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The Phase Response of Nonlinear Systems

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Abstract: Phase shift is very important in the frequency response analysis of nonlinear systems and in this paper the phase response function is analysed and interpreted to provide a clear understanding of the problem. Nonlinear systems which include delay elements are also discussed and simulations of both continuous and discrete time nonlinear systems are included to demonstrate the concepts involved.

1. Introduction

The frequency domain analysis of nonlinear systems is characterised by a series of transfer functions which are defined as the multidimesional Fourier transform of the Volterra kernels. The estimation and analysis of these higher order frequency response functions has recently been studied by several authors(Vinh et al, 1987; Billings and Tsang, 1989a, 1989b; Cho et al, 1992). However, before the method is accepted by engineers and scientists and applied to solve various practical problems a full understanding and interpretation of the complex-valued multi-dimensional functions, both in magnitude and phase, must be formulated. So far some general studies on nonlinear transfer functions have been reported and an interpretation which is mainly in terms of magnitude has been obtained(Zhang and Billings, 1992; Peyton-Jones and Billings, 1990). But very little work has been done on the analysis or interpretation of the phase response. In the present paper, attention will therefore be focused on the phase aspects of nonlinear systems and the important influence of phase on the total systems response will be demonstrated analytically and illustrated using simulated examples.

Initially some results for linear systems are briefly reviewed to provide a foundation which can be extended to the nonlinear case. An expression for the phase response for a wide class of nonlinear systems is then deterived and illustrated using a simple example. In §4 the phase response to a multi frequency input is derived to illustrate how the phase shift of both the input excitation and the phase response of the nonlinear system influence the overall system response. The effects of delay elements on the system are considered in §5 and a simulated example is discussed in detail in §6 to demonstrate in a very simple way the importance of phase in the interpretation of nonlinear frequency response functions. An overview of frequency mixing which plays a prominent part in nonlinear frequency response analysis, is provided in Appendix I.

2. Linear Case - A Review

It is well known that the input/output relationship of a linear time-invariant system is completely characterised by a transfer function $H(s)$. The frequency domain characteristcs can be obtained directly from the frequency(sinusoidal) response function that is the transfer function in which s is replaced by $j\omega$. In general, $H(j\omega)$ will be a complex-valued function of the frequency variable ω. This can be expressed in Polar form as

$$H(j\omega) = |H(j\omega)| \angle H(j\omega) = \Gamma(\omega)\, e^{j\Phi(\omega)} \qquad (1)$$

where $\Gamma(\omega) = |H(j\omega)|$ is the magnitude and $\Phi(\omega) = \angle H(j\omega)$ is the phase angle of $H(j\omega)$. In the frequency domain, the Fourier transforms of input and output, if they exist, are related by a simple product

$$Y(j\omega) = H(j\omega)\,U(j\omega) \tag{2}$$

If a sinusoidal signal $u(t) = A\cdot\cos(\omega t)$ is applied as a stimulus to the system, then the steady state response is

$$\begin{aligned} y_{ss}(t) &= A\,|H(j\omega)|\cos[\omega t + \angle H(j\omega)] \\ &= A\cdot\Gamma(\omega)\cos[\omega t + \Phi(\omega)] \end{aligned} \tag{3}$$

It is apparent from the discussion above that $H(j\omega)$, or equivalently the two real functions $\Gamma(\omega)$ and $\Phi(\omega)$, completely characterise the effect of the linear system on a sinusoidal input signal of any arbitrary frequency. In fact $\Gamma(\omega)$ determines the amplification ($\Gamma(\omega) > 1$) or attenuation ($\Gamma(\omega) < 1$) imparted by the system on the input signal. The phase response $\Phi(\omega)$ determines the amount of phase shift imparted by the system on the input sinusoid. Concequently, a knowledge of $H(j\omega)$ allows the determination of the response of the system to any sinusoidal input signal. If the input to the system consists of more than one sinusoid, the superposition property of linear systems can be used to determine the response(See Appendix I). Since $H(j\omega)$ specifies the response of the system in the frequency domain, it is called the frequency response of the system. Correspondingly, $\Gamma(\omega)$ is called the magnitude response and $\Phi(\omega)$ is the phase response of the system. Clearly, in the linear case, both magnitude and phase response have a clear physical meaning and provide a valuable aid in systems analysis and synthesis.

Nonlinear systems can not be described by the simple transfer function because: (i) the output for a sinusoidal input is not necessarily a sinusoid of the same frequency as the input; and (ii) nonlinear systems do not obey the superposition principle. Employing the Volterra series and multi-dimesional Fourier transform a generalised transfer function or frequency response function for nonlinear systems can be defined. But the interpretation and application of the magnitude and phase response are not as straightforward as in the linear case.

3. Magnitude and Phase Responses of General Nonlinear Systems

For a general nonlinear system, the n-th order generalised frequency response functions, or simply, n-th order transfer functions, are determined by taking the multidimensional Fourier transform of the n-th order Volterra kernels of the system

$$H_n(j\omega_1, \cdots, j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n)\, e^{-j(\omega_1\tau_1 + \cdots + \omega_n\tau_n)}\, d\tau_1 \cdots d\tau_n \qquad (4)$$

Generally the n-th order nonlinear transfer function $H_n(j\omega_1, \cdots, j\omega_n)$ is multidimensional in nature and a complex quantity. In Polar form

$$H_n(j\omega_1, \cdots, j\omega_n) = \Gamma_n(\omega_1, \cdots, \omega_n) e^{j\Phi_n(\omega_1, \cdots, \omega_n)} \qquad (5)$$

where, as in the linear case, $\Gamma_n(\cdot) = |H_n(j\omega_1, \cdots, j\omega_n)|$ is called the magnitude response and

$$\Phi_n(\omega_1, \cdots, \omega_n) = \angle H_n(j\omega_1, \cdots, j\omega_n) = \tan^{-1}\frac{Im[H_n(j\omega_1, \cdots, j\omega_n)]}{Re[H_n(j\omega_1, \cdots, j\omega_n)]} \qquad (6)$$

is the phase respons and $Im[\cdot]$ and $Re[\cdot]$ are used to denote the imaginary and real parts of a complex quantity respectively. Both functions are real and multi-variate. The magnitude response function $\Gamma_n(\cdot)$ can take on both positive and negative values. Since -1 can be expressed as $e^{\pm j\pi}$, the sign changes in $\Gamma_n(\cdot)$ can be accomodated in the phase response $\Phi_n(\cdot)$. In this case the phase jumps of $\pm\pi$ will occur at frequencies where $\Gamma_n(\cdot)$ changes sign. Clearly the linear system is a special case where all the higher than first order transfer functions are zero. Notice that all the frequencies

$\omega_1, \cdots, \omega_n$ run from $-\infty$ to ∞. From the definition eqn.(4), frequency response functions of real systems are conjugated when the signs of all the arguments are changed so that

$$H_n(-j\omega_1, \cdots, -j\omega_n) = H_n^*(j\omega_1, \cdots, j\omega_n) \tag{7}$$

Therefore the phase response $\Phi_n(\cdot)$ is an odd function of all the arguments

$$\Phi_n(-\omega_1, \cdots, -\omega_n) = -\Phi_n(\omega_1, \cdots, \omega_n) \tag{8}$$

In other words, $\Phi_n(\cdot)$ is negatively symmetric along the plane $\omega_1 + \cdots + \omega_n = 0$. Needless to say the magnitude $\Gamma_n(\cdot)$ is an even function. Furthermore, from eqn.(7) it is observed that

$$Im[\, H_{2n}(j\omega, \cdots, j\omega, -j\omega, \cdots, -j\omega)\,] = 0 \qquad n=1,2,... \tag{9}$$

and

$$\Phi_{2n}(\omega, \cdots, \omega, -\omega, \cdots, -\omega) = \begin{cases} 0 & Re[H_{2n}(\cdot)] \geq 0 \\ \pm\pi & Re[H_{2n}(\cdot)] < 0 \end{cases} \tag{10}$$

The d.c. component is known to arise in this case because the frequency produced is zero.

Consider a simple nonlinear circuit which is described by the differential equation

$$\frac{dy(t)}{dt} + c_1 y(t) = c_2 u(t) + c_3 u^2(t) \tag{11}$$

The linear and nonlinear transfer functions for this system are

$$H_1(j\omega) = \frac{c_2}{c_1 + j\omega} \tag{12}$$

and

$$H_2(j\omega_1, j\omega_2) = \frac{c_3}{c_1 + (j\omega_1 + j\omega_2)} \tag{13}$$

respectively. Both $H_1(\cdot)$ and $H_2(\cdot)$ can be expressed as polar form as

$$H_1(j\omega) = \Gamma_1(\omega)\, e^{j\Phi_1(\omega)} \quad \text{and} \quad H_2(j\omega_1, j\omega_2) = \Gamma_2(\omega_1,\omega_2) e^{j\Phi_2(\omega_1,\omega_2)} \tag{14}$$

with

$$\Gamma_1(\omega) = \frac{c_2}{\sqrt{c_1^2+\omega^2}} \quad \text{and} \quad \Phi_1(\omega) = -\tan^{-1}\frac{\omega}{c_1} \tag{15}$$

and

$$\Gamma_2(\omega_1,\omega_2) = \frac{c_3}{\sqrt{c_1^2+(\omega_1+\omega_2)^2}} \quad \text{and} \quad \Phi_2(\omega_1,\omega_2) = -\tan^{-1}\frac{\omega_1+\omega_2}{c_1} \tag{16}$$

For this specific example, the second order phase response is in a rather similar to the first order counterpart but with a dimensional extension. Setting c_1=5, c_2=5 and c_3=−0.8 plots of these four real functions, all within the range of ±5 Hz, are given in Figs.1 and 2. Because of the conjugate symmetry of $H_n(\cdot)$, the plots of $H_1(j\omega)$, $0<\omega<\infty$ and $H_2(j\omega_1, j\omega_2)$ with $0<\omega_1<\infty$ and $-\infty<\omega_2<\infty$ fully characterise the transfer functions. The graphical analysis for the multidimensional phase responses will be presented in the future report.

