k)$ and $[ww_n(k) \;\; ww_d(k)]$ represents all other terms which may include lagged noise terms $e(k-j)$, $j > 0$.

The orthogonal regression matrix W is defined

$$W = \Phi \, T^{-1} \tag{2.13}$$

where

$$W^T = [w^T(1) \cdots w^T(N)]$$
$$= \begin{bmatrix} w_n^T(1) & \ldots & w_n^T(N) \\ w_d^T(1) & \ldots & w_d^T(N) \end{bmatrix} \tag{2.14}$$

Φ is given with reference to eqn (2.7)

$$\Phi = \begin{bmatrix} \phi_n(1) & \phi_d(1) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \phi_n(N) & \phi_d(N) \end{bmatrix} \tag{2.15}$$

and the orthogonal transform matrix T is a unit upper triangular

$$T = \begin{bmatrix} 1 & t_{1\,2} & . & . & . & t_{1\,n} \\ 0 & 1 & . & . & . & . \\ 0 & 0 & 1 & . & . & . \\ . & . & 0 & 1 & . & . \\ . & . & . & 0 & 1 & t_{n-1\,n} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.16}$$

There are several methods of computing the elements of T, such as the Gram-Schmidt, Householder, or Givens, transformations.

The orthogonality of the matrix W yields

$$W^T W = D \tag{2.17}$$

where D is a positive diagonal matrix

$$D = diag\ \{d_{n1}, \cdots, d_{nnum}, d_{d2}, \cdots, d_{dden}\} \tag{2.18}$$

and

$$d_{*j} = \sum_{k=1}^{N} w_{*j}(k)\ w_{*j}(k) \tag{2.19}$$

From eqn (2.12)

$$\frac{d_{*j}}{N} = \frac{\sum_{k=1}^{N} w_{*j}(k)\ w_{*j}(k)}{N} = \overline{ww_{*j}^2(k)} + \overline{e_{*j}^2(k)}\ \sigma_e^2 \tag{2.20}$$

Where the over bar denotes time averaging and * denotes either n or d.

An unbiased estimate of the parameter vector G can now be obtained from

$$\begin{aligned} G &= [W^T W - \sigma_e^2\ \Psi_{orth}]^{-1}\ [W^T \vec{Y} - \sigma_e^2\ \psi_{orth}] \\ &= [D - \sigma_e^2\ \Psi_{orth}]^{-1}\ [W^T \vec{Y} - \sigma_e^2\ \psi_{orth}] \\ &= [[W^T W]_{(k-1)}]^{-1}\ [W^T \vec{Y}]_{(k-1)} \end{aligned} \tag{2.21}$$

where $\vec{Y}$ is defined with reference to eqn (2.10)

$$\vec{Y} = [Y(1), \cdots, Y(N)]^T \tag{2.22}$$

and

$$\begin{aligned} [W^T W]_{(k-1)} &= [W^T W - \sigma_e^2\ \Psi_{orth}] \\ [W^T \vec{Y}]_{(k-1)} &= [W^T \vec{Y} - \sigma_e^2\ \psi_{orth}] \end{aligned} \tag{2.23}$$

All terms involving $e(k)$ appear in $\sigma_e^2\ \Psi_{orth}$ and $\sigma_e^2\ \psi_{orth}$ which are called error terms and the subscript $(k-1)$ indicates that only lagged noise terms (eg $e(k-j)\ j \geq 1$) are present. Finally

$$\Psi_{orth} = diag\ \{\overline{e_{n1}^2}, \cdots, \overline{e_{nnum}^2}, \overline{e_{d2}^2}, \cdots, \overline{e_{dden}^2}\}$$

$$\psi_{orth} = [\overline{p_{d1}(k)e_{n1}(k)}, \cdots, \overline{p_{d1}(k)e_{nnum}(k)}, \overline{p_{d1}(k)e_{d2}(k)}, \cdots, \overline{p_{d1}(k)e_{dden}(k)}]^T \quad (2.24)$$

Inspection of the orthogonal estimator shows that the noise variance σ_e^2 is needed a priori. An estimate of σ_e^2 can be obtained by an iterative procedure (Billings and Zhu 1991, Zhu and Billings 1992) in which parameter estimation and noise variance prediction are recursively updated.

The parameter vector Θ of the original model eqn (3.6) can then be calculated by

$$T\,\Theta = G \quad (2.25)$$

or

$$\Theta = T^{-1}\,G \quad (2.26)$$

Full details of the ORME algorithm were given in Zhu and Billings (1992).

2.3 Structure detection

Structure detection is used to select the relevant rational model terms from a library of possible terms based on measurements of the noisy input and output data. The error reduction ratio (ERR), which is computed as a by-product of the orthogonal estimation algorithm, can be used as a criterion to select model terms. Consider eqn (2.10), squaring this with the assumption that the signals are ergodic gives

$$\frac{1}{\sigma_b^2} = \sum_{j=1}^{num} \frac{g_{nj}^2\,\overline{ww_{nj}^2(t)} + g_{nj}^2\,\overline{e_{nj}^2(t)}\,\sigma_e^2 + 2g_{nj}\,\overline{e_{nj}(t)b(t)}\,\sigma_e^2}{\sigma_Y^2\ \sigma_b^2}$$
$$+ \sum_{j=2}^{den} \frac{g_{dj}^2\,\overline{ww_{dj}^2(t)} + g_{dj}^2\,\overline{e_{dj}^2(t)}\,\sigma_e^2 + 2g_{dj}\,\overline{e_{dj}(t)b(t)}\,\sigma_e^2}{\sigma_Y^2\ \sigma_b^2} + \frac{\sigma_e^2}{\sigma_Y^2} \quad (2.27)$$

