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Qualitative Analysis of Chaos

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Nonlinear Sampled Data Systems

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Abstract

The analysis of nonlinear sampled data systems is considered. A qualitative approach is adopted in which the system is suspended in a parameterised cell state space framework. Application of a global unravelling algorithm reveals the type of behaviour typically found using dynamical systems techniques. Steady state, periodic and aperiodic or chaotic behaviour is detected. System identification techniques are then applied. The resulting nonlinear recursive input output model is also analysed within the above framework and shown to exhibit the self same periodic characteristic as the original nonlinear sampled data system.

1. Introduction

A great variety of nonlinear systems have been reported in the literature as being chaotic. Starting from the early work by Hayashi, 1964, on steady state chaos in nonlinear circuits. The study of global bifurcations in chaotic dynamical systems relies on a rather loose collection of mathematical methods characterised by a coherent geometric way of thinking, Parker and Chua, 1987, Mees and Sparrow, 1987. As such the term qualitative theory of dynamical systems becomes more appropriate.

Most studies of global bifurcation and chaotic phenomena rely heavily on a comprehensive knowledge of the global insets and outsets of the invariant manifolds. Systems such as the Lorenz equations, Van der Pol and Duffing systems are amenable to global analysis, having received much attention indeed as far back as the 19-th century, see for example Cook, 1986. Much is known about the location, stability and dimension of the invariant sets for these systems. Such a priori information is of course rarely available in practical problems. Direct experimental and simulation evidence has however led to a good understanding of the global bifurcation effects that are likely [Ueda, 1980, Kennedy and Chua, 1986].

A great deal of work has been directed towards the study of nonlinear phenomena in simple nonlinear circuits. The driving force behind this work is the ease of construction of the simple systems involved which can then be used, in essence, as analogue computers for the study of chaotic dynamical systems. This has provided three important aspects to the research. Firstly, experimental measurements from real systems. Secondly, a method of validation of associated mathematical models used in simulation studies. Thirdly, a sound basis for the mathematical analysis of bifurcation phenomena observed in the systems.

Work by Chua and coworkers on chaos in simple nonlinear circuits has been stimulated to a large degree by the abundance of new and complex nonlinear phenomena that can be detected experimentally using physically realisable circuits. This has led to a wealth of results being obtained for what has become known as Chua's circuit family and the rigorous proof of the existence of chaos in various circuits, see for example [Chua et al 1986a, 1986b, Pei et al, 1986, Yang and Liao, 1987, Matsumoto et al, 1988].

Two distinct codimension-1 paths in the transition from order to chaos have been identified. The first, the so called *Period Doubling* route involves the *Flip* bifurcation. This cascade to chaos exhibits a sequence of period doubling bifurcations at $\mu_1, \mu_2, ..., \mu_\infty$ terminating at the accumulation point μ_∞ where an infinity of unstable periodic orbits exist. The sequence follows a geometric progression in μ which is easily demonstrated by the logistic map. Similar phenomena have also been demonstrated in the frequency domain. [Fiegenbaum, 1978, 1979 Collet and Eckmann, 1981, Lanford, 1982]. An important role in this development has been played by the so called unimodal maps [Li and Yorke, 1975, Guckenheimer, 1977].

The second route, the *Period Adding* or *Devils Staircase* involves a period adding sequence of bifurcations [Mandelbrot, 1982]. This has been observed experimentally and studied extensively in simple electronic circuits, see for instance Chua, 1987, and Kuo, 1988.

From the control theorists viewpoint the main focus of attention has been the analysis of chaos in feedback systems, see for instance Baillieul et al, 1980, Sparrow, 1980, 1981, Holmes, 1983, Ushio and Hirai, 1983a, 1983b, Cook, 1985, Salam, 1985. Success in this area has not however had such widespread exposure as the other areas of research, in particular nonlinear circuit theory and applied physics. Indeed this may be partly due to there already being a flourishing interest in the algebraic and geometric analysis of nonlinear control systems and stability, see for instance Isidori, 1989, Byrnes and Lindquist, 1986, and Fliess and Hazewinkel, 1986. Notable exceptions that make use of the qualitative approach to bifurcation include the work by Mehra, 1976, on bifurcation free control; by Aeyels, 1985, and Abed, 1986, 1987, on constructing stabilizing feedback control for continuous systems; by Hahn, 1985, using describing functions; by Chang and Chen, 1984, who considered PID control; and by Salam and Bai, 1986, Ydstie and Golden, 1988, and Mareels and Bitmead, 1988, who all have considered bifurcation in adaptive control systems.

In the present study a cell map based analysis is used to provide a graphical interpretation of the stability and asymptotic behaviour of a nonlinear sampled data systems. The utility of this approach is discussed in relation to current analytical methods and it is shown how the cell map analysis can be used successfully to study both local and global behaviour. The power of this qualitative based approach is demonstrated by studying a nonlinear feedback control system and comparing the analysis of this system with equivalent models identified from recorded input/output data. The influence of sampling rate and system set point are investigated and it is shown that the identified models exhibit the same qualitative characteristic as the original system.

2. Problem Definition

Consider the SISO system represented by

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t) \qquad \mathbf{x}\in\mathbf{R}^n \ ; \ \mathbf{u}\in\mathbf{R}^l \ ; \ A{:}n{\times}n \ ; \ B{:}n{\times}l$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \qquad \mathbf{y}\in\mathbf{R}^m \ ; \ C{:}m{\times}n \tag{2.1}$$

For which conventional closed loop control would demand that the input be taken as an error signal derived from a set point, $\mathbf{r}(t)$, and possibly a linear feedback matrix, K, such that

$$\mathbf{u}(t) = \mathbf{r}(t) - K\,\mathbf{y}(t) \qquad r\in\mathbf{R}^l \ ; \ K : l{\times}m \tag{2.2}$$

If we wish to consider this system within the framework of a sampled data system where the input $\mathbf{u}(t)$ is applied via a zero order hold device and the output, $\mathbf{y}(t)$, is sampled such that

$$\mathbf{u}(t) = \mathbf{r}(kT) - K\,\mathbf{y}(kT) \qquad \mathbf{y}(kT) = \mathbf{y}(t)\,\big|_{t=kT} \ ; \ t = kT, (k+1)T,.. \tag{2.3}$$

Then a wealth of linear techniques may be applied to the system. If however we wish to make use a nonlinear feedback the control law then becomes

$$\mathbf{u}(t) = \mathbf{r}(kT) - \mathbf{g}(y\,(kT)) \qquad \mathbf{g}{:}\mathbf{R}^l{\rightarrow}\mathbf{R}^m \tag{2.4}$$

Finally, if it is deemed that the linear model (2.1) no longer satisfactorily reflects the systems behaviour then a nonlinear model may become necessary such that

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) \qquad \mathbf{f}{:}\mathbf{R}^n{\times}\mathbf{R}^l{\rightarrow}\mathbf{R}^n \tag{2.5}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t)) \qquad \mathbf{h}{:}\mathbf{R}^n{\rightarrow}\mathbf{R}^m$$

Obviously making these and other possible changes to the basic model structure (2.1), has profound effects on the type of analysis that can be carried out. For instance we have not specified the functional forms of either **f**, **g** or **h**. We seek to adopt an approach that attempts to overcome any constraints on analysis caused by limiting ourselves to one particular model structure. At the same time as which we seek to provide the type of qualitative information required to detect, analyse and exploit the behaviour typically found in association with bifurcation and chaos.

The motivation behind this work is two fold. Firstly, it allows the characterisation of both continuous and discrete, or sampled data, nonlinear systems. Thus enabling the effects of, for example, sampling rate selection or set point changes to be gauged. Secondly, the type of models produced as a result of system identification studies can be handled. In this later case it is especially important to be flexible in terms of the type of model structure that may be considered.

The augmentation of the system model with one or more parameters enables the study of the systems behaviour over a specified range of parameter values. It is important to see how the behaviour of a system or model changes if the equations that make up that representation change in some manner, if only because such models are seldom known accurately. In the general nonlinear setting this problem is placed within the framework of structural stability and bifurcation theory. Detailed knowledge of the models solution structure is then required in order to classify the complete *unfolding* of behaviour to be expected [Golubitsky et al, 1985, 1988]. In this paper we utilize the essential qualitative aspects of the dynamical systems approach. In particular the concepts of periodic motions, domains of attraction and aperiodic or chaotic phenomena are adopted. At the same time we seek to maintain a perspective that is realistic and which minimizes the amount of apriori information required about the detailed solution structure of our system.

3. Global Analysis

The stability of a model, both absolute and structural, is dependent on the location and distribution of degenerate singularities on the solution manifold of that system. Analytical methods meet some of the requirements for certain classes of models. Such methods however, tend to be valid for only one particular model structure or type and tend to rely on constructing a reduced order system around a point where the *interesting* dynamics are deemed to be situated. In addition most analytical approaches tend to rely on a reduction method in order to reduce the dimension of the problem under consideration. Such methods, when employed, not only rely heavily on a priori knowledge of the solution structure of the problem, but are only locally valid. Additionally the center manifold system itself is not unique. In discrete dynamical systems, where a profusion of periodic behaviour is all too common, constructing such a system is even more cumbersome. For application to the type of models encountered in sampled data systems and system identification, a flexible approach is essential that does not suffer such constraints. Furthermore, it is desirable to add a global aspect to the analysis to enable both the state space, and parameter space, of the model to be probed for *interesting* behaviour.

In order to achieve these aims a dual approach has been adopted. This combines the essentially qualitative ideas of Bifurcation theory with a simple yet attractive numerical algorithm. The analysis of nonlinear systems using the *Cell Map* approximation was first carried out by Hsu, see for example Hsu and Guttalu, 1980. This method has been extended to the qualitative analysis of nonlinear parameterised models of the type commonly encountered in bifurcation studies [Haynes and Billings, 1991b]. The approach proves attractive for a number of reasons. Firstly, it provides a method of enumerating both the stationary and periodic solution structure of the system over a given parameter range. Secondly, it has been shown to detect all of the typically found bifurcation phenomena [Haynes and Billings, 1991a]. Thirdly, and more importantly, information of a global nature is provided on the extent of the systems stability domains. As a result both local and global behaviour can be studied.

3.1. Cell Mappings

The process of analysing a system using cell map analysis comprises of a number of steps. The first is the suspension of the nonlinear system, say (2.5), in a *cell state space*, $\mathbf{Z}^n$. This is an n-dimensional space whose elements are n-tuples of integers. Each element is called a *cell vector*, or simply a *cell*, and is denoted by $\mathbf{z}$.