4. Computation of the Phase Response to a Multi-Frequency Input.

A phase-shift value is always connected to a particular sinusoidal signal. The phase response function is actually used to determine the phase-shift of each particular frequency component in the output. Consider an input composed of K sinusoids with different frequencies and phase shifts:

$$u(t) = \sum_{i=1}^{K} |A_i| \cos(\omega_i t + \angle A_i) = \sum_{i=-K}^{K} \left[\frac{A_i}{2} e^{j\omega_i t}\right] = \sum_{i=-K}^{K} \left[\frac{A_i}{2} e^{j2\pi f_i t}\right] \tag{17}$$

where ω_k is k-th frequency with amplitude $|A_k|$ and phase shift $\theta_k = \angle A_k$. A_k is a complex number which gives the amplitude and phase of the kth frequency with the properties that $A_0 = 0$ and $A_{-k} = A_k^*$. The total response of the nonlinear system can be expressed as

$$y(t) = \sum_{n=1}^{N} y_n(t) \tag{18}$$

where $y_n(t)$ is n-th order output generated by the nth order nonlinearity of the system. It can be shown that subject to the multi-tone input $u(t)$ of eqn.(17),

$$y_n(t) = \frac{1}{2^n} \sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K} \left[A_{k_1} \cdots A_{k_n} H_n(j\omega_{k_1}, \cdots, j\omega_{k_n}) \right] e^{j(\omega_{k_1} + \cdots + \omega_{k_n})t} \tag{19}$$

Writing all the complex quatities in Polar form yields

$$y_n(t) = \frac{1}{2^n} \sum_{k_1=-K}^{K} \cdots \sum_{k_n=-K}^{K} \left[|A_{k_1} \ldots A_{k_n}| \cdot \Gamma_n(\omega_{k_1}, \cdots, \omega_{k_n}) \right]$$

$$\times e^{j(\omega_{k_1} + \cdots + \omega_{k_n})t + (\theta_{k_1} + \cdots + \theta_{k_n}) + \Phi_n(\omega_{k_1}, \cdots, \omega_{k_n})} \tag{20}$$

where θ_{k_i} are the phase shifts associated with the input sinusoids and $\Phi_n(\cdot)$ is the phase response of the system. It is seen that when a sum of K sinusoids is applied to a nonlinear system, the output consists of all possible combinations of the input frequencies $-\omega_K, \cdots, -\omega_{-1}, \omega_1, \cdots, \omega_K$ taken n at a time. The phase angle of each particular term(a phasor) at frequency $(\omega_{k_1} + \cdots + \omega_{k_n})$ is determined by the phase response $\Phi_n(\omega_{k_1}, \cdots, \omega_{k_n})$, as well as by the relevant combination of phase shifts associated with the input frequencies. Namely, the phase shift for this particular frequency term is $(\theta_{k_1} + \cdots + \theta_{k_n}) + \Phi_n(\omega_{k_1}, \cdots, \omega_{k_n})$. This is a very important and intuitive conclusion for the nonlinear phase response to a multi-tone input.

Notice that eqn.(20) is only a general mathemetical expression in which many terms are identical if the symmetric transfer function is used. In order to derive an expression for a specific frequency component, define $M = (m_{-K}, \cdots, m_{-1}, m_1, \cdots, m_K)$ as a frequency mix vector(Chua and Ng, 1979), where m_k, $k = 1, \cdots, K$, are nonnegative integers which denote the number of times the frequency $f_k = \frac{\omega_k}{2\pi}$ appears in the frequency mix. An arbitrary frequency mix is then represented by the vector as

$$f_M = \sum_{\substack{k=-K \\ k \neq 0}}^{K} m_k f_k = \sum_{k=1}^{K} (m_k - m_{-k}) f_k \tag{21}$$

and the sum of all terms with frequency f_M in the nth order output component $y_n(t)$ is given(Billings and Tsang, 1989) as

$$\bar{y}_n(t; f_M) = \frac{n!}{2^n} \left[\prod_{\substack{i=-K \\ i \neq 0}}^{K} \frac{A_i^{m_i}}{m_i!} \right] \times H_n(m_{-K}\{f_{-K}\}, \cdots, m_{-1}\{f_{-1}\}, m_1\{f_1\}, \cdots, m_K\{f_K\}) \times e^{j2\pi f_M t} \tag{22}$$

where $m_k(f_k)$ denotes m_k consecutive arguments with the same frequency f_k and the overhard bar '–' denotes that the term is still a complex phasor rather than a real sinusoidal function. Using $\{Mf\}$ to denote the argument list of frequencies the frequency response function can be expressed in polar form as

$$H_n(Mf) = H_n(m_{-K}\{f_{-K}\}, \cdots, m_{-1}\{f_{-1}\}, m_1\{f_1\}, \cdots, m_K\{f_K\}) = \Gamma_n(Mf) e^{j\Phi_n(Mf)} \tag{23}$$

The real sinusoidal component at frequency f_M can then be obtained by combining two complex conjugated phasors

$$y_n(t; f_M) = \bar{y}_n(t; f_M) + \bar{y}_n(t; -f_M) = 2Re[\bar{y}_n(t; f_M)]$$

$$= \frac{n!}{2^{(n-1)}(m_{-K}!) \cdots (m_{-1}!)(m_1!) \cdots (m_K!)} \left(|A_1|^{m_1+m_{-1}} | \cdots |A_K|^{m_K+m_{-K}} | \cdot \Gamma_n(Mf) \right)$$

$$\times \cos[\, 2\pi f_M t + \Theta_M + \Phi_n(Mf)\,] \tag{24}$$

with

$$\Theta_M = \sum_{\substack{k=-K \\ k\neq 0}}^{K} m_k \theta_k = \sum_{k=1}^{K} (m_k - m_{-k})\theta_k \tag{25}$$

The phase shift at frequency f_M is therefore $\Theta_M + \Phi_n(Mf)$. This is a generalisation of the linear case where now the frequency is a mixture of the input frequencies rather than an individual input frequency. For example, for a two frequency input $u(t) = \cos(2\pi f_1 + \theta_1) + \cos(2\pi f_2 + \theta_2)$, the component at the intermodulation frequency $(f_1 + 2f_2)$ in the third order output will be

$$y_3(t;0,0,1,2) = \frac{3}{4}\Gamma_3(f_1,f_2,f_2)\cos[2\pi(f_1+2f_2) + \theta_1+2\theta_2 + \Phi_3(f_1,f_2,f_2)]$$

Observe that $y_n(t;f_M)$ is only one sinusoidal component at frequency f_M contained in the total response. It is important to realise that several different frequency mixes are capable of contributing to the response at the same output frequency. In other words, there may be several sinusoids, each of which has the same frequency but with a different phase-shift, mixing together to form a frequency component at the output. In this sense the phase response is of equal importance to the magnitude in determining the total response at a particular output frequency. To illustrate this remark, consider a nonlinear system for which the highest order response is fifth order. Let the input consist of two sinusoids at frequencies f_1 and f_2. The output contains a frequency component at $2f_2-f_1$ that is generated by three frequency mixes represented by the vectors $M =$ (0,1,0,2), (1,1,0,3) and (0,2,1,2), that is

$$y(t;2f_2-f_1) = y_3(t;0,1,0,2) + y_5(t;1,1,0,3) + y_5(t;0,2,1,2)$$

$$= 0.75|H_3(-f_1,f_2,f_2)|\cos[2\pi(2f_2-f_1)+2\theta_2-\theta_1 + \Phi_3(-f_1,f_2,f_2)]$$

$$+ 1.25|H_5(-f_1,-f_2,f_2,f_2,f_2)|\cos[2\pi(2f_2-f_1)+2\theta_2-\theta_1 + \Phi_5(-f_1,-f_2,f_2,f_2,f_2)]$$

$$+ 1.875|H_5(-f_1,-f_1,f_1,f_2,f_2)|\cos[2\pi(2f_2-f_1)+2\theta_2-\theta_1 + \Phi_5(-f_1,-f_1,f_1,f_2,f_2)]$$

Clearly, the frequency component $(2f_2-f_1)$ in the output is generated by three sinusoids of the same frequency but with different phase-shifts and different magnitudes. The differences between the phase shifts of the three sinusoids only result from the phase response functions. The phasor diagram of Fig.3 illustrates this effect for this example and shows the importance of the phase response in determining the final response.

5. Delays in Nonlinear Systems

Nonlinear systems with delay elements which are often encountered in practical situations can be represented by the configuration shown in Fig.4. The output of the delay element is

$$v(t) = u(t-d) \tag{26}$$

where d accounts for the delay time. The transfer function of the delay element is

$$H_d(j\omega) = e^{-j\omega d}$$

Although the delay element is linear, the overall system with the delay is still nonlinear. Suppose that the original nonlinear system can described by nth order kernels $\hat{h}(\cdot)$ and transfer functions $\hat{H}_n(\cdot)$, $n=1,2....$ By use of the Volterra functionals, the nth order output $y_n(t)$ can be expressed as

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} v(t-\tau_i)\, d\tau_i \qquad n > 0 \tag{27}$$

Substitution of eqn.(26) yeilds

$$y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{h}_n(\tau_1, \cdots, \tau_n) \prod_{i=1}^{n} u(t-d-\tau_i)\, d\tau_i$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{h}_n(\tau_1-d, \cdots, \tau_n-d) \prod_{i=1}^{n} u(t-\tau_i)\, d\tau_i \qquad n > 0 \qquad (28)$$

This suggests that the nth order Volterra kernel of the overall system is

$$h_n(t_1, \cdots, t_n) = \hat{h}_n(t_1-d, \cdots, t_n-d) \qquad n=1,2,... \qquad (29)$$

where $\hat{h}_n(\cdot)$ is the kernel with no delay. By definition, the overall transfer function is

$$H_n(j\omega_1, \cdots, j\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{h}_n(\tau_1-d, \cdots, \tau_n-d)\, e^{-j(\omega_1\tau_1+ \cdots +\omega_n\tau_n)}\, d\tau_1 \cdots d\tau_n$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \hat{h}_n(\tau_1, \cdots, \tau_n)\, e^{-j(\omega_1\tau_1+ \cdots +\omega_n\tau_n)}\, e^{-j(\omega_1+ \cdots +\omega_n)d}\, d\tau_1 \cdots d\tau_n$$

$$= e^{-j(\omega_1+ \cdots +\omega_n)d}\, \hat{H}_n(j\omega_1, \cdots, j\omega_n) \qquad (30)$$

As expected that the delay has no effect on the magnitude of the system, which remains $\hat{\Gamma}_n(\cdot)$, but it does modify the phase response of the various order transfer functions.