Define the error reduction ratio (ERR) as

$$e\hat{r}r_{nj} = \frac{g_{nj}^2\,\overline{ww_{nj}^2(k)} + g_{nj}^2\,\overline{e_{nj}^2(k)}\,\sigma_e^2 + 2g_{nj}\,\overline{e_{nj}(k)b(k)}\,\sigma_e^2}{\sigma_Y^2\ \sigma_b^2}$$

$$e\hat{r}r_{dj} = \frac{g_{dj}^2\,\overline{ww_{dj}^2(k)} + g_{dj}^2\,\overline{e_{dj}^2(k)}\,\sigma_e^2 + 2g_{dj}\,\overline{e_{dj}(k)b(k)}\,\sigma_e^2}{\sigma_Y^2\ \sigma_b^2} \quad (2.28)$$

where

$$\sigma_Y^2 = \overline{Y^2(k)}$$
$$\sigma_b^2 = \overline{b^2(k)} \tag{2.29}$$

Introduce

$$err_{nj} = \frac{g_{nj}^2\, \overline{ww_{nj}^2(k)}}{\sigma_Y^2\ \sigma_b^2}$$
$$err_{dj} = \frac{g_{dj}^2\, \overline{ww_{dj}^2(k)}}{\sigma_Y^2\ \sigma_b^2} \tag{2.30}$$

as the ERR estimates that would arise if $e(k) = 0$, and

$$Bias\ [\ err_{nj}\ ] = \frac{g_{nj}^2\, \overline{e_{nj}^2(k)}\ \sigma_e^2 + 2g_{nj}\ \overline{e_{nj}(k)b(k)}\ \sigma_e^2}{\sigma_Y^2\ \sigma_b^2}$$
$$Bias\ [\ err_{dj}\ ] = \frac{g_{dj}^2\, \overline{e_{dj}^2(k)}\ \sigma_e^2 + 2g_{dj}\ \overline{e_{dj}(k)b(k)}\ \sigma_e^2}{\sigma_Y^2\ \sigma_b^2} \tag{2.31}$$

as the biases which are induced in the ERR estimates for the realistic case of $e(k) \neq 0$.

An unbiased estimate of ERR for the rational model can therefore be estimated using

$$err_{nj} = e\hat{r}r_{nj} - Bias\ [\ err_{nj}\ ]$$
$$err_{dj} = e\hat{r}r_{dj} - Bias\ [\ err_{dj}\ ] \tag{2.32}$$

where err_{nj}, $Bias\ [\ e\hat{r}r_{nj}\ ]$, err_{dj}, and $Bias\ [\ e\hat{r}r_{dj}\ ]$ are obtained directly from the computations.

With reference to the definitions in eqns (2.28) and (2.31), eqn (2.27) can alternatively be written as

$$\frac{1}{\sigma_b^2} = \sum_{j=1}^{num} e\hat{r}r_{nj} + \sum_{j=2}^{den} e\hat{r}r_{dj} + \frac{\sigma_e^2}{\sigma_Y^2}$$
$$= \sum_{j=1}^{num} err_{nj} + \sum_{j=2}^{den} err_{dj} + \sum_{j=1}^{num} bias\ [\ err_{nj}\ ] + \sum_{j=2}^{den} bias\ [\ err_{dj}\ ] + \frac{\sigma_e^2}{\sigma_Y^2} \tag{2.33}$$

Enq (2.33) can be used as a criterion for determining the number of terms to be included in the model, it therefore determines the model structure. The larger the value of ERR associated with a specific term the more the ratio $\frac{\sigma_e^2}{\sigma_Y^2}$ would be reduced if that term were included in the model. Hence terms can be ordered based upon that ERR value. Insignificant terms can be rejected by defining a cut off value of $1 - \sum err_{*j}$ below which terms are deemed to be negligible. As a criterion ERR attempts to balance the prediction accuracy and complexity of the final model.

2.4 Validation

Model validation is used to test if the identified model adequately describes the data set. In this paper a set of correlation tests (Leontaritis and Billings 1987) are adopted for rational model validation and these can be expressed as

$$
\begin{aligned}
\gamma_{\varepsilon\varepsilon}(\tau) &= \delta(\tau) \\
\gamma_{u\varepsilon}(\tau) &= 0 \quad for\ all\ \ \tau \\
\gamma_{\varepsilon(\varepsilon u)}(\tau) &= 0 \quad for\ all\ \ \tau\geq 0 \\
\gamma_{u^{2'}\varepsilon}(\tau) &= 0 \quad for\ all\ \ \tau \\
\gamma_{u^{2'}\varepsilon^2}(\tau) &= 0 \quad for\ all\ \ \tau
\end{aligned}
\tag{2.34}
$$

where $\gamma_{xy}(\tau)$ indicates the cross correlation function between $x(k)$ and $y(k)$, $\tau = 0, \pm 1, \pm 2, \cdots$ are the correlation distances, $\varepsilon(k)$ is the residual or prediction error, $u^{2'}(k) = u^2(k) - E[\ u^2(k)\ ]$, and δ is an impulse function. If at least one of the correlation functions has values well outside the confidence limits this indicates that the model is inadequate or biased. Experience has shown that if these tests are used in conjunction with the estimation algorithm the experimenter can often infer a great deal of information regarding deficiencies in the fitted model. Indeed the tests equation (2.34) frequently indicate which terms should be included in the model to improve the fit.

