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The Effects of Membrane Thickness and Asymmetry in the Topology Optimization of Stiffeners for Thin Shell Structures

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Study into the Effects of Membrane Thickness and Asymmetry in the Topology Optimization of Stiffeners for Thin Shell Structures

Thin walled shell structures are extensively used in architecture and engineering due to their lightweight and ease of shaping. But they suffer from poor overall stiffness, something addressed by the strategic addition of stiffeners. The optimization of stiffeners is divided between those which include shell membrane in the optimization and those which do not. However, no studies were found which indicate when it is necessary or valid to use either approach. In most cases it was also found that symmetry was forced in the optimization of stiffening layers. These two effects were investigated, concluding that: For membranes with thicknesses less than 20% of the structure, they do not affect the final topology; and for membrane thicknesses greater than 20%, they must be included in the optimization as they have a considerable effect. When bending and membrane loadings are present, symmetry should not be forced, otherwise sub-optimal designs are generated.

Keywords: membrane thickness, asymmetry, thin shell, stiffener layout, topology optimization

1. Introduction

A shell is a three-dimensional (3D) structure bounded primarily by two arbitrary curved surfaces a relatively small distance apart, Zingoni (1997). Such structures are used extensively in architecture and in aeronautical, civil, marine and mechanical engineering (Knopf-Lenoir et al. 1987, Suhuan and Zhijun 2005, Bechthold 2010). The broad definition of a thin shell structure is one where the ratio of its thickness to one of its other two dimensions is less than 10% (Oñate 1995). They are commonly analyzed with the Finite Element Method (FEM) using two-dimensional (2D) surface elements (i.e. shell Finite Elements (FE)) which consist of a combination of a plane stress (membrane) and a plate element (bending) (Bernadou 1996, Bandyopadhyay 1998). They have been optimized in one of three ways: 1) The thickness of the shell was optimized, whilst maintaining the original shape of the structure (Rao and Hinton 1993, Li et al. 1999, Lam et al. 2000); 2) The shape of the shell was optimized by moving the control points which defined it, but keeping its thickness unchanged (Bletzinger and Ramm 1993, Lindby and Santos 1999, Uysal et al. 2007); and 3) The topology of the shell was optimized using topology optimization where both its shape and thickness could be modified (Luo and Gea 1998, Li et al. 1999, Querin et al. 1999, Belblidia and Bulman 2002, Ansola et al. 2002). The application of topology optimization to shell structures has been the least researched of the three. This can be attributed to two consequences of topology optimization: 1) Cavities are introduced into the structural domain; and 2) The perimeter of the structure can be significantly modified. Since a primary use of shell structures is to cover, shield or enclose a space or volume, the two consequences mentioned would severely affect the applicability of a topologically optimized shell structure.

A reason for using shell structures is that they are lightweight and can be easily manipulated into the desired shape. But they suffer from poor overall stiffness, something which can be addressed by the strategic addition of stiffeners. A very effective means of determining the location of stiffeners in a structure is to use topology or layout design optimization. Gea and Fu (1997) introduced a new method to design the layout of shell stiffeners using a new microstructure-based Design Domain Method (DDM) to solve the optimal topology problems with frequency considerations. Belblidia et al. (1999, 2001) combined topology and size optimization to obtain optimum stiffener designs for shell structures using a three-layer finite shell element. The central layer represented the unstiffened shell and the symmetrically located upper and lower layers were potential stiffener zones. Lee et al. (2000) presented a methodology for shell topology optimization using an isotropic multi-layer material model which allowed the formation of holes or stiffener zones. De Souza and Ono Fonseca (2003) introduced a two-level strategy for the optimization of laminated shells. The two levels consisted of: 1) The optimization of the principal material orientation in each ply by minimizing the structural compliance; and 2) The use of topology optimization to minimize the volume of each ply. Ansola et al. (2004)

presented an optimization procedure to simultaneously find the shape and reinforcement layout on shell structures. Afonso et al. (2005) used an integrally stiffened shell formulation for the design of optimally stiffened shells to overcome the drawbacks of applying topology and structural shape optimization methods individually. They used both the single-layer and three-layer models, in the latter the middle layer represented the solid membrane material. Suhuan and Zhijun (2005) introduced a layout optimization of stiffeners for plate-shell structures. The stiffeners were placed at the locations of high strain energy/stress and material was removed from regions with small strain energy/stress. Stegmann and Lund (2005) investigated nonlinear effects of optimal stiffened topologies using the SIMP method (Bendsøe 1989). An anisotropic multi-layer shell model was employed to allow the formation of through-the-thickness holes or stiffening zones.

Three general observations can be made of these studies: 1) The contribution of the shell membrane was considered in the optimization (Lee et al. 2000; Belblidia et al. 2001; Stegmann and Lund 2005; Long et al. 2009); 2) The optimization was carried out without the shell membrane (Gea and Fu 1997; Swam and Kosaka 1997; Luo and Gea 1998; Lam et al. 2000; Suhuan and Zhijun 2005); and 3) Symmetry was forced when the stiffeners above and below the shell membrane were simultaneously optimized (Lee et al. 2000; Belblidia et al. 2001; Afonso et al. 2005; Stegmann and Lund 2005).

Solutions which include the shell membrane produce more realistic stiffener topologies whereas not including the shell membrane provides results more expediently. No studies were found which indicate when the membrane should be included in the optimization process. When a shell structure is loaded in pure bending and provided the material has the same mechanical behaviour in tension and compression, the generated stiffeners topology will be symmetrically about the neutral plane of the structure. But when both membrane and bending loads of equivalent magnitudes are present or the material behaves differently in tension and compression, there is no guarantee that the optimal stiffeners configuration will be symmetrical. This paper investigates when it is appropriate or not to include the shell membrane or force symmetry in the topology optimization of stiffeners. This paper presents: 1) The definition of a thin shell structure; 2) The commonly used layered models for stiffener optimization; 3) The optimization problem; and 4) Three commonly published stiffener optimization models to examine the above mentioned effects.

2. Definition of a thin-shell structure

There are currently two theories which can be used to model and analyse shell structures: 1) Kirchhoff-Love for the analysis of thin shell structures (Timoshenko and Woinowsky-Krieger 1940); and 2) Reissner-Mindlin for the analysis of moderately-thick shell structures (Reissner 1945, Mindlin 1951). Although the Reissner-Mindlin shell theory can be used to analyse thin shell structures, it is prone to shear locking (Bathe and Dvorkin 1985, 1986). In order to know which theory, and hence FE to use for the optimization of a thin shell structure, what is meant by a thin shell needs to be clearly defined. Unfortunately, there is no definitive rule which clearly specifies what a thin shell is. However, three definitions were found in the literature, these are: 1) A thin-flat shell is one where the ratio of its thickness to one of its other two dimensions is less than 10% (Oñate 1995), Equation (1); 2) A thin-curved shell is one where the ratio of its thickness to the minimum radius of curvature is: a) 5% (Novozhilov 1970), Equation (2) or; b) 3.33% (Vlasov 1944), Equation (3). In this work, a structure is considered a thin shell if it satisfies at least two of the criteria of Equations (1)-(3).

$$t/w \leq 1/10 \tag{1}$$
$$t/r_m \leq 1/20 \tag{2}$$
$$t/r_m \leq 1/30 \tag{3}$$

where: t is the shell thickness, w is the minimum in-plane dimension of the shell, and r_m is the minimum radius of curvature of the middle surface of the shell.

3. Layered models for shell optimization

There are five ways commonly use to represent a shell structure in the topology design of stiffeners (Lee et al. 2000, Stegmann and Lund 2005). They consist of the following five models, the: 1) Single-layer; 2) Double-layer; 3) Three-layer stiffening; 4) Three-layer voided; and 5) Multilayer-asymmetric.

The single-layer model represents classical topology optimization which allows the introduction of cavities within the shell (Figure 1a). The two-layer model allows the introduction of asymmetric stiffeners and consists of one solid and one virtual material layer (Figure 1b). The three-layer stiffening model has an inner solid layer with the two outer layers only containing virtual material. This model allows the introduction of symmetric ribs on both sides of the surface (Figure 1c). The three-layer voided or honeycomb model has the inner or middle layer made of virtual material and the two outer layers consisting of solid material (Figure 1d). This model permits the design of the inside or core of the material. The multilayer-asymmetric model consists of several solid and voided material layers in any order (Figure 1e). This model allows any combination of lay-up to be treated which means that the core material and symmetric or asymmetric stiffeners can be generated.