6. Simulations

In a linear time-invariant system the frequency content at the output is always identical to that of the input. But this is not true for nonlinear systems which can generate new frequencies which are not contained in the input. These new frequencies can be divided into two catogories: harmonics and intermodulations. Each frequency component will have an associated phase-shift which depends on the nonlinear transfer functions as well as the phase-shifts associated with the input frequencies. In this section, an example nonlinear system is simulated to illustrate the importance of the the phase response to a single and a multi-frequency sinusoidal input. It is shown how the output is affected by both the phase-response and the phase shifts in the input. The simulated nonlinear system is described by the following discrete-time model

$$y(k) = 0.9y(k-1) + 0.1u(k-1) + 0.1u^2(k-1) \tag{31}$$

The first and second order frequency response functions of the system are given by

$$H_1(f) = \frac{0.1e^{-j2\pi f}}{1 - 0.9e^{-j2\pi f}} \tag{32}$$

and

$$H_2(f_1,f_2) = \frac{0.1e^{-j2\pi(f_1+f_2)}}{1 - 0.9e^{-j2\pi(f_1+f_2)}} \tag{33}$$

respectively. Notice that for this specific example all the higher than second order frequency response functions are zero, thus all the nonlinear effects are caused by $H_2(\cdot)$, namely by the presence of nonlinear term $0.1u^2(k-1)$. The plots for $H_1(\cdot)$ and $H_2(\cdot)$ in form of magnitude and phase are given in Fig.5 and Fig.6, respectively. The first and second order phase response functions are obtained as

$$\Phi_1(f) = -\tan^{-1}\left[\frac{\sin 2\pi f}{\cos 2\pi f - 0.9}\right] \tag{34}$$

and

$$\Phi_2(f_1,f_2) = -\tan^{-1}\left[\frac{\sin 2\pi(f_1+f_2)}{\cos 2\pi(f_1+f_2) - 0.9}\right] \tag{35}$$

<u>**Response to a single frequency input**</u> Initially consider a single frequency sinusoidal input

$$u(t) = A\cos(2\pi f t + \theta) \tag{36}$$

It can be shown that the output will contain a fundamental frequency and the second order harmonic

$$y(t;f) = A\,|H_1(f)|\cos[2\pi f t+\theta+\Phi_1(f)] \tag{37}$$

and

$$y(t;2f) = 0.5A^2|H_2(f,f)|\cos[2\pi(2f)t + 2\theta + \Phi_2(f,f)] \tag{38}$$

respectively, as well as some d.c. components. From the results in Appendix I it is known that the response $y(t)$, which consists of $y(t;f)$ and $y(t;2f)$, is periodic with frequency f and the waveform is dependent on both the amplitudes and phase of both components. Notice how the nonlinear transfer function $H_2(\cdot)$ and input phase-shift θ affect the phase-shift of the output. Initialy set A=2, f=0.01 and θ=0, to give the simulated output in Fig.7. Clearly the output which is given by

$$y(t) = 1.718\cos[\,2\pi(0.01)t - 0.5697\,] + 1.286\cos[\,2\pi(0.02)t - 0.9369\,] \tag{39}$$

plus some d.c. component is highly distorted by the harmonic effect.

If the amplitude A is decreased from 2 to 1, the waveform is changed, see Fig.8(a). This is simply because the magnitude of $H_2(f,f)$ is not changed in the same scale as that of $H_1(f)$. If the input frequency f is now increased to 0.02, then the change in the ouput waveform, Fig.8(b), is not only caused by the magnitude of $H_2(f,f)$, but also by the phase value $\Phi_2(f,f)$ at (f,f). However, the shape of the output waveform is not affected by the phase shift θ in the input. Setting $\theta = -\pi/5$, say, the output, Fig.8(c), exhibits the same waveform shape as Fig.7 but with a shift. This is discussed further in Appendix I.

<u>Response to a multi–frequency input</u> Consider an input composed of two frequencies

$$u(t) = A_1\cos(2\pi f_1 t + \theta_1) + A_2\cos(2\pi f_2 t + \theta_2) \tag{40}$$

The system response for the above input will contain seven frequencies which include two single input frequencies f_1 and f_2; two harmonics $2f_1$ and $2f_2$; two intermodulations $|f_1 - f_2|$ and $f_1 + f_2$, and zero frequency(or d.c.) components. Notice that all the extra(new) frequencies are produced by $H_2(\cdot)$. Table 1 lists all the amplitudes and phases for each sinusoidal component contained in the output. This illustrates that each sinusoidal component in the nonlinear response is determined by the magnitude and phase response functions. Let $\theta_1 = \theta_2 = 0$, $A_1 = A_2 = 2$, $f_1 = 0.02$ and $f_2 = 0.035$, so that the input is

$$u(t) = 2\cos[\ 2\pi(0.02)t\ ] + 2\cos[\ 2\pi(0.035)t\ ] \tag{41}$$

An evaluation of the terms in Table 1 for this input are given in Table 4. If all the seven sinusoids defined in Table 2 are artificially generated the response will be given by

$$\begin{aligned} y(t) \ = \ & 4 + 1.2858\cos[2\pi(0.02)t - 0.9370] + 0.8658\cos[2\pi(0.035)t - 1.2359] \\ & + 0.7752\cos[2\pi(0.04)t - 1.3017] + 0.4696\cos[2\pi(0.07)t - 1.5595] \\ & + 1.1724\cos[2\pi(0.055)t - 1.4507] + 2.9824\cos[2\pi(0.015)t - 0.7777] \end{aligned} \tag{42}$$

The resultant waveform together with the input $u(t)$ of eqn.(41) is illustrated in Fig.9. It can be seen that the signal composed from eqn.(42) is basicly the same with the actual response which is given by Fig.10, except for the different phase. This confirms the analytical analysis. The power spectral densisty given in Fig.11 clearly shows the frequency content of the response.

If the 0.035 sinusoid is shifted by $\pi/3$ (i.e., set $\theta_1 = 0$, $\theta_2 = -\pi/3$) then most output phase components will change. As a result, the output waveform is changed, as shown in Fig.12, although all the magnitudes are exactly as before. As expected, the power spectral density reveals exactly the same frequency content.

Finally, if the parameter 0.1 associated with the nonlinear term is set to 0, then the system becomes linear. With the same input $u(t)$ given by eqn.(41) the system output is illustrated in Fig.13 and the power spectral density, Fig.14, clearly shows that there are now only two frequencies included in the output, 0.02 and 0.035.

7. Conclusions

The importance of the phase response of nonlinear systems has been invetigated by both analytically studying the higher order frequency response functions and by considering simulated examples. It has been demonstrated that phase has a significant influence on the total system response and suggests that a combined interpretation of both gain and phase will be required to unravel the frequency domain behaviour of nonlinear systems.

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Appendix I: Superposition of Sinusoidal Signals

Introduction Sinusoidal signal excitation and response (sine waves or harmonic oscillation) plays a prominent part in many branches of science and engineering. Any signal, periodic or otherwise, can be represented to an arbitrary degree of accuracy by a sum of sinusoidal waves and the Frequency Response Analysis (FRA) of linear systems is now well established and widely applied. For nonlinear systems the study of sinusoidal superposition is of particular importance since the nonlinear transfer functions are obtained by probing the system with a 'multi-tone' input. The output will consist of various frequencies, some of which are generated by the energy transfer mechanism associated with nonlinear systems.

A unit sinusoidal signal may be in the form of $\sin(\omega t)$, $\cos(\omega t)$ or $e^{j\omega t}$ in the complex plane (i.e., a phasor). Consider a real phase-shifted sinusoidal signal

$$x(t) = A\cos(\omega t + \theta) = A\cos(2\pi f t + \theta) \qquad \text{(A-1)}$$

This can also be described as

$$x(t) = A_1\cos(2\pi f t) + A_2\sin(2\pi f t) \qquad \text{(A-2)}$$

where $A_1 = A\sin\theta$ and $A_2 = A\cos\theta$. This result is useful for some derivations since it suggests that any phase-shifted sinusoidal signal can be expressed as a sum of a sine and a cosine wave, both of which have no phase shift.

Superposition Sinusoids. If two or more sinusoids of the same frequency are added the resultant is another sinusoid at the same frequency, with a different amplitude and phase. So that if

$$x_1(t) = A_1\cos(2\pi f t + \theta_1) \quad \text{and} \quad x_2(t) = A_2\cos(2\pi f t + \theta_2) \qquad \text{(A-3)}$$

then

$$x(t) = x_1(t) + x_2(t) = A \cos(2\pi f t + \theta) \qquad \text{(A-4)}$$

with

$$\begin{cases} A = \sqrt{(A_1\sin\theta_1 + A_2\sin\theta_2)^2 + (A_1\cos\theta_1 + A_2\cos\theta_2)^2} \\ \theta = \tan^{-1}\left[\dfrac{A_1\cos\theta_1 + A_2\cos\theta_2}{A_1\cos\theta_1 + A_2\cos\theta_2}\right] \end{cases} \qquad \text{(A-5)}$$

The superposition of two or more sinusoids of the same frequency may be computed more conveniently by the use of phasors. These complex numbers or phasors can be added to yeild the amplitude and phase of the summed sinusoid.