There are many ways to obtain a cell structure over a given euclidean state space. The simplest way is to construct a cell structure consisting of rectangular parallelepipeds of uniform size (squares, cubes etc). Let x_i , $i = 1,..,n$ be the state variables and let each coordinate axis of the state variable be divided into a number N_i, of intervals of uniform size h_i. The interval z_i along the x_i axis is defined such that it covers all the x_i of interest and

$$(z_i - \frac{1}{2})h_i \le x_i \le (z_i + \frac{1}{2})h_i \qquad z_i = 1,2,..,N_i \tag{3.1}$$

The n-tuple z_i , $i = 1,..,n$ is then called the cell vector, denoted by $\mathbf{z}$. A point $\mathbf{x}$ belongs to a cell $\mathbf{z}$ iff

x_i and z_i satisfy (3.1) $\forall i \in [1,n]$. Each cell $\mathbf{z}$ is considered as an entity and the entire collection of cells as the *cell state space*. Consider now the mapping between two cells $z(j)$ and $z(j+1)$, where $j = 1,2,..$ is used to denote a sequence in the same manner as the iterates of a mapping. The *cell map*, $\mathbf{C}(\mathbf{z})$, is a mapping of a set of integers $\{N+\}$ such that

$$\mathbf{z}(j+1) = \mathbf{C}(\mathbf{z}(j)) \qquad \mathbf{z}(j) \in \mathbf{Z}^n \subset \mathbf{S} \tag{3.2}$$

The *cell function*, $\mathbf{F}(\mathbf{z},\mathbf{C})$, is then defined as

$$\mathbf{F}(\mathbf{z},\mathbf{C}) = \mathbf{C}(\mathbf{z}) - \mathbf{z} \tag{3.3}$$

If $\mathbf{C}^K(\mathbf{z})$ denotes $\mathbf{C}(\mathbf{z})$ applied K times, then the K-th step ahead *cell function is*

$$\mathbf{F}^K(\mathbf{z},\mathbf{C}^K) = \mathbf{C}^K(\mathbf{z}) - \mathbf{z} \tag{3.4}$$

A *singular cell* $\mathbf{z}^*$ as a cell satisfying the relationship

$$\mathbf{F}(\mathbf{z}^*,\mathbf{C}) = 0 \qquad \text{or} \qquad \mathbf{z}^* = \mathbf{C}(\mathbf{z}^*) \tag{3.5}$$

A *periodic cell cycle,* given that $\mathbf{C}^0(\mathbf{z})$ denotes the identity mapping, is a sequence of K distinct cells $\mathbf{z}^*(j)$, $j = 1,..,K$, K being the minimum value which satisfies

$$\mathbf{z}^*(m+1) = \mathbf{C}^m(\mathbf{z}^*(1)) \qquad m = 1,..,K-1 \qquad \mathbf{z}^*(1) = \mathbf{C}^K(\mathbf{z}^*(1)) \tag{3.6}$$

Each element of the periodic cycle is a *periodic cell*. The complete cell cycle is labelled as **P-K**. Additionally those cells eventually mapped onto the **P-K** cycle by (3.2) are defined as within the *domain of attraction*, or **DOA**, of the cell cycle and labelled as the **DOA-K** cells.

The size of the cell state space is determined by the system itself. For most practical systems there are ranges of values of the state variable beyond which we are no longer interested. This means that there is only a finite region of the state space which is of concern. Similarly for a dynamical system governed by a cell mapping there is only a finite region of cell space of interest, and correspondingly a finite number, N_c, of cells, named the *regular cells*. The *sink cell* is used to encompasses all possible cells outside the region of interest. If the mapped image of a regular cell lies outside the region of interest it is then said to be mapped into the sink cell. The regular cells are labelled by positive integers $\{1,2,..,N_c\}$. The sink cell is labelled as $\{0\}$, the zero cell. This makes the total number of cells N_c+1, such that $\mathbf{S} = \{N_c+\}$. The set, $\mathbf{S}$, is closed under the mapping described by

$$\begin{aligned} \mathbf{z}(j+1) &= \mathbf{C}(\mathbf{z}(j)) \qquad & \mathbf{z}(j),\ \mathbf{z}(j+1) \in \mathbf{Z}^n \subset \mathbf{S} \\ \mathbf{C}(0) &= 0 \qquad & \mathbf{S} = \{N_c+\} \end{aligned} \tag{3.7}$$

The sink cell, C(0), is a **P-1** cell. The set of regular cells within the influence domain of the sink cell, these being eventually mapped to C(0), are in the domain of attraction of the sink cell, and labelled the **DOA-Sink** cells.

3.2. Cell Mapping Discretisation

The cell map, $\mathbf{C}(\mathbf{z})$, system may be considered as a discrete system similar to the point mapping

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k) \qquad \mathbf{x} \in \mathbf{R}^{n_x} \tag{3.8}$$

In order to make use of the qualitative ideas provided by dynamical systems theory it is necessary to extend the algorithm of Hsu to the analysis of a more general parameterised nonlinear system

$$\mathbf{x}_{k+1} = \mathbf{g}(\mathbf{x}_k, \mu) \qquad \mathbf{x} \in \mathbf{R}^{n_x} \quad \mu \in \mathbf{R}^{n_\mu} \tag{3.9}$$

Applying the center point method of discretisation requires the division of $\mathbf{R}^{n_x}$ into a collection of cells according to (3.1) and the calculation of each cells center point $\mathbf{x}^{(d)}(j)$ such that

$$x_i^{(d)}(j) = x_i^{(l)} + h_i z_i(j) - \frac{h_i}{2} \qquad z_i(j) = 1,..,N_i \quad j = 1,..,N_i \quad i = 1,..,n_x \tag{3.10}$$

where h_i is the cell size and $x_i^{(l)}$ the lower bound defining the region of interest such that $\mathbf{x}^{(l)} \le \mathbf{x} \le \mathbf{x}^{(h)}$, where $\mathbf{x}^{(l)} = (x_1^{(l)},..,x_{n_x}^{(l)})$ and $\mathbf{x}^{(h)} = (x_1^{(h)},..,x_{n_x}^{(h)})$. Similarly $\mathbf{R}^{n_\mu}$ is discretised using

$$\mu_i^{(d)}(j) = \mu_i^{(l)} + g_i z_i(j) - \frac{g_i}{2} \qquad z_i(j) = 1,..,N_i \quad j = 1,..,N_i \quad i = n_x + 1,..,n_x + n_\mu \tag{3.11}$$

where g_i is the cell size over the region defined by $\mu^{(l)} \le \mu \le \mu^{(h)}$ where $\mu^{(l)} = (\mu_1^{(l)},...,\mu_{n_\mu}^{(l)})$ and $\mu^{(h)} = (\mu_1^{(h)},..,\mu_{n_\mu}^{(h)})$. The point mapping or image of the center point $\mathbf{x}^{(d)}(j)$ is then calculated using (3.9) such that

$$\mathbf{x}_{k+1}^{(d)} = \mathbf{g}(\mathbf{x}_k^{(d)}, \mu^{(d)}) \tag{3.12}$$

and the cell map $\mathbf{C}(\mathbf{z})$ is constructed by determining the image of each cell within **S** using

$$C_i(z_i(j)) = z_i(j+1) = \mathrm{INT}\left[\frac{x_i^{(d)}(k+1) - x_i^{(l)}}{h_i} + 1\right] \qquad i = 1,..,n_x \quad j = 1,..,N_i$$

$$C_i(z_i(j)) = z_i(j) \qquad i = n_x + 1,..,n_x + n_\mu \quad j = 1,..,N_i \tag{3.13}$$

Note we are in effect constructing n_μ separate cell mappings of dimension n_x, each representing a *slice* within the parameter space $\mu = (\mu_1,..,\mu_{n_\mu})^T$.

3.3. Cell Map Analysis

Having constructed **C(z)** the classification of all cells within **S** is carried out using a modified version of the *Unravelling Algorithm* [Hsu and Guttalu, 1980]. The algorithm involves calling up each cell in turn and processing it in order to determine its global characteristics. Each cell is then classified accordingly.

A cell may be a singular cell, **P-1**, or a periodic cell cycle, **P-K**, satisfying (3.5) or (3.6). The set of all such cells make up the invariant orbits within $\mathbf{Z}^n$. When $K = 1$ a fixed point has been located, when $K > 1$ a periodic solution or limit cycle has been detected.

Alternatively a cell may simply be a regular cell in the **DOA** of a **P-K** cell. Each such cell is then said to be in the same *group* and have the same *periodicity* as that cell cycle and is labelled accordingly.

Finally a cell may be mapped by (3.2) outside the region of interest into the Sink cell. Such a cell is then said to be in the domain of attraction of the sink cell, the **DOA-Sink**, and is labelled accordingly. For more detail on the Unravelling algorithm and is variants see Hsu, 1987 or Haynes and Billings, 1991a.

The advantage of the parameterised cell map approach is that it combines the qualitative aspects of *Dynamical Systems Theory* with the global aspect afforded by the *Unravelling Algorithm*. In implementing the above algorithm many variations are possible. The version developed in this work focuses on the problems of enumerating the global characteristics of the parameterised nonlinear systems. One aim throughout this work has been to maintain an interactive aspect to the algorithm. Thus enable the analyst to probe the dynamics of a problem in an iterative manner by varying both the extent and form of the cell state space.

3.4. Constructing the Parameterised Cell Map for a Nonlinear Sampled Data System

In order to apply the cell map algorithm to a nonlinear system it is first necessary to recast the model in the form of (3.9). Consider first a linear MIMO system with state space equations

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}(t)$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \qquad (3.14)$$

Given that the plant under digital control with sampling period T and an implied ZOH element, Fig. (1), within the $(k+1)^{th}$ sampling interval (3.14) becomes

$$\dot{\mathbf{x}}(t) = A\,\mathbf{x}(t) + B\,\mathbf{u}_k \qquad \mathbf{u}(t_k) = \mathbf{u}_k$$

$$\mathbf{y}(t) = C\,\mathbf{x}(t) \qquad t \in [t_k, t_{k+1}[\tag{3.15}$$

and under the usual assumption that the plant is time invariant

$$\mathbf{x}_{k+1} = e^{AT}\mathbf{x}_k + \int_0^T e^{A(t-\tau)} B\,\mathbf{u}_k\,d\tau \tag{3.16}$$

Thus an exact recursive relationship has been obtained in the form

$$\mathbf{x}_{k+1} = \Phi\,\mathbf{x}_k + \Delta\,\mathbf{u}_k$$

$$\mathbf{y}_k = C\,\mathbf{x}_k \tag{3.17}$$

where Φ and Δ are constants dependent upon T. Assume now that feedback is applied such that

$$\mathbf{u}_k = \mathbf{r}_k - \mathbf{g}(\mathbf{y}_k) \tag{3.18}$$

where g(.) is either a linear or nonlinear function of the output. If g(.) is linear the system (3.17-18) can be analysed using well known techniques. If g(.) is nonlinear (3.17-18) can in general be rewritten in the form

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k, \mu) \qquad \mathbf{x} \in \mathbf{R}^{n_x}\ ;\ \mu \in \mathbf{R}^{n_\mu} \tag{3.19}$$

where the parameter μ is taken here to be the set point $\mu = \mathbf{r}_k$, $n_x = n$ and $n_\mu = l$.

If the plant is instead a general nonlinear system the state space model (3.16) is no longer valid and we have

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad \mathbf{f}: \mathbf{R}^n \times \mathbf{R}^l \rightarrow \mathbf{R}^n \tag{3.20}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t)) \qquad \mathbf{h}: \mathbf{R}^n \rightarrow \mathbf{R}^m$$

And an exact discrete model of the plant is impossible to formulate. However, all that is necessary for the system to be analysed using a parameterised cell state space is that the feedforward and feedback elements can be combined into the form

$$\mathbf{x}_{k+1} = \mathbf{F}(\mathbf{x}_k, \mu) \tag{3.21}$$

or

$$\dot{\mathbf{x}}(t) = \mathbf{F}(\mathbf{x}(t), \mu) \tag{3.22}$$

In the first case the plant is assumed to be already in the form of a nonlinear discrete map. That is the feedforward element in Fig. (1) is linear. The construction of the cell state space system, C(z), the proceeds using (3.9-13). In the second case, when the plant itself exhibits a nonlinear continuous characteristic it is necessary to use an approximate discretisation to create the mapping (3.9) from (3.22).

