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Formal Concept Analysis over Graphs and Hypergraphs

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Abstract. Formal Concept Analysis (FCA) provides an account of classification based on a binary relation between two sets. These two sets contain the objects and attributes (or properties) under consideration. In this paper I propose a generalization of formal concept analysis based on binary relations between hypergraphs, and more generally between pre-orders. A binary relation between any two sets already provides a bipartite graph, and this is a well-known perspective in FCA. However the use of graphs here is quite different as it corresponds to imposing extra structure on the sets of objects and of attributes. In the case of objects the resulting theory should provide a knowledge representation technique for structured collections of objects. The generalization is achieved by an application of work on mathematical morphology for hypergraphs.

1 General Introduction

1.1 Formal Concept Analysis

We recall the basic notions of Formal Concept Analysis from [GW99] with some notation taken from [DP11].

Definition 1 *A* ***Formal Context*** $\mathbb{K}$ *consists of two sets U, V and a binary relation $R \subseteq U \times V$. The elements of U are the* ***objects*** *of $\mathbb{K}$ and the elements of V are the* ***properties****.*

The relation can be written in the infix fashion $a\ R\ b$ or alternatively as $(u, v) \in R$. The converse of R is denoted $\breve{R}$.

Definition 2 *Given a formal context (U, R, V) then $R^{\vartriangle} : \mathscr{P}U \to \mathscr{P}V$ is defined for any $X \subseteq U$ by*

$$R^{\vartriangle}(X) = \{v \in V : \forall x \in X\ (x\ R\ v)\}.$$

This operator provides the set of properties which every one of the objects in X possesses. Using the converse of R, the operator $\breve{R}^{\vartriangle} : \mathscr{P}V \to \mathscr{P}U$ can be described explicitly by

$$\breve{R}^{\vartriangle}(Y) = \{u \in U : \forall y \in Y\ (u\ R\ y)\}$$

for any $Y \subseteq V$.

Definition 3 *A* ***formal concept*** *belonging to the context (U, R, V) is a pair (X, Y) where $X \subseteq U$, $Y \subseteq V$ and such that $R^{\vartriangle}(X) = Y$ and $\breve{R}^{\vartriangle}(Y) = X$. The sets X and Y are respectively the* ***extent*** *and the* ***intent*** *of the concept.*

1.2 Importance of graphs and hypergraphs

The question posed in this paper is what happens when we replace the sets U and V in a formal context by graphs, or by hypergraphs or more generally still by pre-orders? The question is not straightforward, partly because there is more than one possible meaning for a relation between graphs. First, however, we need to consider why the question might be worth answering.

The idea that properties can have structure is well established within FCA, see for example the treatment of scales and many-valued contexts in [GW99, p36]. Suppose however our collection of objects also has structure (rather than each individual object being structured).

We now consider some examples.

Assemblages An assemblage of individuals where the individuals have independent existence. For example, a marriage between two people or a group of four friends who take a holiday together. Further examples would be alliances of nations, twinnings of towns. In all these cases we can consider attributes of the individuals and also attributes of the groupings. A particular grouping can have attributes which may not depend solely on the individuals, and the same set of individuals may participate in a number of different groupings.

Granules Such groupings could also appear in granulation, equivalence classes are a simple case but other kinds of granules are also well known. A granule can have attributes (say some viewpoint that engenders the clustering) and again the same individuals may appear in different granules for different viewpoints. This indicates the need to attach attributes to the granules themselves as well as to the individuals.

Links In a different kind of example the collection of objects could make up both the stations and lengths of track joining stations to be found in a railway system. Similarly there might be cities linked by roads or nodes and links in a social network. In the case of a railway network it is clear that the railway line between two stations can require different attributes from the stations themselves.

All of these examples can be modelled as graphs or hypergraphs, and we shall restrict to undirected graphs in the present paper.

2 Theoretical Background

2.1 Graphs and Hypergraphs

A hypergraph can be defined as consisting of a set N of nodes and a set E of edges (or hyperedges) together with an incidence function $i : E \to \mathscr{P}N$ from E to the powerset of N. This approach allows several edges to be incident with the same set of nodes, and also allows edges incident with the empty set of nodes. An example is shown in Figure 1 where the edges are drawn as curves enclosing the nodes with which they are incident.