<u>Superposition of Different Frequences.</u> If two or more sinusoids of diffrerent frequencies are added then the resultant signal will no longer be a sinusoid in gereral. Suppose $x(t)$ is composed of N sinusoids

$$x(t) = \sum_{n=1}^{N} A_n \cos(2\pi f_n t + \theta_n) \qquad \text{(A-6)}$$

Consider the periodicity initially and assume $x(t)$ is periodic with period time $T = 1/f$ then

$$\begin{aligned} x(t+T) &= \sum_{n=1}^{N} A_n \cos[2\pi f_n (t+T) + \theta_n] \\ &= \sum_{n=1}^{N} A_n \cos[2\pi f_n t + \theta_n + 2\pi f_n / f] \end{aligned} \qquad \text{(A-7)}$$

Clearly if f_n/f are all integers for any f_n, n=1,2...N, then $x(t+T)=x(t)$ for all t. In other words, if $f_1, f_2 \cdots f_N$ are all multiples of some highest common factor, say, f, then $x(t)$ will repeat itself after a time $T = 1/f$. the following three examples illustrate this

$$\begin{aligned} x_1(t) &= 2\cos(6\pi t) + 2\cos(12\pi t) \\ x_2(t) &= \cos(2\pi t) + 2.5\cos(4\pi t) + 2\sin(14\pi t) \\ x_3(t) &= \cos(2\pi t) + 2.5\cos(\sqrt{19}\pi t) + 2\sin(\sqrt{177}\pi t) \end{aligned} \qquad \text{(A-8)}$$

where x_1 and x_2 have fundamental frequencies of 3 Hz and 1 Hz, respectively. The frequencies involved in x_3, however, have no common factor. As a result, x_3 is not periodic in time and the signal goes on forever without repeating.

Now consider how the amplitudes and phases of the summed sinusoids affect the waveform of the resultant signal. If the amplitude of any of the summed sinusoids is changed, the waveform of the resultant signal will change accordingly, unless all the amplitudes are changed by the same multiplier. In this case the resultant waveform will retain the same basic pattern but expand or contract in scale.

Similarly, if the phase-shift of any summed sinusoid is changed the resultant waveform will also change. But there is also a special case which arises if all the summed sinusoids are shifted proportionally according to frequency. In this case the resultant waveform will only shift forward or backwards without any distortion in the waveform, so that for example

$$x(t) = \sum_{n=1}^{N} A_n \cos[2\pi f_n t + \theta_n + \varepsilon f_n]$$

$$= \sum_{n=1}^{N} A_n \cos[2\pi f_n (t + \frac{\varepsilon}{2\pi}) + \theta_n]$$

$$= y(t + \tau)$$

where $\tau = \dfrac{\varepsilon}{2\pi}$ is the shift of the resultant waveform.

To illustrate these simple concepts consider the response of a linear system to a sum of sinusoids, or a multi-tone signal. If $x(t)$ of Eqn.(A-6) is input into a linear time-invariant system with a transfer function $H(j2\pi f)$ ($H(j\omega)$) the output will be

$$y(t) = \sum_{n=1}^{N} A_n |H(j2\pi f_n)| \cos[2\pi f_n t + \theta_n + \angle H(j2\pi f_n)] \qquad \text{(A-9)}$$

where $|H(\cdot)|$ and $\angle H(\cdot)$ are the magnitude and phase responses, respectively, imparted by the system to the individual frequency component of the input signal. It is clear that depending on the frequency response $H(j2\pi ft)$ of the system input sinusoids of different frequencies will be affected differently by the system, both in magnitude and phase. Hence the linear system may change the shape of the periodic input by scaling the amplitude and shifting the phase of the input sinusoidal component (the Fourier series components) but it does not affect the periodicity of the periodic input signal. To transmit $x(t)$ to $y(t)$ therefore without introducing distortion, from the above analysis, the system transfer function $H(j2\pi f)$ should satisfy the following two conditions: (i) the magnitude $|H(j2\pi f_n)|$ is constant regardless of f_n (ii) the phase response $\angle H(j2\pi f)$ is a linear function of f_n (i.e., $\angle H(j2\pi f) = k \cdot f_n$). Under these conditions the output $y(t)$ would exhibit the same waveform as the input $x(t)$ and the systems would be distortion free.

A second example concerns the harmonic response of a nonlinear system. Using the nonlinear transfer function method the response of a second order nonlinear system to a single frequency input $u(t) = A\cos(2\pi ft + \theta)$ is

$$y(t) = A|H_1(j2\pi f)|\cos[2\pi ft + \theta + \angle H_1(j2\pi f)]$$
$$+ 0.5A^2 |H_2(j2\pi f, j2\pi f)|\cos[4\pi ft + 2\theta + \angle H_2(j2\pi f, j2\pi f)] \qquad \text{(A-10)}$$

The output $y(t)$ is a distortion of the pure sinusoidal wave $u(t)$. If either the frequency f or the amplitude A of input sinusoid are changed, the waveform of $y(t)$ will generally change since the amplitudes and phases of the output components will be changed. However, if only the phase-shift θ is altered, $y(t)$ will be shifted but the waveform will not be changed. This is because the phase-shift imparted by the input phase θ are proportional to the order of the harmonic components in the output. The conclusion holds true even for the case where more than two harmonics are included in the output.

Frequncies	Magnitudes	Phases
f_1	$A_1 \vert H_1(f_1)\vert$	$\theta_1+\Phi_1(f_1)$
f_2	$A_2 \vert H_1(f_2)\vert$	$\theta_2+\Phi_1(f_2)$
$2f_1$	$0.5A_1^2 \vert H_2(f_1,f_1)\vert$	$2\theta_1+\Phi_2(f_1,f_1)$
$2f_2$	$0.5A_2^2 \vert H_2(f_2,f_2)\vert$	$2\theta_2+\Phi_2(f_2,f_2)$
f_1+f_2	$A_1A_2 \vert H_2(f_1,f_2)\vert$	$\theta_1+\theta_2+\Phi_2(f_1,f_2)$
$-f_1+f_2$	$A_1A_2 \vert H_2(-f_1,f_2)\vert$	$\theta_1-\theta_2+\Phi_2(-f_1,f_2)$
d.c.	$0.5A_1^2 \vert H_2(f_1,-f_1)\vert+0.5A_2^2 \vert H_2(f_2,-f_2)\vert$	0

Table 1. Amplitudes and phases of various output components.

Frequncies	Magnitudes	Phases(in redians)
0.02	1.2858	-0.9370
0.035	0.8658	-1.2359
0.04	0.7752	-1.3017
0.07	0.4696	-1.5595
0.055	1.1724	-1.4507
0.015	2.9824	-0.7777
d.c.	4	0

Table 2. Amplitudes and phases of the output components with the two-tone input defined by eqn.(41).

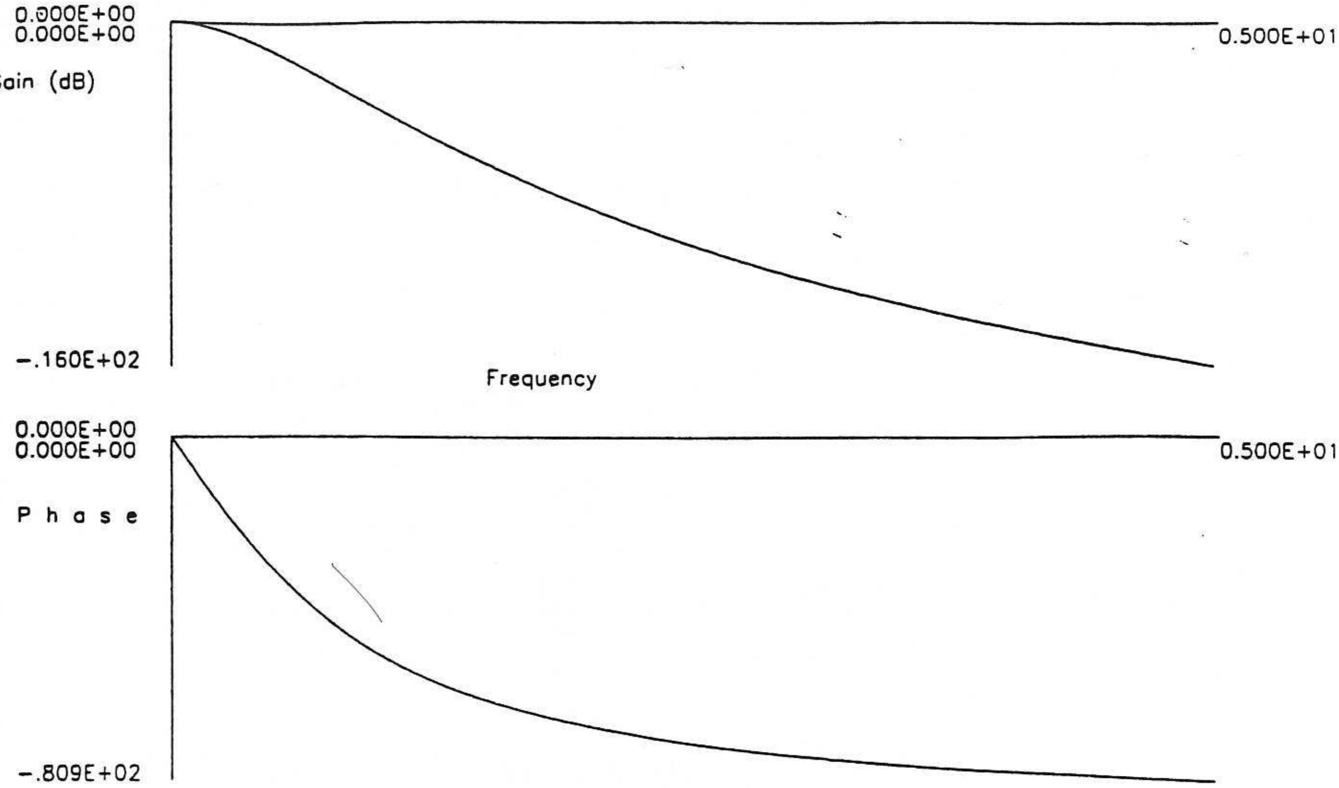

Fig.1 Gain and phase of $H_1(\cdot)$ for Eqn.(11).