3 Modelling of Morison's equation

Morison's equation (Morison, O'Brien, Johnson, and Schaf 1950) has been extensively used to predict wave forces on slender cylinders in fluid loading engineering. The equation takes the form

$$y(t) = \frac{1}{2}\rho D C_d u(t)|u(t)| + \frac{1}{4}\pi\rho D^2 C_m \dot{u}(t) \tag{3.1}$$

where t is a continuous time index, $y(t)$ is the force per unit axial length, $u(t)$ is the instantaneous flow velocity, ρ is a water density and D is the diameter of the cylinder, C_d and C_m are the dimensionless drag and inertia coefficients respectively which are determined by the characteristics of the flow. Eqn (3.1) can alternatively be written as

$$y(t) = a_1 u(t)|u(t)| + a_2 \dot{u}(t) \tag{3.2}$$

where

$$a_1 = \frac{1}{2}\rho D C_d$$

$$a_2 = \frac{1}{4}\pi\rho D^2 C_m \tag{3.3}$$

In the present paper, no attempt is made to discuss the advantages and disadvantages of the Morison's equation in the prediction of wave forces the objective here is considering the approximation to the Morison's equation using an identified rational model.

The discrete Morison's equation can be transformed using the backward difference scheme

$$\dot{u}(t) = \frac{u(k) - u(k-1)}{h} \tag{3.4}$$

where $k = 1, 2, 3, \cdots$ is a discrete time index and h=1 is the sampling period. Therefore eqn (3.2) may be written with an added noise term

$$y(k) = a_1 u(k)|u(k)| + a_2(u(k)-u(k-1)) + e(k) \tag{3.5}$$

This will be refered to as the stochastic Morison's equation where the noise $e(k)$ is any error induced by either uncertainty in the measurements or instrument errors, etc.

A zero mean uniform random sequence with amplitude ±5 (variance $\sigma_u^2 = 8.33$) was used as the input $u(k)$ and a zero mean Gaussian white sequence with variance $\sigma_e^2 = 1.0$ was used as the noise $e(k)$. One thousand data pairs were used in the simulation. With reference to eqn (3.1) the parameters were set as $\rho = 1$, $D = 2$ (Worden, Stansby, Tomlinson, and Billings 1991), $C_d = 0.7$, and $C_m = 2.0$ (Barltrop and Adams 1991) which are typical values for circular cylinders under fatigue loading, such that the parameters in eqn (3.3) become

$$a_1 = 0.7 \qquad a_2 = 2\pi \tag{3.6}$$

The ORME parameter estimator was used with an initial model consisting of 112 terms as the full model with specifications *numerator degree* = 3, *denominator degree* = 2, and *input lag* = *output lag* = *noise lag* = 2. Notice that this includes polynomial models as a subset. The input and output data sequences for this simulation are shown in Fig. 3.1. The final identified model is given in Table 3.1 which shows that the approximating equation is

$$y(k) = 7.39u(k) + 0.10u^3(k) - 6.29u(k-1) - 0.03u(k-1)e(k-2) + e(k) \tag{3.7}$$

The one step ahead predictions and residuals are illustrated in Fig. 3.2. The model validity tests are shown in Fig. 3.3.

All these results indicate that a proper approximation to Morison's equation has been obtained. It should be noted that although the estimator searched over the class of rational models it has correctly set the denominator to one and estimated a polynomial description.

4 Modelling of an exponential time series

A representative nonlinear exponential time series is selected to demonstrate the approximation properties of the rational model. A total of 1000 observation samples were generated using the model

$$\begin{aligned} y(k) &= (0.8 - 0.5\exp(-y^2(k-1)))y(k-1) \\ &- (0.3 + 0.9\exp(-y^2(k-1)))y(k-2) \\ &+ 0.1\sin(3.1415926y(k-1)) + e(k) \end{aligned} \tag{4.1}$$

where $k = 1, 2, 3, \cdots$ is a discrete time index and the noise $e(k)$ was a Gaussian white sequence with zero mean and variance $\sigma_e^2 = 0.01$.

The ORME estimation algorithm was applied with an initial model consisting of 56 terms as the full model with specifications *numerator degree* = *denominator degree* = 2, and *input lag* = 0, *output lag* = 2, and *noise lag* = 4. This includes the class of polynomial models as a subset. The output data sequence for this simulation is shown in Fig. 4.1. The final identified model is given in Table 4.1 and can be expressed as

$$y(k) = \frac{a(k)}{b(k)} + e(k) \tag{4.2}$$

where

$$\begin{aligned} a(k) &= 0.98y(k-1) - 1.14y(k-2) + 0.64e(k-1) + 0.15e(k-2) - 0.09e(k-3) - 0.34e(k-4) \\ b(k) &= 1 + 0.59y^2(k-1) - 0.39y(k-1)y(k-2) + 0.44y(k-1)e(k-3) \\ &- 0.74y(k-2)e(k-1) + 0.35y(k-2)e(k-4) + 0.34e^2(k-2) \end{aligned} \tag{4.3}$$

The one step ahead predictions and residuals are illustrated in Fig. 4.2. The model validity tests are shown in Fig. 4.3. All these results indicate that a proper approximation to the nonlinear exponential time series has been obtained.

From previous studies (Chen and Billings 1991) the simulated system eqn (4.1) gives rise to a stable limit cycle without noise $e(t)$ and this is shown in Fig. 4.4. The identified rational approximation model without noise terms

$$y(k) = \frac{0.98y(k-1) - 1.14y(k-2)}{1 + 0.59y^2(k-1) - 0.39y(k-1)y(k-2)} \tag{4.4}$$

produces a similar limit cycle as Fig. 4.5 and this suggests that the rational approximation model captures the underlying dynamics of the simulated nonlinear system very well even with noise contamination.