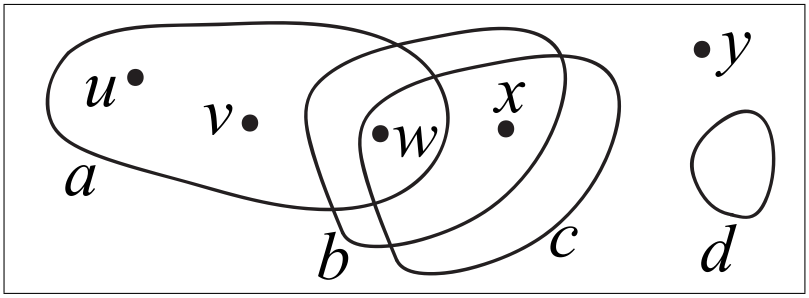


Fig. 1. Hypergraph with edges $E = \{a, b, c, d\}$, and nodes $N = \{u, v, w, x, y\}$.

When studying relations on these structures it is more convenient to use an equivalent definition, in which there is a single set comprising both the edges and nodes together. This has been used in [Ste07,Ste10] and is based on using a similar approach to graphs in [BMSW08]. This is illustrated in Figure 2 and set out formally in Definition 4.

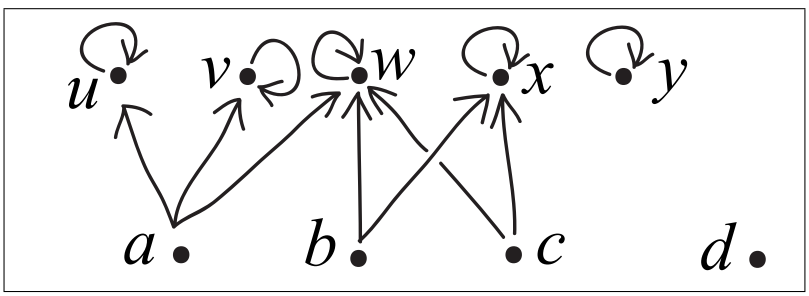


Fig. 2. Hypergraph from Figure 1 represented as a relation φ on the set U of all edges and nodes.

Definition 4 *A* ***hypergraph*** *consists of a set U and a relation $\varphi \subseteq U \times U$ such that for all $x, y, z \in U$,*

1. *if $(x, y) \in \varphi$ then $(y, y) \in \varphi$, and*
2. *if $(x, y) \in \varphi$ and $(y, z) \in \varphi$ then $y = z$.*

From a hypergraph described in this way we can obtain the edges as those $u \in U$ for which $(u, u) \notin \varphi$, whereas the nodes satisfy $(u, u) \in \varphi$. A **sub-hypergraph** of (U, φ) is defined as a subset $K \subseteq U$ for which $k \in K$ and $(k, u) \in \varphi$ imply $u \in K$. We will usually say sub-graph rather than sub-hypergraph for brevity.

It is technically convenient to replace the relation φ by its reflexive closure $\varphi^{\circlearrowleft}$. This loses some information, as we cannot distinguish between nodes incident with no edges and edges incident with no nodes. However, this loss is not significant as Stell [Ste12] shows that the relations we need to define on hypergraphs are the same in each case.

This means that a hypergraph can be regarded as a special case of a structure (U, H) where U is a set and H is a reflexive transitive relation on U. In the case of a graph every edge is incident with at most two nodes, this corresponds to the set $\{v \in U : u \; H \; v\}$ having at most three elements. The need to deal with hypergraphs and not only graphs comes from their duality. Interchanging the edges and nodes in hypergraph yields the **dual hypergraph**. For the structure (U, H) this corresponds to $(U, \breve{H})$ where $\breve{H}$ is the converse of H.

2.2 Mathematical Morphology on Sets

Mathematical morphology [Ser82,BHR07] has been used widely in image processing and has an algebraic basis in lattice theory. It can be seen as a way of approximating images so as to emphasise essential features. At the most abstract level it depends on the action of binary relations on subsets. Although various authors [CL05,ZS06,DP11] have analysed several connections between different approaches to granulation and approximation, the exploration of the links between mathematical morphology and formal concept analysis is only beginning to be developed [Blo11].