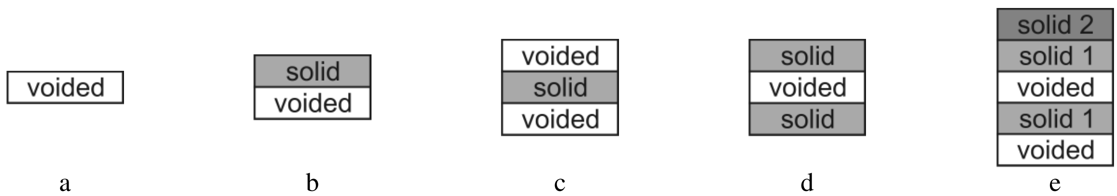


Figure 1. Typical layered model for shell optimization: **a** Single-layer; **b** Double-layer; **c** Three-layer stiffening; **d** Three-layer voided; **e** Multilayer-asymmetric.

Two methods exist of representing within the FEM the double-, three- and multi-layer models: 1) Using laminated shell FE where the properties of each layer is incorporated into the FE stiffness matrix; and 2) Using overlapping layers of thin-shell FE with each layer corresponding to a single FE, where: a) The nodes are shared by all of the overlapping FE; or b) The degrees of freedom (DOF) of the nodes of the overlapping FE are coupled using multi-point constraints. Depending on the FE software used, either of the two methods can be used so that the asymmetrical characteristic of the stiffener can be considered.

4. Optimization problem

The Isolines Topology Design (ITD) method (Victoria et al. 2009, 2010, 2011) was used in this work. The problem (Equations 4 - 7) consisted of minimizing the difference between the structure volume and a target volume subject to: 1) Structural equilibrium (Equation 5); 2) An equal Minimum Stress Threshold (MST) in each of the optimized layers (Equation 6); and 3) The inequality condition (Equation 7) where all points in and on the boundary of the structure must have a stress value greater than or equal to the MST. The stress used was the maximum through-the-thickness von Mises Stress (MVMS) from those calculated at the top and bottom of each layer using the FE nodal values.

Minimize

$$V - V_{\mathrm{Target}} \quad (4)$$

Subject to:

$$\mathbf{Ku} - \mathbf{P} = 0 \quad (5)$$

$$\sigma_{\mathrm{MST},j}^{\mathrm{MVMS}} = \sigma_{\mathrm{MST}}^{\mathrm{MVMS}} \quad (6)$$

$$\sigma_{\mathrm{MST,j}}^{\mathrm{MVMS}} - \sigma_{i,j}^{\mathrm{MVMS}} \leq 0 \quad (7)$$

where:

V	is the volume of the structure which can be optimized
V_{Target}	is the target volume of the optimized structure
$\mathbf{K}$	is the structure stiffness matrix
$\mathbf{u}$	is structure displacement vector
$\mathbf{P}$	is the nodal load vector
i	i^{th} point which lies inside or on the boundary of the structure in the j^{th} stiffener layer
j	j^{th} stiffener layer

$\sigma_{\mathrm{MST}}^{\mathrm{MVMS}}$ is the value of the minimum stress threshold

$\sigma_{\mathrm{MST},j}^{\mathrm{MVMS}}$ is the value of the minimum stress threshold in the j^{th} stiffener layer

$\sigma_{i,j}^{\mathrm{MVMS}}$ is the maximum through-the-thickness von Mises stress in the i^{th} point of the j^{th} stiffener layer

5. Numerical studies to determine the effects of membrane thickness and asymmetry in the topology optimization of stiffeners

Three centrally loaded shell structures were studied and are presented here: 1) The clamped square plate; 2) The corner hinged cylindrical shell; and 3) The corner clamped spherical shell (cap) with square platform. The first example studies the effect of membrane thickness on the emerging stiffener topology. The second and third examples investigate the effect of not forcing symmetry when optimizing stiffeners.

In all of the models, the elasticity modulus of the material was 210 GPa and its Poisson's ratio was 0.3. A quarter of the domain of each example was analyzed using a mesh of 50×50 thin-shell FE, although the full design domains are shown here. The figures which show the resulting stiffener designs have the top layer in light grey, the bottom layer in dark grey and the solid membrane (or middle layer) is transparent. The FE used was the ANSYS SHELL63 (Swanson Analysis System 2013), which is based on the Kirchhoff-Love theory. This FE includes membrane and bending stiffness and consists of four nodes, each with six DOF (three displacements and three rotations) and four Gauss integration points. In this study, the single-layer (Figure 2a), voided double-layer (Figure 2b), and three-layer stiffening models (Figure 2c) were used. However, unlike previous studies, the voided double-layer, and three-layer stiffening models were allowed to generate asymmetrical topologies in the upper and lower voided layers. In these two models, multi-point constraints were used to connect the nodal DOF of the overlapping FE and the solid membrane was treated as non-design domain and therefore did not take part in the optimization process.

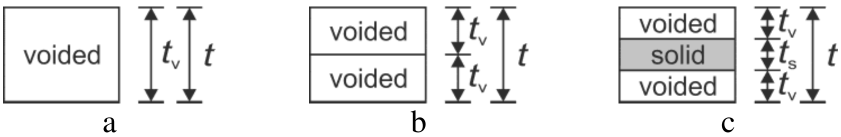


Figure 2. The layer models used in this work: **a** Single-layer; **b** Voided double-layer; **c** Three-layer stiffening. t_s is the solid non-design membrane thickness, t_v is the voided stiffening layers, and t is the total thickness.

5.1. Clamped square plate

The design domain, shown in Figure 3a, consists of a square of length L = 1000 mm and thickness t = 100 mm subjected to a vertically downwards load applied to the centre of the plate P = 100 kN, with all for edges clamped (displacements and rotations equal to zero).

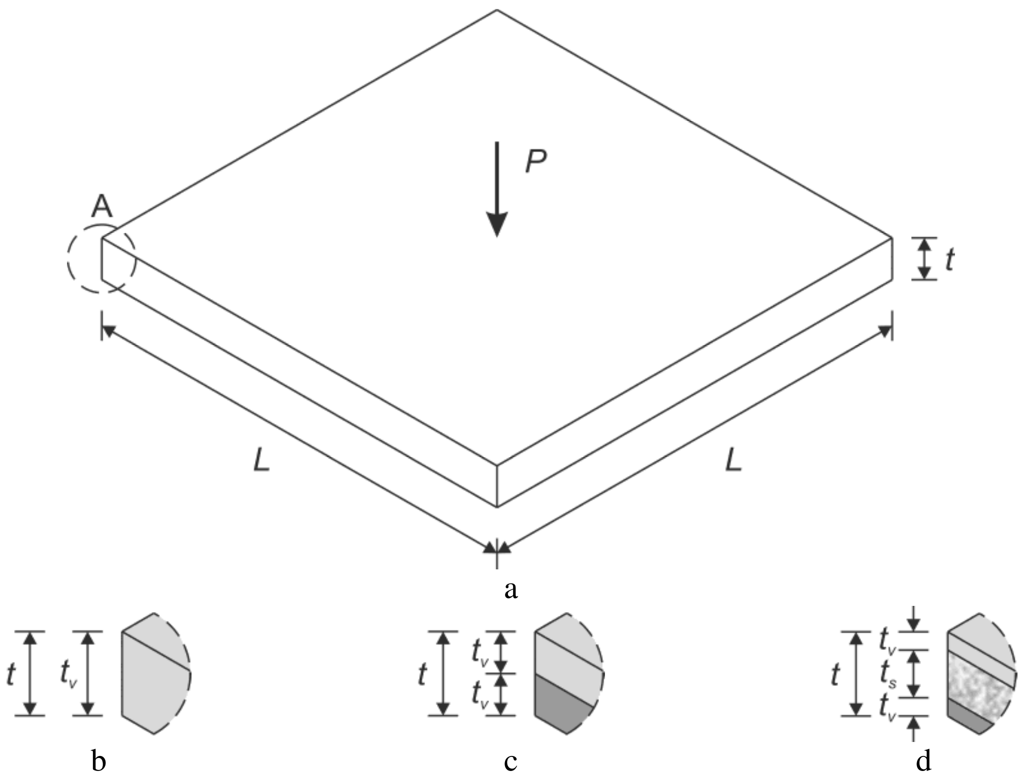


Figure 3. Clamped square plate: **a** Design domain. Detail-A shows the: **b** Single-layer model; **c** Voided double-layer model; **d** Three-layer stiffening model.

Figure 4a shows the resulting stiffener design using the single-layer model (Figure 3b) for a final volume fraction of 45%. This is in excellent agreement with the solutions obtained by Belblidia et al. (2001), Belblidia and Bulman (2002) and Liang (2005). The volume fraction and MVMS history vs. iteration number is given in Figure 4b. An observation from these results is that the stiffener material is placed in regions of high stress levels (Suhuan and Zhijun 2005).