4. Analysis of a Nonlinear Sampled Data System

It is well known the importance of the correct choice of sampling interval, T, in the design of a sampled data control system. Sampling rate may also be of prime concern when analysing a nonlinear sampled data system [Zheng et al, 1990]. This is easily demonstrated by considering a simple example based on the normal form for the pitchfork bifurcation [Guckenheimer and Holmes, 1983]

$$\dot{x}(t) = \alpha x - x^3 \tag{4.1}$$

application of the Euler forward difference discretisation, with interval h, results in

$$x_{k+1} = x_k + h\,(\alpha x_k - x_k^3) \tag{4.2}$$

For which the location of **P-1** fixed points, $\overline{x}$, are given by the solution to the equation

$$\overline{x} - f(\overline{x}) = 0 \qquad h\,(\alpha \overline{x}_k - \overline{x}_k^3) = 0 \quad \therefore \quad \overline{x} = 0,\ \pm\sqrt{\alpha} \tag{4.3}$$

The linearised eigenvalue, $\lambda(h,\alpha)$, dependent upon both the interval h and the parameter μ, is

$$\lambda(h,\alpha) = D_x f(x,\mu)\,\Big|_{x=\overline{x}} = 1 - 3h\overline{x}^2 + h\,\alpha \tag{4.4}$$

And the **P-K** cycles are given by the solution to

$$\overline{x} - f^K(\overline{x}) = 0 \qquad\qquad K = 2,3,.. \tag{4.5}$$

Obviously even for a simple form of $f(\overline{x})$, and low values of K, this may prove troublesome to solve [Gumowski and Mira, 1980]. About the origin $(x,\alpha) = (0,0)$ the expected *Pitchfork* bifurcation occurs, $\lambda(0,0) = 1$. However, away from the origin the Euler method introduces a complex sequence of *ghost* bifurcations, some of which may be located by hand. For example if $h = 0.2$ then at $(x,\alpha) = (0,-10)$ a *Flip* bifurcation is found when $\lambda(0,0) = -1$. If the interval h changes such that $h = 0.4$ then this bifurcation moves to $(x,\alpha) = (0,-5)$. Outside this range a sequence of period doubling flip bifurcations occur.

4.1. Bifurcation and Chaos in Sampled Data System

Consider now the effect of sampling interval on the qualitative behaviour of the nonlinear sampled data system depicted in Fig. (1). Assume that the plant, is a linear system, as in (3.15), and the feedback g(.) nonlinear such that (3.18) applies. Setting $n=2$, $l=1$, $m=1$ and A, B, C and $g(x)$ such that

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad C = \begin{bmatrix} c_1 & c_1 \end{bmatrix} \quad g(x) = x^3 \tag{4.6}$$

and taking $\lambda_1 = 1$, $\lambda_2 = -1$, $b_1 = b_2 = 1$ and $c_1 = c_2 = 1$, completes the system definition.

The adoption of this particular block structure may seem restrictive, bearing in mind the flexibility of the Cell Map approach. However this system was originally considered by Ushio and Hirai, 1983a, 1983b in order to demonstrate chaos using an analytical approach. There the linear form of the feed forward element played an important role in the analysis. The same structure is adopted here purely as a basis for comparison. It should be born in mind that there is no such linear restriction on the form of the plant.

Suspending the system in the Cell Map framework, (3.17-19), the input set point r is selected as the bifurcation parameter μ. Define a cell state space such that $x_1 \in [-2,2]$, $x_2 \in [-2,2]$ and $\mu \in [-2,2]$ over a mesh of 150×150×30 cells and applying (3.9-13) to construct the cell map, C(z), over the region of interest. Subsequent application of the Unravelling Algorithm for the cases $T=0.1$, $T=0.2$ and $T=0.3$ produced the Cell diagram, Fig. (2).

The diagrams produced exhibit a number of interesting features. For $T=0.1$ the system exhibits an asymptotically stable fixed point over the range of input set point considered, $\mu = r \in [-2,2]$. This in indicated by the curve of **P-1** periodic cells on the graph. For $T=0.2$ a stable **P-2** cycle is produced at $r = \pm 0.4$ which is in agreement with the behaviour predicted by Ushio and Hirai. For values of r outside this range a sequence of periodic **P-K** cycles are detected which grow in amplitude as r increases. Note that the *course* cell size used in this analysis results in the plots in Fig. (2) appearing to have gaps that distort the cone like characteristic typically found in association with a growing periodic cycle. This situation can easily be improved by adjusting the number of cells used, see later. We choose to leave the results in this first-cut form, preferring to emphasize the ease of analysis of the method as a quick route to obtaining a broad global picture.

Notice the Cell diagrams show the origin, at $r=0$, as being unstable for this system. This is due to two unstable manifolds emanating from the *fold* bifurcations located asymmetrically either side of the origin.

For $T = 0.3$ this trend increases with **P-K** cycles evident over the full range of set point values considered. Indeed this periodic behaviour is so pronounced that it deserves further attention.

Fig. (3) shows the response for a step input $r = 0.4$ applied to the system for the cases $T = 0.1$, $T = 0.2$ and $T = 0.3$. The resulting outputs confirm the **P-1**, **P-2** and **P-K** behaviour predicted in Fig. (2). A similar pattern of cyclic behaviour is displayed if the sampling interval is fixed and the set point r is varied. In the lower plot in Fig. (3) the behaviour for $T = 0.3$ and at $r = 4$, appears to be aperiodic or chaotic. Fig. (4) shows the phase plane plot of this response plotted over 2000 points. This confirms the behaviour to be chaotic, the outline of the characteristic invariant strange attractor being clearly visible.

Due to the finite number of Cells used, the Unravelling algorithm detects chaotic phenomena simply as cyclic behaviour of lengthy period. The simple cell map algorithm can not in itself detect chaos [Hsu, 1982]. However, it is usually a straight forward matter to decide whether periodic, **P-K**, behaviour with K large, is actually periodic or indeed aperiodic/chaotic. Typically either a phase plane, frequency domain or Poincare Map based analysis can be used to ascertain this.

5. Analysis of a Nonlinear Identified System

Now consider the identification of the nonlinear system under study. Two distinct approaches to this task may be adopted. The first, which we call open loop identification, involves fitting a model to the feedforward elements in Fig. (1), that is the plant and ZOH element. With the aim of utilising such a model in for example the design of a feedback compensator.

The second approach, and the one we adopt here, is to consider the complete system, both feedforward and feedback elements, as one closed loop system to which a nonlinear model is to be fitted. It is then possible to consider how the qualitative behaviour detected above will be reflected in the identified model. In particular how do the stationary periodic and even aperiodic characteristics carry over to the parametric input/output model fitted to data collected.

5.1. Nonlinear Model Structure

As part of the identification process a number of decisions have to be made. In particular what model structure is to be adopted. The most popular model structure used in linear system identification to date is the *Auto Regressive* model with *eXogenous* inputs, or **ARX** model

$$y(t) = -\sum_{i=1}^{n_y} a_i\, y(t-i) + \sum_{j=1}^{n_u} b_j\, u(t-j-d) \tag{5.1}$$

Here $y(t)$ and $u(t)$ are discrete input output values at the t^{th} sample time, n_y is the order of the noise free output, n_u is the order of the input and d is an input delay.

It the system is nonlinear, as is the case here, one must first decide on the class of model to be fitted. In this study we adopt the NARMAX model introduced by Billings and Leontaritis, 1981. This takes the form of a nonlinear input output model

$$y(t) = F^l(y(t-1),...,y(t-n_y)\,,\, u(t-d),...,u(t-d-n_u)\,) \tag{5.2}$$

The *input output* description (5.2) expands the current output in terms of past inputs and outputs providing a model structure which will represent a broad class of nonlinear systems and is a valid realisation of the general nonlinear state space realisation (2.5) [Leontaritis and Billings, 1985].

As part of the process of system identification is is necessary to construct estimates of the parameters, or coefficients, in (5.2). Expanding $F^l(.)$ as a polynomial of order l and defining $V_1 = y(t-1)$,..., $V_{n_y} = y(t-n_y)$, $V_{n_y+1} = u(t-d)$,..., $V_S = u(t-d-n_u)$, where $S = n_y + n_u$, enables (5.2) to be expressed in the form

$$y(t) = F^l(\,V_1\,, V_2\,,..., V_S\,) \tag{5.3}$$

which when expanded yields

$$y(t) = \sum_{i_1=1}^{S} \theta_{i_1} V_{i_1} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{i_1} \theta_{i_1 i_2} V_{i_1} V_{i_2} + \cdots + \sum_{i_1=1}^{S} \cdots \sum_{i_l=1}^{i_{l-1}} \theta_{i_1 \cdots i_l} V_{i_1} \cdots V_{i_l} \tag{5.4}$$

where θ_i represents the coefficients of the linear, quadratic, cubic ... terms up to l. Assuming that the system output $y(t)$ is corrupted by zero mean additive noise $e(t)$ such that $y(t)$ becomes $y(t) + e(t)$ then the input output model can be rewritten as [Leontaritis and Billings, 1985]

$$y(t) = F^l(y(t-1),..,y(t-n_y)\,,\, u(t-d),..,u(t-d-n_u)\,,\, e(t-1),..,e(t-n_e)\,) + e(t) \tag{5.5}$$

where n_e is the order of the noise model. Notice that although the model is linear in the parameters the inclusion of lagged process outputs introduces cross product terms between the noise and the process input and output signals. This model is referred to as a Nonlinear

AutoRegressive Moving Average model with eXogenous inputs, or NARMAX model. A NARMAX model with first order dynamics expanded as a second order polynomial nonlinearity would for example be represented as

$$y(t) = \theta_1 y(t-1) + \theta_2 u(t-1) + \theta_{11} y^2(t-1) + \theta_{12} y(t-1)u(t-1) + \theta_{22} u^2(t-1)$$
$$+ e(t) - \theta_3 e(t-1) - \theta_{13} y(t-1)e(t-1) - \theta_{23} e(t-1)u(t-1) + \theta_{33} e^2(t-1)$$

Much has been achieved in developing parameter estimation techniques for the NARMAX and other models. In this paper we are interested only in using the general model fitting procedure. Details of the algorithms used may be found in Billings and Voon, 1984, 1986, and Korenberg et al, 1988.

5.2. Suspension of the NARMAX Model

To analyse a NARMAX model within the framework outlined above it is first necessary to suspend the model, appropriately parameterised in the cell state space. Rewriting (5.5) assuming the input $u(t)$ to be a quasi static parameter and neglecting the noise terms $e(t),..,e(t-n_e)$ gives

$$y(t) = F^l \left[y(t-1),..,y(t-n_y) \, , \, \mu \right] \tag{5.6}$$

where $u(t)$ is assumed to be a quasi static constant such that

$$\mu = u(t) = u(t-d) = u(t-d-1) = \cdots = u(t-d-n_u) \tag{5.7}$$

The NARMAX model can now be written in the form of a parameterised discrete system

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k \, , \mu \, , \alpha) \qquad \mathbf{x} \in \mathbf{R}^{n_x} \quad \mu \in \mathbf{R}^{n_\mu} \quad \alpha \in \mathbf{R}^{n_\alpha} \tag{5.8}$$

where $\mathbf{x}_{k+1} = y(t)$, $\mathbf{x}_k = (y(t-1),..,y(t-n_y))^T$, $\mu = u$, and $\alpha = (\theta_1,..,\theta_{n_\theta})$, such that $n_x = n_y$, $n_\mu = 1$ and $n_\alpha = n_\theta$. The convention adopted here when considering multiparameter problems is to label μ as the primary parameter and α as an auxiliary parameter vector. The auxiliary parameter vector α is only considered when taking into account perturbations to the basic problem.