The fundamental operations used in mathematical morphology have been described with different names and notions in several fields. The basic setting consists of two sets, U and V, and a binary relation $R \subseteq U \times V$. Given any subsets, $X \subseteq U$ and $Y \subseteq V$ we can define

$$X \oplus R = \{v \in V : \exists u \; (u \; R \; v \text{ and } u \in X)\}$$
$$R \ominus Y = \{u \in U : \forall v \; (u \; R \; v \text{ implies } v \in Y)\}$$

These operations are known respectively as **dilation** and **erosion**. Dilation by R acts on the right and erosion acts on the left. This means that given further relations composable with R, say $S \subseteq U' \times U$ and $T \subseteq V \times V'$ then $X \oplus (R ; T) = (X \oplus R) \oplus T$ and $(S \, ; R) \ominus Y = S \ominus (R \ominus Y)$.

The basic image processing context of these constructions is usually that the sets U and V are both $\mathbb{Z}^2$ thought of as a grid of pixels and the relation R is generated by a particular pattern of pixels. Extensions of the operations to graphs have been discussed from various viewpoints [HV93,CNS09], and approaches to morphology on hypergraphs include [Ste10,BB11].

The operation $R^{\vartriangle}$, introduced in Definition 2, is related to dilation and erosion as follows. In these constructions $\hat{R}$ is the converse complement of R, that is $\hat{R} = \breve{\overline{R}} = \overline{\breve{R}}$, where $\overline{R}$ denotes the complement of the relation R. The complement of a set X is denoted $-X$. The proof is a straightforward calculation from the definitions.

Theorem 1 *Let $R \subseteq U \times V$ be a relation and $X \subseteq U$ and $Y \subseteq V$. Then*

$$R^{\vartriangle}(X) = \hat{R} \ominus (-X),$$
$$\breve{R}^{\vartriangle}(Y) = -(Y \oplus \hat{R}).$$

The reason for establishing this result is that it provides a guide for generalizing formal concept analysis from sets to graphs including the potential of extending the treatment of decomposition of contexts in [DP11]. It also provides a foundation for developing correspondences between formal concept analysis and mathematical morphology. This is clear once the operations of (morphological) opening and closing, which are derived from combinations of dilation and erosion are introduced.

Corollary 2 (X, Y) *is a formal concept iff*

$$-X = (\hat{R} \ominus -X) \oplus \hat{R}, \text{ and}$$
$$Y = \hat{R} \ominus (Y \oplus \hat{R}).$$

In other words, $-X$ is open (equal to its own morphological opening) and Y is closed (equal to its own morphological closing) with respect to the relation $\hat{R}$.

The calculus of erosion and dilation makes particularly clear the well-known fact that the intent of a concept determines the extent and vice versa. Given Y where $Y = \hat{R} \ominus (Y \oplus \hat{R})$ we can define $X = -(Y \oplus \hat{R})$ and immediately obtain $Y = \hat{R} \ominus (-X)$. Similarly if we are just given the extent X We shall see in Section 3 how the theory of mathematical morphology on hypergraphs allows us to formulate a generalization of formal concept analysis.

2.3 The algebra of graphs and hypergraphs

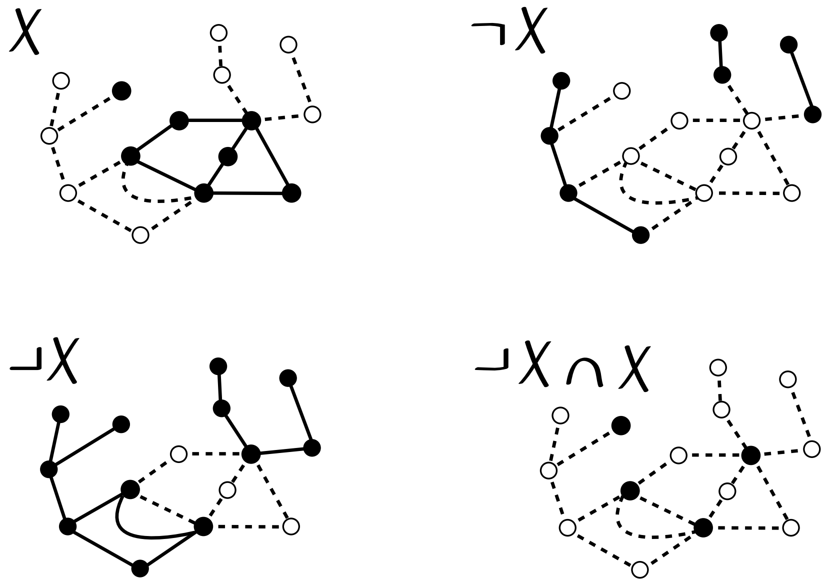
Given a graph, by which we mean an undirected multigraph with possible loops, the set of all subgraphs forms a bi-Heyting algebra, generalizing the Boolean algebra of subsets of a set. This bi-Heyting structure has been described in [Law86,RZ96], but we review it here, in the more general setting of hypergraphs, as it provides motivation for later sections.