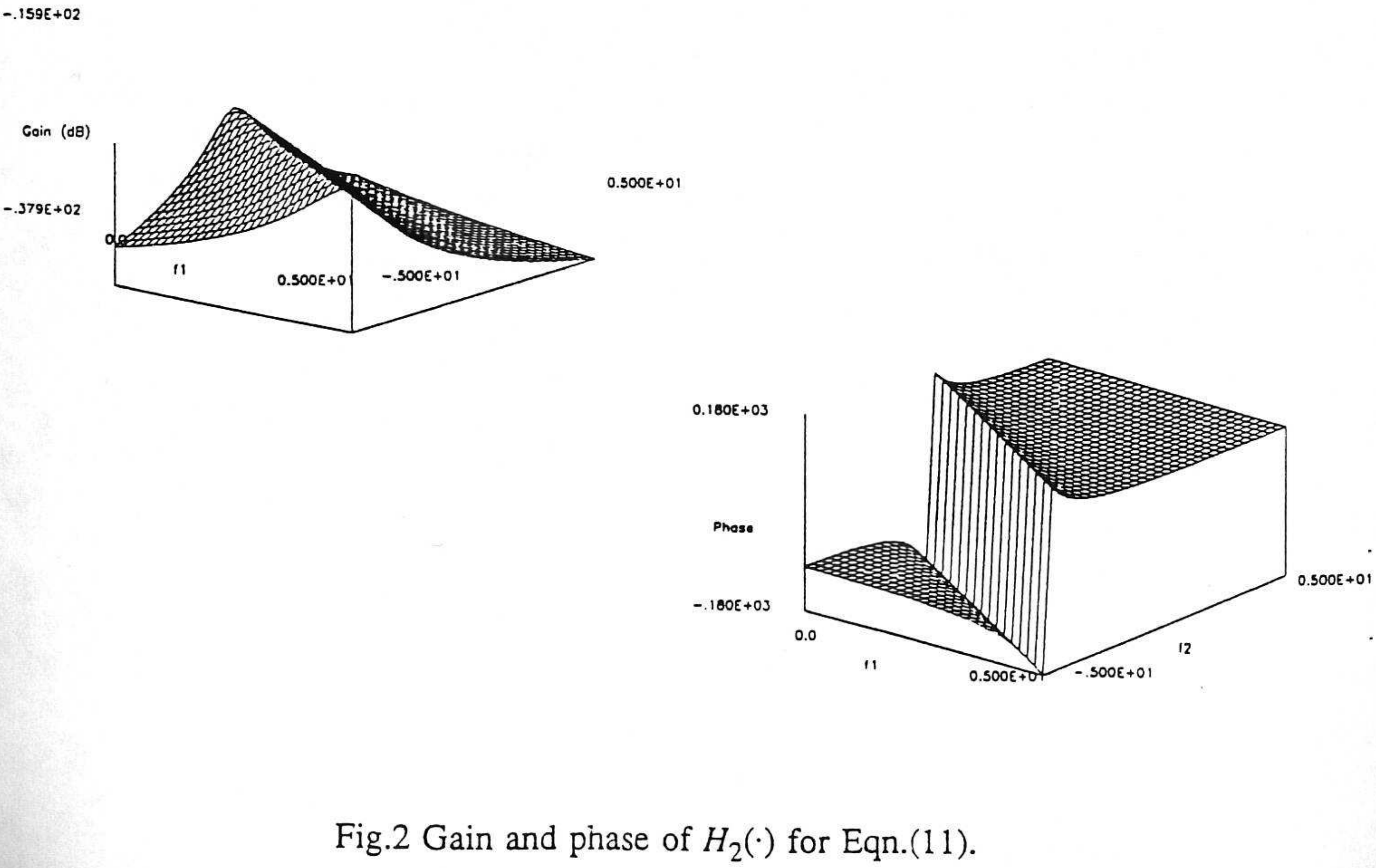

Fig.2 Gain and phase of $H_2(\cdot)$ for Eqn.(11).

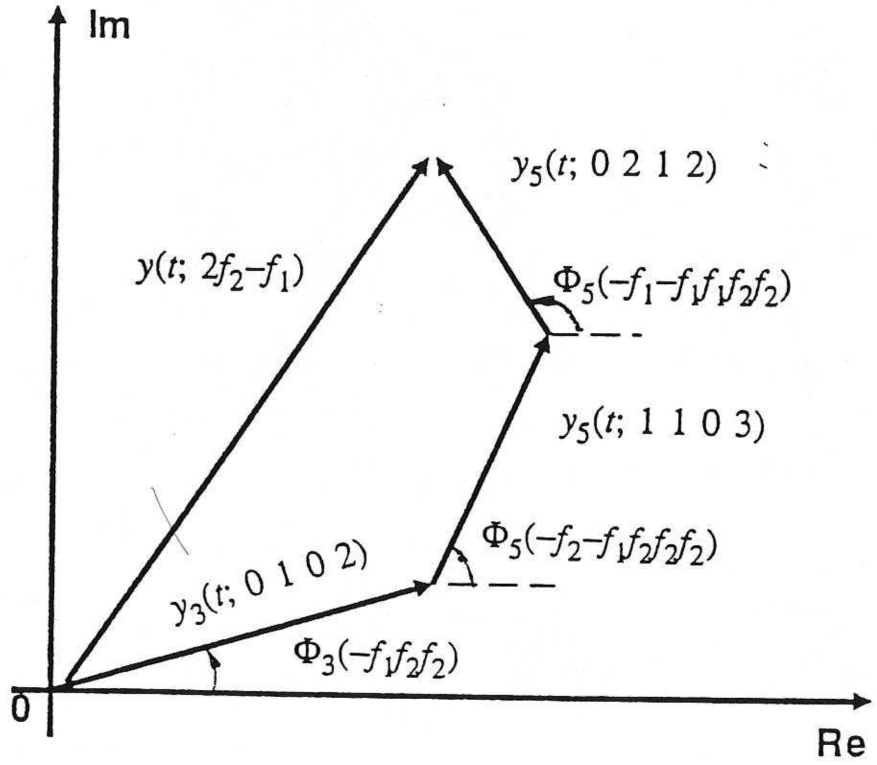

Fig.3 Phasor diagram for a particular frequency component

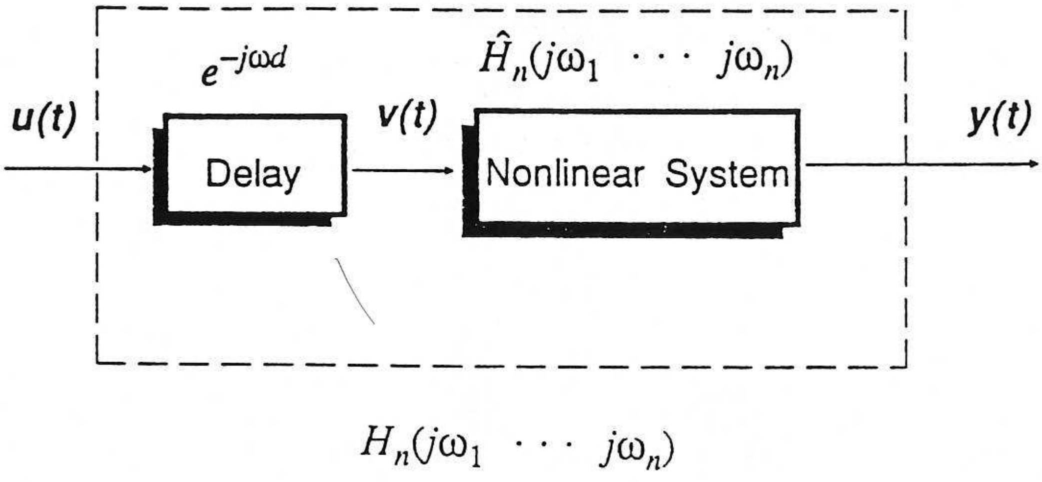

Fig.4 Schematic of nonlinear system with delay.