5.3. Identification of the nonlinear sampled data system

Separate identification of the system is required at each of the sampling intervals $T = 0.1$, $T = 0.2$ and $T = 0.3$. Input Output data sets were generated for the system using a gaussian noise input signal with zero mean and rms = 0.3. To excite the system over a range of operating points a square wave was superimposed on the input, see Fig. (5a-c).

Structure detection and identification using the generated input output data at each of the sampling intervals, with $n_u = n_y = 2$, $d = 1$, $l = 3$ produced the following NARMAX models.

For $T = 0.1$

$$y(t) = 0.880y_{t-1} - 0.156u_{t-2} + 0.115y_{t-2} - 0.027y_{t-1}^2u_{t-1} + 0.114u_{t-1} + 0.047u_{t-2}y_{t-1}^2$$
$$+ \left\{ -0.377e_{t-1} + 0.248e_{t-1}u_{t-2}y_{t-1} \right\} \quad (5.9)$$

For $T = 0.2$

$$y(t) = 1.044y_{t-2} - 2.629y_{t-2}^3 - 0.412u_{t-2} + 0.865y_{t-2}y_{t-1}u_{t-1} + 6.971y_{t-2}^2 - 1.007y_{t-2}^2u_{t-1}$$
$$+ 0.303y_{t-2}^2u_{t-2} + 0.258u_{t-1} - 0.034y_{t-1} - 0.074y_{t-1}^2u_{t-1} - 6.357y_{t-2}y_{t-1}^2 + 2.000y_{t-1}^3$$
$$+ \left\{ 0.482e_{t-2}y_{t-2}u_{t-1} + 0.371e_{t-1}y_{t-2}u_{t-2} - 0.578e_{t-2}^2y_{t-2} - 0.394e_{t-1}^2y_{t-1} \right\} \quad (5.10)$$

For $T = 0.3$

$$y(t) = 1.053y_{t-2} - 2.637y_{t-2}^3 - 0.420u_{t-2} + 0.872y_{t-2}y_{t-1}u_{t-1} + 6.966y_{t-2}^2y_{t-1} - 1.017y_{t-2}^2u_{t-1}$$
$$+ 0.308y_{t-2}^2u_{t-2} + 0.265u_{t-1} - 0.044y_{t-1} - 0.079y_{t-1}^2u_{t-1} - 6.331y_{t-2}y_{t-1}^2 + 1.988y_{t-1}^3$$
$$+ \left\{ 0.471e_{t-2}y_{t-2}u_{t-1} + 0.374e_{t-1}y_{t-2}u_{t-2} - 0.556e_{t-2}^2y_{t-2} + 0.379e_{t-1}^2y_{t-1} \right\} \quad (5.11)$$

Where $y_t = y(t)$, $u_t = u(t)$ and $e_t = e(t)$. For the case $T = 0.1$ the identified NARMAX model (5.9) produces good predictions and passes all the *model validity tests*, MVT tests, Fig. (6a). These are used to check that the estimated model is unbiased. That is the residues contain no predictable linear or nonlinear dynamics [Billings and Voon, 1983]. For $T = 0.2$ the NARMAX model (5.10) produces reasonable predictions and fails one MVT test, Fig. (6b), that is the correlation test lies just outside the dotted 95% confidence interval. This failure is probably due to a poor noise model having been fitted over the prediction data set. For $T = 0.3$ the NARMAX model (5.11) produces poor predictions and fails the MVT tests, Fig. (6c), indicating that some small but still significant nonlinear dynamics are present in the modelling residuals. This suggests that the initial model structure specification is not sufficient when $T = 0.3$, the input is not persistently exciting for this case, or more likely, that the output prediction over time for this essentially chaotic system is not going to be accurate.

5.4. Cell Map Analysis

Defining a cell state space, **C**, separately for each of the above models such that $x_1, x_2 \in [-2,2]$ and $\mu \in [-2,2]$ over 200×200×30 cells and applying the Unravelling algorithm produced the Cell diagrams, Fig. (7a-c). Notice how the increased number of cells used has given rise to a smoother cell map characteristic.

For the case $T=0.1$ the NARMAX model exhibits only a fixed point or **P-1** cell characteristic, Fig. (7a), no other periodic behaviour is detected over the range of input $\mu = r \in [-2,2]$. This agrees well with the system characteristic given in Fig. (2), apart from some distortion in the Cell diagram around the origin. This is due to the strong repelling influence of the unstable manifold located at the origin, it being very difficult to design a persistently exciting input that will exercise a system in the vicinity of an unstable limit point or cycle. It was partly for this reason that a square wave was superimposed upon the input in order to excite the system fully across the known characteristic. Without which, a decidedly *one sided* model would result valid over either just the +ve or -ve input ranges.

Notice that in the first and third quadrants a number of extraneous cell singularities are detected. These cell cycles delineate part of the boundary between the stable attracting cell space and the unstable sink cell region. Such cell singularities commonly occur on the boundary marking the edge of a **DOA** for two main reasons. Firstly, when the stable/unstable manifold of a saddle point form part of the boundary of a **DOA** one or more limit points will occur actually on the boundary. Secondly, at points on the edge of a **DOA**, where the vector field is weak, the discrete nature of the cell map algorithm may result in spurious cell cycles being detected as part of the analysis.

For the case $T=0.2$ the NARMAX model (5.10) exhibits a prominent **P-2** cycle along with a small number of higher order **P-K** cycles, Fig. (7b).

For the case $T=0.3$ the NARMAX model posses a larger number of **P-2** and higher **P-K** cycles. Notice that despite its poor predictive behaviour the identified model still manages to capture a significant part of the qualitative characteristic of the NLSDS, as indicated by the Cell diagram, Fig. (7c). Work is in hand aimed at a more complete characterisation and identification of plants operating in a chaotic mode.

6. Discussion

It has been shown how a qualitative approach based around using the parameterised cell map can provide useful information on periodic, and to some extent aperiodic, characteristics within a system. Furthermore this study has served to illustrate just how important parameters such as sampling interval, set point and input excitation can be in determining the qualitative characteristic in a nonlinear sampled data system. The analysis of both periodic and to some extent aperiodic behaviour has been attempted in a setting that allowed both a nonlinear sampled data system and an equivalent nonlinear input output representation, the NARMAX model, to be analysed under a similar frame of reference. The qualitative approach adopted allows both a broad based *coarse probing* of the nonlinear dynamics as well as more detailed or *focused* analysis around particular points of interest.

Obviously a number of problems need still to be resolved. In particular, from the dynamical systems viewpoint, at what point exactly do the periodic cells detected actually reflect aperiodic or chaotic behaviour in the underlying system, see for instance Hsu, 1982. And from the control viewpoint can the identification methods be adapted to specifically distinguish chaos from measurement noise.