Given a hypergraph (U, H), a **sub-graph** is any $X \subseteq U$ for which $X \oplus H \subseteq X$. The sub-graphs form a complete sub-lattice $\mathscr{L}$ of the lattice of all subsets of U. The meet and join operations in $\mathscr{L}$ are the usual intersections and meets of subsets. If X is a sub-hypergraph, then its complement $U - X$ is not necessarily a sub-hypergraph. There are, however, two weaker operations on sub-hypergraphs: the pseudocomplement, $\neg X$, and the dual pseudocomplement, $\urcorner X$. These satisfy both $X \cup \urcorner X = U$ and $X \cap \neg X = \varnothing$ but not necessarily $X \cup \neg X = U$ or $X \cap \urcorner X = \varnothing$.

These can be expressed using dilation and erosion with respect to H:

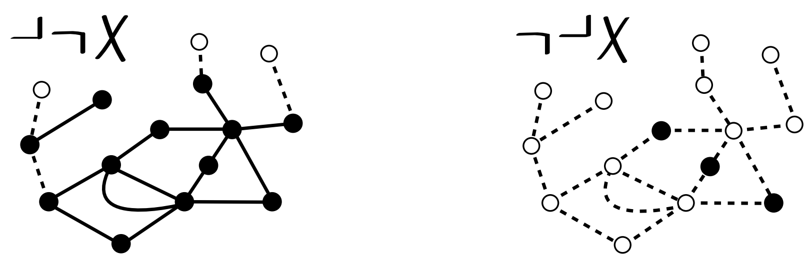
$$\neg X = H \ominus (-X),$$
$$\urcorner X = (-X) \oplus H.$$

These two weaker forms of the usual Boolean complement provide an expressive language allowing us to define notions such as boundary, expansion and contraction for subgraphs. These are illustrated in the following diagrams where we have taken a subgraph X of a graph for simplicity, but the constructions have similar interpretations for subgraphs of a hypergraph.

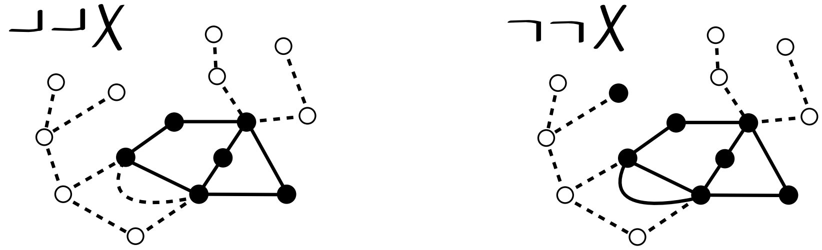


The **boundary** of X consists of those nodes incident with some edge which is not in X. The boundary is $\lrcorner X \cap X$.

The **expansion** of X contains all of X together with any edges incident with a node in X and any nodes incident with these edges. Intuitively, this is X expanded by anywhere accessible along an edge from X. The expansion is $\lrcorner\neg X$. In the other direction the **contraction** contains only the nodes which are not incident with anything in the boundary, and it also contains any edges in X which are exclusively incident with these nodes.



The other two combinations of the operations, in which one of the weaker converses is repeated twice, provide the **node opening**, $\lrcorner\lrcorner X$, and the **edge closing**, $\neg\neg X$. The node opening removes from X any nodes which are incident only with edges which are not in X. The edge closing adds to X any edges all of whose incident nodes are in X.



When a subgraph X forms the extent of a formal concept the availability of the four, potentially distinct (as in the above example), subgraphs: $\neg\lrcorner X$, $\neg\neg X$, $\lrcorner\neg X$, $\lrcorner\lrcorner X$ can be expected to produce a much richer theory than the classical when X is a set. In this classical case of course all four of these reduce to $--X$ which is just X itself.