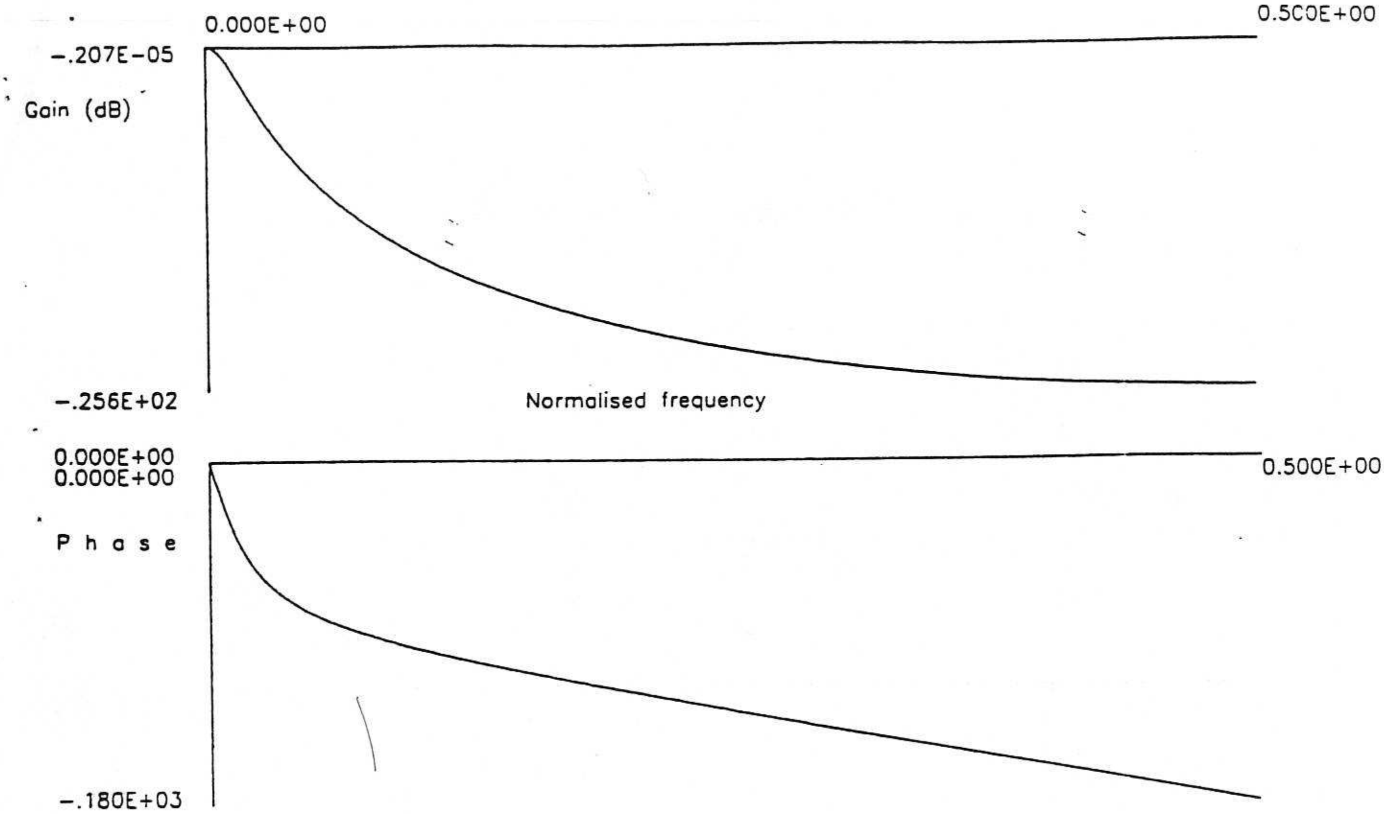

Fig.5 Gain and phase of $H_1(\cdot)$ for the simulated example.

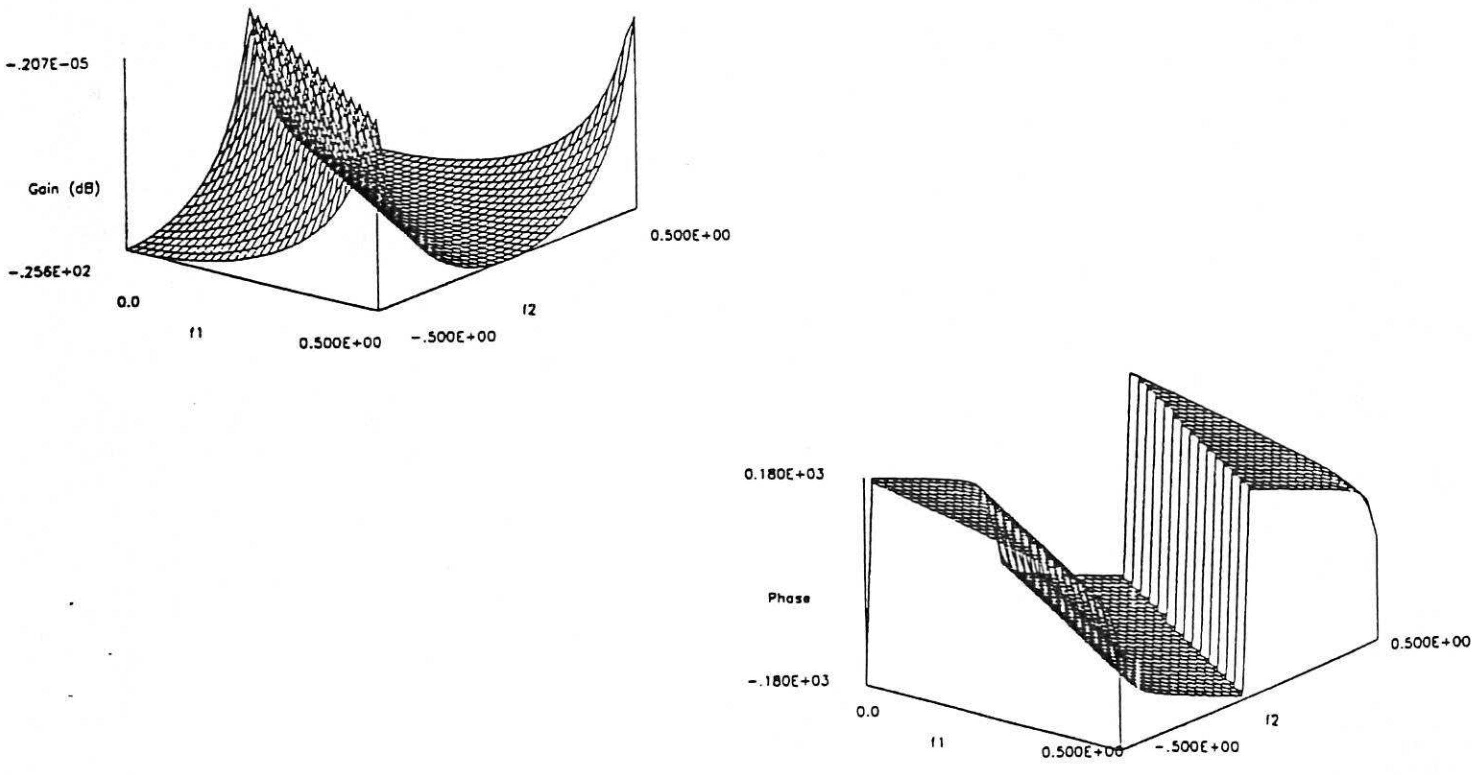

Fig.6 Gain and phase of $H_2(\cdot)$ for the simulated example.

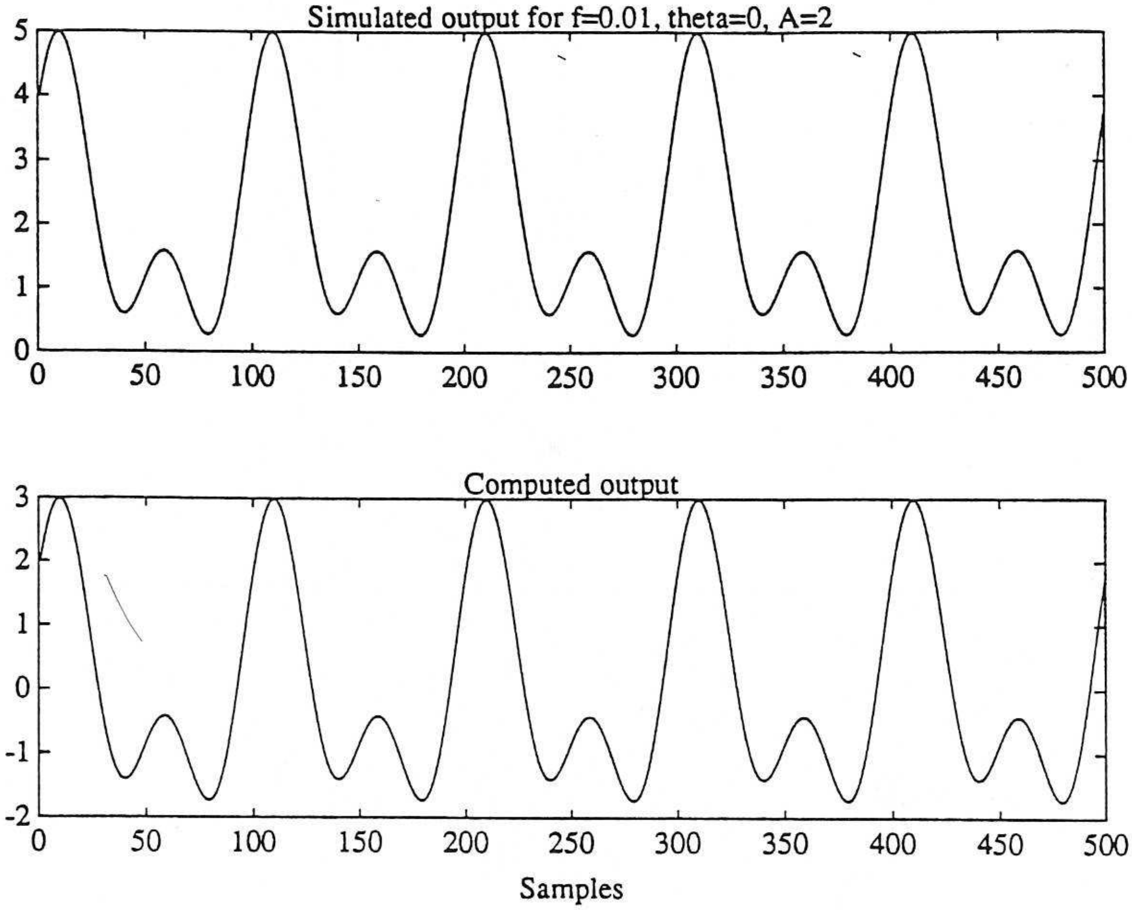

Fig.7 Simulated and computed output to the single frequency input with f=0.01, A=2 and $\theta = 0$.