5 Modelling of dynamics with dead zone and saturation

A typical nonlinear model with dead zone and saturation is shown in figure 5.1. This system can be described by

$$x_1(k) = 0.7x_1(k-1) + u(k-1)$$

$$x_2(k) = \begin{cases} 0.0, & |x_1(k)| \le 1.0 \\ 1.0 * sign(x_1(k)), & |x_1(k)| \ge 2.0 \\ x_1(k), & otherwise \end{cases}$$

$$y(k) = -0.5y(k-1) + x_2(k-1) + e(k) \tag{5.1}$$

and has a structure which is common in many branches of mechanical and electrical engineering (Ogata 1970). A zero mean uniform random sequence with amplitude ±5 (variance $\sigma_u^2 = 8.33$) was used as input $u(k)$ and a zero mean Gaussian white sequence with variance $\sigma_e^2 = 1.0$ was used as noise $e(k)$. 1000 data pairs were used in the simulation.

The ORME estimator was applied with an initial model consisting of 552 terms defining the full model with specifications *numerator degree = denominator degree* = 2, *input lag = noise lag* = 10, and *output lag* = 2. Again this model includes the polynomial models as a subset. The input and output data sequences for this simulation are shown in Fig. 5.2. The final identified model is given in Table 5.1. The approximating equation is

$$y(k) = \frac{a(k)}{b(k)} + e(k) \tag{5.2}$$

where

$$a(k) = -0.51y(k-1) + 0.21u(k-1) + 0.14u(k-3) + 0.09u(k-4) + 0.07u(k-5)$$
$$+ 0.05u(k-6) + 0.01u(k-5)u(k-9) + 0.04u(k-5)e(k-3) + 0.162e(k-6)e(k-10)$$

$$b(k) = 1 - 0.07y(k-2)e(k-5) \quad (5.3)$$

The one step ahead predictions and residuals are illustrated in Fig. 5.3. and the model validity tests are shown in Fig. 5.4.

This simulation study demonstrates that the rational model can be used to approximate discontinuous nonlinear elements. Notice that additive noise on the system often induces nonlinear noise terms in the model. These terms have to be included as part of the estimation because if they were omitted this would induce bias.

6 Conclusions

The stochastic rational model and an associated parameter estimation algorithm have been introduced and shown to provide excellent approximations to three quite different nonlinear models.

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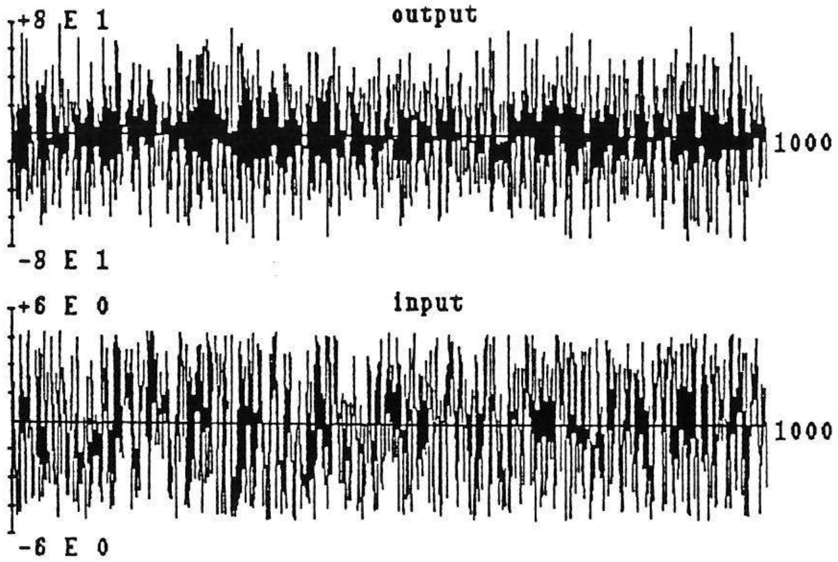

Figure 3.1 Input & output for Morison's equation

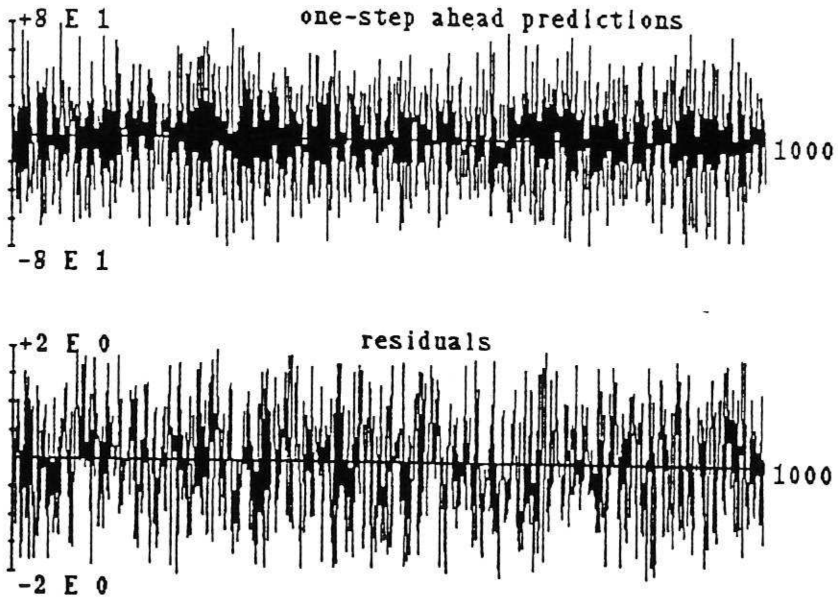

Figure 3.2 One step ahead predictions & residuals for the identified Morison's equation