Acknowledgment

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Figures

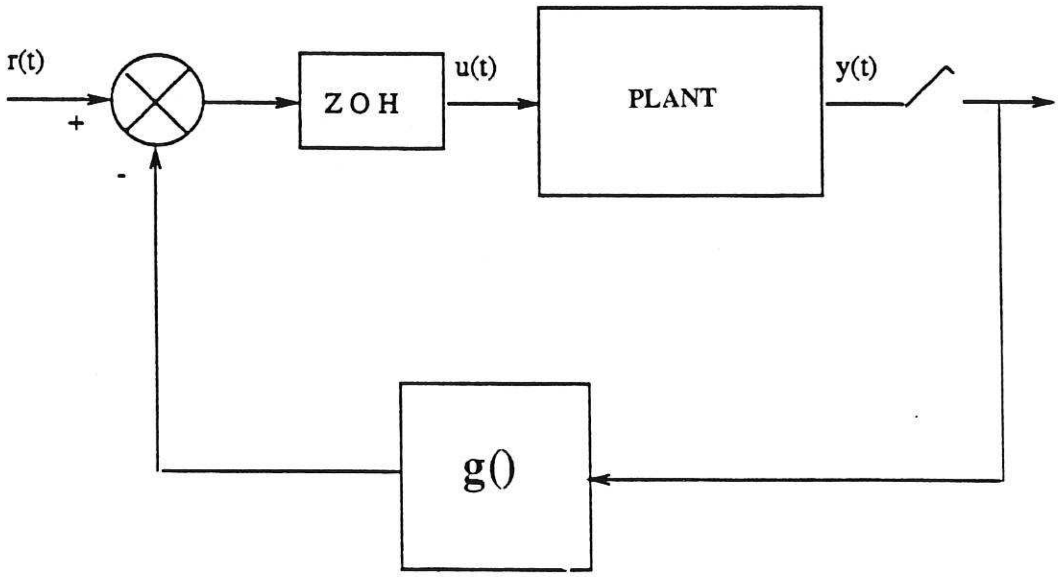


Fig. 1 Nonlinear Sampled Data System

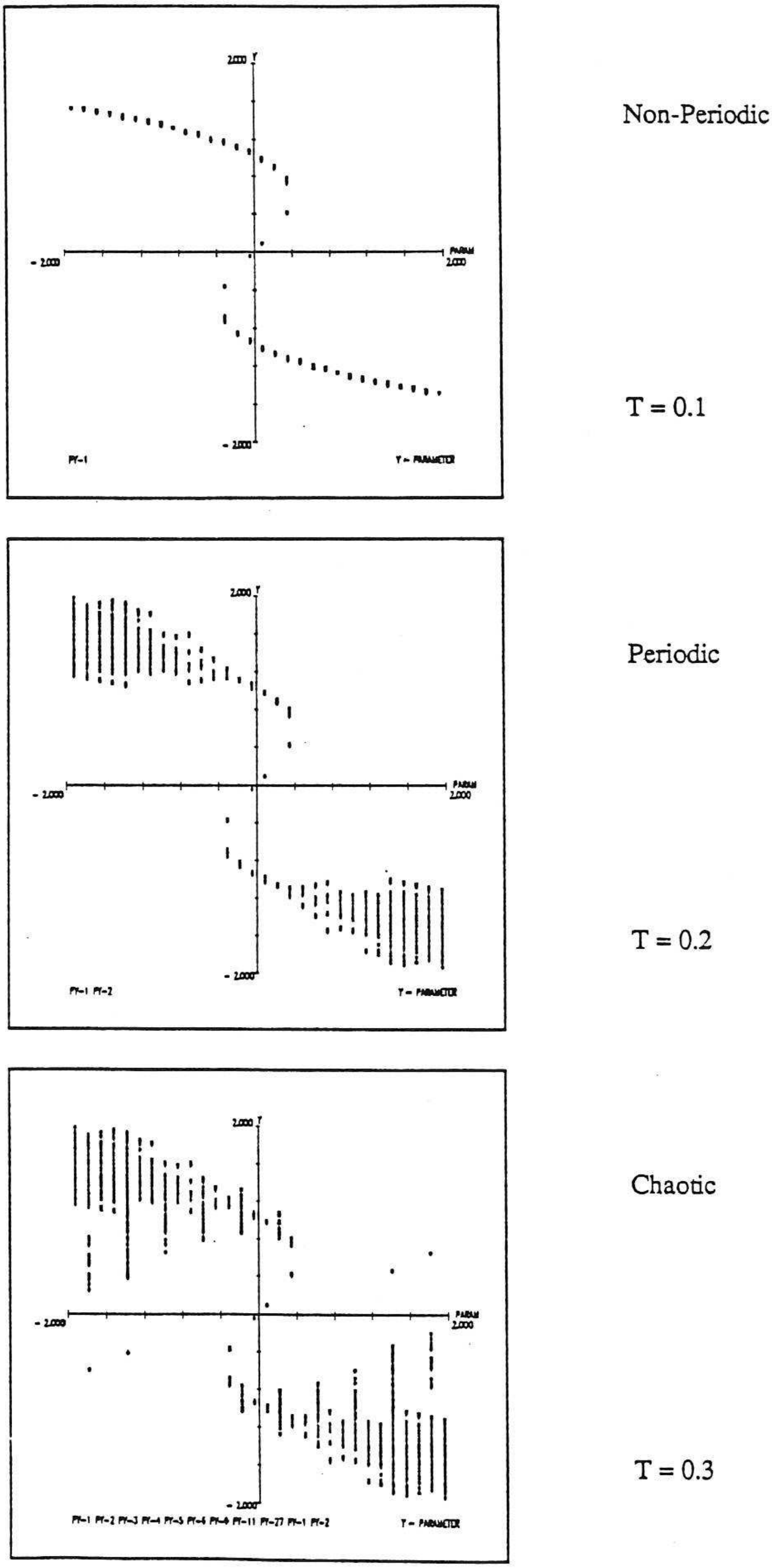


Fig. 2 **CELL** Diagram System - $\mu = r$ for $T = 0.1$, $T = 0.2$ and $T = 0.3$

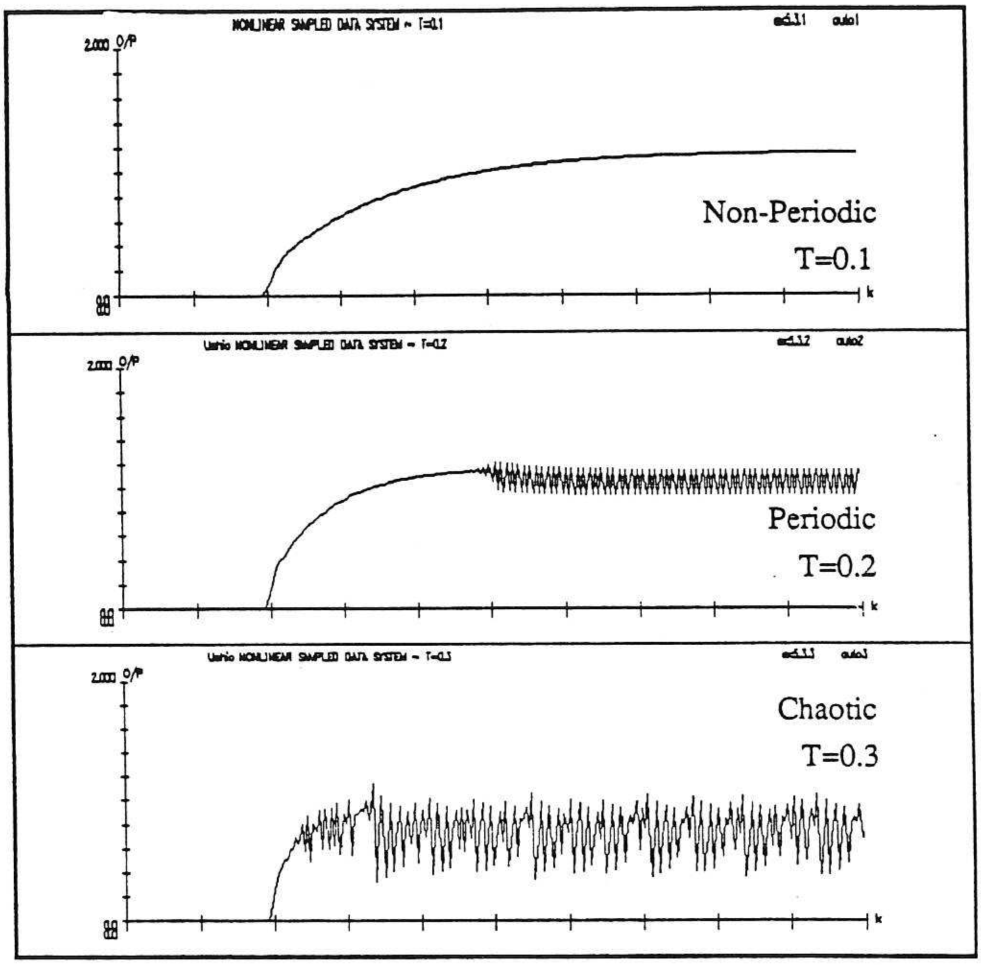
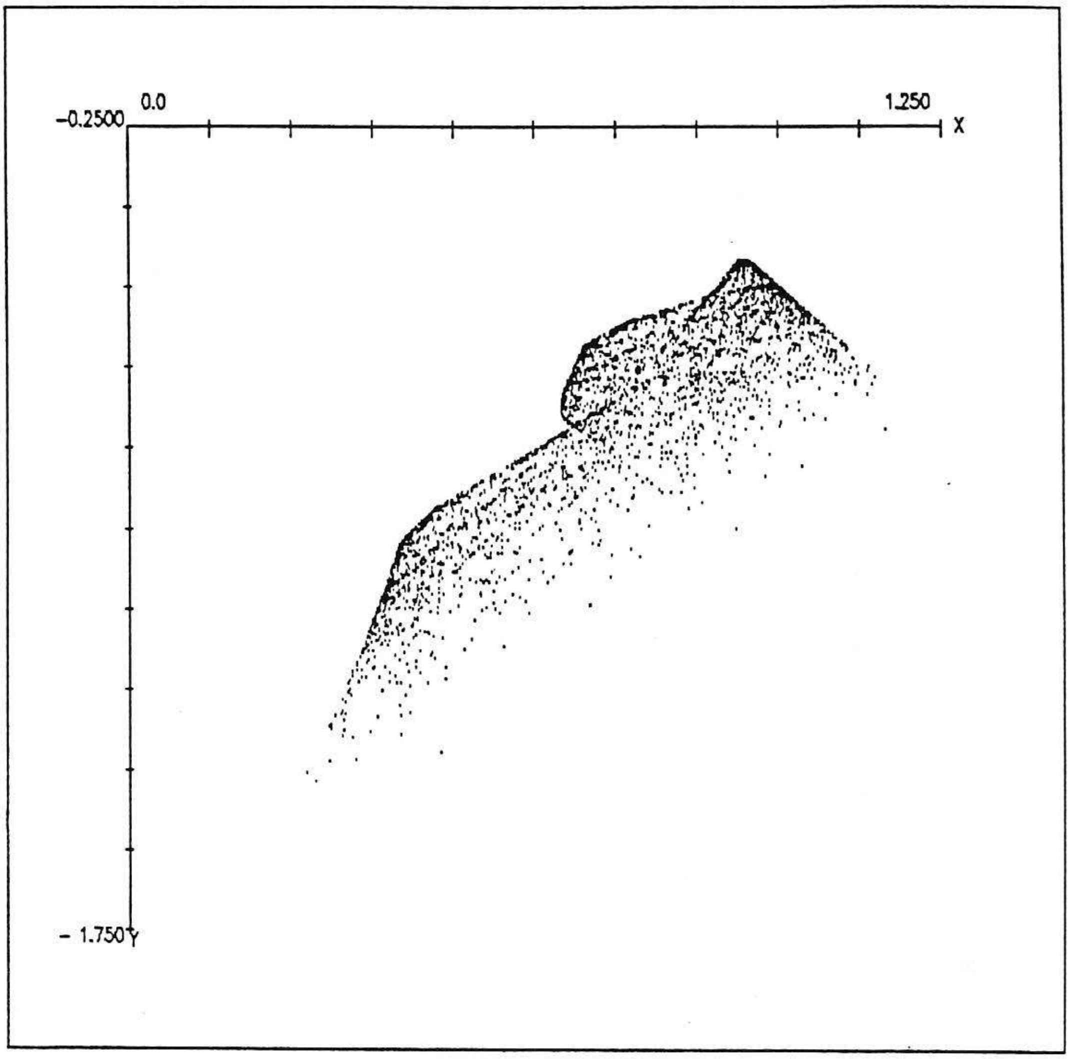