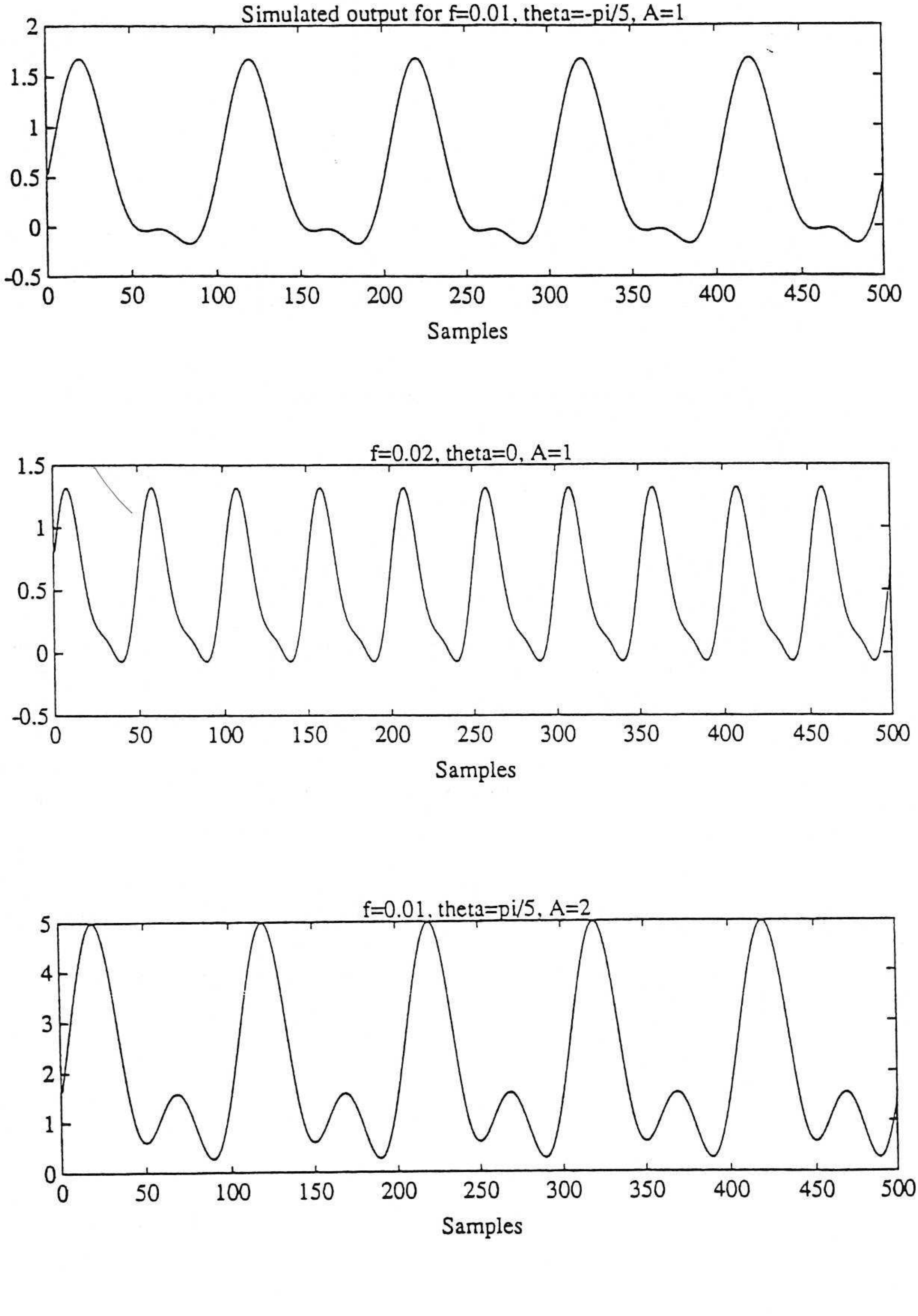

Fig.8 Responses to the single frequency inputs with various settings.

(a) f=0.01, A=1 $\theta=-\pi/5$

(b) f=0.01, A=2 $\theta=\pi/5$

(c) f=0.02, A=1 $\theta=0$

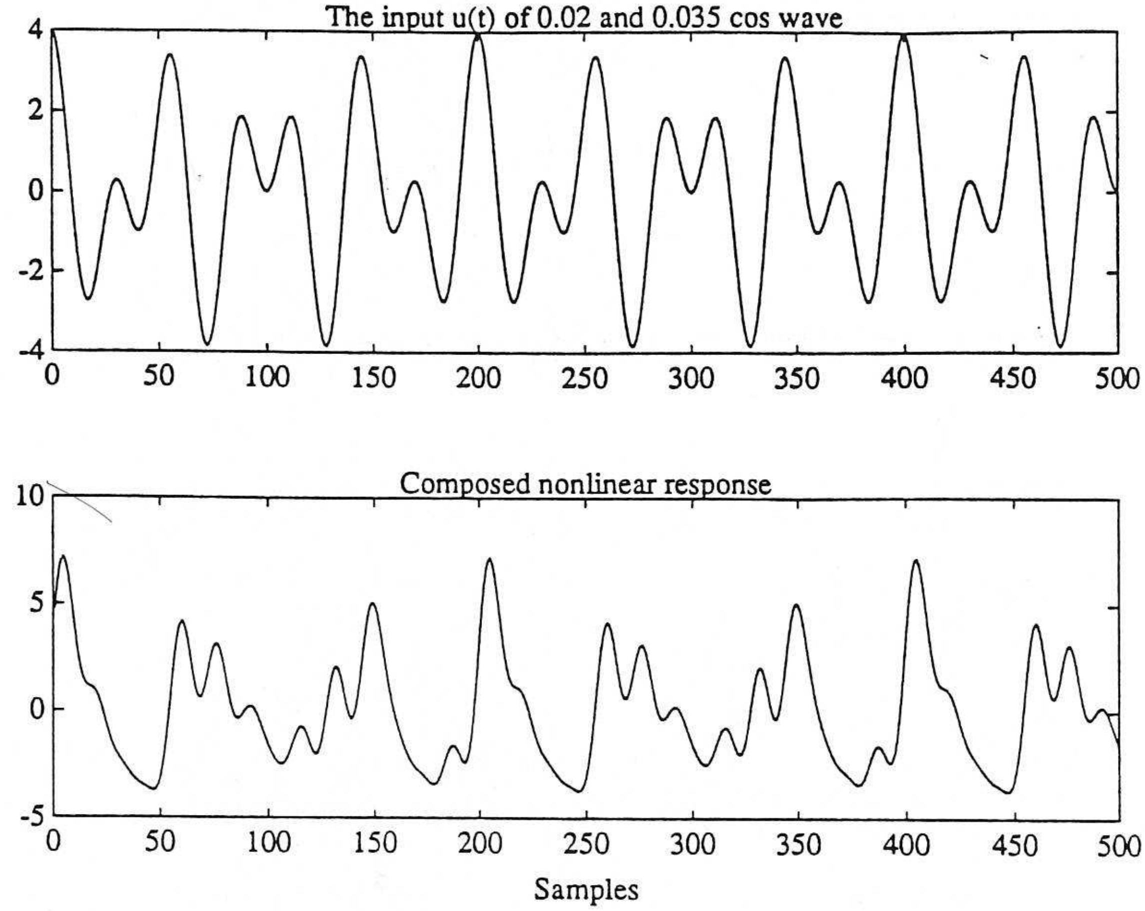

Fig.9 The two-tone input and the composed nonlinear output by Eqn.(42).

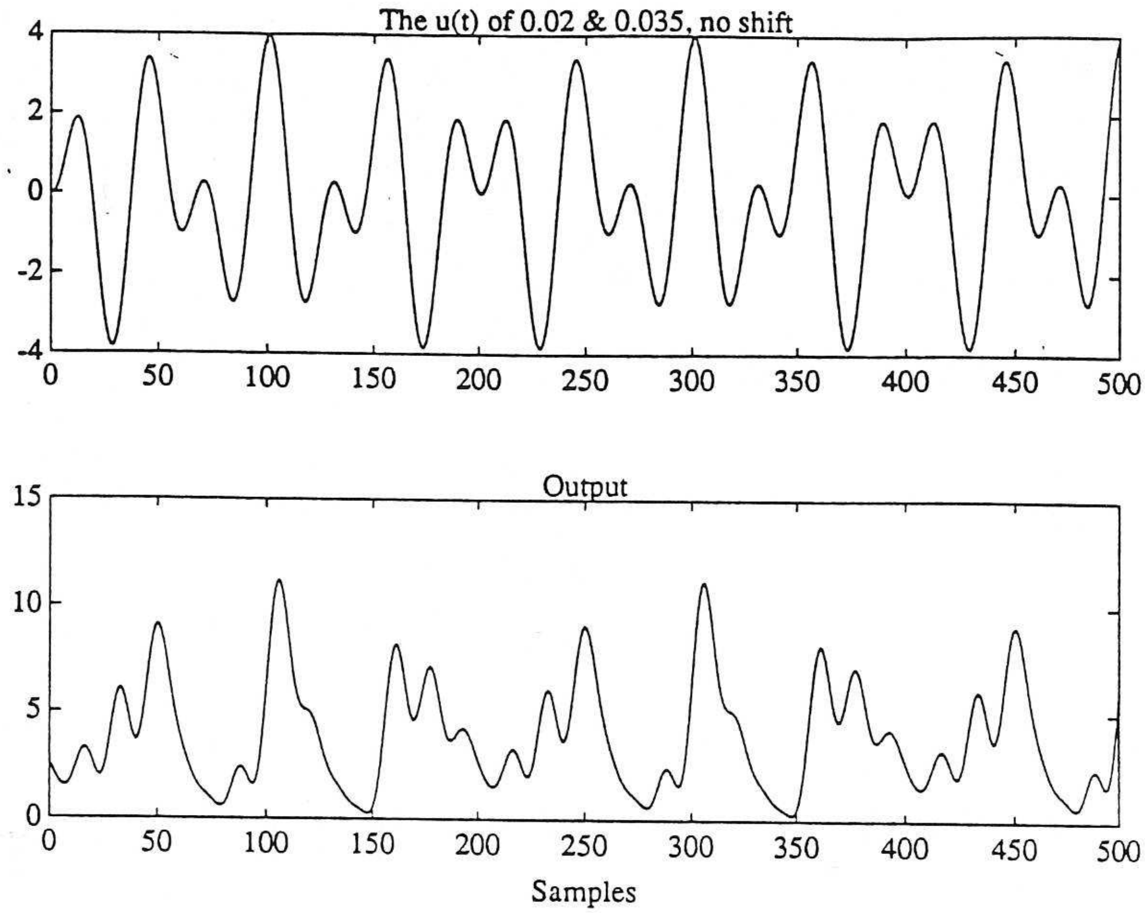

Fig.10 The actual nonlinear output to the two-tone input.