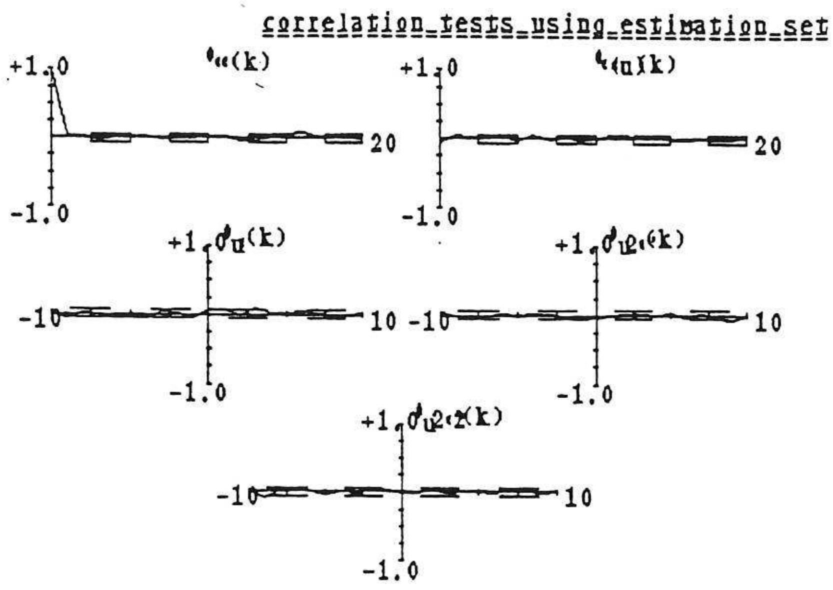

Figure 3.3 Model validation for Morison's equation

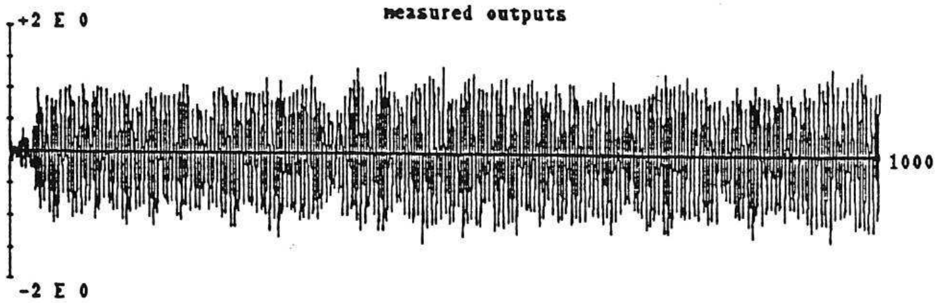

Figure 4.1 Output for exponential time series

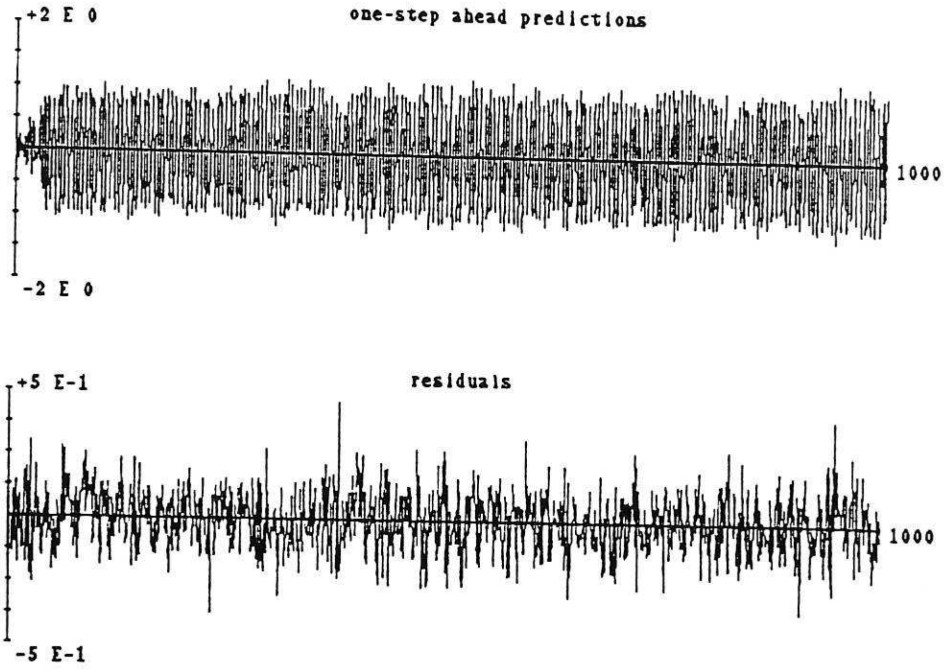

Figure 4.2 One step ahead predictions & residuals for the exponential time series approximation