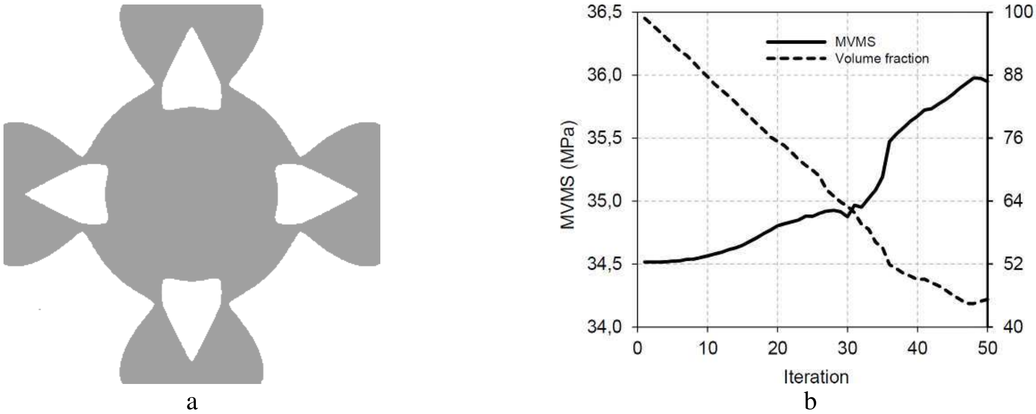


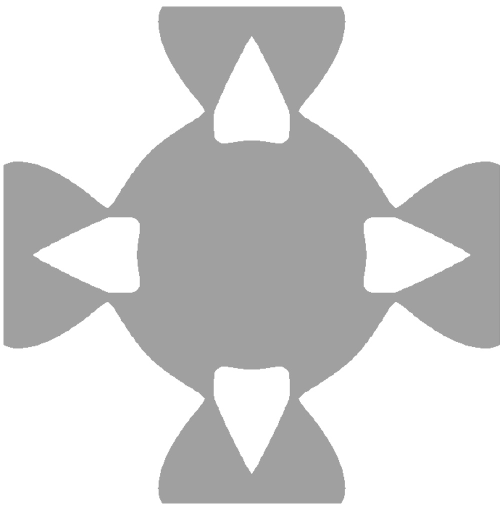
Figure 4. Clamped square plate: **a** Final design (top view) for the single-layer model; **b** Topology optimization history of the design criterion and volume fraction.

The effect of varying the membrane thickness on the optimization of the stiffeners was studied using six models. In these, the membrane thickness was varied from 0 to 50 mm, whilst maintaining the overall thickness of the structure constant at 100 mm, Table 1. The membrane thickness is defined as the solid non-design domain (t_s), and the stiffener is defined as the voided stiffening layers (t_v).

Table 1. Thickness of the solid membrane and voided stiffeners for the six cases analyzed

Case number	t_v (mm)	t_s (mm)	t_s (%)
1	50	0	0
2	45	10	10
3	40	20	20
4	35	30	30
5	30	40	40
6	25	50	50

The resulting stiffener designs using the voided double-layer (Figure 3c) and the three-layer stiffening models (Figure 3d) for different solid and voided thicknesses (Table 1) are shown in Figure 5. The target final volume fraction is 45% of the volume within the voided layers. Since this example is only subjected to pure bending, it has a stress distribution through the thickness which is symmetrical and therefore the top and bottom stiffener layers are identical. The volume fraction and MVMS history vs. iteration number is given in Figure 6 for the different percentages of membrane thicknesses (t_s (%)).



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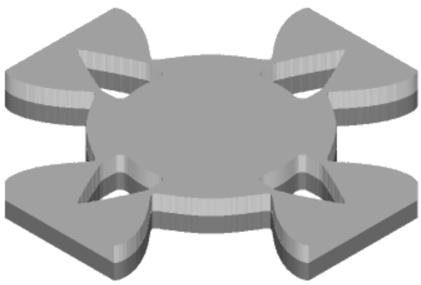

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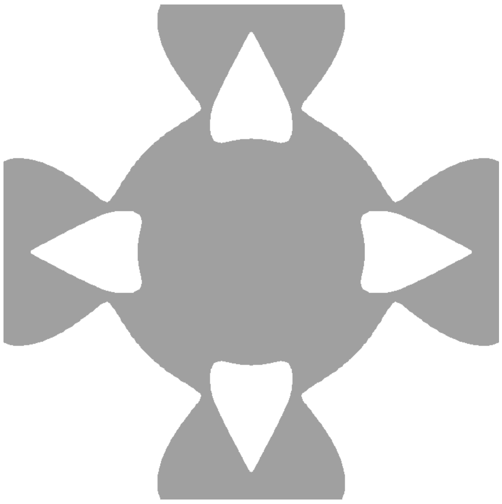

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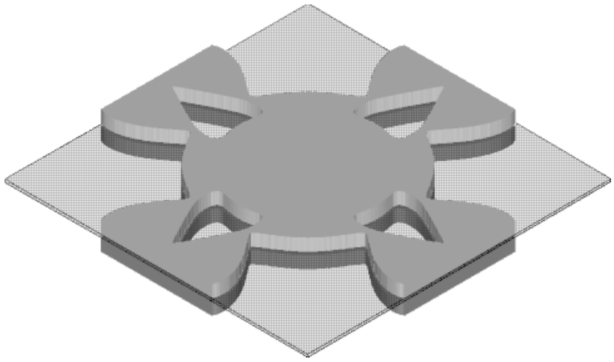

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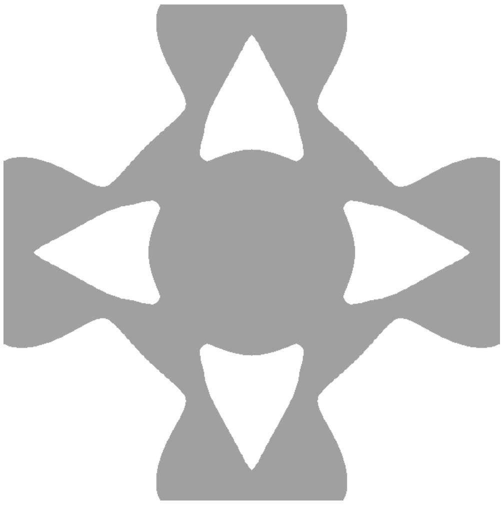

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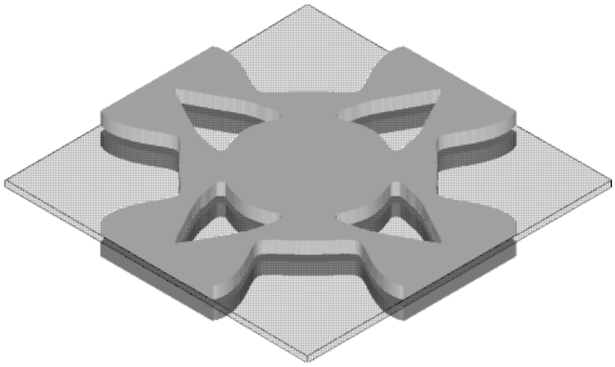

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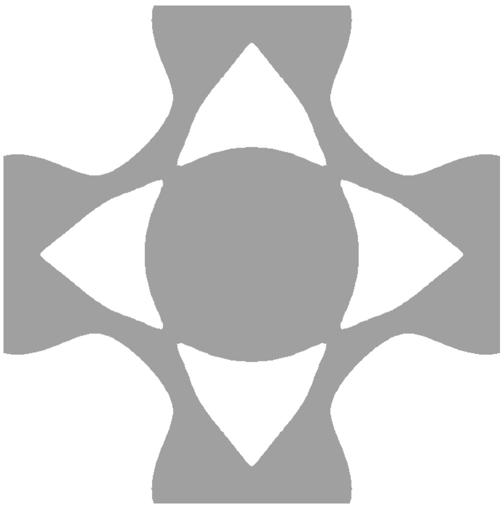

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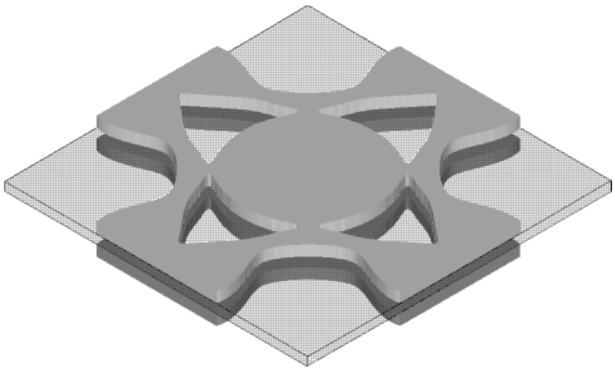