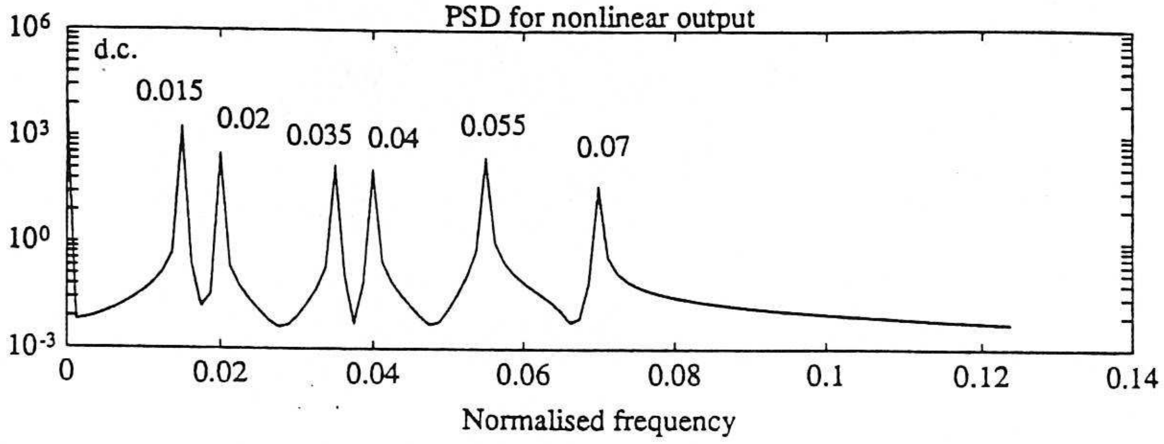

Fig.11 The power spectral density of the nonlinear output.

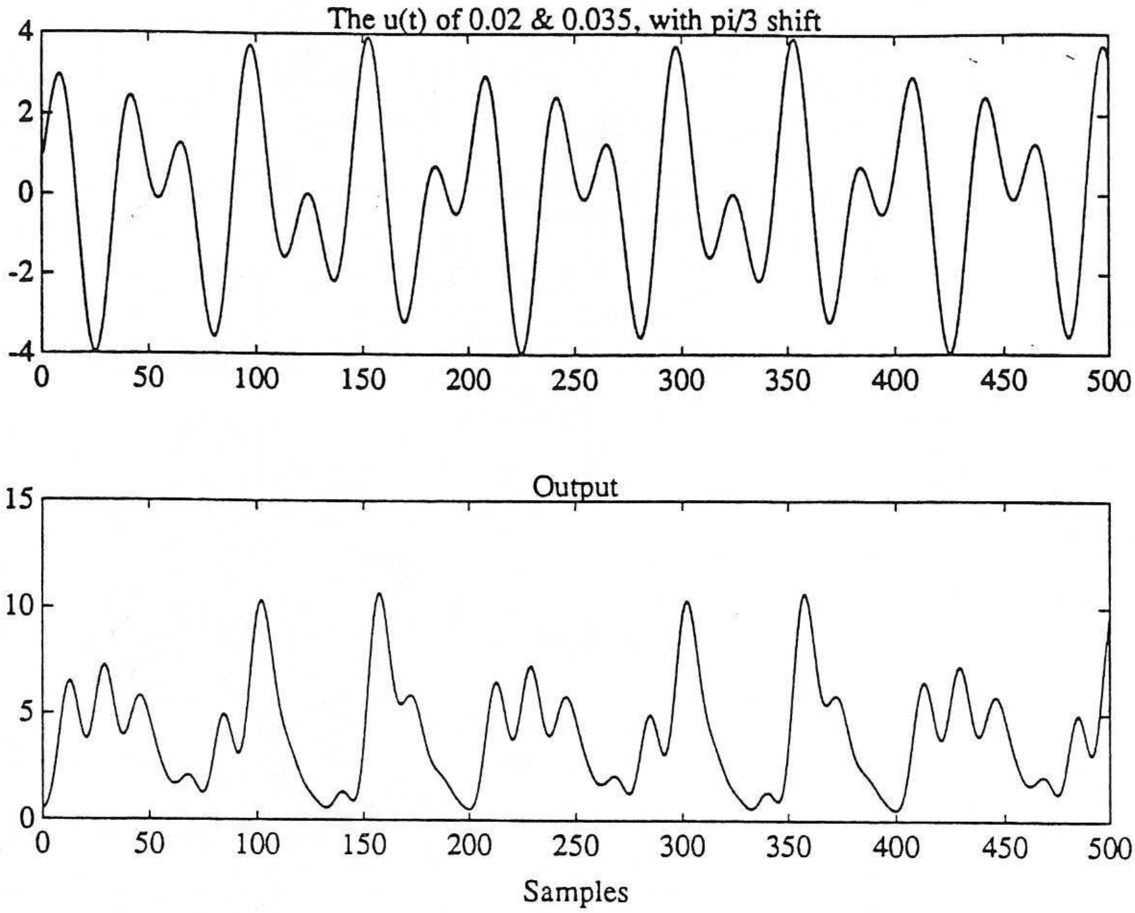

Fig.12 The nonlinear output to a two-tone input with phase shift.

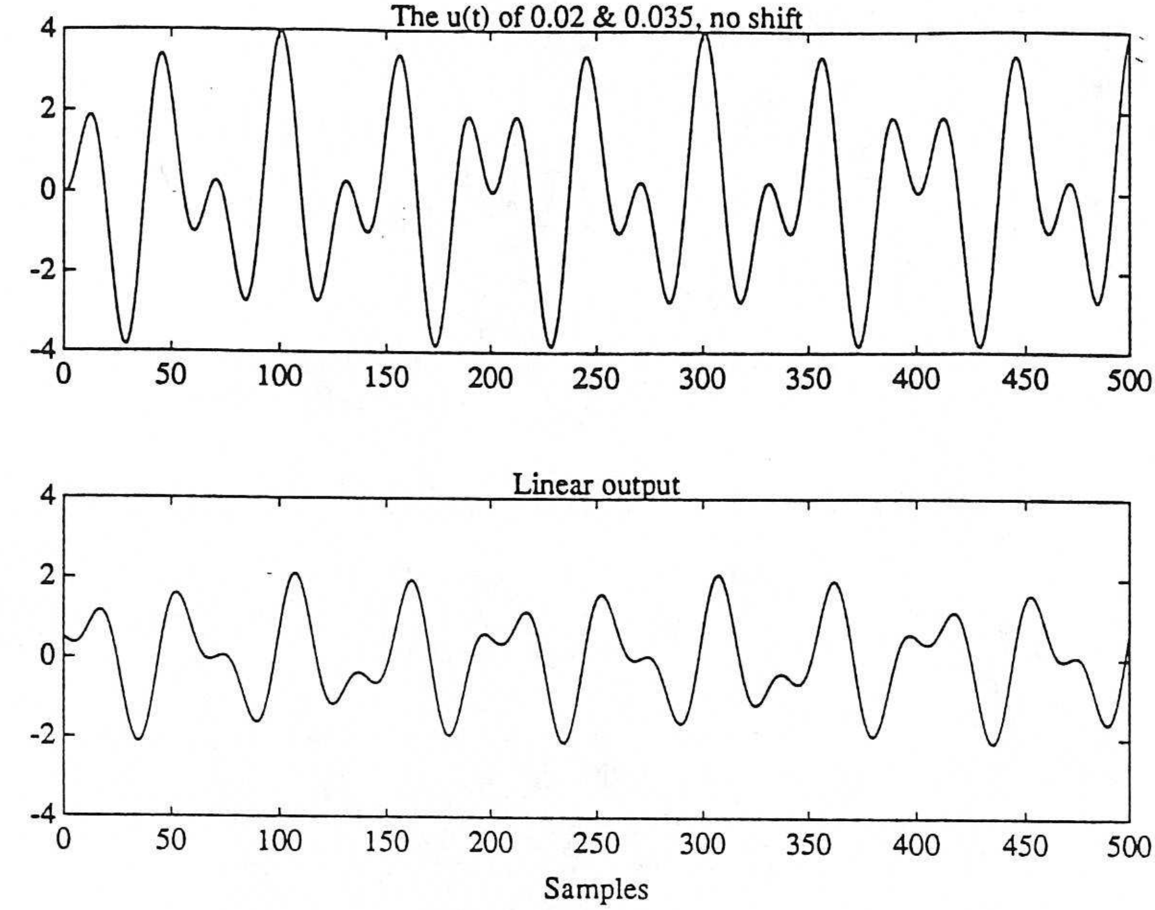

Fig.13 The linear output to the two-tone input.

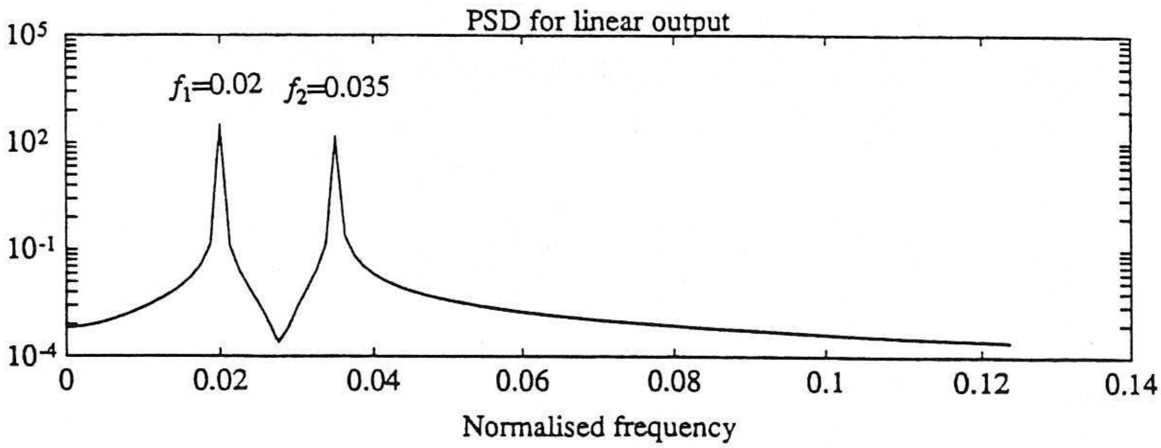

Fig.14 The power spectral density of the linear output.