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Why Mathematical Morphology Needs Quantales

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Abstract

The importance of complete lattices in mathematical morphology is well-known. There are some aspects of the subject that can, however, be clarified by using complete lattices equipped with the additional structure of a binary operation subject to conditions making them quantales. This extended abstract shows how quantales, and more generally quantale modules, can be used to capture a number of constructions for dilations in a uniform way. The treatment is partly an exposition of material that is not technically novel, but which provides a valuable conceptual perspective on mathematical morphology.

Index Terms— Quantale, Complete Lattice, Dilation, Relation

1 Introduction

Complete lattices have been used in mathematical morphology as a framework which unifies several important examples which originally appeared to be distinct. Having such a framework not only allows a deeper understanding of these examples, it has provided and continues to provide a foundation for advances in the subject. Additionally, as Heijmans and Ronse made clear [5, p253]

> "an abstract approach provides a deeper insight into the essence of the theory (e.g., about what assumptions are minimally required to have certain properties) and links it to other, sometimes rather old, mathematical disciplines."

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The purpose of this extended abstract is to explain why another algebraic structure, called a quantale, also deserves the attention of researchers interested in the essence of mathematical morphology. The title is a deliberate allusion to that of [7]. While mathematical morphology does still need complete lattices, I will argue that for some purposes it also needs complete lattices with some extra structure, namely quantales.

Before giving the formal details, I will indicate briefly why these structures are relevant to morphology. Some further motivation appears in the penultimate paragraph of Section 4. A quantale is a complete lattice $\mathscr{L}$ where, besides the usual lattice operations, there is a binary operation (satisfying certain conditions) on the elements of the lattice. This enables us to regard the elements of the lattice both as carrying the partial order structure, and simultaneously as being dilations on $\mathscr{L}$. (I will follow the practice of taking a dilation on $\mathscr{L}$ to mean a sup-preserving operation.) The binary operation, in this view, models the composition of dilations. This general situation is familiar to anyone with experience of mathematical morphology. Taking E to be a Euclidean space, such as $\mathbb{R}^2$, the subsets of E function both as binary images and as structuring elements which have associated dilations. In this case the powerset $\mathscr{P}(E)$ is a quantale with the binary operation being the usual Minkowski addition, typically denoted $\oplus$.

For a full treatment of quantales, including the history of the development of the concept, the book by Rosenthal [8] may be consulted. The relevance of quantales to mathematical morphology has already been observed by Russo [9] in a paper that includes some of the examples I present below. My aim here is partly to draw the attention of the morphology community to these known applications of quantales. However, the full range of applications of equation (1) below, including in particular relations, does not appear have been noted before.

2 Quantales

Definition 1. *A* ***quantale*** *is a complete lattice $\mathscr{L}$ equipped with a binary operation, &, which is associative and for every $A \subseteq \mathscr{L}$ and every $x \in \mathscr{L}$ satisfies $(\bigvee A) \,\&\, x = \bigvee(A \,\&\, x)$ and $x \,\&\, (\bigvee A) = \bigvee(x \,\&\, A)$ where $A \,\&\, x$ denotes $\{a \,\&\, x \mid a \in A\}$ and analogously for $x \,\&\, A$.*

The use of & to denote the binary operation follows [8]. We have already mentioned one example of a quantale, the set of subsets of a Euclidean space E, with dilation as the binary operation. Three further examples are as follows.

1. A Boolean algebra with the usual ordering and with & as $\wedge$.
2. The completed reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ with $+$ provided we require $x + (-\infty) = -\infty + x = -\infty$ for all $x \in \overline{\mathbb{R}}$.
3. The set of all relations on a set A, with composition of relations.

Given a quantale $(\mathscr{L}, \&)$ and $x \in L$ we can define an operation δ_x by $\delta_x(y) = y \,\&\, x$. We see that δ_x is a dilation and $\delta_x \delta_y(z) = \delta_{y \& x}(z)$. It follows by well-known properties of adjunctions between complete lattices that each δ_x has an associated upper adjoint (also called right adjoint in the literature) denoted ε_x and that $\varepsilon_x \varepsilon_y = \varepsilon_{x \& y}$. We can define a second binary operation on $\mathscr{L}$ by $x \multimap y = \varepsilon_y(x)$ and a short calculation will show that $(x \multimap y) \multimap z = x \multimap (z \,\&\, y)$. If we were to write & as $\oplus$ and $\multimap$ as $\ominus$ we would obtain the equations $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and $(x \ominus y) \ominus z = x \ominus (z \oplus y)$ where the unfamiliar order of y and z in the last equation is required as the binary operation need not be commutative.

It is also possible to define another family of dilations by $\delta'_x(y) = x \,\&\, y$ with associated erosions ε'_x. This situation has already been noted in mathematical morphology by Roerdink [6] who explains the connection with residuated lattices, which are closely related to quantales.