3 Contexts and Concepts for Pre-Orders

3.1 Relations between preorders

A formal context is a relation between two sets. To generalize this to relations between graphs or hypergraphs we have to consider what should be meant by a relation between such structures. We find it is convenient to work in the more general setting of pre-orders, but it is especially useful to work with examples where the pre-order defines the incidence structure in a graph or a hypergraph.

There are several different possibilities for how to define a relation between graphs. As relations are subsets of cartesian products we might try defining a relation between graphs as a subgraph of a product of graphs, where 'product of graphs' would need to be understood in a suitable category of graphs. While it is straightforward to define products of directed graphs, the undirected case is more subtle [BL86,BMSW08] requiring some adjustment to the usual notions of undirected graph and of morphism between such structures. Instead of developing this approach, we use the notion studied in [Ste12] which generalizes the characterization of relations R between sets U and V with $R \subseteq U \times V$ as union-preserving functions from $\mathscr{P}U$ to $\mathscr{P}V$. This means we use a notion of relation between hypergraphs, (U, H) and (V, K), equivalent to union-preserving functions from the lattice of subgraphs of (U, H) to the lattice of subgraphs of (V, K).

We assume in this section that U and V are sets and that $H \subseteq U \times U$ and $K \subseteq V \times V$ are both reflexive and transitive relations (often called preorders). The appropriate generalization of subgraph to this situation is as follows.

Definition 5 *If $X \subseteq U$ is any subset, then X is a H-**set** if $X \oplus H \subseteq X$.*

The lattice of H-sets on (U, H) will be denoted $\mathscr{L}(U, H)$.

Proposition 3 *The following are equivalent*

1. X is an H-set.
2. $-X$ is an $\breve{H}$-set

Definition 6 *Let $R \subseteq U \times V$ be a relation. Then R is an (H, K)**-relation** if $H \,;R \subseteq R$ and $R\,; K \subseteq R$.*

This extends the notion of a relation on a hypergraph studied in [Ste12] which is a special case of the situation when $(U, H) = (V, K)$. The significance of these relations lies in the following property.

Theorem 4 *There is a bijection between the set of (H, K)-relations and the set of join-preserving functions from $\mathscr{L}(U, H)$ to $\mathscr{L}(V, K)$.* □

The result is proved by a straightforward generalization of the well-known case of H and V just being the identity relations on U and V respectively (that is when R is an ordinary relation between sets). The bijection preserves both the compositional structure and the way relations act on H-sets in U. The appropriate setting for the correspondence would be in terms of quantaloids [Ros96] but this is beyond the scope of this paper.

For the applications to formal concept analysis, we need to consider relations which are $(\breve{H}, K)$-relations, while also considering H-sets rather than $\breve{H}$-sets in U. It is straightforward to show that the converse of an $(\breve{H}, K)$-relation is not necessarily a $(K, \breve{H})$-relation, and the complement of an $(\breve{H}, K)$-relation is not necessarily a $(\breve{H}, K)$-relation. However, the operations of converse and complement do have the properties stated in Proposition 6. In order to establish these properties we need to recall the properties of the residuals for relations.

Definition 7 *Let $Q \subseteq U \times V$ and $R \subseteq V \times W$ be relations, and let $S \subseteq U \times W$ also be a relation.*
*The **Left Residual**, $\backslash$, is defined by $Q \backslash S = \overline{\breve{Q}\,; \overline{S}}$.*
*The **Right Residual**, $/$, is defined by $R \,/\, S = \overline{\overline{R}\,; \breve{S}}$.*

The following well-known property of these residuals is important.

$$Q\,; R \subseteq S \quad \text{iff} \quad R \subseteq Q \backslash S \quad \text{iff} \quad Q \subseteq R \,/\, S \tag{1}$$

Lemma 5 *Let $R \subseteq U \times V$ and let H and K be pre-orders on U and V respectively. Then*

1. $H\,; \overline{R} \subseteq \overline{R}$ iff $\breve{H}\,; R \subseteq R$,
2. $R\,; \breve{K} \subseteq \overline{R}$ iff $R\,; K \subseteq R$. □

Proof. We have $H\,; \overline{R} \subseteq \overline{R}$ iff $\overline{R} \subseteq H \backslash \overline{R}$ by (1). But we also have $\breve{H}\,; R \subseteq R$ iff $\overline{R} \subseteq \overline{\breve{H}\,; R} = H \backslash \overline{R}$. This gives the first part, and the second is similar using the other residual.