Fig. 3 Step Responses - $r = 0.4$ for $T = 0.1$, $T = 0.2$ and $T = 0.3$,



Chaos in Nonlinear Sampled Data System, Strange Attractor

Fig. 4 Phase Plane Plot of Strange Attractor - $r = 0.4$ for $T = 0.3$

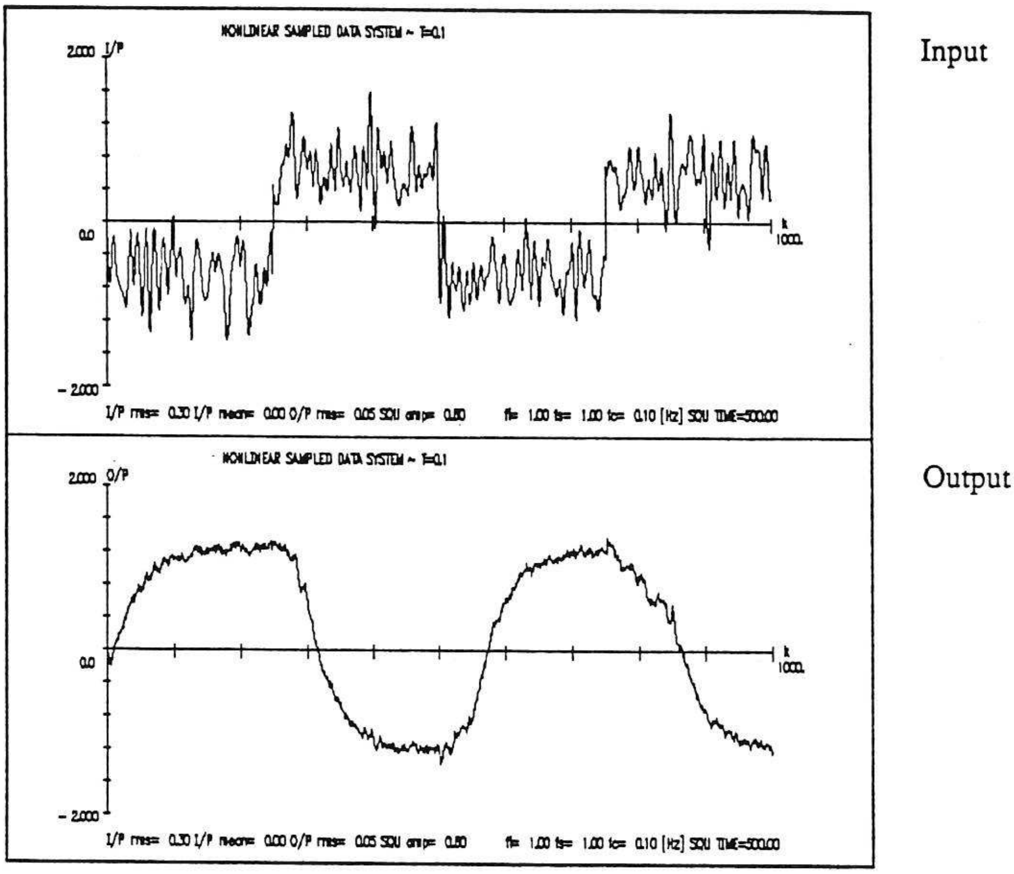


Fig. 5a **I/O data** - $T = 0.1$

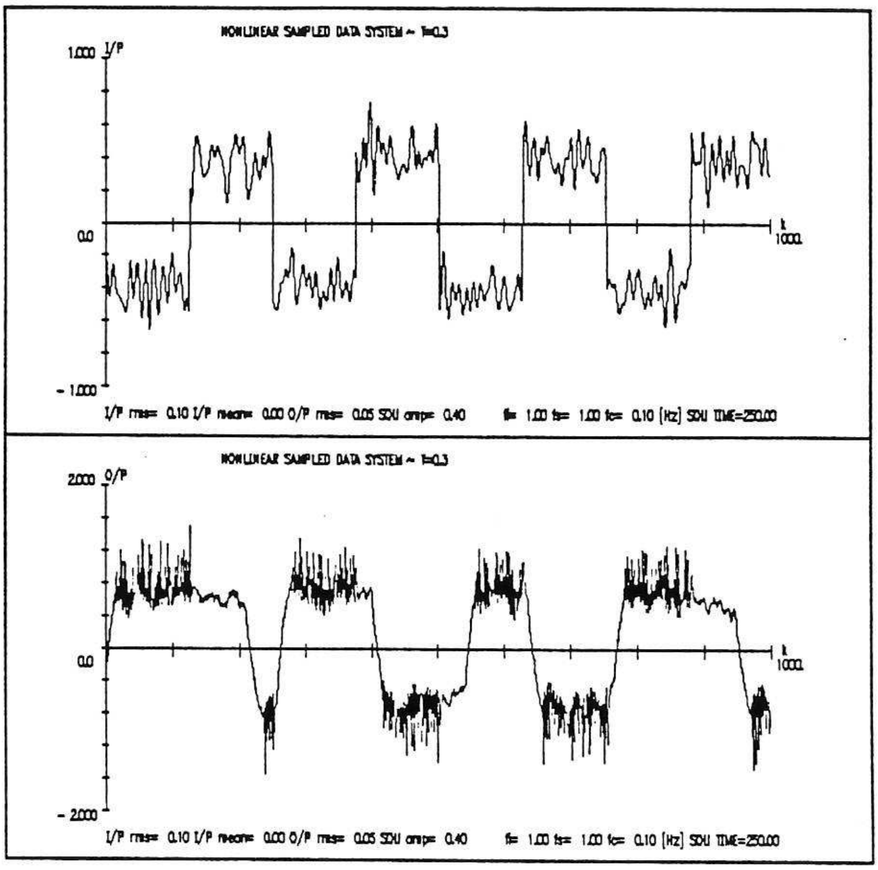
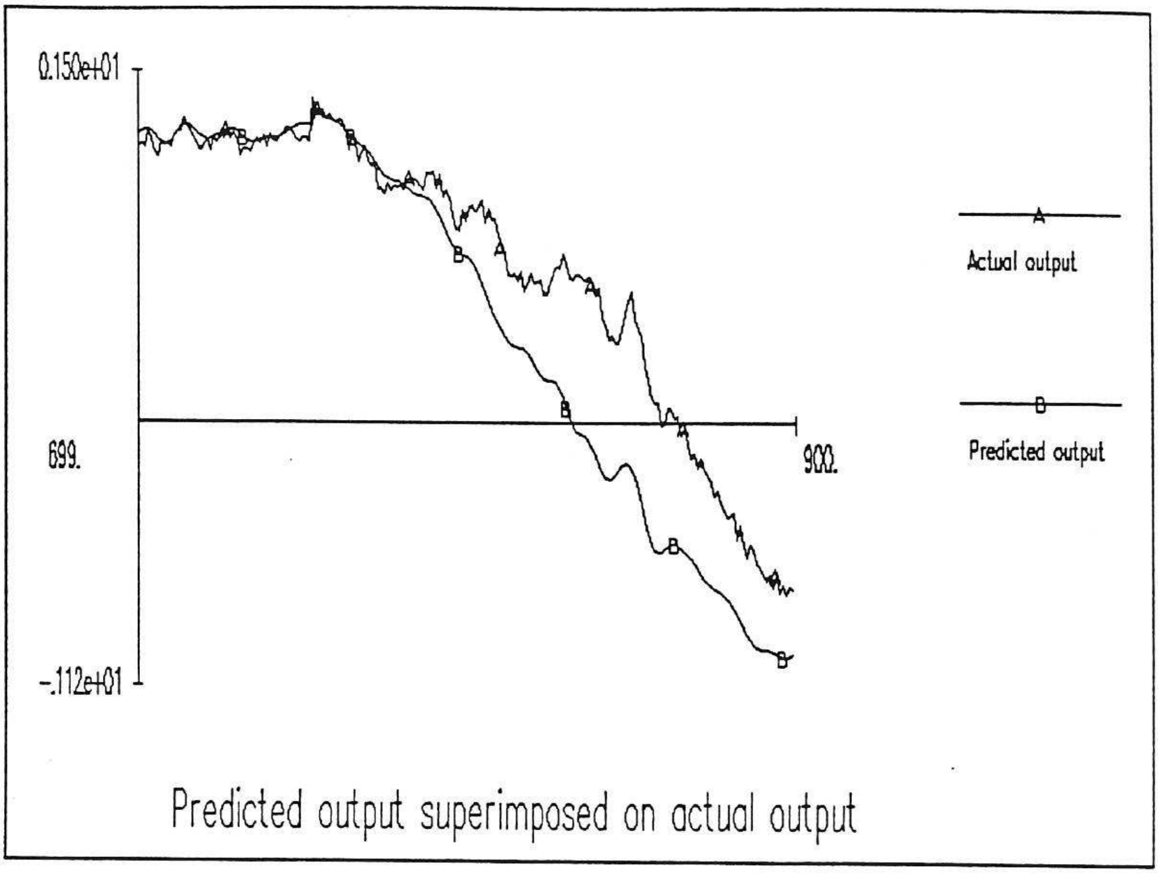
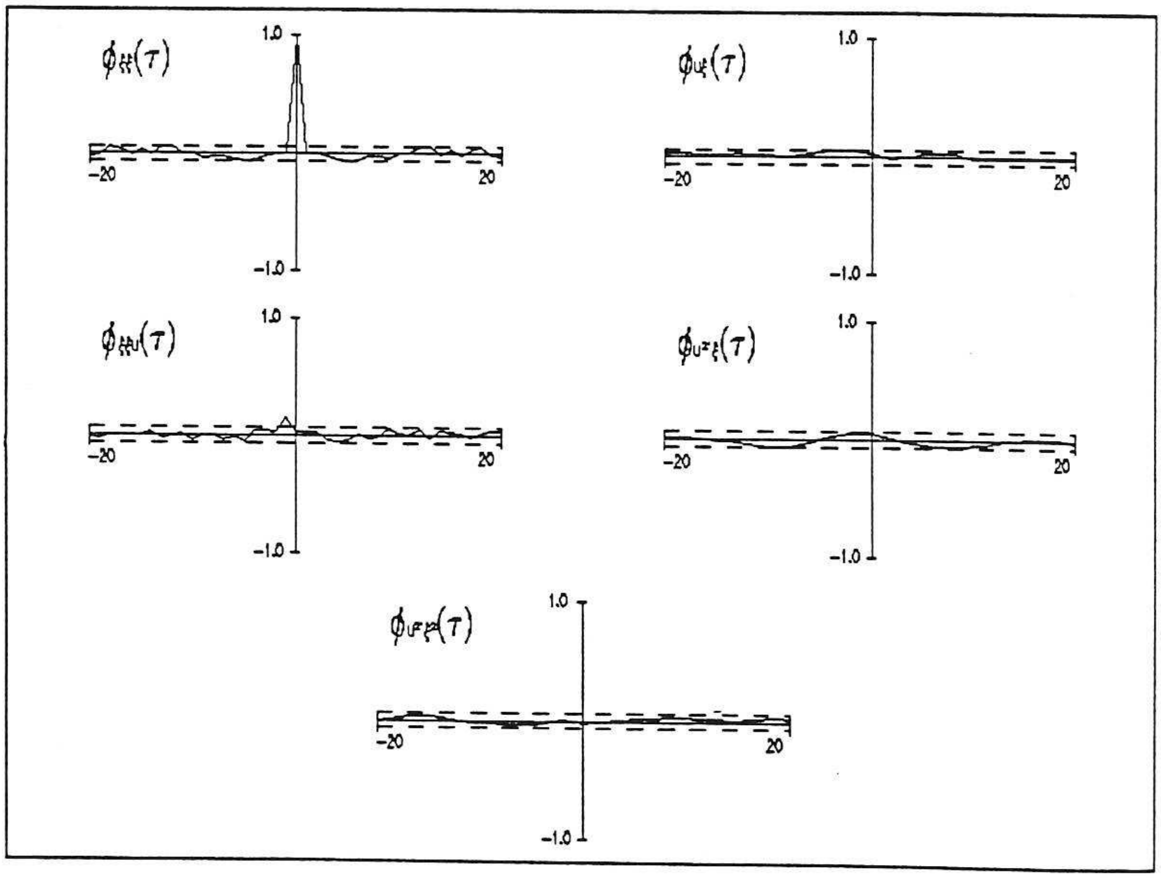