3 Quantale-valued Functions

In mathematical morphology the lattice theoretic approach has been especially useful in situations where we study functions from a set X to a complete lattice. In general the set X carries some algebraic structure, such as the group of translations of Euclidean space.

However, a substantially weaker structure than an abelian group is adequate for the most basic parts of the theory. We shall say that X is a partial semigroup if it has a binary operation, $\Box$, not necessarily defined for all $x, y \in X$, such that whenever $x, y, z \in X$, the two expressions $x \Box (y \Box z)$ and $(x \Box y) \Box z$ are either (a) both undefined or (b) both defined and equal to each other. For a lattice $\mathscr{L}$, the lattice of functions from X to $\mathscr{L}$ with the usual 'pointwise' ordering will be denoted $\mathscr{L}^X$.

Theorem 1. *Let $(\mathscr{L}, \&)$ be a quantale, and $(X, \Box)$ a partial semigroup. Then $\mathscr{L}^X$ is a quantale with a binary operation $\&^X$ where for all $F, G \in \mathscr{L}^X$,*

$$(F \,\&^X\, G)(x) = \bigvee_{y \Box z = x} F(y) \,\&\, G(z). \quad (1)$$

The theorem is proved by a straighforward verification of the necessary properties. Equation (1) provides a unified treatment of several definitions of dilation, as the following examples demonstrate.

3.1 Binary morphology

Here $\mathscr{L}$ is the two element Boolean algebra, and X is a Euclidean space with $\Box$ being the usual addition. The condition $y \Box z = x$ becomes $y = x - z$. In this case $\&^X$ is the usual dilation of binary images by binary structuring elements.

3.2 Greyscale morphology

We can take X to be $\mathbb{R}^n$ or $\mathbb{Z}^n$ with $\Box$ as the usual vector addition. By taking $(\mathscr{L}, \&)$ to be $(\overline{\mathbb{R}}, +)$, equation (1) gives the familiar operation of greyscale dilation, as found for example in [5, p250].

To use a bounded set of grey levels, whether an interval in $\mathbb{R}$ or in $\mathbb{Z}$, requires specifying the binary operation on the set of grey levels. It is now well-known, and was pointed out in [7], that using truncated addition with arbitrary functions in this context can lead to problems. The fact that since truncated addition is not associative we fail to have a quantale, is a useful way of understanding why these problems arise.

3.3 Fuzzy morphology

There are several different approaches to fuzzy morphology [2] and not all the proposed definitions of fuzzy dilations will fit into the quantale framework. The central idea in fuzzy morphology is to work with fuzzy sets which are functions from a crisp set X to a complete lattice of truth values which in many formulations is taken to be the interval $[0,1] \subseteq \mathbb{R}$.

In several cases the lattice of truth values together with the fuzzy conjunction will be a quantale, in particular this happens with what Drossos called a "t-norm-like structure" [4]. When $(X, \Box)$ is as in the binary or greyscale cases, and when the fuzzy conjunction makes the lattice of truth values into a quantale, then equation (1) gives the appropriate definition of fuzzy dilation.

3.4 Relations

It is well-known that the lattice of all relations on a set A provides a convenient description of the lattice of all dilations on $\mathscr{P}(A)$. We can give the set $A \times A$ a partial semigroup structure where $(a,b) \Box (c,d)$ is defined iff $b = c$, and when this happens the value is (a,d). By taking the quantale $(\mathscr{L}, \&)$ to be the two-element Boolean algebra and X to be $A \times A$ with $\Box$ as just defined, we see that equation (1) gives the composition of relations.

To observe that the lattice of relations on A is isomorphic to the lattice of dilations on $\mathscr{P}(A)$ reveals only part of the picture. The fact that these are isomorphic not merely as lattices but also as quantales is what lies at the heart of many of the manifestations of the equation

$$\delta_x \delta_y = \delta_{y \oplus x} \tag{2}$$

in the mathematical morphology literature.

This is one of the key reasons why mathematical morphology does need quantales. The additional structure provides a way of 'indexing' or parameterizing dilations without which equation (2) cannot be formulated at the appropriate level of abstraction. Although the statement and derivation of equation (2) in the context of relations is entirely straightforward, it provides a useful way of thinking about the role of composition which is not always made explicit in recent accounts such as [3, pp885–890].

3.5 Fuzzy relations

There is a standard notion of a fuzzy relation taking values in $\mathscr{L}$ as a function $A \times A$ to $\mathscr{L}$. If we use a type of fuzzy logic with a conjunction, $\&$ such that $(\mathscr{L}, \&)$ is a quantale then equation (1) gives the usual composition of such relations, taking the partial semigroup structure $\Box$ as in the crisp relation case.

Although the identification of fuzzy relations with functions $A \times A \to \mathscr{L}$ is firmly established, it does mean that fuzzy relations no longer correspond to dilations $\mathscr{L}^A \to \mathscr{L}^A$. This is easily seen when $\mathscr{L}$ is the 3-element chain $\{\bot, 0, \top\}$ and A has one element. In this case $\mathscr{L}^{A \times A}$ has only three elements but there are six sup-preserving mappings $\mathscr{L}^A \to \mathscr{L}^A$.