Proposition 6 *The following are equivalent for $R \subseteq U \times V$.*

1. R is an $(\breve{H}, K)$-relation.

2. $\overline{R}$ *is an* $(H, \breve{K})$*-relation.*

3. $\breve{R}$ *is a* $(\breve{K}, H)$*-relation.*

4. $\hat{R}$ *is a* $(K, \breve{H})$*-relation.*

Proof. The first two parts are equivalent by Lemma 5. The first and third parts are equivalent by basic properties of the converse operation. Finally if R is an $(\breve{H}, K)$-relation then $\overline{R}$ is an $(H, \breve{K})$-relation, and $\breve{\overline{R}}$ is a $(\breve{\breve{K}}, \breve{H})$-relation so the first part implies the fourth, and the fourth implies the first since the converse-complement is an involution. □

Theorem 7 *If* $X \subseteq U$ *and* $Y \subseteq V$ *are any subsets and* R *is an* (H, K)*-relation then*

1. $X \oplus R$ *is a* K*-set, and*
2. $R \ominus Y$ *is an* H*-set.*

Proof. For the first part, we have the calculation

$$(X \oplus R) \oplus K = X \oplus (R \,;\, K) \subseteq X \oplus R.$$

For the second, suppose $x \in R \ominus Y$ and $x \; H \; x'$. If $x' \; R \; v$ where $v \in -Y$ then we have a contradiction as $x \; R \; v$. □

Using a special case of this (when X is an H-set), it can be shown that dilation provides a union-preserving mapping from $\mathscr{L}(U, H)$ to $\mathscr{L}(V, K)$. This dilation then forms a Galois connection with the erosion of K-sets. This confirms that we have established an appropriate generalization of mathematical morphology on sets, and that we can re-capture the usual case by taking H and K to be the identity relations on U and V.

This generalization of mathematical morphology now allows us to use Theorem 1 and Corollary 2 as the basis of a generalization of formal concept analysis from sets of properties and objects to hypergraphs of properties and objects. More generally still the approach works for pre-orders with their associated H-sets as collections of objects and K-sets as collections of properties.

3.2 Defining Formal Concepts

Definition 8 *A* ***pre-order context*** *is a pair* $((U, H),\ (V, K))$*, where* H *and* K *are pre-orders on the sets* U *and* V *respectively, together with an* $(\breve{H}, K)$ *relation* $R \subseteq U \times V$*.*

It is sometimes convenient to denote a context as $R : (U, H) \to (V, K)$. Here the arrow is to be understood as indicating not a function between sets, but as a morphism in the category where the objects are pre-orders and the morphisms are relations with the appropriate properties.

Because the bi-Heyting algebra of H-sets is more complex than the Boolean algebra of all subsets of U, as noted at the end of Section 2.3 above, there is more than one possible definition of a formal concept in the generalized setting. These will need to be evaluated against potential applications, some of which are indicated in the following section.

The following is a corollary to Theorem 7 established using Proposition 6.

Corollary 8 *Suppose we have a formal context, with notation as in the above definition. If X and Y are respectively subsets of U and V then*

1. *$\hat{R} \ominus X$ is a K-set in (V, K),*
2. *$-(Y \oplus \hat{R})$ is an H-set in (U, H).* □

This means that we can make the following definition for what should be meant by a formal concept when we have pre-orders and not merely sets.

Definition 9 *A **formal pre-order concept** in the formal context $R : (U, H) \to (V, K)$ is a pair (X, Y) where X is an H-set, Y is a K-set, and the following hold:*

$$X = -(Y \oplus \hat{R})$$

$$Y = \hat{R} \ominus (-X).$$

However, this is not the only possibility because both X and Y have some spatial structure. The condition on Y characterizes it as

'all those properties which only objects *outside* X do not have'.

In Definition 9 the *outside* of X is expressed by $-X$, but as Section 2.3 showed there are other notions of *outside*, namely $\neg X$ and $\urcorner X$. The next result shows that the second of these does not lead to a different notion of intent.