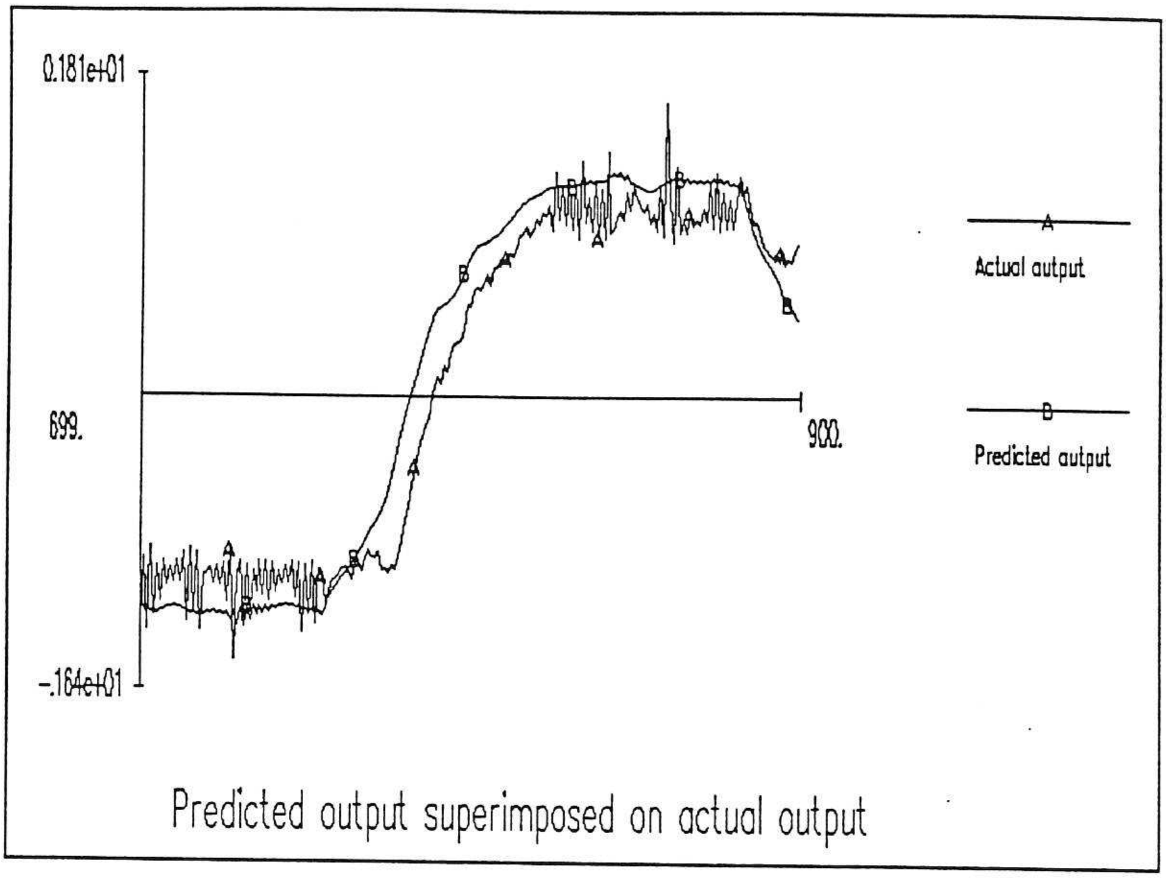
Fig. 5c **I/O data** - $T = 0.3$

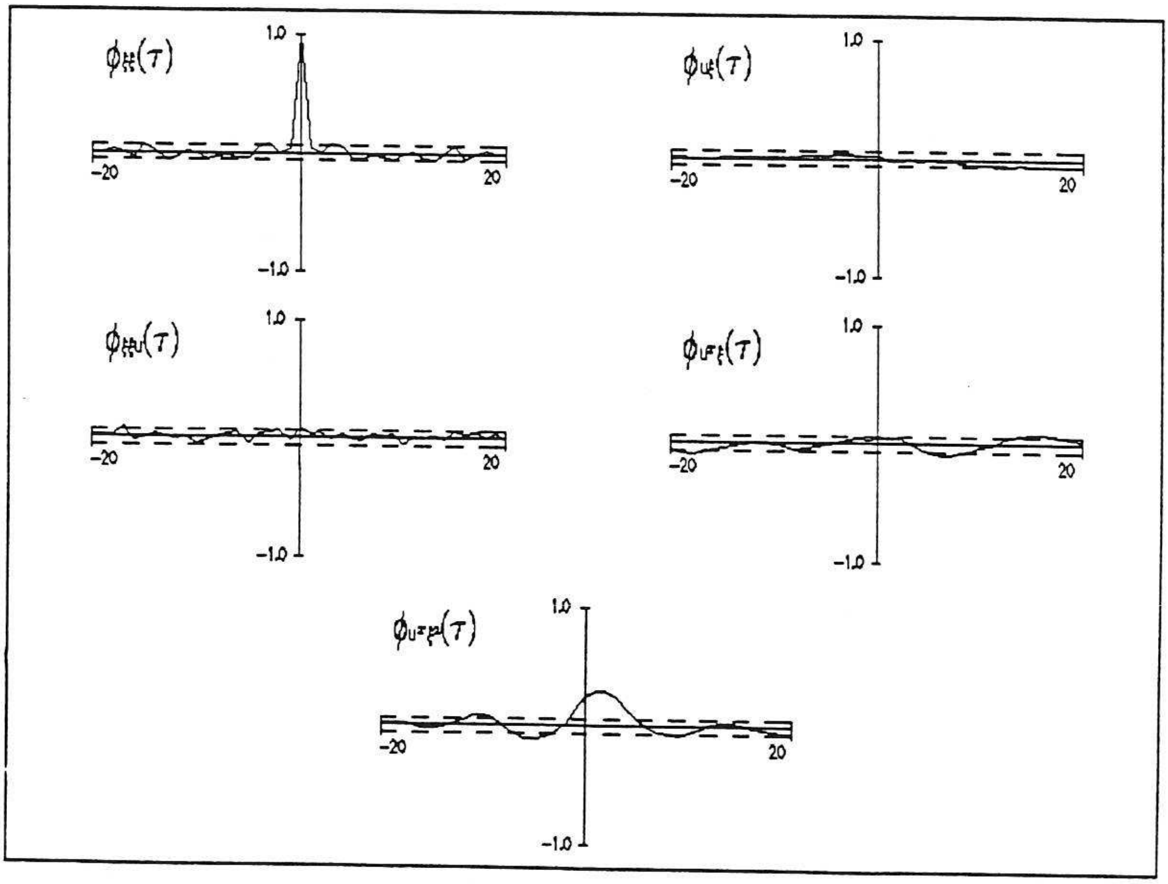




Linear and Nonlinear Model Validity Correlation Tests

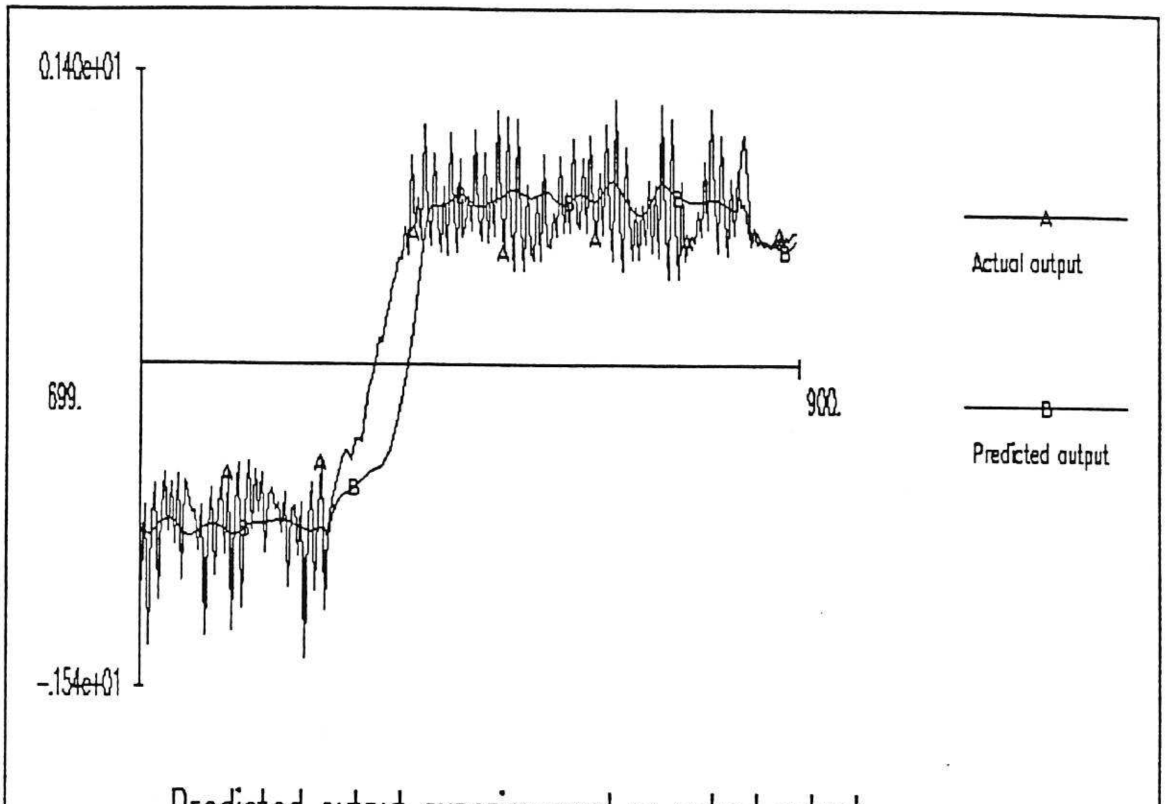
Fig. 6a **PRED** data and **MVT** test - $T = 0.1$