Heijmans and Ronse showed [5, p270] that dilations on $\mathscr{L}^A$ are in a one-to-one correspondence with functions which assign to each element of $A \times A$ a dilation on $\mathscr{L}$. This suggests it would be worth exploring the possibility that an $\mathscr{L}$-fuzzy relation on A could be defined as an asssignment of this form. In this case the equation (1) provides us with the definition of composition of relations.

4 Quantale Modules

We can think of a quantale as a complete lattice, $\mathscr{L}$, where the elements of the lattice also serve as dilations on $\mathscr{L}$. This is a rather special case, because we may well have a lattice of dilations which act on a structure other than this lattice itself. The quantale of relations has elements that we think of as being dilations on a powerset rather than on the set of relations.

Definition 2. *A quantale module consists of a quantale, $(\mathscr{L}, \&)$, and a complete lattice, M, together with a binary operation $\cdot : M \times \mathscr{L} \to M$ such that* *the following hold for all $A \subseteq M$, all $X \subseteq \mathscr{L}$, all $x, y \in \mathscr{L}$ and all $a \in M$*

1. $a \cdot (x \,\&\, y) = (a \cdot x) \cdot y$,
2. $(\bigvee A) \cdot x = \bigvee\{a \cdot x \in M \mid a \in A\}$,
3. $a \cdot (\bigvee X) = \bigvee\{a \cdot x \in M \mid x \in X\}$.

The operation $m \mapsto m \cdot x$ provides a dilation $\delta_x : M \to M$, and we have $\delta_x \delta_y = \delta_{y\&x}$. Erosions can be obtained from these dilations and many well-known properties of erosions and dilations follow in this abstract setting.

To treat the most essential features of dilations and erosions in an algebraic setting that uses only complete lattices means that we are unable to capture the full structure. Just having the lattice M we can consider dilations and erosions on M and their composites and the order relation between them, but although we may form $\delta_x \delta_y$ we cannot talk about $y \,\&\, x$ or $\delta_{y\&x}$.

Abramsky and Vickers [1] introduce quantale modules as models where the quantale $\mathscr{L}$ is a set of observations of computational processes, and the lattice M is a set of entities which change under these observations. This is closely related to the common description of mathematical morphology as a body of techniques in which an image is 'probed' by a structuring element to obtain information. We can view a dilation $\delta \in \mathscr{L}$ as an observation which may be made of an image $I \in M$ and the result of making the observation δ on I is $I \cdot \delta \in M$. The $\cdot$ of the module models the action of observing, while the $\&$ in $\mathscr{L}$ models the ability to build compound observations.

Understanding morphology via quantales and quantale modules can be expected to lead to useful links with other topics, especially in the area of logic, where quantales have important applications to linear logic for example. In morphology, the graph morphology in [10] can usefully be studied in this setting and other examples can be found in [9].

References

[1] S. Abramsky and S. Vickers. Quantales, observational logic and process semantics. *Mathematical structures in computer science*, 3:161–227, 1993.

[2] I. Bloch. Duality vs. adjunction for fuzzy mathematical morphology and general form of fuzzy erosions and dilations. *Fuzzy Sets and Systems*, 160:1858–1867, 2009.

[3] I. Bloch, H. J. A. M Heijmans, and C. Ronse. Mathematical morphology. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of Spatial Logics*, chapter 14, pages 857–944. Springer, 2007.

[4] C. A. Drossos. Generalized t-norm structures. *Fuzzy Sets and Systems*, 104:53–59, 1999.

[5] H. J. A. M Heijmans and C. Ronse. The algebraic basis of mathematical morphology. I. Dilations and erosions. *Computer Vision Graphics and Image Processing*, 50:245–295, 1990.

[6] J. B. T. M. Roerdink. Mathematical Morphology with Non-Commutative Symmetry Groups. In E. R. Dougherty, editor, *Mathematical Morphology in Image Processing*, chapter 7, pages 205–254. Marcel Dekker, 1993.

[7] C. Ronse. Why mathematical morphology needs complete lattices. *Signal Processing*, 21:129–154, 1990.

[8] K. I. Rosenthal. *Quantales and their applications*, volume 234 of *Pitman Research Notes in Mathematics*. Longman, 1990.

[9] C. Russo. Quantale modules and their operators with applications. to appear in Journal of Logic and Computation. Preprint at `www.dmi.unisa.it/people/russo/www/publications.html`, 2008.

[10] J. G. Stell. Relations in Mathematical Morphology with Applications to Graphs and Rough Sets. In S. Winter et al., editors, *Conference on Spatial Information Theory, COSIT 2007*, volume 4736 of *Lecture Notes in Computer Science*, pages 438–454. Springer Verlag, 2007.