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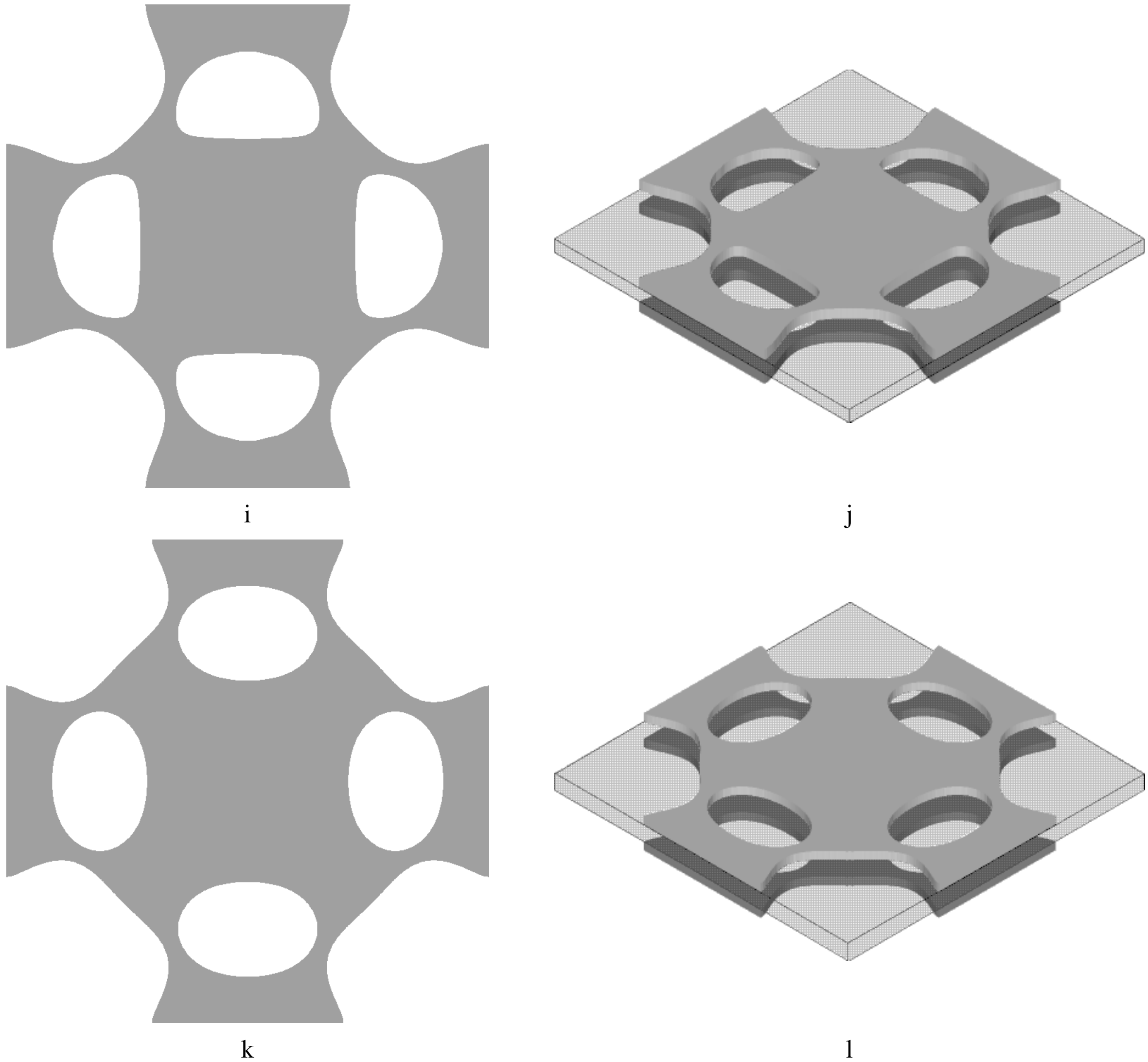


Figure 5. Final stiffener designs for different percentages of t_s (%). Case number: 1) **a**, **b** 0%; 2) **c**, **d** 10%; 3) **e**, **f** 20%; 4) **g**, **h** 30%; 5) **i**, **j** 40%; 6) **k**, **l** 50%.

Two important observations can be made from these results: 1) For membrane thicknesses of less than 20%, the single layer model produces the same stiffener designs as the voided double-layer model; and 2) For membrane thicknesses greater than 20%, the three-layer stiffening model produces significantly different topological stiffener designs. It can therefore be concluded that the membrane should be included in the topology design of such stiffeners.

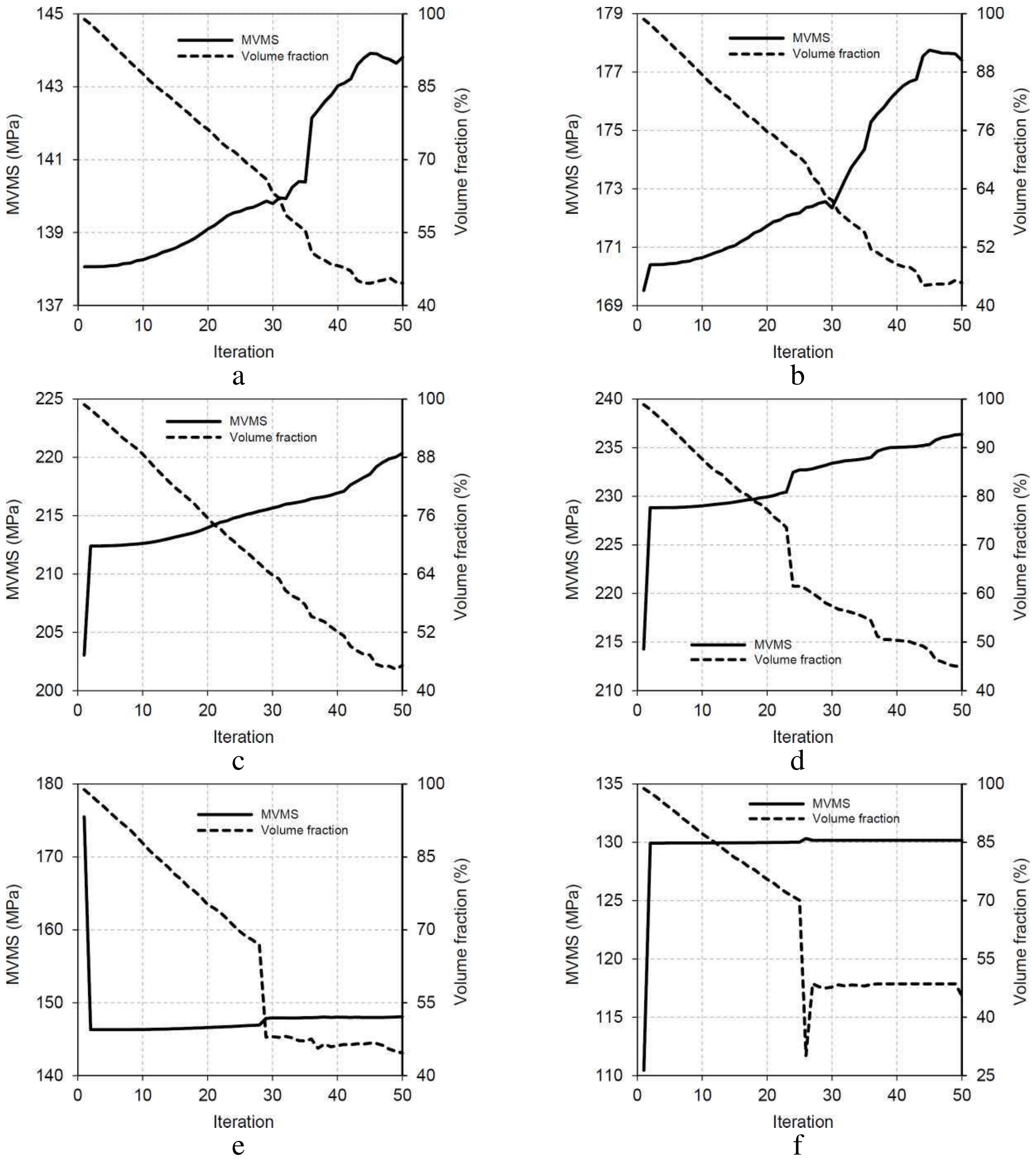


Figure 6. Topology optimization history of the design criterion and volume fraction for different percentages of t_s (%): **a** 0%; **b** 10%**;** **c** 20%; **d** 30%; **e** 40%; **f** 50%.

5.2. Corner hinged cylindrical shell

The design domain is the cylindrical shell shown in Figure 7. All edge lengths are L = 1000 mm with the curved edge spanning an angle θ = 1/0.86 rad of a circle with a radius R = 860 mm and thickness t = 40 mm. The shell rises above the base by the distance h = 141.3 mm and has a width w = 944.6 mm. The load P = 15 kN is applied vertically downwards at the centre of the shell, with the four corners simply supported (displacements equal to zero).