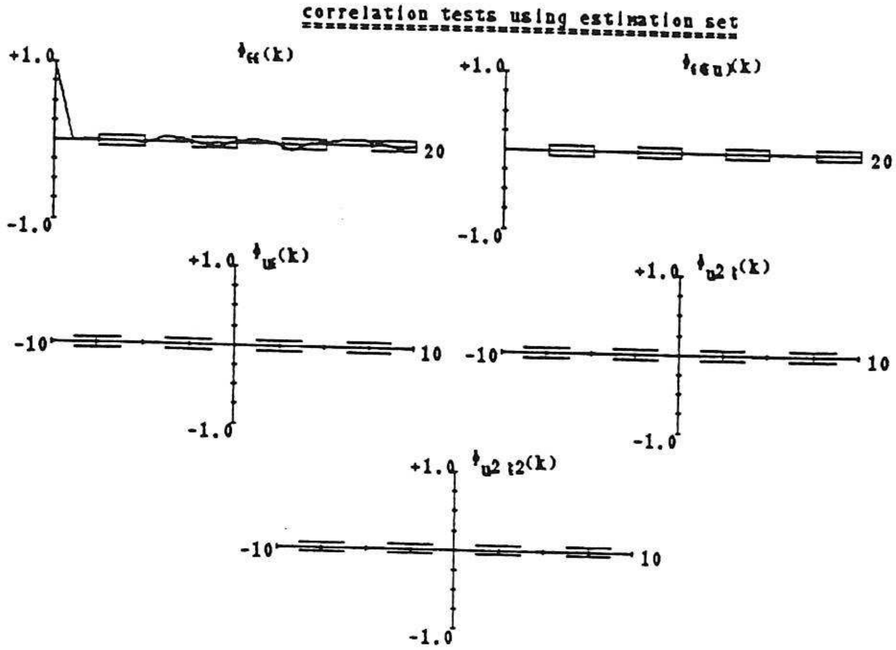

Figure 4.3 Model validation for the exponential time series

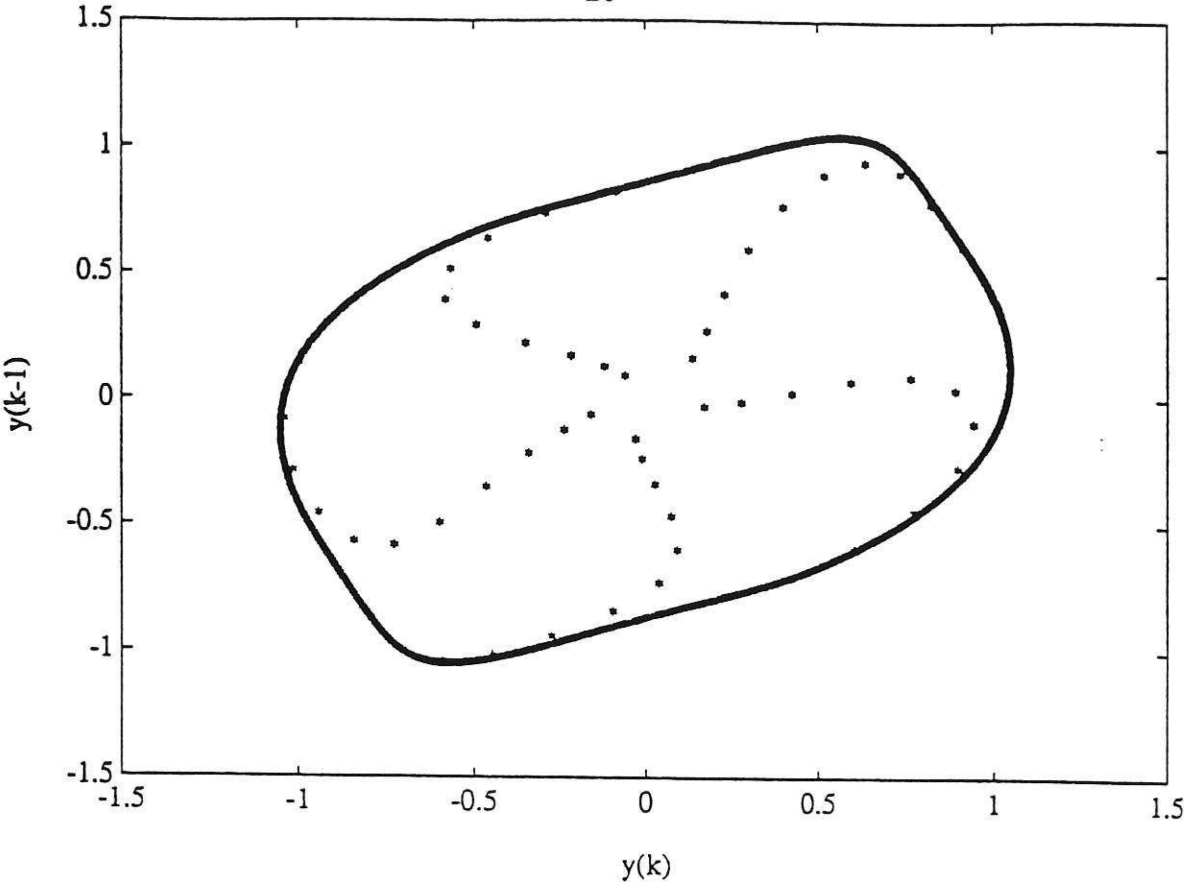

Figure 4.4 Limit cycle for the exponential time series with initial condition: $y(-1)=y(0)=0.1$