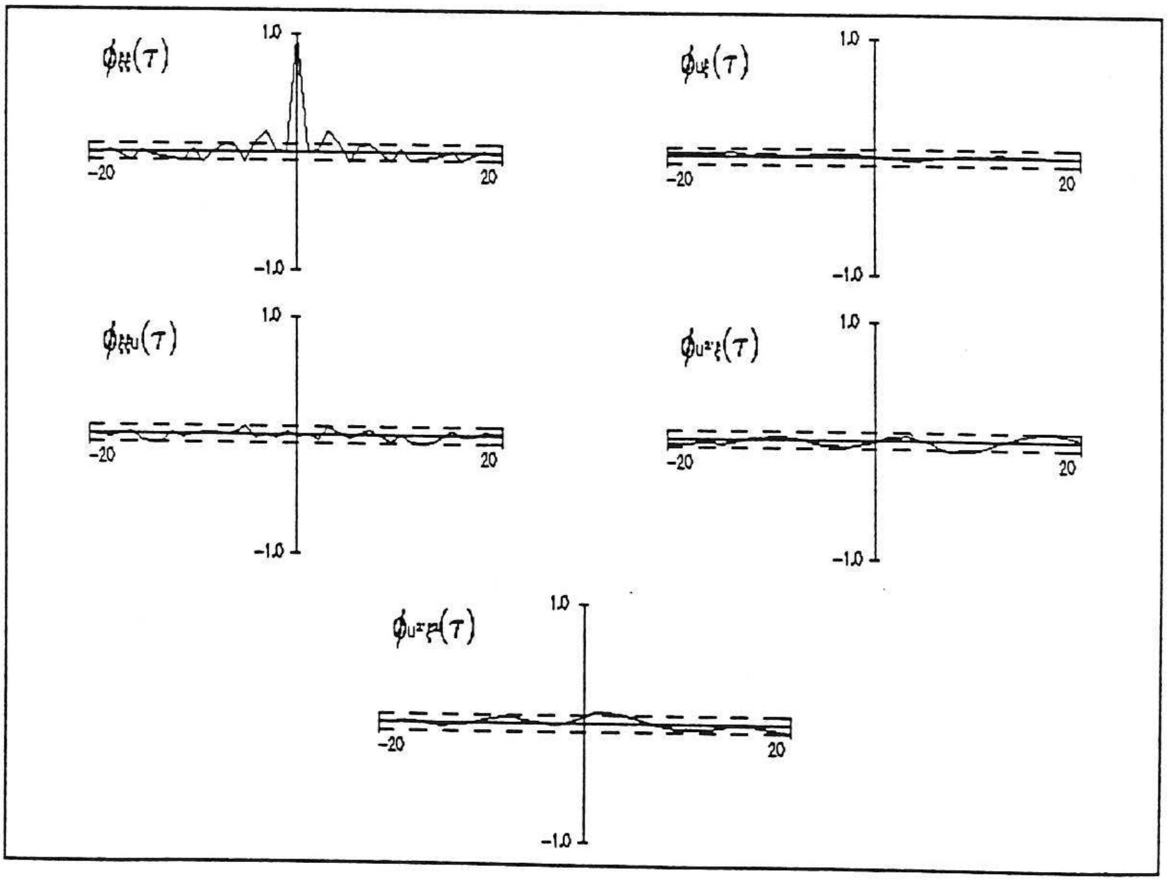


Linear and Nonlinear Model Validity Correlation Tests

Fig. 6b **PRED** data and **MVT** test - $T = 0.2$



Predicted output superimposed on actual output



Linear and Nonlinear Model Validity Correlation Tests

Fig. 6c **PRED** data and **MVT** test - $T = 0.3$

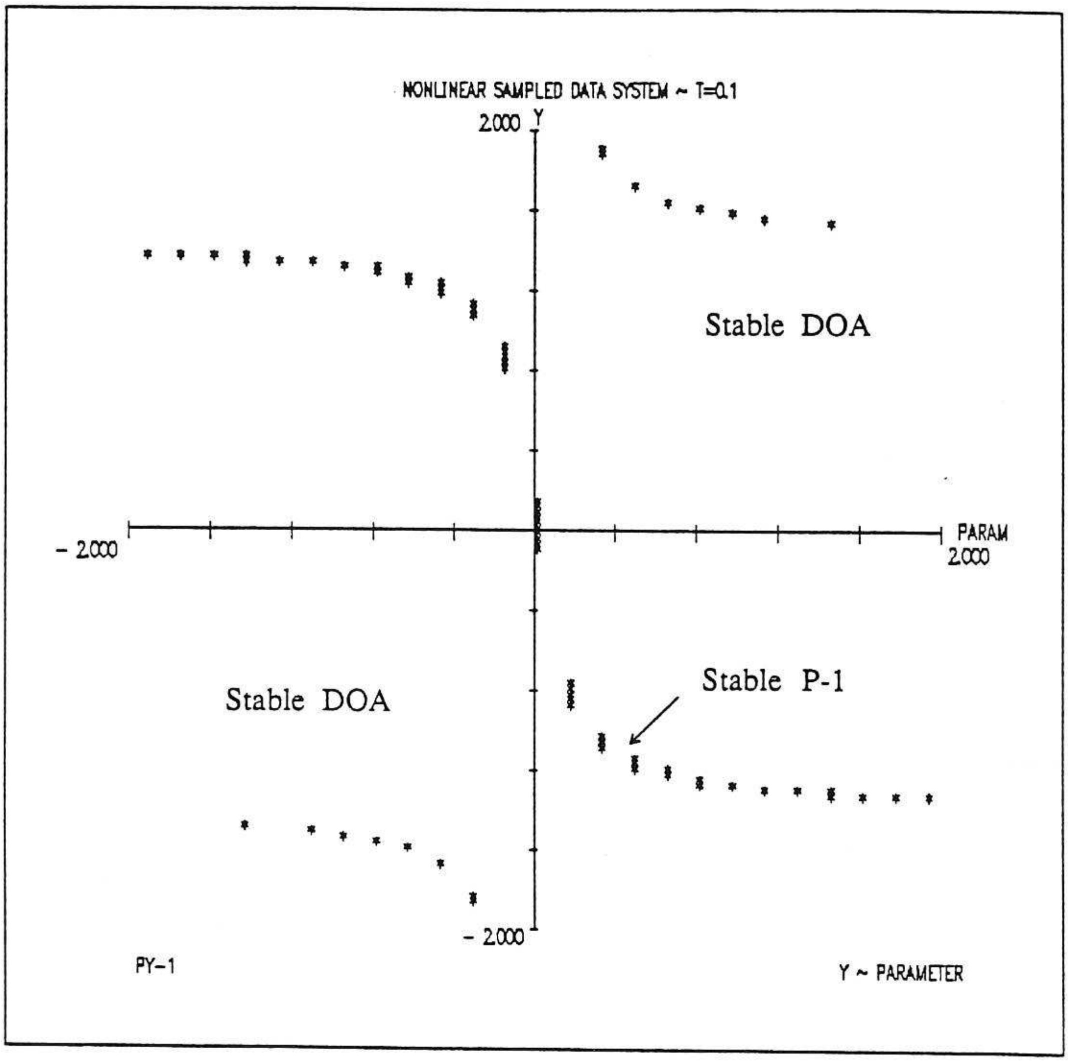


Fig. 7a CELL Diagram - $\mu = u = r$ for $T = 0.1$

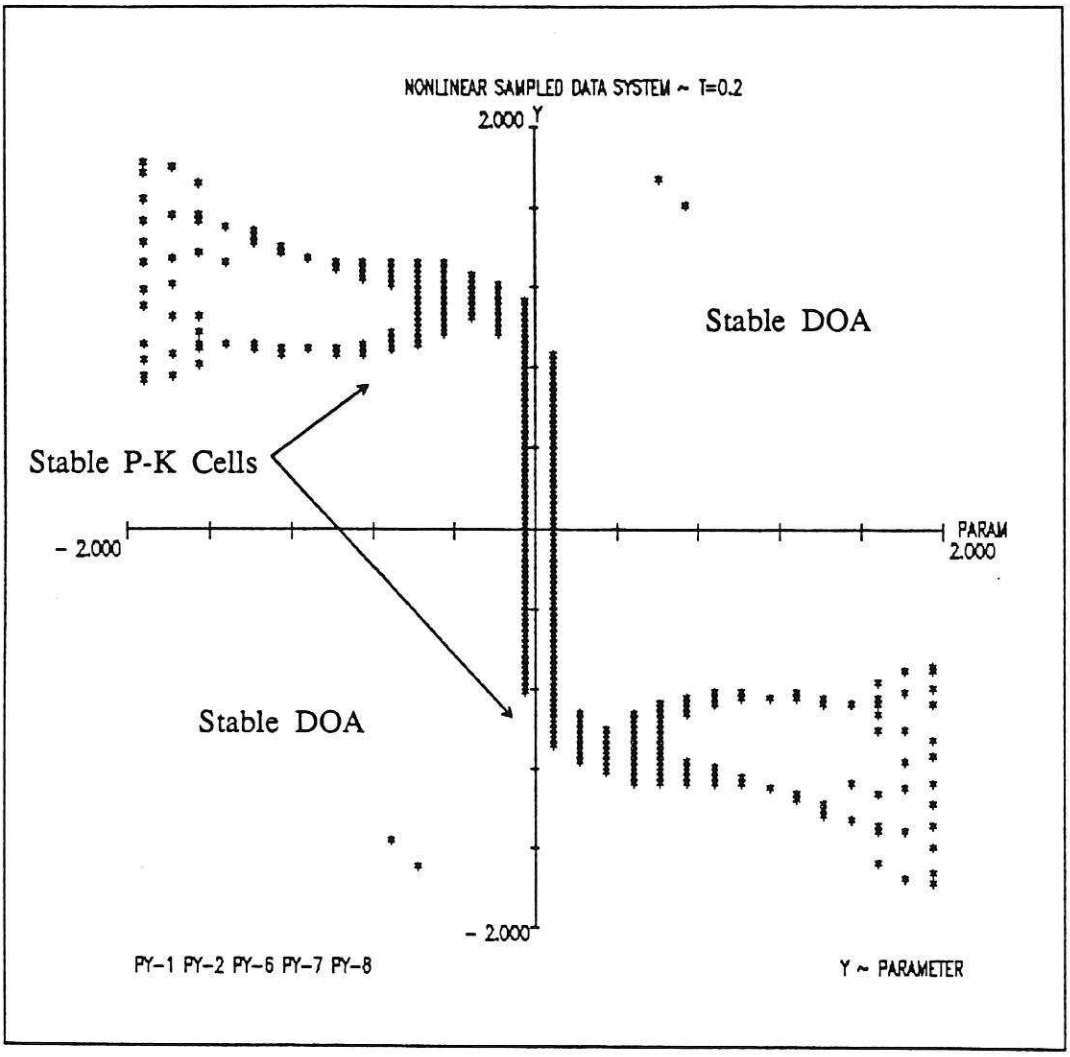


Fig. 7b CELL Diagram - $\mu = u = r$ for $T = 0.2$

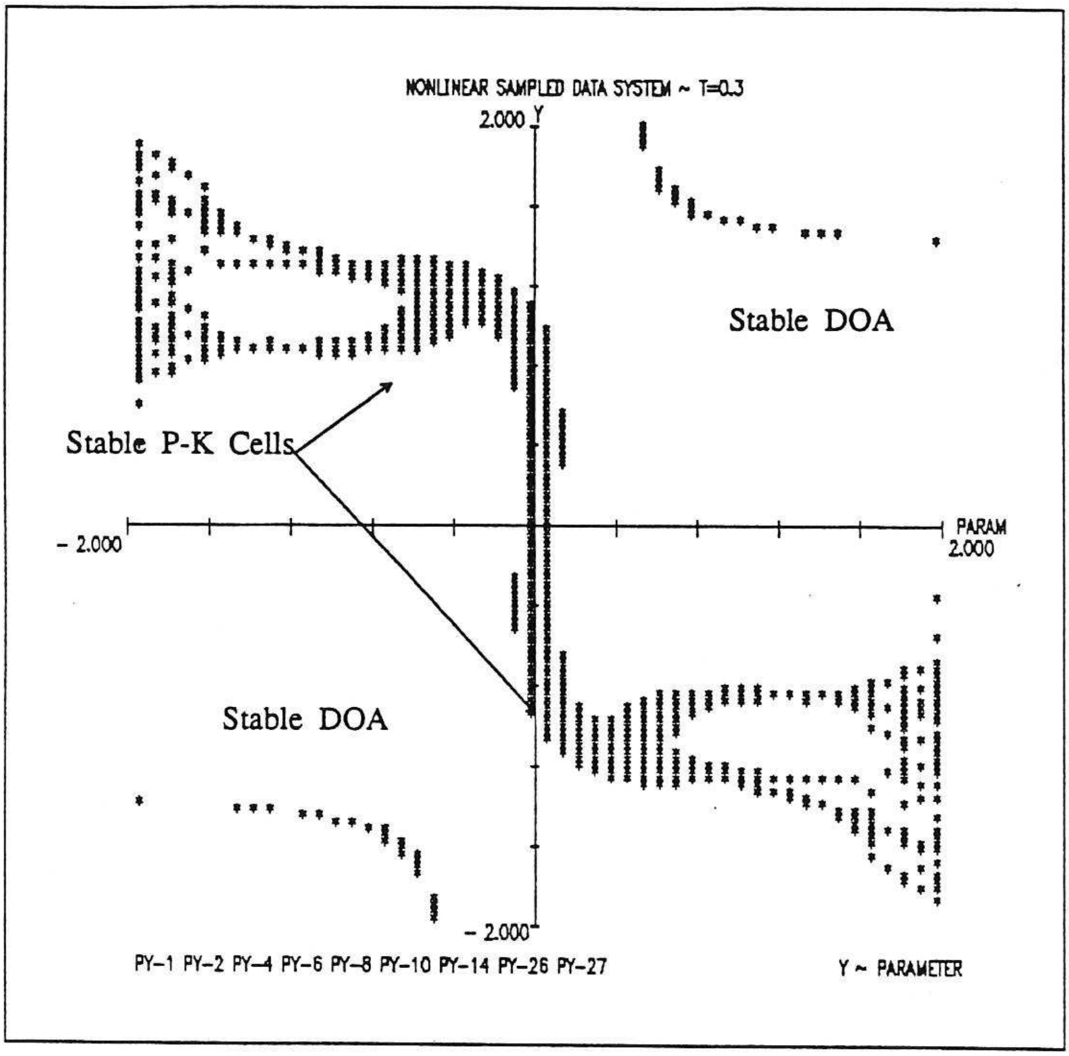


Fig. 7c CELL Diagram - $\mu = u = r$ for $T = 0.3$