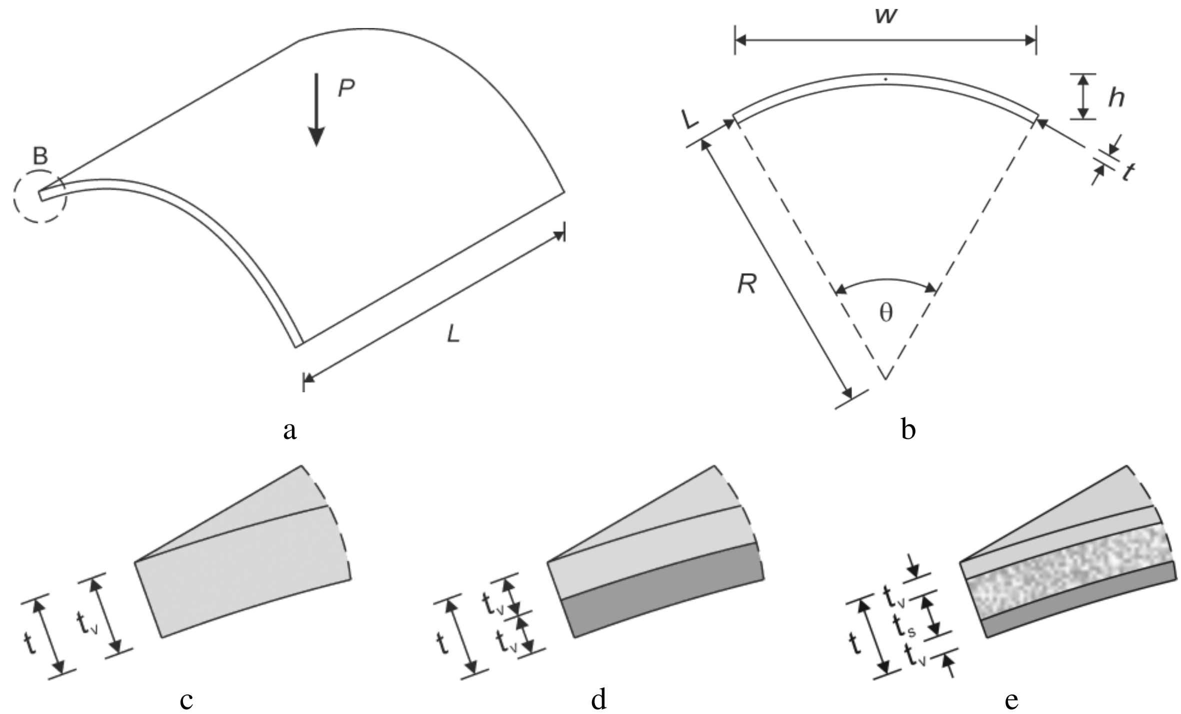


Figure 7. Corner hinged cylindrical shell. Design domain: **a** Isometric view; **b** Front view. Detail-B shows the: **c** Single-layer model; **d** Voided double-layer model; **e** Three-layer stiffening model.

Figure 8 shows the resulting stiffener design using the single-layer model (Figure 7c) for a final volume fraction of 20%. This agrees with the solutions obtained by Belblidia et al. (2001), Stegmann and Lund (2005), and Long et al. (2009). Note that all stiffener regions are connected to each other by a cross-shaped topology. The volume fraction and design criterion history vs. iteration number is given in Figure 8c.

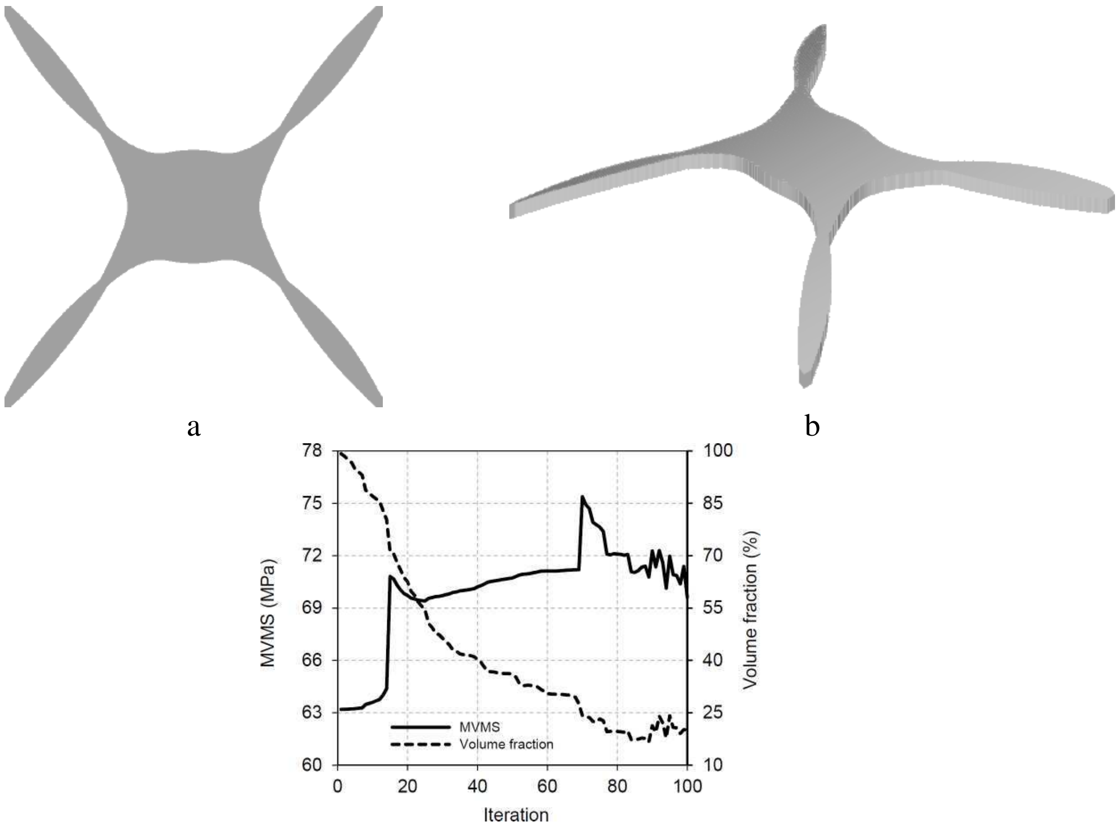


Figure 8. Corner hinged cylindrical shell. Final design using a single-layer model: **a** Top view; **b** Isometric view; **c** Topology optimization history of the design criterion and volume fraction.

To study the effect asymmetry, this example was optimized again using both the voided double- (Figure 7d) and the three-layer stiffening model (Figure 7e), the latter with a 20% membrane thickness (t_s = 8mm). The resulting stiffener designs are shown in Figure 9 and 10, respectively. The volume fraction and MVMS history vs. iteration number for the voided double-layer model and the three-layer stiffening models are given in Figure 11.

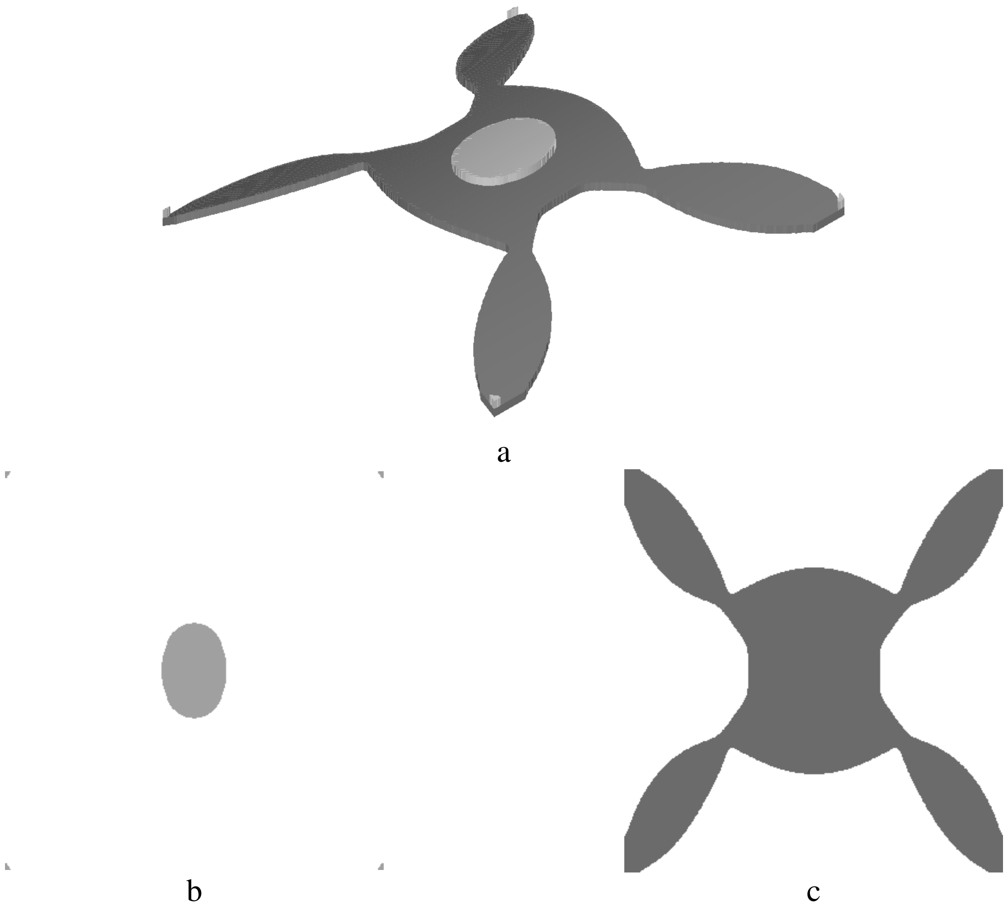
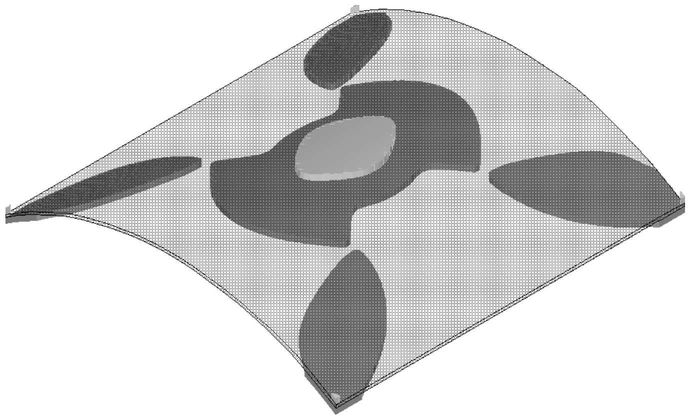