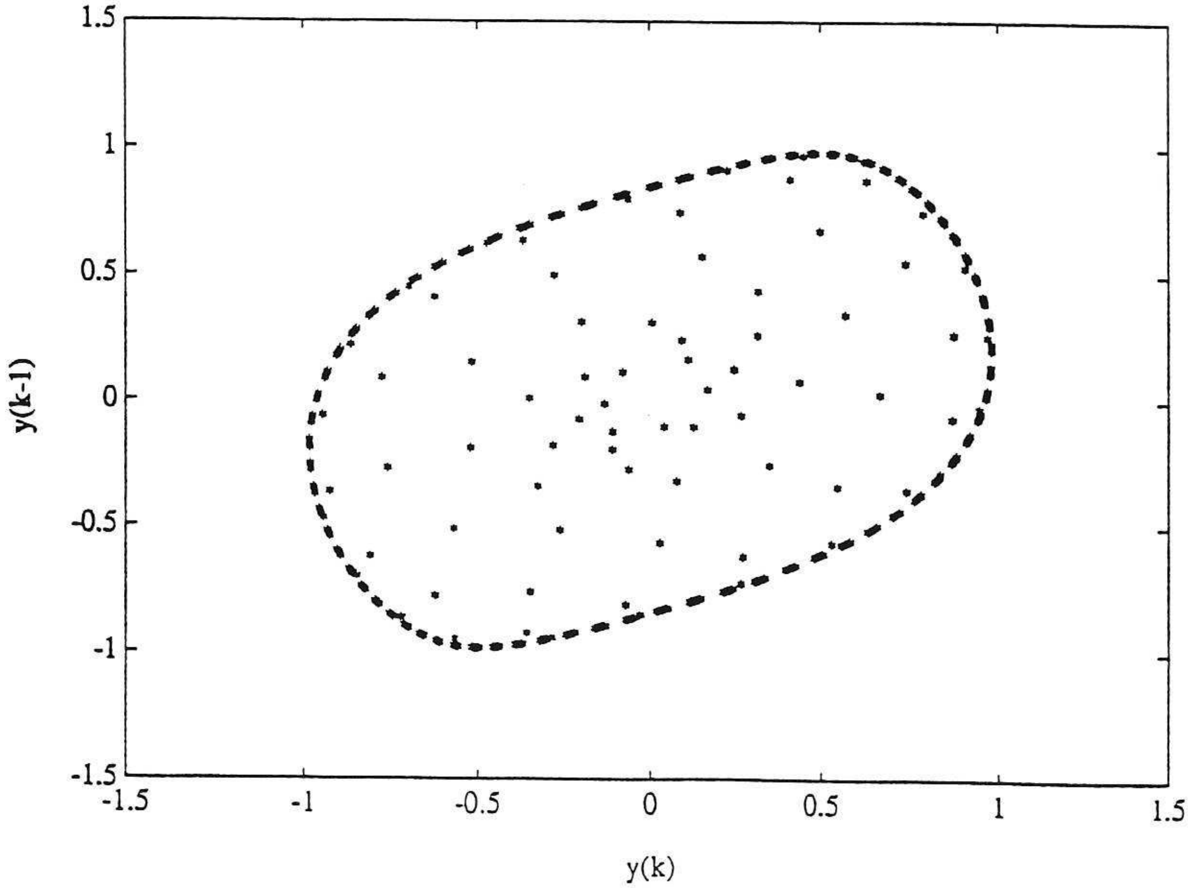

Figure 4.5 Limit cycle from the identified model with initial condition: $\hat{y}(-1)=\hat{y}(0)=0.1$