Theorem 9 *If X is an H-set then*

$$\hat{R} \ominus (\urcorner X) = \{v \in V : \forall u\ (u \in X \textit{ implies } u\ R\ v)\} = \hat{R} \ominus (-X).$$

Proof. In one direction we have $\hat{R} \ominus (-X) \subseteq \hat{R} \ominus (\urcorner X)$ since $-X \subseteq \urcorner X$.

For the reverse inclusion we give a proof using the special case of the preorder being a graph so it can be visualized using diagrams as in Section 2.3. Suppose that $v \in \hat{R} \ominus (\urcorner X)$, and that $u \in X \cap \urcorner X$. There must be some edge, w, which lies in X and is incident with u. As $w \notin \urcorner X$ it is impossible that $v\ \hat{R}\ w$ so $w\ R\ v$ holds. Since R is an $\breve{H}$-relation, this implies $u\ R\ v$. This argument can be readily extended to the general case of any preorder. □

The following result shows that using the pseudocomplement $\neg X$ as the formalization of the outside of the extent X does yield a notion of intent as consisting of those properties holding everywhere in $\urcorner\neg X$, the expansion of X.

Theorem 10 *If X is an H-set then*

$$\hat{R} \ominus (\neg X) = \{v \in V : \forall u\ (u \in \urcorner\neg X \textit{ implies } u\ R\ v)\}.$$

Proof. As with Theorem 9 we use the special case of graphs which makes the underlying ideas easily visualized. The case of an arbitrary pre-order is similar. Suppose that $v \in \hat{R} \ominus (\neg X)$ and that $u \in \ulcorner\neg X$, the expansion of X. If it is not true that $u \mathrel{R} v$ then we have $v \mathrel{\hat{R}} u$ so that $u \in \neg X$, and hence $u \in \neg X \cap \ulcorner\neg X$. Some edge, say z, must be incident with u and be in $\ulcorner\neg X$ but not in X. From $\breve{H} \mathbin{;} R \subseteq R$ we deduce that $z \mathrel{R} v$ cannot hold. But we then have $v \mathrel{\hat{R}} z$ with $z \notin \neg X$ which is a contradiction, so $u \mathrel{R} v$.

Conversely, assume that for every $u \in \ulcorner\neg X$ we have $u \mathrel{R} v$ for a particular v. If $v \notin (\neg X) \ominus \hat{R}$ there is some $w \notin \neg X$ for which $v \mathrel{\hat{R}} w$. Since $-\neg X \subseteq \ulcorner\neg X$ we get $w \in \ulcorner\neg X$ so that $w \mathrel{R} v$ which contradicts $v \mathrel{\hat{R}} w$. □

In the simplest approach to formal concept analysis, an object either has or does not have a particular property. The use of many-valued attributes and scaling is one way to model properties which may hold to some degree. In the case of graphs, hypergraphs and more generally pre-orders, we can distinguish degrees of possession even for single valued attributes.

Definition 10 *Let $x \in U$ and $y \in V$ in the formal context $R : (U, H) \to (V, K)$. Then we say object x* ***strongly possesses*** *property y if for all x' where $x \mathrel{H} x'$ or $x' \mathrel{H} x$ we have $x' \mathrel{R} y$.*

In the special case of a graph, a node will posses a property strongly just when it together with any incident edges possess the property. It follows from the requirement that R is an $(\breve{H}, K)$-relation that edges which possess a property must possess it strongly.

Theorem 10 allows us to define a new kind of formal concept.

Definition 11 *Let $R : (U, H) \to (V, K)$ be a formal context. A* ***strong concept*** *is a pair (X, Y) where X is an H-set, Y is a K-set, and the following hold:*

$$\ulcorner\neg X = -(Y \oplus \hat{R})$$
$$Y = \hat{R} \ominus (\neg X).$$

Conceptually the intent, Y, of a strong concept consists of all those properties which hold strongly throughout X. The extent X, of such a concept then consists of those objects which not only have all the properties in Y, but have them strongly. Another view of the intent is that it consists of the properties not merely holding throughout X but holding anywhere 'near' X.

The ability to define both ordinary concepts and strong concepts for a context $R : (U, H) \to (V, K)$ provides a further indication of the expressive power of the algebraic operations from Section 2.3. Clearly this style of investigation can be carried further by considering, for example, the role of the contraction of X in place of the expansion.

4 Application Domains

We discuss how the structure of an $(\breve{H}, K)$-relation R can correspond to useful situations. For the first examples it is sufficient to assume that K is just the identity relation on V.