Figure 9. Final stiffener designs for the voided double-layer model showing: **a** An isometric view of all layers; **b** The top stiffener; **c** The bottom stiffener.



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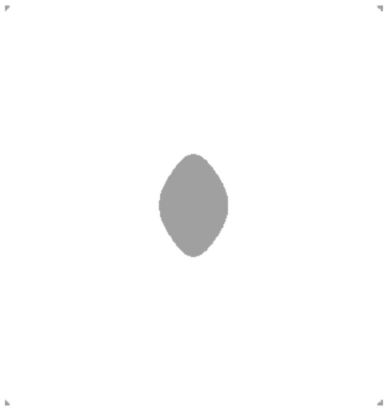

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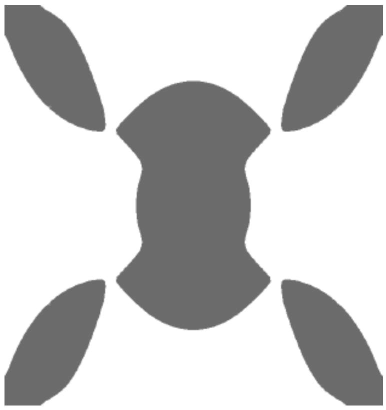
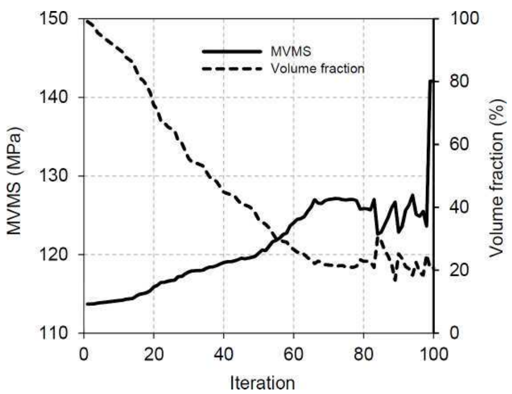
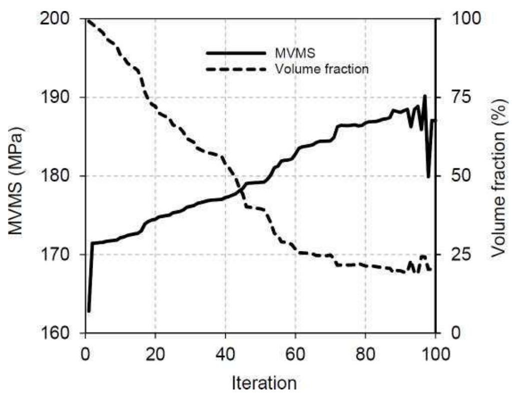


c

Figure 10. Final stiffener designs for the three-layer stiffening model with an 8 mm membrane showing: **a** An isometric view of all layers; **b** The top stiffener; **c** The bottom stiffener.



a



b

Figure 11. Topology optimization history of the design criterion and volume fraction using: **a** The voided double-layer model; **b** The three-layer stiffening model.

Two observations can be made from these results: 1) When a membrane layer is present, the cylindrical shell does not require a bottom stiffener along its edge extending between the four corners; and 2) When bending and membrane effects are present in a shell structure, there is a significant asymmetrical effect in the topological design of the stiffeners. Therefore no matter which model is used (double-layer or three-layer stiffening), both voided layers must be optimized individually without artificially forcing symmetry, otherwise a sub-optimal design is generated.

5.3. Corner clamped spherical shell (cap) with square platform

The equation which represents the height $z(x,y)$ of the spherical shell as a function of the x and y directions (Stegmann and Lund 2005) is given by Equation (8) and the design domain is shown in Figure 12a. The edge-length is L = 1000 mm, the height of the central point is h = 100 mm and the thickness is t = 60 mm. A vertical downward load P = 15 kN was applied at the centre of the shell, with the four corner nodes clamped.

$$z(x,y) = h - \frac{2h}{L^2}\left(x^2 + y^2\right) \quad (8)$$

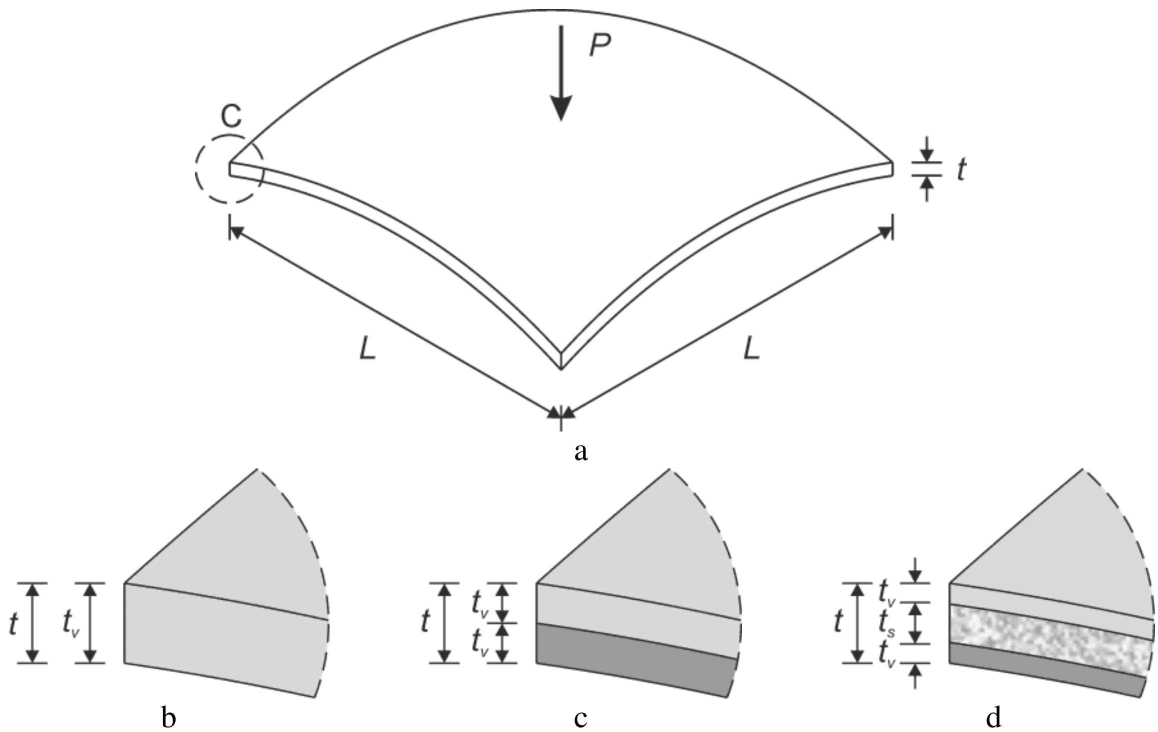


Figure 12. Corner clamped spherical shell (cap) with square platform: **a** Design domain. Detail-C shows the: **b** Single-layer model; **c** Voided double-layer model; **d** Three-layer stiffening model.

Figure 13 shows the resulting stiffener design using the single-layer model (Figure 12b) for a final volume fraction of 20%. Note that again all of the stiffening regions are connected to each other by a cross-shaped topology. The volume fraction and design criterion history vs. iteration number is given in Figure 13c.