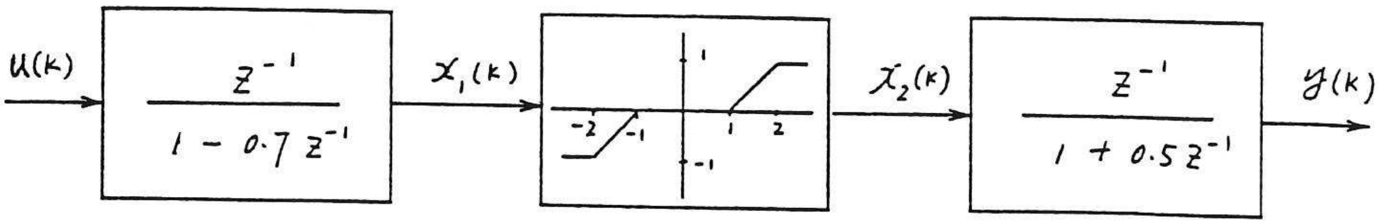

Figure 5.1 Dynamic system with dead zone & saturation

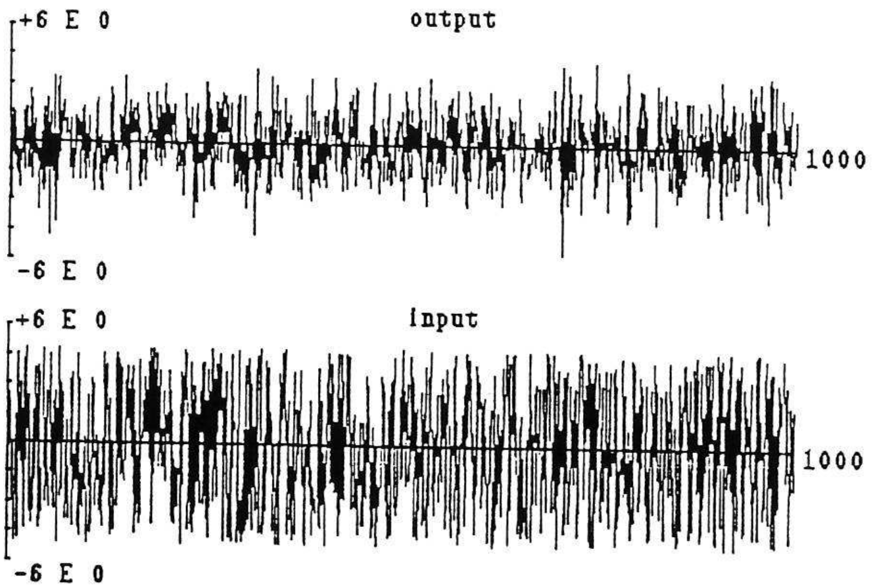

Figure 5.2 Input & output for the dynamic system with dead zone & saturation

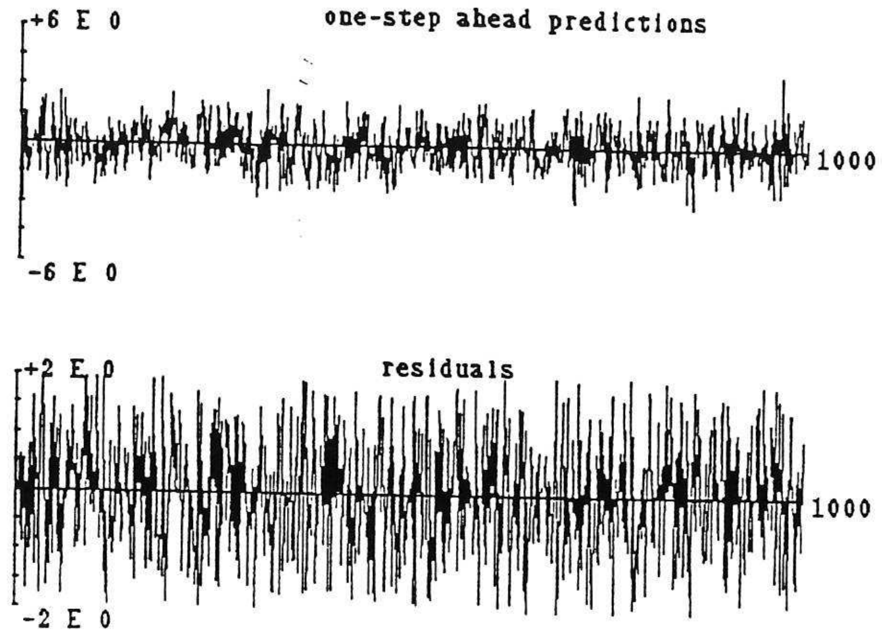

Figure 5.3 One step ahead predictions & residuals for the identified model of the dynamic system with dead zone & saturation

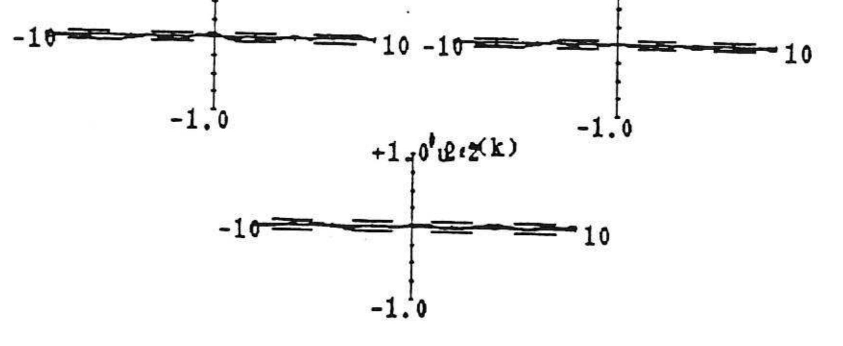

Figure 5.4 Model validation for the dynamic system with dead zone & saturation

terms	estimates	e.r.r.s	st.de.s	o.s.
numerator:				
(4) u(t- 1)**1=	0.7389E+01	0.6610E+00	0.2923E-01(	1)
(5) u(t- 2)**1=	-0.6292E+01	0.3331E+00	0.1161E-01(	2)
(65) u(t- 1)**3=	0.1015E+00	0.3740E-02	0.1762E-02(	3)
(25) u(t- 2)**1*e(t- 2)**1=	-0.2694E-01	0.4681E-05	0.1322E-01(	4)

Table 3.1 Identified model for Morison's equation

terms	estimates	e.r.r.s	st.de.s	o.s.
numerator:				
(3) y(t- 2)**1=	-0.1141E+01	0.5423E+00	0.1126E-01(	1)
(2) y(t- 1)**1=	0.9777E+00	0.4065E+00	0.1533E-01(	2)
(4) e(t- 1)**1=	0.6381E+00	0.2744E-02	0.6489E-01(	5)
(7) e(t- 4)**1=	-0.3395E+00	0.8780E-03	0.6268E-01(	8)
(5) e(t- 2)**1=	0.1493E+00	0.7421E-03	0.4198E-01(	10)
(6) e(t- 3)**1=	-0.9130E-01	0.2637E-03	0.4490E-01(	12)
denominator:				
(36) y(t- 1)**2*y(t)=	-0.5925E+00	0.2247E-01	0.2566E-01(	3)
(37) y(t- 1)**1*y(t- 2)**1*y(t)=	0.3880E+00	0.1123E-01	0.2483E-01(	4)
(43) y(t- 2)**1*e(t- 1)**1*y(t)=	0.7393E+00	0.2248E-02	0.1105E+00(	6)
(40) y(t- 1)**1*e(t- 3)**1*y(t)=	-0.4408E+00	0.1306E-02	0.1064E+00(	7)
(46) y(t- 2)**1*e(t- 4)**1*y(t)=	-0.3522E+00	0.8755E-03	0.1132E+00(	9)
(51) e(t- 2)**2*y(t)=	-0.3431E+00	0.2648E-03	0.3755E+00(	11)

Table 4.1 Identified model for the exponential time series

numerator:				
(5) u(t- 2)**1=	0.2127E+00	0.1890E+00	0.1273E-01(	1)
(2) y(t- 1)**1=	-0.5079E+00	0.1142E+00	0.2744E-01(	2)
(6) u(t- 3)**1=	0.1406E+00	0.5494E-01	0.1406E-01(	3)
(7) u(t- 4)**1=	0.8921E-01	0.2924E-01	0.1270E-01(	4)
(8) u(t- 5)**1=	0.7361E-01	0.2146E-01	0.1287E-01(	5)
(9) u(t- 6)**1=	0.4822E-01	0.6958E-02	0.1275E-01(	6)
(266) e(t- 6)**1*e(t-10)**1=	0.1629E+00	0.7268E-02	0.4332E-01(	7)
(149) u(t- 5)**1*e(t- 3)**1=	0.3681E-01	0.4396E-02	0.1407E-01(	8)
(145) u(t- 5)**1*u(t- 9)**1=	0.1147E-01	0.4475E-02	0.4346E-02(	10)
denominator:				
(337) y(t- 2)**1*e(t- 5)**1*y(t)=	0.7418E-01	0.4050E-02	0.2048E-01(	9)

Table 5.1 Identified model for the dynamic system with dead zone & saturation