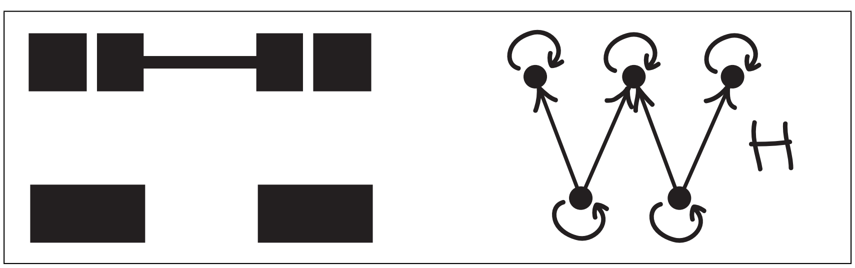
1. Take the edges of a graph to be intervals of time. Every interval has two nodes as its endpoints, but it is also permitted to have nodes (time instants) that are not the endpoint of any interval. This structure gives us an H-set in which we use $u \; H \; u'$ to model the idea that edge u is incident with node u'.
   Now consider the principle:

   > If a property holds throughout a time interval then it must hold at the end points of the interval.

   This principle is just the requirement that $\breve{H} \,;\, R \subseteq R$.
2. Instead of intervals of time forming a graph, consider edges in a hypergraph that correspond to geographic regions. A region here is an abstract entity that may contain specific point locations, modelled by nodes in the hypergraph. It is permitted to have points that lie in no region. Some (but not all) properties will adhere to this principle:

   > If a geographical region as a whole has a property then any points in the region have the property.

   To take a simple example, the property of being above a certain altitude, or the property of lying within a certain administrative region. In such cases the requirement $\breve{H} \,;\, R \subseteq R$ formalizes the principle correctly.
3. Granularity provides another example. Suppose we have data about objects at two levels of detail or resolution. These might be obtained by some remote-sensing technology. A high-level object can be represented as a node and a low level one as an edge in a hypergraph. The incidence structure records which low-resolution objects resolve into what sets of high-level ones. Consider the following example in which passage from more detail to less can involve closing up small gaps and ignoring narrow linkages.

   In the diagram the higher resolution version appears at the top. The principle here is

   > If low-resolution object α is known to have property v then a higher resolution version of α still has v.

   Again, the requirement $\breve{H} \,;\, R \subseteq R$ formalizes the principle correctly.

Considering the case of K not just being the identity on V, the most evident is when the properties are hierarchical, so that $v \; K \; v'$ means that v is more specific than v'. To model the principle:

> if an object has a specific property then it also has any more general version of the property

we need to impose the condition $R \,;K \subseteq R$. Of course this in itself can already be handled by existing techniques in formal concept analysis, but the ability to combine it with the three examples just mentioned appears to be a strength of this approach.

5 Conclusions and Further Work

We have combined the extension of mathematical morphology to graphs and hypergraphs with formal concept analysis to establish a foundation for concepts where instead of a set of objects having properties there is a graph or a hypergraph. Although mathematical morphology has been considered in connection with hypergraphs and separately with formal concept analysis before, the study of the interaction between these three models in a single setting is novel. The whole framework depends on the notion of relation between hypergraphs which is a special case of the notion of relation between pre-orders.

It should be possible to build on this foundation in terms of relations and extend some of the work by Priss [Pri06,Pri09] to the relation algebra operations for relations on hypergraphs described by Stell [Ste12]. As the results in Section 3 show, there are a number of potential definitions of formal concept in our generalized setting, and analysing the properties of all of these will take a significant effort.

Another direction is to extend the studies which have investigated connections between rough set theory and formal concept analysis. These studies include [DG03,Yao04,YM09]. The notion of an 'object-oriented concept' [Yao04, secn 2.3] in a context $R \subseteq U \times V$ can readily be expressed in terms of the morphological operations as a pair (X, Y) where $X = Y \oplus \breve{R}$ and $Y = \breve{R} \ominus X$. However, if we have a context with pre-orders (U, H) and (V, K) and R is an $(\breve{H}, K)$-relation, we find that object-oriented concepts provide an H-set X together with a $\breve{K}$-set Y. This contrasts with the situation for Definition 9 where Y is a K-set, so a full account would appear to need to consider the dual subgraphs as well as the subgraphs.

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