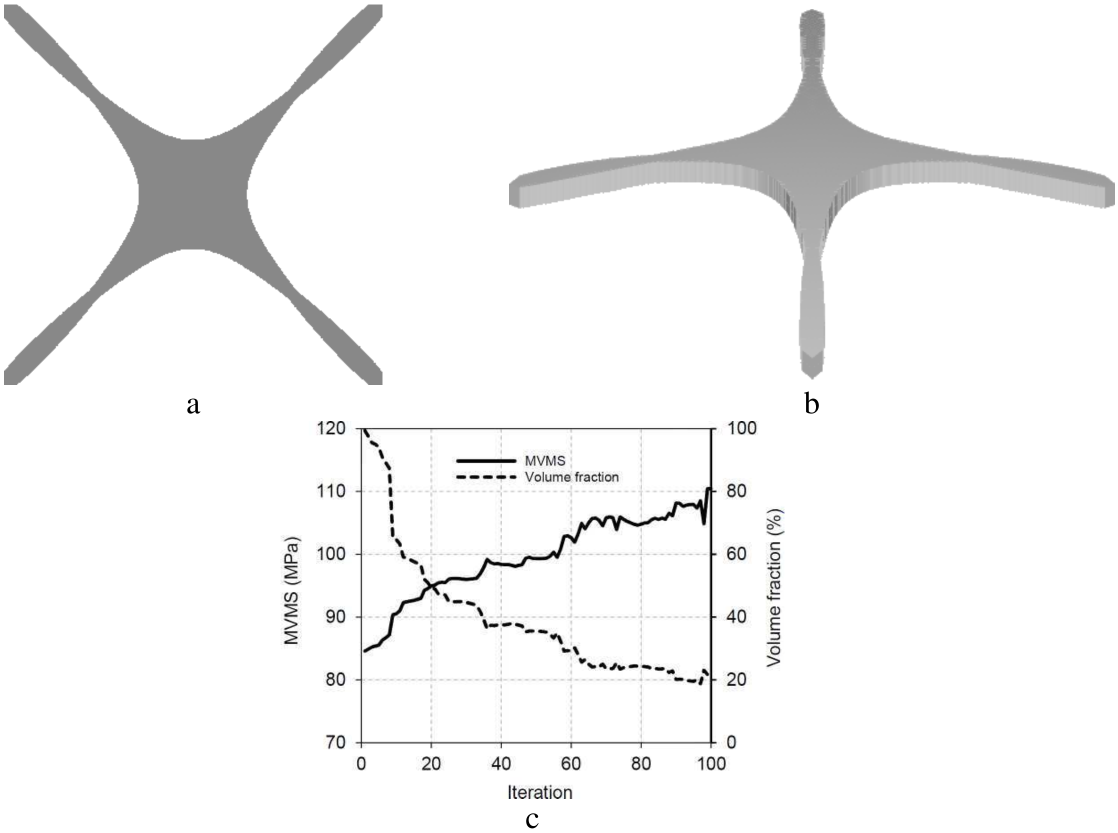
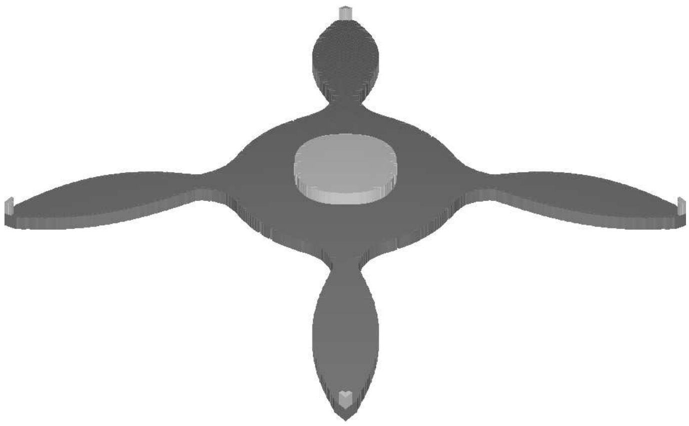


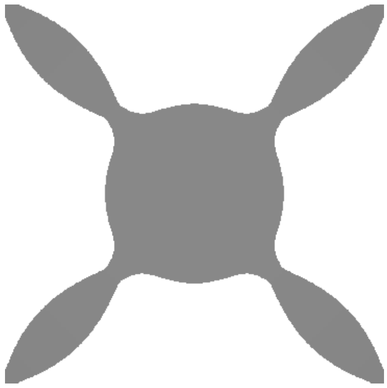
Figure 13. Corner clamped spherical shell (cap) with square platform. Final design using a single-layer model: **a** Top view; **b** Isometric view; **c** Topology optimization history of the design criterion and volume fraction.

To study the effect of asymmetry, this example was also optimized using both the voided double- (Figure 12c) and the three-layer stiffening model (Figure 12d), the latter with a 20% membrane thickness (t_s = 12mm). The resulting stiffener designs are shown in Figure 14 and 15, respectively. The volume fraction and MVMS history vs. iteration number for the voided double-layer model and the three-layer stiffening model are given in Figure 16.



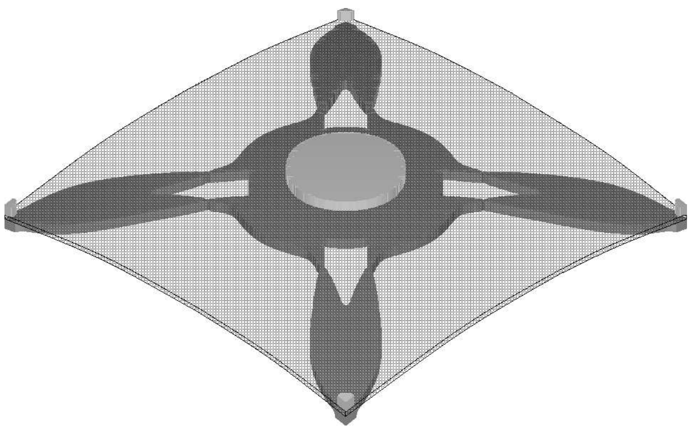
a

b

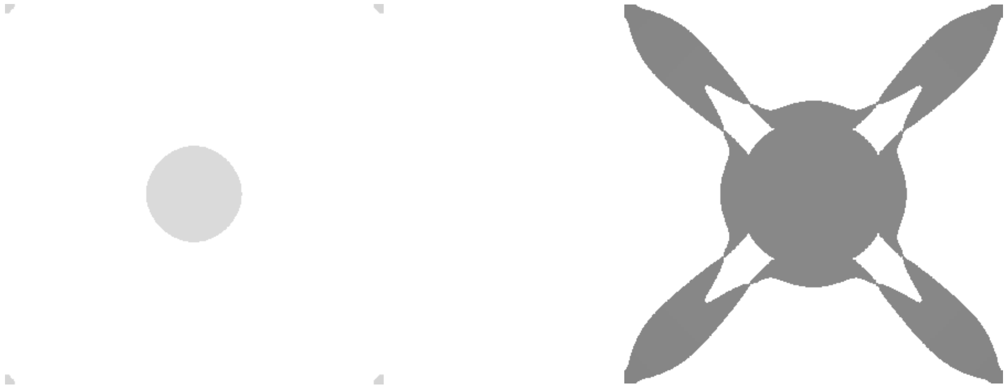


c

Figure 14. Final stiffener designs for the voided double-layer model showing: **a** An isometric view of all layers; **b** The top stiffener; **c** The bottom stiffener.



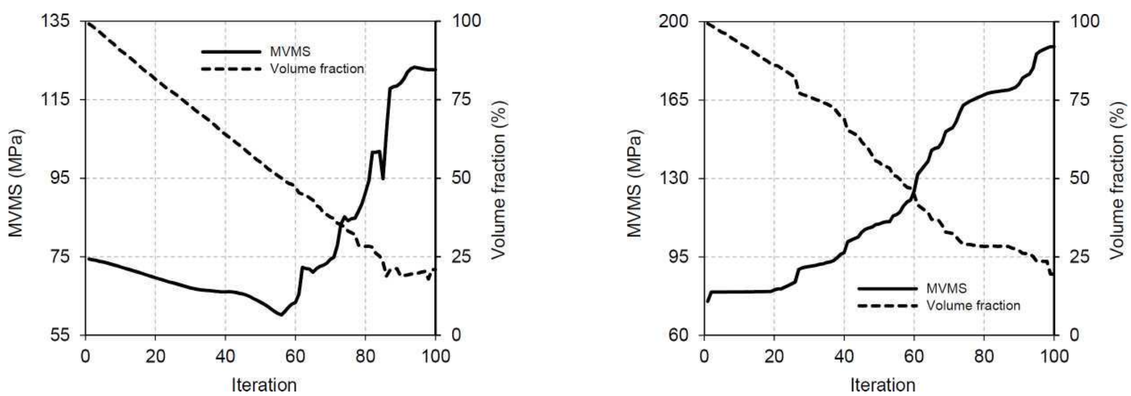
a



b c

Figure 15. Final stiffener designs for the three-layer stiffening model with a 12 mm membrane showing: **a** An isometric view of all layers; **b** The top stiffener; **c** The bottom stiffener.

As was the case for example 5.2, these results show that there is a significant asymmetrical effect in the topological design of the stiffeners when both a membrane and a bending loading are present. They also show that symmetry should not be artificially forced on this type of topology optimization problems, otherwise sub-optimal designs are produced.



a b

Figure 16. Topology optimization history of the design criterion and volume fraction using: **a** the voided double-layer model, **b** the three-layer stiffening model.

6. Conclusions

This paper presents a study to determine the effect of membrane thickness and asymmetry in the topology optimization of stiffeners for thin shell structures. To obtain the stiffener layout, the shell structure was modelled using overlapping layers of thin-shell FE whose nodes were coupled using multi-point constraints. The ITD method was used to carry out the optimization using the single-, voided double- and three-layer stiffening models, but each voided layer could be optimized independently in order to generate asymmetrical designs. Three examples of 2D and 3D shell structures were used and the following three conclusions were reached from the results: 1) For membrane with thicknesses less than 20% of the thickness of the structure, the single-layer model produced the same stiffener designs as the voided double-layer model, hence there is no need to include the membrane in the optimization model; 2) For membrane thicknesses greater than 20%, the three-layer stiffening model produced significantly different topological designs of the stiffener. So the membrane must be included in the topology design of such stiffeners; and 3) When a shell structure experiences both bending and membrane loads there is a significant asymmetrical effect in the topological design of the stiffeners, so no matter which model is used (double- or three- layer stiffening), both voided layers need to be optimized individually without artificially forcing symmetry, otherwise a sub-optimal design will be generated.

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References

Afonso, S.M.B., Sienz, J. and Belblidia, F., 2005. Structural optimization strategies for simple and integrally stiffened plates and shells. *Engineering Computations*, 22 (4), 429−452.

Ansola, R., Canales, J., Tarrago, J.A. and Rasmussen, J., 2002. An integrated approach for shape and topology optimization of shell structures. *Computers and Structures*, 80, 449−458.

Ansola, R., Canales, J., Tarrago, J.A. and Rasmussen, J., 2004. Combined shape and reinforcement layout optimization of shell structures. *Structural and Multidisciplinary Optimization*, 27, 219−227.

Bandyopadhyay, J.N., 1998. *Thin shell structures classical and modern analysis*. India: New Age International (P) Ltd.

Bathe, K.J. and Dvorkin, E.N., 1985. A four node plate bending element based on Mindlin−Reissner plate theory and a mixed interpolation. *International Journal for Numerical Methods in Engineering*, 21, 367−383.

Bathe, K.J. and Dvorkin, E.N., 1986. A formulation of general shell elements - the use of mixed interpolation of tensorial components. *International Journal for Numerical Methods in Engineering*, 22, 697−722.

Bechthold, M., 2010. On shells and blobs: structural surfaces in the digital age. *In*: Princeton Architectural Press. *Fabrication architecture: selected readings in digital design and manufacturing*. New York: Robert Corser.

Belblidia, F., Afonso, S.M.B., Hinton, E. and Antonio, G.C.R., 1999. Integrated design optimization of stiffened plate structures. *Engineering Computations*, 16(8), 934−951.

Belblidia, F. and Bulman, S., 2002. A hybrid topology optimization algorithm for static and vibrating shell structures. *International Journal for Numerical Methods in Engineering*, 54, 835−852.

Belblidia, F., Lee, J.E.B., Rechak, S. and Hinton, E., 2001. Topology optimization of plate structure using a single- or three-layered artificial material model. *Advances in Engineering Software*, 32, 159−168.

Bendsøe, M.P., 1989. Optimal shape design as a material distribution problem. *Structural Optimization*, 1, 193−202.

Bernadou, M., 1996. *Finite element methods for thin shell problems*. Chichester: Wiley-Blackwell.

Bletzinger, K.U. and Ramm, E., 1993. Form finding of shells by structural optimization. *Engineering with Computers*, 9(1), 27−35.

De Souza, C.E. and Ono Fonseca, J.S., 2003. A two-level strategy for topology and orientation optimization of laminated shell structures. *Mecánica Computacional*, 22, 1852−1865.

Gea, H.C. and Fu, Y., 1997. Optimal 3D stiffener design with frequency considerations. *Advances in Engineering Software*, 28, 525−531.

Knopf-Lenoir, C., Taverniere, P. and Cochet, J.F., 1987. Optimization of a three dimensional shell structure. *Engineering Optimization*, 11, 265−279.

Lam, Y.C., Manickarajah, D. and Bertolini, A., 2000. Performance characteristics of resizing algorithms for thickness optimization of plate structures. *Finite Elements in Analysis and Design*, 34, 159−174.

Lee, S.J., Bae, J.E. and Hinton, E., 2000. Shell topology optimization using the layered artificial material model. *International Journal for Numerical Methods in Engineering*, 47, 843−867.

Li, Q., Steven, G.P. and Querin, O.M., 1999. Optimization of thin shell structures subjected to thermal loading. *Structural Engineering and Mechanics*, 7(4), 401−412.

Liang, Q.Q., 2005. *Performance-based Optimization of Structures. Theory and applications*, Spon Press. Taylor & Francis Group, London and New York.

Lindby, T. and Santos, J.L.T., 1999. Shape optimization of three-dimensional shell structures with the shape parametrization of a CAD system. *Structural and Multidisciplinary Optimization*, 18(2−3), 126−133.

Long, C.S., Loveday, P.W. and Groenwold, A.A., 2009. Effect of finite element formulation on optimal plate and shell structural topologies. *Finite Elements in Analysis and Design*, 45, 817−825.

Luo, J.H. and Gea, H.C., 1998. A systematic topology optimization approach for optimal stiffener design. *Structural Optimization*, 16, 280−288.

Mindlin, R.D., 1951. Influence of rotatory inertia and shear in flexural motions of isotropic elastic plates. *Journal of Applied Mechanics*, 18, 31−38.

Novozhilov, V.V., 1970. *Thin shell theory*. Groningen: Wolters-Noordhoff.

Oñate, E., 1995. *Cálculo de estructuras por el método de los elementos finitos. Análisis estático lineal*. Barcelona (Spain): CIMNE.

Querin, O.M., Steven, G.P. and Xie, Y.M., 1999. Morphing Evolutionary Structural Optimisation (MESO). In: *Proceedings of the 1st ASMO UK/ISSMO Conference on Engineering Design Optimization*, 311−316.

Rao, N.V.R. and Hinton, E., 1993. Structural optimization of variable-thickness plates and free-form shell structures. *Structural Engineering Review*, 5, 1−21.

Reissner, E., 1945. The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics*, 12, 69−76.

Stegmann, J. and Lund, E., 2005. Nonlinear topology optimization of layered shell structures. *Structural and Multidisciplinary Optimization*, 29, 349−360.

Suhuan, C. and Zhijun, Y., 2005. The layout optimization of stiffeners for plate-shell structures. *Acta Mechanica Solida Sinica*, 18(4), 365−373.

Swan, C.C. and Kosaka, I., 1997. Voigh-reuss topology optimization for structures with linear elastic material behaviours. *International Journal for Numerical Methods in Engineering*, 40, 3033−3057.

Swanson Analysis System, 2013. *ANSYS User's manual and theory reference*. Version 14.5. Houston: Swanson Analysis Systems, Inc.

Timoshenko, S. and Woinowsky-Krieger, S., 1940. *Theory of plates and shell*. 1st ed. McGraw-Hill.

Uysal, H., Gul, R. and Uzman, U., 2007. Optimum shape design of shell structures. *Engineering Structures*, 29(1), 80−87.

Victoria, M., Martí, P. and Querin, O.M., 2009. Topology design of two-dimensional continuum structures using isolines. *Computer and Structures*, 87, 101−109.

Victoria M., Querin O.M. and Martí P., 2010.*Topology design for multiple loading conditions of continuum structures using isolines and isosurfaces*, Finite Elements in Analysis and Design, 46, 229–237.

Victoria M., Querin O.M. and Martí P., 2011. *Topology Design of Three-Dimensional Continuum Structures Using Isosurfaces*, Advances in Engineering Software, Advances in Engineering Software, vol. 42, 671–679.

Vlasov, V.Z., 1944. Osnovnye differentsialnye uravnenia obshche teorii uprugikh obolochek. *Prikladnaia Matematika I Mekhanika*, 8, 109−140. (English translation, 1951: NACA Technical Memorandum, 1241. Basic differential equations in general theory of elastic shells. *National Advisory Committee for Aeronautics*).

Zingoni, A., 1997. *Shell structures in civil and mechanical engineering: theory and closed-form analytical solutions*. London: Thomas Telford Publishing.